The 37th IEEE Conference on Decision and Control, Tampa, Florida, December 1998.

On the Equivalence of Different Types of Local Minima in Sub-Riemannian Problems

Andrei A. Agrachev
Steklov Mathematical Institute, 8 ul. Gubkina, Moscow 117966
E-mail: agrachev@mi.ras.ru

Abstract—**Sub-Riemannian problems are under intensive study in 90-th. They are typical optimal control problems admitting singular and abnormal minimizers (see [1 – 5]). Moreover, any singular geodesic in the sub-Riemannian problem is abnormal and vice versa.**

So minimizers may be singular geodesics, but it is not clear for today, may they have singularities as curves in the state space or not. Until now, all known minimizers were smooth. In this paper we compare different types of local minimality for smooth admissible curves. Surprisingly, the smoothness of the trajectory implies the equivalence of local minimality in rather different topologies.

Index terms—**Optimal control, sub-Riemannian geometry, strong minimum.**

Let M be a smooth manifold endowed with a sub-Riemannian structure, $\dim M = n$. We suppose that the sub-Riemannian structure is extended to a Riemannian structure g on M: $g(v_1, v_2)$ is the scalar product and $|v_i|_g = \sqrt{g(v_i, v_i)}$ is the length of $v_i \in T_qM$, $i = 1, 2$, $q \in M$.

The sub-Riemannian structure is thus the restriction of g to a subbundle $\Delta \subset TM$. Recall that a Lipschitzian curve $\zeta : [0,1] \to M$ is called *admissible* if $\dot\zeta \in \Delta$ for almost all $t \in [0,1]$. Let $\gamma : [0,1] \to M$ be a smooth curve and

$$\dot\gamma(t) \in \Delta, \ |\dot\gamma(t)|_g = l > 0, \quad \forall t \in [0,1].$$

Suppose $V_\gamma \subset M$ is an open neighborhood of the subset $\{\gamma(t) : t \in [0,1]\}$ of M and $\Phi : V_\gamma \to R^m$ is an imbedding of V_γ in R^m.

The curve γ is an *H^1-local length minimizer* if there exists $\varepsilon > 0$ such that $l \le \int_0^1 |\dot\zeta(t)|_g \, dt$ for any admissible curve $\zeta : [0,1] \to V_\gamma$ satisfying the boundary conditions $\zeta(0) = \gamma(0), \zeta(1) = \gamma(1)$ and the inequality

$$\int_0^1 |\Phi_*\dot\zeta(t) - \Phi_*\dot\gamma(t)|^2 \, dt < \varepsilon. \tag{1}$$

It is easy to check that given definition of the H^1-local length minimizers doesn't depend on the choice of V_γ and Φ. Moreover, we get an equivalent definition if replace $|\Phi_*\dot\zeta(t) - \Phi_*\dot\gamma(t)|_g^2$ by $|\Phi_*\dot\zeta(t) - \Phi_*\dot\gamma(t)|_g^p$ in the above inequality, where p is an arbitrary real number greater or equal to 1. We obtain the definition of a *C^0-local length minimizer* if replace (1) by the inequality

$$\sup_{0\le t\le 1} |\Phi(\zeta(t)) - \Phi(\gamma(t))| < \varepsilon. \tag{2}$$

Of course, any C^0-local length minimizer is automatically a H^1-local length minimizer. The opposite implication is not obvious since it is easier to satisfy inequality (2) than (1). We'll however prove that these two kinds of local minima are equivalent.

Theorem 1 *If γ is an H^1-local (strict) length minimizer, then it is a C^0-local (strict) length minimizer.*

Proof Let $\exp : \mathcal{U} \to M$ be the exponential mapping associated to the Riemannian structure g; here $\mathcal{U} \subset TM$ is an open neighborhood of the zero section of the vector bundle $TM \to M$. Suppose M is imbedded in TM as the zero section. We have

$$\exp(q) = q, \ \exp_* v = v, \quad \forall q \in M, v \in T_q(T_qM) = T_qM.$$

Let γ be an H^1-local (strict) length minimizer. Take a regular smooth curve $\bar\gamma : [-1,2] \to M$ such that $\bar\gamma|_{[0,1]} = \gamma$. Fix a smooth with respect to $t \in [-1,2]$ orthonormal frame $\frac{1}{l}\dot{\bar\gamma}, v_1(t), \ldots, v_{n-1}(t)$ in $T_{\bar\gamma(t)}M$. Consider an open $n-1$-dimensional Euclidean ball $B = \{y = (y_1, \ldots, y_{n-1}) : |y| < 1\}$. If $\delta > 0$ is small enough, then the mapping $f_\delta : (-1,2) \times B \to M$,

$$f_\delta(x,y) = \exp\left(\delta \sum_{i=1}^{n-1} y_i v_i(\bar\gamma(x))\right)$$

is well defined and satisfies the following:

- $\gamma(t) = f_\delta(t, 0)$.
- f_δ is a local diffeomorphism of $(-1,2) \times B$ onto an open neighborhood of $\{\gamma(t) : t \in [0,1]\}$ in M.
- The matrix function $A_\delta = \{\delta^{-2}(f_\delta^* g)(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})\}_{i,j=1}^{n-1}$ tends uniformly to the unit $(n-1) \times (n-1)$-matrix as $\delta \longrightarrow 0$.
- The vector-function $a_\delta = \{\delta^{-1}(f_\delta^* g)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y_i})\}_{i=1}^{n-1}$ tends uniformly to 0 as $\delta \longrightarrow 0$.
- The real function $(f_\delta^* g)(\frac{\partial}{\partial x}, \frac{\partial}{\partial x})$ tends uniformly to the constant l^2 as $\delta \longrightarrow 0$.

Set $\mathcal{A}_\delta$ the set of all $(\xi, \eta) : [0,1] \to (-1,2) \times (\delta B)$ such that $(\xi,\eta)(0) = (0,0)$, $(\xi,\eta)(1) = (1,0)$, and $t \mapsto f_\delta(\xi(t), \frac{1}{\delta}\eta(t))$ is an admissible curve. We have $\mathcal{A}_{\delta'} \subset \mathcal{A}_\delta$ if $\delta' < \delta$. The curve γ is a (strict) H^1-local length minimizer. Hence there exists $\varepsilon > 0$ such that for any δ and any $(\xi,\eta) \in \mathcal{A}_\delta$ the inequality

$$\int_0^1 (l^2|\dot\xi(t) - 1|^2 + |\dot\eta(t)|^2) \, dt < 2\varepsilon$$

implies that the length of the curve $t \mapsto f_\delta(\xi(t), \frac{1}{\delta}\eta(t))$, $t \in [0,1]$ is no less (greater) than l.

Now take $\bar{\delta} > 0$ such that

$$2\bar{\delta}^2 > \|A_{\bar{\delta}}\| > \frac{\bar{\delta}^2 l^2}{(l^2+\varepsilon)},$$

$$2l^2 > \left|(f_{\bar{\delta}}^* g)(\frac{\partial}{\partial x}, \frac{\partial}{\partial x})\right| > \frac{l^4}{(l^2+\varepsilon)}, \qquad (3)$$

$$|a_{\bar{\delta}}| \leq \frac{\bar{\delta}\varepsilon}{8l^3(l^2+\varepsilon)}.$$

Let $(\xi, \eta) \in \mathcal{A}_{\bar{\delta}}$ and

$$\int_0^1 (l^2|\dot{\xi}(t) - 1|^2 + |\dot{\eta}(t)|^2)\, dt \geq 2\varepsilon.$$

Set $\zeta(t) = f_\delta(\xi(t), \frac{1}{\delta}\eta(t))$. To complete the proof of the theorem it is enough to show that $\int_0^1 |\dot{\zeta}(t)|_g\, dt > l$. Without loss of generality we may assume that $|\dot{\zeta}(t)|_g \equiv \int_0^1 |\dot{\zeta}(t)|_g\, dt$; then $\left(\int_0^1 |\dot{\zeta}(t)|_g\, dt\right)^2 = \int_0^1 g(\dot{\zeta}(t), \dot{\zeta}(t))\, dt$. Suppose $|\dot{\zeta}(t)|_g \leq l$, then $|\dot{\xi}(t)| \leq \sqrt{2}l^2$, $|\dot{\eta}(t)| \leq 2l$ (see (3)) and we have

$$\int_0^1 g(\dot{\zeta}(t), \dot{\zeta}(t))\, dt =$$

$$\int_0^1 \left(\bar{\delta}^{-2}\langle A_{\bar{\delta}}\dot{\eta}(t), \dot{\eta}(t)\rangle + 2\bar{\delta}^{-1}\langle a_{\bar{\delta}}, \dot{\eta}(t)\rangle\dot{\xi}(t) + (f_{\bar{\delta}}^* g)(\dot{\xi}, \dot{\xi})\right) dt$$

$$> \frac{l^2}{l^2+\varepsilon}\int_0^1 (|\dot{\eta}(t)|^2 + l^2|\dot{\xi}(t)|^2 - \varepsilon)\, dt =$$

$$\frac{l^2}{l^2+\varepsilon}\left(\int_0^1 (|\dot{\eta}(t)|^2 + l^2|\dot{\xi}(t) - 1|^2)\, dt + l^2 - \varepsilon\right) \geq l^2.$$

Hence $|\dot{\zeta}(t)|_g > l$ and the theorem has been proved by contradiction.

The C^0-local minimizers are called *strong* minimizers in the classical calculus of variations. One obtains the definition of a *weak* minimizer replacing (2) by the inequality

$$\sup_{0\leq t\leq 1} |\Phi_*\zeta(t) - \Phi_*\gamma(t)| < \varepsilon.$$

The natural question is: could we strengthen the statement of Theorem 1 working with weak minimizers instead of H^1-local ones? The answer is negative. The following example of a weak but not a strong real-analytic sub-Riemannian minimizer was studied in [6, Append. B]. Let M be a neighbourhood of 0 in $R^3 = \{(x,y,z)\}$, Δ be the annihilator of the differential form

$$\omega = dz - x^6(y - x^2)^2 dx,$$

and

$$\|(\dot{x}, \dot{y}, \dot{z})\|_g^2 = \dot{x}^2 + \dot{y}^2 + \langle\omega, (\dot{x}, \dot{y}, \dot{z})\rangle^2.$$

Then the curve $t \mapsto (\varepsilon t - \varepsilon), \varepsilon^2(t-\varepsilon)^2, 0)$, $t \in [0,1]$, is a weak but not a strong minimizer (see [6]).

So, in general, weak and strong minimalities are not equivalent. It happens however that some natural regularity assumptions make them equivalent even for abnormal geodesics (see [3]).

The space $H^1([0,1]; M)$ of all (not necessary admissible) H^1-curves in M enjoys the standard structure of a smooth Banach manifold modelled on the Banach space $H^1([0,1]; R^n)$. A subset $\mathcal{U} \subset H^1([0,1]; M)$ is called a submanifold if any point $\xi \in \mathcal{U}$ belongs to a coordinate neighbourhood

$$\Psi : \mathcal{O}_\xi \to H^1([0,1]; R^n) \qquad (4)$$

such that $\Psi(\mathcal{U} \cap \mathcal{O}_\xi)$ is a relatively open convex subset of $H^1([0,1]; R^n)$. We say that a convex set is relatively open if it is open in its own linear hull. Note that the linear hull may be nonclosed in $H^1([0,1]; R^n)$; in particular, it may be everywhere dense in $H^1([0,1]; R^n)$. A subset $\mathcal{U}_0 \subset \mathcal{U}$ is called a submanifold of $\mathcal{U}$ if any point $\xi \in \mathcal{U}$ belongs to a coordinate neighbourhood (4) such that both $\Psi(\mathcal{U}_0 \cap \mathcal{O}_\xi)$ and $\Psi(\mathcal{U} \cap \mathcal{O}_\xi)$ are relatively open convex subsets of $H^1([0,1]; R^n)$.

For example, the space of C^k-curves in M, $1 \leq k \leq \infty$, is an everywhere dense submanifold of $H^1([0,1]; M)$. If M is real–analytic, then the space of all real–analytic curves in M is also an everywhere dense submanifold of $H^1([0,1]; M)$.

Let $\Omega(\Delta)$ be the space of admissible curves; this space is a submanifold of $H^1([0,1]; M)$. The space of smooth admissible curves is an everywhere dense submanifold of $\Omega(\Delta)$. If M and Δ are real–analytic, then the space of real–analytic admissible curves is also an everywhere dense submanifold of $\Omega(\Delta)$.

Suppose that $\mathcal{U}$ is a submanifold of $H^1([0,1]; M)$ and N is a smooth finfite-dimensional manifold. A mapping $F : \mathcal{U} \to N$ is called smooth if F is the restriction to $\mathcal{U}$ of a smooth mapping from $H^1([0,1]; M)$ to N. The mapping F is a submersion at $\xi \in \mathcal{U}$ if the differential of F at ξ is a surjective linear map. Any submersion is a locally open mapping. Let $\xi \in \mathcal{U}_0 \subset \mathcal{U}$, where $\mathcal{U}_0$ is an everywhere dense submanifold of $\mathcal{U}$. If $F : \mathcal{U} \to N$ is a submersion at ξ, then $F\big|_{\mathcal{U}_0}$ is also a submersion at ξ. Indeed, the image of the differential of $F\big|_{\mathcal{U}_0}$ at ξ is an everywhere dense subspace of the finite–dimensional space $T_{F(\xi)}N$. Hence this image coincides with $T_{F(\xi)}N$.

Proposition 1 *Suppose that $F : \mathcal{U} \to N$ is a submersion at $\xi \in \mathcal{U}$, $F(\xi) = q$. Then $F^{-1}(q) \cap \mathcal{U}_0 \neq 0$ for any everywhere dense submanifold $\mathcal{U}_0$ of $\mathcal{U}$.*

Proof. Let

$$\Psi : \mathcal{O}_\xi \to H^1([0,1]; R^n)$$

be local coordinates such that $\Psi(\mathcal{U}_0 \cap \mathcal{O}_\xi)$ and $\Psi(\mathcal{U} \cap \mathcal{O}_\xi)$ are relatively open convex subsets of $H^1([0,1]; R^n)$,

$\Psi(\xi) = 0$. Then there exists an open n-dimensional symplex $S \subset \Psi(\mathcal{U} \cap \mathcal{O}_\xi)$, $0 \in S$, such that $F \circ \Psi^{-1}\big|_S$ is a diffeomorphism. A small perturbation of the vertices of S puts the vertices into $\Psi(\mathcal{U}_0 \cap \mathcal{O}_\xi)$. The convexity property implies that the perturbed symplex S' is contained in $\Psi(\mathcal{U}_0 \cap \mathcal{O}_\xi)$. The mapping $F \circ \Psi^{-1}\big|_{S'}$ is still a diffeomorphism and the image of this diffeomorphism contains q. The details are left to the reader.

Let us consider the "boundary values" mapping

$$\partial : H^1([0,1]; M) \to M \times M, \quad \partial(\xi) = (\xi(0), \xi(1)).$$

This is, obviously, a smooth mapping and a submersion at every point. At the same time, $\partial\big|_{\Omega(\Delta)}$ is not a submersion at every point: critical points of $\partial\big|_{\Omega(\Delta)}$ are exactly the singular (abnormal) geodesics of the distribution Δ. The following impotant fact is a corollary of the classical "local controllability" techniques.

Theorem 2 *Suppose that Δ is a bracket generating distribution. Then the mapping $\partial\big|_{\Omega(\Delta)}$ is a submersion at any point of an open everywhere dense subset of $\partial^{-1}(q_0, q_1) \cap \Omega(\Delta)$, $\forall (q_0, q_1) \in M \times M$.*

Proof. Let $\xi \in \Omega(\Delta)$. Then there exists a bounded measurable with respect to t and smooth with respect to q nonautonomous vector field $(t, q) \mapsto V_t(q)$, $t \in [0,1]$, $q \in M$, $V_t(q) \in \Delta$, such that $\dot\xi(t) = V_t(\xi(t))$ for almost all $t \in [0,1]$ (see [2]). The differential equation $\dot q = V_t(q)$ defines a family of diffeomorphisms

$$P_{\tau,t} : M \to M, \quad P_{t,t}(q) = q,$$

$$\frac{\partial}{\partial t} P_{\tau,t}(q) = V_t(P_{\tau,t}(q)), \quad \tau, t \in [0,1],\ q \in M.$$

Let $\alpha \in (0,1)$; we define a mapping $I_\alpha : \Omega(\Delta) \to \Omega(\Delta)$ by the formula

$$I_\alpha(\zeta)(t) = \begin{cases} \zeta(t) & , \quad 0 \le t \le \alpha \\ P_{\alpha,t}(\zeta(\alpha)) & , \quad \alpha \le t \le 1 \end{cases}$$

and a mapping

$$\partial_\alpha : H^1([0,1]; M) \to M \times M$$

by the formula $\partial_\alpha(\zeta) = (\zeta(0), \zeta(\alpha))$.

Lemma 1 *Suppose that $\partial_\alpha\big|_{\Omega(\Delta)}$ is a submersion at ξ. Then $\partial\big|_{\Omega(\Delta)}$ is a submersion at ξ as well.*

Indeed, $\partial(I_\alpha(\zeta)) = (\zeta(0), P_{\alpha,1}(\zeta(\alpha)))$, $I_\alpha(\xi) = \xi$. Since $P_{\alpha,1}$ is a diffeomorphism, then $\partial \circ I_\alpha$ is a submersion at ξ. Hence $\partial\big|_{\Omega(\Delta)}$ is a submersion at ξ as well.

For $\xi, \zeta \in \Omega(\Delta)$, $\xi(0) = \zeta(0)$, we denote

$$(\xi \cdot \zeta)_\alpha = \begin{cases} \xi(t) & , \quad 0 \le t \le \alpha \\ \xi(2\alpha - t) & , \quad \alpha \le t \le 2t \\ \zeta\left(\frac{t - 2\alpha}{1 - 2\alpha}\right) & , \quad 2\alpha \le t \le 1\ . \end{cases}$$

We have $(\xi \cdot \zeta)_\alpha(0) = \zeta(0)$, $(\xi \cdot \zeta)_\alpha(1) = \zeta(1)$; besides that, $(\xi \cdot \zeta)_\alpha$ tends to ζ in H^1-topology as the length of ξ tends to 0 while ζ remains fixed.

Suppose that $\partial_\alpha\big|_{\Omega(\Delta)}$ is a submersion at ξ. It follows from the lemma that $\partial_\alpha\big|_{\Omega(\Delta)}$ is a submersion at $(\xi \cdot \zeta)_\alpha$. To prove the theorem it is enough to construct an arbitrary short admissible curve ξ started from given point $q_0 \in M$ and such that $\partial_\alpha\big|_{\Omega(\Delta)}$ is a submersion at ξ.

The construction is classical. Let V_1 be a smooth vector field with the values in Δ, $V_1(q_0) \ne 0$, and $t \mapsto e^{tV_1}$ be the (local) flow in M generated by V_1. Take $\varepsilon > 0$; there exist a smooth vector field V_2 with the values in Δ and a moment $t_1^1 \in (0, \varepsilon)$ such that V_1 and V_2 are linearly independent at $e^{t_1^1 V_1}(q_0)$. Indeed, in the opposite case the distribution Δ would be tangent to the one-dimensional submanifold $\{e^{\tau V_1}(q_0) \mid \tau \in (0, \varepsilon)\}$ and could not be bracket generating. Moreover, there exist a field V_3 with values in Δ and moments $t_2^1, t_2^2 \in (0, \varepsilon)$ such that the mapping

$$(\tau_1, \tau_2) \mapsto e^{\tau_2 V_2} \circ e^{\tau_1 V_1}(q_0)$$

is an immersion at (t_2^1, t_2^2) and V_3 is transversal to the image of this immersion at the point $e^{t_2^2 V_2} \circ e^{t_2^1 V_1}(q_0)$. The reason is the same: in the opposite case the distribution Δ would be tangent to the two-dimensional submanifold consisting of the point $e^{\tau_2 V_2} \circ e^{\tau_1 V_1}(q_0)$ (where τ_1 is close to t_2^1, τ_2 is positive close to 0) and could not be bracket generating. We may continue and find fields $V_1, \ldots, V_n$ with values in Δ and moments $t_{n-1}^1, \ldots, t_{n-1}^{n-1} \in (0, \varepsilon)$ such that the mapping

$$(\tau_1, \ldots, \tau_{n-1}) \mapsto e^{\tau_{n-1} V_{n-1}} \circ \cdots \circ e^{\tau_1 V_1}(q_0)$$

is an immersion at $(t_{n-1}^1, \ldots, t_{n-1}^{n-1})$ and V_n is transversal to the image of this immersion at the point $e^{t_{n-1}^{n-1} V_{n-1}} \circ \cdots \circ e^{t_{n-1}^1 V_1}(q_0)$. It is easy to show that the mapping $\partial_{n\varepsilon}\big|_{\Omega(\Delta)}$ is a submersion at $\hat\xi$, where $\hat\xi(\tau) = e^{\tau V_1}(q_0)$ for $0 \le \tau \le t_{n-1}^1$, $\hat\xi(\tau) = e^{\tau V_2} \circ e^{t_{n-1}^1 V_1}(q_0)$ for $t_{n-1}^1 \le \tau \le t_{n-1}^2, \ldots$, $\hat\xi(\tau) = e^{\tau V_n} \circ e^{t_{n-1}^{n-1} V_{n-1}} \circ \cdots \circ e^{t_{n-1}^1 V_1}(q_0)$ for $\sum_{i=1}^{n-1} t_{n-1}^i \le \tau$. We are done since $V_1, \ldots, V_n$ and ε can be chosen arbitrary small.

Corollary 1 *Let $\mathcal{U}_0$ be an everywhere dense submanifold of $\Omega(\Delta)$ and $\gamma \in \Omega(\Delta)$. Suppose that γ realises the minimum of the length among all admissible curves $\zeta \in \mathcal{O}_\gamma \cap \mathcal{U}_0$ with the boundary conditions $\zeta(0) = \gamma(0)$, $\zeta(1) = \gamma(1)$, where $\mathcal{O}_\gamma$ is a neighbourhood of γ in $\Omega(\Delta)$. Then γ is a H^1-local length minimizer.*

The statement of the corollary follows directly from Theorem 2 and Proposition 1.

Summing up we conclude that it is enough to deal only with smooth (or analytic in the case of a real–analytic sub-Riemannian structure) perturbations of the referenced curve in order to check the H^1-local minimality.

References

[1] A. A. Agrachev, *Methods of control theory in nonholonomic geometry*, Proceed. ICM-94, Zürich. Birkhäuser, 1995, pp. 1473–1483.

[2] A. A. Agrachev, R. V. Gamkrelidze, *Feedback–invariant optimal control theory – I. Regular extremals*, J. Dynamical and Control Systems, 1997, vol. 3, pp. 343–389.

[3] A. A. Agrachev, A. V. Sarychev, *Strong minimality of abnormal geodesics for 2-distributions*, J. Dynamical and Control Systems, 1995, vol. 1, pp. 139–176.

[4] I. Kupka, *Abnormal extremals*, Preprint, 1992.

[5] R. Montgomery, *Abnormal minimizers*, SIAM J. Control and Optimization, 1994, vol. 32, pp. 1605–1620.

[6] H. J. Sussmann, W. Liu, *Shortest paths for sub-Riemannian metrics on rank-2 distributions*, Mem. Amer. Math. Soc. vol. 118, 1995.