Morse Theory and Optimal Control Problems

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1 Introduction

It is well known that *Morse Theory* is a very flexible tool for dealing with nonlinear problems of analysis and topological problems. The main purpose of the present paper is to describe a modification of this theory which can be used for the study of optimal control problems. The necessity of such a modification is related to the fact that for these problems the inequality constraints are typical (for example, control constraints, phase constraints, etc.) The inequalities destroy the smooth structure and hence the necessity to construct the theory for spaces with singularities. We encounter this situation in the case of optimal control problems.

However, note that the classical theory also can be used for the study of some optimal control problems. So we begin by pointing out this class of problems.

Let us remember that the basic facts of Morse theory are so called *Morse inequalities* which relate the topological characteristic of manifold M with the number of critical points of a smooth function $f : M \to R$ with definite index (or co-index).

Let $b_i(M)$ be the i-th Betti number of the Riemannian manifold M, $X(M)$ be the (homological) Euler characteristic of $M, f : M \to R$ be a *Morse function* on M (i.e. the smooth function with compact level sets $f^a = \{x \in M \mid f(x) \geq a\}$ which has only nondegenerate critical points), $c_i(f)$ be the number of critical points of this function of co-index i. Then for arbitrary $m = 0, 1, ..., \dim M$, the following inequalities hold:

$$\sum_{i=0}^{m}(-1)^{m-i}b_i(M) \leq \sum_{i=0}^{m}(-1)^{m-i}c_i(f),$$

$$b_i(M) \leq c_i(f), \quad i = 0, 1, ..., m, \tag{1.1}$$

$$X(M) = \sum_{i=0}^{\dim M}(-1)^i c_i(f).$$

Let us remember that the *index* (*co-index*) of the critical point x_o of the function f is the maximal dimension of subspaces in $T_{x_o}M$ where the *Hessian* $H_{x_o}(f)$ is negatively (positively) defined. Of course, inequalities (1.1)

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can be written in the usual "index" form if we replace the function f by $-f$.

Similar inequalities are valid if M is an infinite dimensional manifold and f a smooth function with only nondegenerate critical points satisfying condition (C) (*by Palais and Smale* see [6], [7]):

(C) if $\inf_{x\in S}\| \nabla f(x)\|_x = 0,$ then there exists the critical point p of the function f which belongs to the closure $\overline{S}$ of the set S .

Here $\|p\|_x = \sqrt{\langle p,p\rangle_x}$ and $\langle\cdot,\cdot\rangle_x$ is the Riemann structure on M. More precisely, let $\tilde{c}_i(f)$ be the number of critical points of f lying on the level set $f^a = \{x \in M \mid f(x) \geq a\}$ and $b_i(f^a)$ the i-th Betti number of the space f^a. Then, if f is bounded from above, satisfying the Palais-Smale condition (C) and having only nondegenerate critical points, then the following *Morse inequalities holds:*

$$\sum_{i=0}^{m}(-1)^{m-1}b_i(f^a) \leq \sum_{i=0}^{m}(-1)^{m-i}\tilde{c}_i(f), \qquad X(f^a) - \sum_{i}(-1)^i\tilde{c}_i(f)$$

(where $X(f^a)$ is the homological Euler characteristics of f^a). In these inequalities the number $\tilde{c}_i(f)$ denotes the number of critical points of f in f^a with finite co-index. In particular, for all $m = 0,1,2,....$

$$b_i(f^a) \leq \tilde{c}_i(f), \qquad i = 0,1,2,...,m$$

The last relations are valid even in the case when f is not bounded from above. In this case $\tilde{c}_i(f)$ and $b_i(f^a)$ may be infinite.

Now let us consider the smooth control system

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x), \qquad x \in M, u \in R^m, \tag{1.2}$$

on the smooth manifold M isometrically embedded into the Euclidean space R^d. The smooth vector fields $A, B_i, i = 1,...,m$, can be identified with α-dimensional vector functions. We propose that these functions satisfy the following *growth conditions*:

$$(x, A(x)) \leq k_1(1 + |x|^2), \qquad (x, B_i(x)) \leq k_2(1 + |x|^2), \qquad i = 1,...,m,$$

where $k_1, k_2 =$ const $\geq 0; (\cdot,\cdot)$ is the inner product in R^d and $|\cdot|$ is the corresponding Euclidean norm in R^d.

With the system (1.2) one can relate the input-output map $F_{x_o,T} : L_2^m[0,T] \to M$ assigning the right endpoint $x(T) = x(T; x_o, u(\cdot))$ of the trajectory $x(\cdot)$ of system (1.2) to an arbitrary admissible control $u(\cdot)$.

If for every $x_o \in M, T > 0$ the rank of this map is *constant* (i.e. the dimension of the image $\mathcal{I}_m F'_{x_o,T}(u)$ of the differential $F'_{x_o,T}(u)$ doesn't depend on $u(\cdot)$, then the system (1.2) is called a *system of constant rank.* This class of systems was introduced in [1] in the context of studying conditions for an extremal control to be bang-bang. The full theory on such systems can be found in [8].

These systems have a lot of remarkable properties, in particular the *reachable set*

$$\mathcal{R}_{x_0}(T) = \{x(T; x_0 u(\cdot)); u(\cdot) \in L_2^m[0,T]\}$$

is a smooth submanifold in M.

There exists sufficient conditions characterizing this class of systems. They can be described as follows. Let in some neighbourhood of each point $x_o \in M$ the condition of *finite definitness* hold, i.e. there exists an integer $s \geq 0$ such that

$$ad^{s+1}AB_i = \sum_{\alpha=0}^{s}\sum_{\beta=1}^{m} a_{\alpha\beta}^{si} ad^{\alpha} AB_{\beta}, \qquad i = 1, ..., m, \tag{1.3}$$

and the following *bang-bang condition* holds $i, j = 1, ..., m, k = 0, 1, ...,$

$$[B_i, ad^k AB_j] = \sum_{\alpha=0}^{k}\sum_{\beta=1}^{m} a_{\alpha\beta}^{ijk} ad^{\alpha} AB_{\beta} \tag{1.4}$$

with the functions $a_{\alpha\beta}^{si}, a_{\alpha\beta}^{ijk}$ being smooth in that neighborhood. Then the system (1.2) is of constant rank.

The condition of finite definiteness one can replace by the *real analycity condition.* The general necessary and sufficient conditions are described in [1], [2] [8].

The following proposition is true.

Proposition 1.1 *Let the system (1.2) be of constant rank and $x_o \in M, T > 0$ be given. Then for an arbitrary smooth submanifold $N \subset M$ that is transversal to the reachable set $\mathcal{R}_{x_0}(T)$ of this system, the set*

$$\mathcal{H} = F_{x_0,T}^{-1}(N) \subset L_2^m[0,T]$$

is the Hilbert submanifold in $L_2^m[0,T]$. If N is a closed submanifold in M then $\mathcal{H}$ can be equipped with the structure of a complete Riemannian manifold.

Let us now introduce the class of functionals

$$f(u) = \int_o^T \varphi(x, u)dt \tag{1.5}$$

defined on the set of trajectories of the system (1.2). These are smooth functions on $\mathcal{H}$ for which the condition (C) is valid.

Proposition 1.2 *Suppose that* $\varphi : R^d \times R^m \to R$ *is a smooth integrand for which the following conditions hold:*

1. $-\varphi(x, u) \geq k|u|^2, \quad k = \text{const} > 0;$
2. $|\varphi_x(x, u) - \varphi_x(y, u)| + |-\varphi_u(x, u) - \varphi_u(y, u)| \leq L|x - y|, L = \text{const} \geq 0;$
3. $|\varphi_x(x, u)| + |\varphi_u(x, u)| \leq b(x) + a|u|$, *where* $a = \text{const} \geq 0$ *and* $b(x) \leq 0$ *is bounded on bounded subsets of* R^d*;*
4. $-(\varphi_{uu}(x, u)\zeta, \zeta) \geq \mu(x)|\zeta|^2$, *where* $0 \leq \mu(x)$ *is bounded away from zero on compact subsets of* R^d.

Then the functional (1.5) is a smooth function on $\mathcal{H}$ *satisfying* (C) *condition by Palais and Smale.* [1]

We have no place to deal with the concrete examples of the theory. One can find these examples in [10].

It is necessary to note that for the infinite dimensional case the finite dimensional theory is also important. Namely, in the simplest case when the path space

$$\mathcal{P}_y = \{u(\cdot) \in L_2^m[0, T] \mid x(T; x_0 u(\cdot)) = y\}$$

is contractable for every $y \in N$ the homology groups of $\mathcal{H}$ and $N \cap \mathcal{R}_{x_0}(T)$ conincide (and for its calculation the finite dimensional theory is used.) In the general case the homology groups of $\mathcal{H}$ can be calculated by means of spectral sequences if the homology groups of $N \bigcap \mathcal{R}_{x_0}(T)$ and $\mathcal{P}_y$ are known.

Taking into account the previous remark, we first construct the Morse theory for the spaces with singularities in the finite dimensional case.

[1] Note that the (C) condition by Palais and Smale is valid for the function f iff it is valid for the function $-f$. The sign "-" in the conditions 1), 4) of proposition 1.2 appears since we consider the "co-index" form of the Morse inequalities.

2 Morse theory for manifolds with corners (the finite dimensional case)

Manifolds with corners are objects which can be described locally in terms of a system of smooth nonlinear inequalities.

Let M be a compact finite dimensional smooth Riemannian manifold. The closed subset $V \subset M$ is called a *submanifold with corners* iff for every $x_0 \in V$ there exists a coordinate chart (O_{x_0}, φ) of M and a convex polyhedral cone $K_{x_0} \subset R^n, n = \dim M$ with vertex at the origin such that $\varphi(x_0) = 0$ and

$$\varphi(O_{x_o} \cap V) = K_{x_o}.$$

We can define the notion of *tangent cone* T_xV to V at the point $x \in V$ in the usual way. This cone is isomorphic to the cone K_x and the set-valued map $x \mapsto T_xV : M \longrightarrow 2^{TM}$ is lower semicontinuous. Moreover, there exists a complete integrable distribution $\prod_x, x \in M$, of constant dimension such that

$$\prod\nolimits_x = \widehat{T_xV} \stackrel{\text{def}}{=} T_xV - T_xV \qquad \forall x \in V.$$

Hence there exists a smooth manifold of *minimal* dimension that contains the submanifold V with corners. If $T_xM = \widehat{T_xV}$ for all $x \in V$ then we say that submanifold V with corners is *solid.* In the following we consider only solid submanifolds with corners.

The maximal (with respect to inclusion) smooth submanifold $\Gamma \subset M$ is called the *open face* of submanifold V with corners iff $\Gamma \subset V$. Every point of submanifold V with corners in a unique open face of maximal dimension. Moreover, the family $\{\Gamma\}$ of all open faces defines on V the structure of *stratified space* (in the sense of Whitney).

Let $f : V \longrightarrow R$ be the *smooth function* on submanifold V with corners (i.e. f is the restriction to V of some smooth function $\tilde{f} : M \longrightarrow R$.) The point $x_o \in V$ is called the *critical point* of this function if the differential, $d_{x_o}f$, of function f at point x_o belongs to the *negative polar cone* $T^*_{x_o}V$ to the tangent cone $T_{x_o}V$ at the same point, i.e. the following relation holds:

$$d_{x_0}f \in T^*_{x_0}V \stackrel{\text{def}}{=} \{\omega \in T^*_{x_0}M \mid \langle \omega, X_{x_0} \rangle \leq 0 \quad \forall X_{x_0} \in T_{x_0}V\}.$$

The critical point x_0 is called a *nondegenerate critical point* iff

(i) $d_{x_0}f \in$ rel int $T^*_{x_0}V$;

(ii) the point x_0 is a nondegenerate critical point (in the usual sense) for the restriction $f|_\Gamma$ of function f to the open face Γ of V, having maximal dimension and containing this point in its relative interior.

As in classical theory the nondegenerate critical points are isolated.

The real number $c \in R$ is called the *critical value* of the function $f : V \rightarrow R$ iff the set $f^{-1}(C) = \{x \in V \mid f(x) = C\}$ contains the critical points. All other values are called *regular.*

In the following by *Morse function* we mean smooth functions which have only nondegerate critical points.

Now, our goal is to study the change of homotopic type of the level sets $f^a = \{x \in V \mid f(x) \geq a\}$ when the parameter $a \in R$ varies. For that purpose we need the important notion of *transversality* which generalizes the corresponding notion of smooth analysis to the case considered.

Let N be another smooth manifold, $W \subset N$ be the submanifold with corners in N and the smooth map $g : V \rightarrow N$ is given. We call this map *transversal* to W iff for all $x \in V, g(x) \in W$, the following condition holds:

$$-g_{*x}T_xV + T_{g(x)}W = T_{g(x)}N.$$

Here g_{*x} is the *differential* (or *tangent map*) of $g : V \rightarrow N$. The following *transversality theorem* is true:

Theorem 2.1 *Let $f_t, t \in R$, be the family of smooth maps from V to N smoothly depending on parameter $t \in R$, and each of these maps transversal to submanifold $W \subset N$ with corners. Then for arbitrary $t', t'' \in R$ the sets $f_{t'}^{-1}(W)$ and $f_{t''}^{-1}(W)$ are of the same homotopic type.*

In fact, we can state the strong conclusion: there exist smooth maps

$$F : f_{t'}^{-1}(W) \rightarrow f_{t''}^{-1}(W), \quad G : f_{t''}^{-1}(W) \rightarrow f_{t'}^{-1}(W)$$

(not "onto") such that the maps $F \circ G$ and $G \circ F$ are *smoothly isotopic to the identity.*[2]

Let us give a sketch of the proof of this theorem. We can assume that $t' = 0, t'' = 0$. First we show that there exists a flow $P_t, t \in R$, on manifold M such that $P_t f_o^{-1}(W) \subset f_t^{-1}(W)$ for all $t, 0 \leq t \leq 1$, i.e. the condition $f_o(x) \in W$ implies the relation $f_t(P_t(x)) \in W, \quad 0 \leq t \leq 1$. For an arbitrary flow $P_t, t \in R$, on M we have the relation

$$\frac{d}{dt}f_t(P_t(x)) = \frac{\partial}{\partial t}f_t(P_t(x)) + f_{t*P_t(x)}X_t(x),$$

where $X_t, t \in R$, is the (nonstationary) smooth vector field on M which generates this flow. Since $\frac{\partial}{\partial t}f_t(P_t(x)) \in T_{f_t(P_t(x))}N$, we can use the transversality condition, and find nonstationary smooth vector field $X_t, t \in R$, on M such that the corresponding flow $P_t, t \in R$, verifies the condition

[2] i.e. there exist the extensions $\tilde{F}$ and $\tilde{G}$ on M of the maps F and G such that $\tilde{F} \circ \tilde{G}$ and $\tilde{G} \circ \tilde{F}$ are smoothly isotopic to identity id_M. Let us remember that the smooth map $h : M \rightarrow M$ is *smoothly isotopic to identity* iff there exists the smooth map $H : [0,1] \times M \rightarrow M$ such that $H_t = H(t, \cdot)$ is a diffeomorfism for all $t, 0 \leq t \leq 1$, and $H_o = id_M, H_1 = h$.

$$\frac{d}{dt}f_t(P_t(x)) \in T_{f_t(P_t(x))}W \quad \forall t, 0 \leq t \leq 1.$$

Hence, using the condition $f_o(x) \in W$, we have $f_t(P_t(x)) \in W \quad \forall t, 0 \leq t \leq 1$. Replacing the family $f_t, t \in R$, by the family $f_{1-t}, t \in R$, and repeating the previous reasoning, we find the flow $Q_t, t \in R$, on M such that

$$Q_t f_1^{-1}(W) \subset f_{1-t}^{-1}(W), \quad \forall t, 0 \leq t \leq 1$$

Now, putting $F = P_1$ and $G = Q_1$ we finished the proof.

From Theorem 2.1 we immediately receive the following result: If on the interval $[a,b]$ there are no critical values of the function f, then the sets f^a and f^b are of the same homotopic type.

For the proof let us consider the family $f_t = f - t, t \in R$, and the submanifold with corners $W = R_t \stackrel{\text{def}}{=} \{s \mid s \geq 0\} \subset R$. Since in $[a,b]$ there are no critical values of f, then this family is transversal to W. So, using the theorem 2.1, we have the sets $f_a^{-1}(R_+) = f^a, f_b^{-1}(R_+) = f^b$ are of the same homotopic type.

It is necessary to emphasize that Theorem 2.1 is true for such closed subsets $V \subset M$ which have (convex) tangent cones T_xV with only two properties: (1) the set-valued map $x \mapsto T_xV : M \rightarrow 2^{TM}$ is lower semi-continuous; (2) the planes $\widehat{T_xV} \stackrel{\text{def}}{=} T_xV - T_xV, x \in V$, are of constant dimensions. This remark permits a more general definition of submanifold with corners to be given. Namely, call the closed subset $V \subset M$ as *submanifold with corners* iff for every $x \in V$ there exists the coordinate chart (O_x, ψ) of manifold M, the convex polyhedral cone $K_x \subset R^m$ with vertex at the origin and the smooth map $F : R^n \rightarrow R^m, \quad n = \dim M$, that preserves the origin and is transversal to K_x such that $\psi(x) = 0$ and $\psi(O_x \cap V) = F^{-1}(K_x)$

With this definition the class of submanifolds with corners becomes *closed* under operation of *transversal intersection*. In the present paper we do not consider these objects. The theory of "general" submanifolds with corners, the smooth control systems on such submanifolds. Morse theory and other topics will appear in a forthcoming paper. Here we only note that a great deal of the results in geometrical control theory are also true for the smooth control systems on such manifolds with corners.

Now let us consider the case when interval $[a,b]$ contains the critical values of the smooth function $f : V \rightarrow R$. Changing the homotopic type of the level sets as in classical theory can be described in terms of co-indexes (positive indexes) of critical points which correspond to those critical values.

In the present context the *co-index of critical point* $x_o \in V$ is defined as co-index of x_o for the restriction $f|_\Gamma$ of function f to the open face $\Gamma \subset V$ of maximal dimension which contains x_o in its relative interior.

The following theorem is true.

Theorem 2.2 *Let f be the Morse function on submanifold V with corners and $c, a < c < b$, the unique critical value of this function on $[a, b]; x_1, ..., x_k$ the correspondent critical points and $r_1, ..., r_k$ the co-indexes of these points. Then the level set f^a have the homotopic type of the set f^b with the* cells *$D^{r_1}, ..., D^{r_x}$ have been attached. In particular, V has a* cell complex structure.

This theorem can be proved using the *stratified Morse theory* [4]. However the authors have proved this theorem without using this theory.

From Theorem 2.2 one can derive the Morse inequalities following Milnor approach [5]. However for deriving the Morse inequality it is sufficient to only have some variant of *Morse lemma* (see [10]). Let us formulate the *"homotopic"* variant of this lemma.

Let f be the smooth function on convex polyhedral cone $K \subset R^n$ with vertex at origin $f(0) = 0$ and zero the nondegenerate critical points of this function. Put $P = K \cap (-K)$ and let Q be the orthogonal complement of $P : P \oplus Q = R^n$. Let us represent the point $x \in R^n$ in the form $x = (p, q), p \in P, q \in Q$, and define the function $\hat{f} : k \to R$ by the formula

$$\hat{f}(p, q) = \frac{\partial f(0,0)}{\partial q} q + \frac{\partial^2 f(0,0)}{\partial q \partial p}(p, p).$$

Theorem 2.3 *There exists a neighbourhood U of origin in R^n such that the sets $\{(p, q) \in K \cap U \mid \hat{f}(p, q) \geq 0\}$ and $\{(p, q) \in K \cap U \mid f(p, q) \geq 0\}$ are of the same homotopic type.*

Theorem 2.3 can be proved by means of the transversality Theorem 2.1. From Theorem 2.3 (or Theorem 2.2) follows the *main result* of the present paper:

Theorem 2.4 *Let f be the Morse function on submanifold V with corners. Then inequalities (1.1) hold with M replaced by V.*

3 Palais-Smale theory for manifolds with corners

The following abstract theory can be applied in the study of optimal control problems with inequality constraints on the right end point of a trajectory.

Let the finite dimensional submanifold $V \subset M$ with corners, Hilbert manifold $\mathcal{H}$ and a surjective submersion $F : \mathcal{M} \to M$ be given. The inverse image $\mathcal{V} = P^{-1}(V)$ is called *a Hilbert submanifold with corners.*

Since locally V is a convex polyhedral cone, the Hilbert submanifold $\mathcal{V}$ with corners can be represented as a direct product of convex polyhedral cone and a Hilbert space. Hence, the elementary geometry of Hilbert submanifold with corners is similar to those of finite dimensional submanifolds V with corners which modelled $\mathcal{V}$. In particular, the tangent cone $T_x\mathcal{V}$ to Hilbert submanifold $\mathcal{V}$ with corners have the following properties: (1) the set-valued map $x \mapsto T_x\mathcal{V} : \mathcal{M} \to 2^{T^{\mathcal{M}}}$ is lower semicontinuous; (2) the space $\widehat{T_x\mathcal{V}} \stackrel{\text{def}}{=} T_x\mathcal{V} - T_x\mathcal{V}, x \in \mathcal{V}$, are of the constant co-dimension in $T_x\mathcal{M}$. Hilbert submanifolds with corners are also stratified spaces in the Whitney sense: this stratification is given by means of open faces. The *open face* of Hilbert submanifold $\mathcal{V}$ with corners is the maximal (with respect to inclusion) submanifold $\mathcal{J} \subset \mathcal{M}$ such that $\mathcal{J} \subset \mathcal{V}$. Of course, the open faces of $\mathcal{V}$ can be described in terms of open faces of the finite dimensional submanifold V with corners, which modelled $\mathcal{V}$. Every point $x \in \mathcal{V}$ is contained in a unique open face $\mathcal{J}$ which has the minimal co-dimension in $\mathcal{M}$.

In the following we assume that manifold $\mathcal{M}$ is *complete Riemannian manifold.* By $\langle\cdot,\cdot\rangle_x$ we denote the *Riemannian structure* on $\mathcal{M}$ and set $||p||_x = \sqrt{\langle p,p\rangle_x}, p \in T_x\mathcal{M}$.

The following analog of the (C) condition by Palais and Smale is an important consideration. To formulate this condition let us introduce the function $m : \mathcal{V} \to R_+$ by the formula:

$$m(x) = \sup_{\substack{X_x \in T_x\mathcal{V} \\ ||X_x||_x \leq 1}} \langle \nabla f(x), X_x \rangle x.$$

Generalized (C) Condition: if $S \subset \mathcal{V}$ is the subset on which the function $f : \mathcal{V} \to R$ is bounded and

$$\inf_{x \in S} m(x) = 0,$$

then there exists the critical point of the function f which belongs to $\bar{S}$, the closure in $\mathcal{M}$ the subset S.
Note that if $\mathcal{V} = \mathcal{M}$ we have the usual (C) condition by Palais and Smale.

For smooth function $f : \mathcal{V} \to R$ on Hilbert submanifold $\mathcal{V}$ with corners as in finite dimensional case one can define the notions of *critical points, nodegenerate critical points, its index and co-index, critical values* , etc. For example, the point $x_o \in \mathcal{V}$ is called the *critical point* of function $f : \mathcal{V} \to R$ iff the differential $d_{x_o}f$ of this function at point x_o satisfies the condition:

$$d_{x_o}f \in T^*_{x_o}\mathcal{V} \stackrel{\text{def}}{=} \{\omega \in T_{x_o}\mathcal{M} \mid \langle \omega, X_{x_o}\rangle \leq 0 \forall X_{x_o} \in T_{x_o}\mathcal{V}\}.$$

By *Morse function* we now mean the smooth function which has only nondegenerate critical points and satisfies the generalized (C) condition.

The following theorem is true:

Theorem 3.1 *Let $\mathcal{V}$ be the connected Hilbert submanifold with corners in complete Riemannian manifold $\mathcal{M}$, and $f : \mathcal{V} \to R$ be bounded from above Morse function on $\mathcal{V}$. Then f achieves on $\mathcal{V}$ its maximal values.*

Theorem 3.2 *If $f : \mathcal{V} \to R$ is bounded from above Morse function on Hilbert submanifold $\mathcal{V}$ with corners then the following* Morse inequalities *are valid: $\forall m = 0, 1, 2, ..., a \in R$,*

$$\sum_{i=0}^{m}(-1)^{m-i}b_i(f^a) \leq \sum_{i=0}^{m}(-1)^{m-i}\tilde{c}_i(f) \qquad \chi(f^a) = \sum_{i}(-1)^i\tilde{c}_i(f).$$

In particular for all $m = 0, 1, ...$

$$b_m(f^a) \leq \tilde{c}_m(f).$$

Here $\tilde{c}_m(f)$ is the number of critical points of the function f lying on the level set f^a and having *finite* co-index i. The full proof of these statements can be found in [10].

Now let us consider the control system (1.2).

Proposition 3.1 *Let (1.2) be the control system of constant rank and $V \subset M$ is the submanifold with corners such that $V \subset \mathcal{R}_{x_0}(T)$. Then the set $\mathcal{V} = F^{-1}_{x_o,T}(V)$ is the Hilbert submanifold with corners in $L_2^m[0,T]$.*

Proposition 3.2 *Let the conditions of Proposition 1.2 be valid. Then the functional (1.5) is a smooth function on $\mathcal{V}$ which satisfies the generalized (C) condition.*

Here $F_{x_o,T}$ is the input-output map of system (1.2).

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