

STRUCTURE OF THE ENDPOINT MAP NEAR NICE SINGULAR CURVES

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ABSTRACT. Given a rank-two sub-Riemannian structure (M, Δ) and a point $x \in M$, a singular curve is a critical point of the endpoint map $F : \gamma \mapsto \gamma(1)$ defined on the space of horizontal curves starting at x . The typical least degenerate singular curves of these structures are often called *nice singular curves*; another name is “regular abnormal geodesics”. The main goal of this paper is to show that locally around a nice singular curve γ , once we choose a suitable topology on the control space we can find a normal form for the endpoint map, in which F writes as a sum of a linear map and a quadratic form. We also study the restriction of F to the level sets of the action functional and give a Morse-like formula for the inertia index of its Hessian at γ .

1. INTRODUCTION

1.1. Horizontal path spaces and singular curves. Let M be a smooth m dimensional manifold and consider a smooth, totally nonholonomic distribution $\Delta \subset TM$ of rank 2. Given a point $x \in M$ (which we will assume fixed once and for all) the horizontal path space Ω of *admissible* (also called *horizontal*) curves starting at x is defined by:

$$\Omega = \{\gamma : I \rightarrow M \mid \gamma(0) = x, \gamma \text{ is absolutely continuous, } \dot{\gamma} \in \Delta \text{ a.e. and is } L^2\text{-integrable}\}.$$
¹

The $W^{1,2}$ topology endows Ω with a Hilbert manifold structure, locally modeled on $L^2(I, \mathbb{R}^2)$. The *endpoint map* $F : \Omega \rightarrow M$ is the smooth map assigning to each curve its final point $F(\gamma) = \gamma(1)$; given $y \in M$ we will denote

$$\Omega(y) := F^{-1}(y)$$

to be the set of all horizontal curves joining x and y . Given an *energy* functional $J : \Omega \rightarrow \mathbb{R}$, the sub-Riemannian *length-minimizing problem* consists into characterising the admissible curves realizing $\min\{J(\gamma) \mid \gamma \in \Omega(y)\}$, and to solve this problem it is crucial to understand the local geometry of $\Omega(y)$.

If y is a *regular* value of F , then the space $\Omega(y)$ is a smooth Hilbert manifold and its geometrical picture can be studied by classical methods; in general however y is not regular and $\Omega(y)$ has singularities. A *singular* curve is a critical point of F . Singular curves are central objects in the theory of nonholonomic distributions, but their study is a difficult problem and many fundamental questions related to their existence are still open. Most of the difficulties come from the fact that the differential of F is not a Fredholm map, which makes the singularities very deep and their local geometry essentially inaccessible as opposed, for example, to the singularities of maps between finite dimensional manifolds. Already in the simplest case when the differential $d_\gamma F$ at a singular curve has corank one, that is the image of $d_\gamma F$ is of codimension one in $T_{F(\gamma)}M$, the Hessian $\text{He}_\gamma(F)$ can be a *degenerate* quadratic form and it might not be possible to find a normal form for the endpoint map near γ (as one could do for finite dimensional maps with a non-degenerate Hessian, using Morse Lemma).

¹Throughout this paper we denote with I the closed interval $[0, 1]$.

1.2. Rank-two-nice singular curves. We concentrate in this paper on singular curves γ that satisfy the following two conditions (we say that such curves are *rank-two-nice* for $\Omega(y)$):

- (1) γ is a corank-one strictly abnormal regular singular curve;
- (2) $y = F(\gamma)$ is not a conjugate point along γ .

Condition (1) means that γ is a critical point of F such that $\text{Im}(d_\gamma F)$ is of codimension one in $T_{F(\gamma)}M$, but also that there is no covector $(\lambda, \lambda_0) \in \mathbb{R}^{m+1}$ with $\lambda_0 \neq 0$ annihilating the differential at γ of the *extended* endpoint map $(F, J) : \Omega \rightarrow M \times \mathbb{R}$. Requiring that γ is a regular singular curve is equivalent to demand that $\lambda_t \in (\Delta_{\gamma_t}^2)^\perp \setminus (\Delta_{\gamma_t}^3)^\perp$ for every $t \in [0, 1]$, where $t \mapsto \lambda_t \in (\Delta_{\gamma_t})^\perp$ is the dual curve of covectors associated with γ , satisfying $\lambda(1) = \lambda$ and such that γ_t is the projection of λ_t onto M for every $t \in I$. The regularity condition on singular curves was introduced in [17], and it reminds very much of the *minimal order condition* of [9]. Corank-one singular curves of minimal order are the only singular curves for the generic choice in the C^∞ -Whitney topology of pairs (Δ, g) (distribution and sub-Riemannian metric on it) by [9, Theorem 2.4 and Proposition 2.7]; however, the regularity condition is much stronger, and indeed it ensures that the given curve is smooth.

Condition (2) concerns the Hessian of F at γ , that is the quadratic form $\text{He}_\gamma(F) : \ker(d_\gamma F) \rightarrow \text{coker}(d_\gamma F) = T_{F(\gamma)}^*M / \text{Im}(d_\gamma F) \simeq \mathbb{R}$. Recall that $\ker(d_\gamma F)$ is a codimension $m - 1$ subspace of the Hilbert space $T_\gamma\Omega$ equipped with the $W^{1,2}$ -topology. The quadratic form $\text{He}_\gamma(F)$ is compact and continuous even for a weaker L^2 -topology on Ω . Nonetheless, $\ker(d_\gamma F)$ is a degenerate quadratic form: any ‘‘tangent vector’’ to the reparameterizations of γ belongs to its kernel.

We say that $y = F(\gamma)$ is not a *conjugate point* along γ , if the kernel of the extension of $\text{He}_\gamma(F)$ to the closure of $\ker(d_\gamma F)$ in the L^2 -topology is equal to the closure of the tangent space to the reparameterizations of γ (see Section 3 for details). Once a rank-two nice curve is chosen, the set of s such that $y = \gamma(s)$ is not conjugate along γ (i.e. the set of times s such that $\gamma|_{[0,s]}$ satisfies also condition (2)) is dense in the interval of definition of γ [23, Lemma 7].

1.3. Local coordinates and the main theorem. Let $\gamma \in \Omega(y)$ be a rank-two nice singular curve. We are going to study the endpoint map in a small neighborhood of γ in the space of horizontal curves and we may assume without loss of generality that γ does not have self-intersections. Indeed γ is a smooth regular curve and, if necessary, we may lift Δ and γ to a covering of a neighborhood of $\{\gamma(t) \mid t \in I\}$ in M . If γ does not have self-intersections, then there exists a pair of smooth vector fields X_1 and X_2 such that γ is an integral curve of the field X_1 and, in a sufficiently small neighborhood $O_\gamma \subset M$ of γ , we have

$$\Delta_z = \text{span}\{X_1(z), X_2(z)\}.$$

With this choice of the frame, we parametrize admissible curves in Ω as integral curves on M of the differential system (this is done in much greater detail in Section 2 below):

$$(1.1) \quad \dot{\xi}_t = (1 + v_1(t))X_1(\xi_t) + v_2(t)X_2(\xi_t) \quad \text{a.e. on } I, \quad \xi(0) = x_0, \quad (v_1, v_2) \in L^2(I, \mathbb{R}) \oplus L^2(I, \mathbb{R}).$$

It is easy to see that the $L^2(I, \mathbb{R}^2)$ topology in the space of controls (v_1, v_2) corresponds to the $W^{1,2}(I, \mathbb{R}^2)$ topology in Ω .

By a slight abuse of notation we can reinterpret the endpoint map on $L^2(I, \mathbb{R}^2)$ as

$$F(v_1, v_2) = F(\xi),$$

where the control (v_1, v_2) is associated to ξ via (1.1). In particular $F(\gamma) = F(0)$. The main theorem of our paper gives a local normal form of F . ‘‘Local’’ in this setting does not just mean: ‘‘in a neighborhood of 0 in $L^2(I, \mathbb{R}^2)$ ’’, but is a bit more delicate. Given a subspace $E \subset$

$L^2(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$, the intersection of E with a neighborhood of the origin in $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$ will be called an $(\infty, 2)$ -neighborhood of the origin in E . An $(\infty, 2)$ -neighborhood of the origin in $E \oplus \mathbb{R}^m$ is the sum of an $(\infty, 2)$ -neighborhood of the origin in E and a neighborhood of the origin in \mathbb{R}^m .

Theorem 1. *Let γ be a rank-two nice singular curve, $\gamma(1) = y$. Then there exist an origin-preserving homeomorphism $\mu : \mathcal{V} \rightarrow \mathcal{V}'$ of $(\infty, 2)$ -neighborhoods of the origin $\mathcal{V} \subset \ker(d_0F) \oplus \text{Im}(d_0F)$ and $\mathcal{V}' \subset L^2(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$, and a diffeomorphism $\psi : \mathcal{O}_y \rightarrow \mathcal{O}_0$ of neighborhoods $\mathcal{O}_y \subset M$ and $\mathcal{O}_0 \subset \mathbb{R} \oplus \text{Im}(d_0F)$, respectively of y and 0 , such that:*

$$\psi \circ F \circ \mu(v, w) = (\text{He}_0F(v), w), \quad \text{for every } (v, w) \in \mathcal{V}.$$

Remark 1. The class of available $(\infty, 2)$ -neighborhoods does not depend on a particular choice of the frame as long as γ is an integral curve of X_1 , since a change of the frame would result in a smooth change of local coordinates in the space of horizontal curves.

Let us stress that the restriction to neighborhoods in $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$ is not by chance, and there is no hope for Theorem 1 to be true on the whole $L^2(I, \mathbb{R}^2)$. Indeed for a rank-two-nice curve γ , the negative eigenspace N of $\text{He}_\gamma(F)$ is of finite dimension (see Proposition 13 below), and it is known (see, e.g. [7, Proposition 2]) that the restriction of F to any subspace of finite codimension is an open map; were Theorem 1 true in $L^2(I, \mathbb{R}^2)$, we would come to an absurd since the projection onto the abnormal direction would have a sign (in fact, we would arrive to the same absurd conclusion choosing any $L^p(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$ control space, with $1 \leq p < \infty$). In this sense, our result can be seen as another instance of the *rigidity phenomenon* of [8]. Due to the presence of μ which is just an homeomorphism, Theorem 1 cannot be interpreted as an instance of a Morse Lemma, but from a topological perspective it essentially reduces $\Omega(y)$ to the infinite-dimensional quadratic cone $\{\text{He}_0(F) = 0\}$, as soon as we choose a proper system of coordinates. Heuristically speaking however, it would not be even reasonable to expect μ to be a diffeomorphism, given the heavy degeneration of the Hessian map (its kernel contains all the reparametrizations of γ); in this sense Theorem 1 is the best result that one can hope for.

To conclude with our introduction, let us lastly comment the appearance of conjugate time moments along γ . We introduce the shorthand notation $q = \text{He}_\gamma(F)$, and denote similarly by \hat{q} the Hessian of the extended endpoint map (F, J) . The corank one assumption and the strict abnormality of γ imply that (here ind denotes the negative inertia index of a given quadratic form):

$$\text{ind}(\hat{q}) \leq \text{ind}(q) \leq \text{ind}(\hat{q}) + 1,$$

as indicated in Figure 1 (notice that $\ker(d_\gamma(F, J)) = \ker(d_\gamma F) \cap \ker(d_\gamma J)$ is of codimension one in $\ker(d_\gamma F)$). Whenever the index of either one of the two forms is zero, Theorem 1 implies the isolation (with respect to the $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$ topology) of γ in the level set $\Omega(y)$ (resp. in $\Omega(y) \cap J^{-1}(J(\gamma))$), whence the local minimality of γ follows. If $\text{ind}(q) = 0$, this means that γ is isolated in $\Omega(y)$, no matter the functional we are trying to minimize; if instead $(\text{ind}(q), \text{ind}(\hat{q})) = (1, 0)$ it is no longer true that γ is isolated in $\Omega(y)$, but still it retains its minimality if we restrict ourselves to a fixed level of the energy functional J .

1.4. An explicit computation of conjugate times. Consider the following example of [25]. Let $M = SO(3) \times \mathbb{R}$, $\mathfrak{m} = \mathfrak{so}(3) \oplus \mathbb{R}$ be its Lie algebra, and let us consider $X_1 = (T_1 + T_2) \oplus 2$, $X_2 = T_1 \oplus 1$, where T_1, T_2, T_3 are the standard generators for $\mathfrak{so}(3)$, that is $[T_1, T_2] = T_3$, $[T_2, T_3] = T_1$, $[T_3, T_1] = T_2$. We define a distribution $\Delta \subset TM$ extending these vectors to vector fields on M by left-translation:

$$\Delta = \text{span}\{X_1, X_2\}.$$

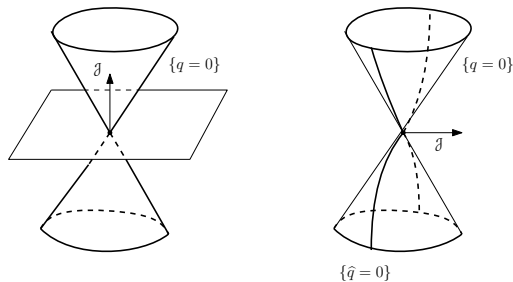


FIGURE 1. The relative positions between $\mathcal{J} := \text{proj}_{\ker(d_\gamma F)}(d_\gamma J)$ and the quadratic cone $\Omega(y) \simeq \{q = 0\}$. In the first case \mathcal{J} belongs to the negative eigenspace of q , and q and \hat{q} have different indexes. In the second case, instead, \mathcal{J} has non-trivial projection onto the positive eigenspace of q , and the two forms have the same index.

A sub-Riemannian metric on Δ is defined by declaring the two fields $\{X_1, X_2\}$ orthonormal at every point. The energy of a horizontal curve is defined by integrating the square of the sub-Riemannian norm of its velocity. We consider the curve γ associated to the control $u(t) = (1, 0)$. This curve satisfies condition (1) above (it is a corank-one strictly abnormal singular curve of minimal order) by the general recipe for producing such kind of curves given in [25, Section 8]. Let us denote by γ_s the restriction of this curve to the interval $[0, s]$, by q_s the Hessian of the endpoint map F at γ_s and by \hat{q}_s its restriction to $\ker(d_{\gamma_s} J)$ (equivalently, \hat{q}_s is the Hessian at γ_s of the extended endpoint map (F, J)). One finds that the conjugate points for q_s and for \hat{q}_s are given by the zeros of the functions

$$a(s) = \frac{1}{\sqrt{2}} \sin(\sqrt{2}s) \quad \text{and} \quad \hat{a}(s) = \frac{1}{2}(1 - \cos(\sqrt{2}s)) - \frac{1}{3}s \frac{1}{\sqrt{2}} \sin(\sqrt{2}s)$$

(the plot of these two functions is as in Figure 2). We will explain later at the end of Section 6.2.1 how to derive such equations. If we pick a point s_0 which is not a zero of neither one of these two functions, the curve γ_{s_0} satisfies also conditions (2) above for the point $y = \gamma(s_0)$ and is a rank-two-nice singular curve. Incidentally, we observe that the sequence of pairs of indexes for the two forms proceeds on the consecutive intervals, separated by the zeros of these functions, as:

$$(\text{ind}(q_s), \text{ind}(\hat{q}_s)) = (0, 0), (1, 0), (2, 1), (2, 2), (3, 2), (4, 3), (4, 4), (5, 4), (6, 5), \dots$$

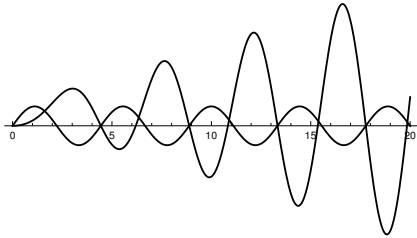
and so on. In fact, along corank-one singular trajectories of minimal order, the Morse index of a control system equals the sum of the multiplicities of the conjugate points along the curve [23, Theorem 1]. In particular, there exist time intervals arbitrarily far from zero on which either the two forms have the same index, or where their indexes differ by one.

Remark 2. It is interesting to reinterpret the previous example in terms of its compact version on $SU(2) \times \mathbb{S}^1 \simeq U(2)$. The isomorphism between $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ is given by

$$2T_1 \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad 2T_2 \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad 2T_3 \leftrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

and therefore, supposing that we start from $x = (\text{Id}, \pm 1)$, the nice singular curve $s \mapsto x \circ e^{sX_1}$, $s \in [0, 1]$, is explicitly computed as

$$x_0 \circ e^{sX_1} = \left(\begin{pmatrix} \cos\left(\frac{s}{\sqrt{2}}\right) & -\sqrt{2}(1+i)\sin\left(\frac{s}{\sqrt{2}}\right) \\ -\sqrt{2}(-1+i)\sin\left(\frac{s}{\sqrt{2}}\right) & \cos\left(\frac{s}{\sqrt{2}}\right) \end{pmatrix}, \pm 1 \right).$$

FIGURE 2. Plot of the functions $a(s)$ and $\hat{a}(s)$

1.5. Structure of the paper. All technical preliminaries needed to prove our main theorem are given in Section 2, where we introduce the geometrical setting of our problem, and we start to investigate first and second-order conditions coming from the expansion of the endpoint map F . Section 3 is devoted to the study of conjugate points along rank-two-nice singular curves, and we develop here all the tools needed to understand their main properties. Section 4 is devoted to the construction of an homeomorphism ρ , which is needed to cut out the kernel of the Hessian map, even though this requires the passage to the space $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$. Section 5 is the core of the paper, where we prove the existence of a normal form for the endpoint map F locally around γ and we give the proof of Theorem 1 in Section 5.2; as a consequence, we are able to discuss some nontrivial isolation properties of rank-two-nice singular curves in $\Omega(y)$. The main tool used here is a generalized version of the Morse Lemma, Proposition 21, and the homeomorphism μ is obtained as the composition of ρ with the diffeomorphism provided by this Proposition. Finally, Section 6 contains the needed details for the computations of conjugate points along rank-two-nice singular curves for a whole family of examples, while we reserve Appendix A for a minor technical proof.

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2. RANK-TWO SUB-RIEMANNIAN MANIFOLDS

2.1. The endpoint map, the energy and the extended endpoint map. Let M be a smooth, connected m -dimensional manifold. A rank-two sub-Riemannian manifold on M is specified by a pair (M, Δ) , where Δ is a rank-two, totally nonholonomic distribution $\Delta \subset TM$ (a more intrinsic characterization can be found, for example, in [1]). For any given $x_0 \in M$, we call Ω the space of all admissible curves starting at x_0 , that is

$$\Omega = \{\gamma : I \rightarrow M \mid \gamma(0) = x_0, \gamma \text{ is absolutely continuous, } \dot{\gamma} \in \Delta \text{ a.e. and is } L^2\text{-integrable}\}.$$

We endow Ω with the $W^{1,2}$ -topology, defining on it a Hilbert manifold structure² locally modelled on $L^2(I, \mathbb{R}^2)$. We call this topology the strong topology in contrast with the weak topology that can also be considered on Ω . We refer to [19, 22, 16] for more details on these topologies. The endpoint map F is the map that gives the final point of a horizontal curve starting at x_0 ,

$$F : \Omega \rightarrow M, \quad F(\gamma) = \gamma(1).$$

² In order to be able to integrate, one should define in principle a metric on Δ . Nevertheless, the property of being integrable is independent on the chosen metric.

We recall in the following proposition some useful properties of F (see [26, 7])

Proposition 2. *The endpoint map $F : \Omega \rightarrow M$ is smooth (with respect to the Hilbert manifold structure on Ω). Moreover if $\gamma_n \rightarrow \gamma$ weakly, then $\gamma_n \rightarrow \gamma$ uniformly on I (in particular $F(\gamma_n) \rightarrow F(\gamma)$), F is continuous for the weak topology) and $d_{\gamma_n}F \rightarrow d_\gamma F$ in the operator norm.*

If $y \in M$, we denote by

$$\Omega(y) = F^{-1}(y)$$

the preimage under F of the point y , that is the set of all horizontal curves joining x_0 and y .

Definition 3. We say that $\gamma \in \Omega$ is a singular curve if γ is a critical point of F , or equivalently if $d_\gamma F : \Omega \rightarrow T_{F(\gamma)}M$ is not a submersion. The corank of γ as a singular curve is then defined as the codimension of the image of $d_\gamma F$ in $T_{F(\gamma)}M$.

The subspace $\Delta^\perp \subset T^*M$ is intrinsically defined in the cotangent space by the condition

$$\Delta^\perp := \{\lambda \in T^*M \mid \langle \lambda, v \rangle = 0, \text{ for every } v \in \Delta\},$$

where the notation $\langle \cdot, \cdot \rangle$ stands for the duality product between vectors and covectors. We also recall that T^*M is canonically endowed with a symplectic form ω , that is a closed non-degenerate differential two form $\omega : M \rightarrow \Lambda^2(T^*M)$. The restriction $\bar{\omega}$ of ω to Δ^\perp no longer needs to be non-degenerate and may admit characteristic lines [19].

Definition 4. An absolutely continuous curve $\lambda : I \rightarrow \Delta^\perp$ is an abnormal extremal if $\dot{\lambda}_t \in T_{\lambda_t} \Delta^\perp$ belongs to $\ker \bar{\omega}_{\lambda_t}$ for every $t \in I$, that is if

$$\bar{\omega}_{\lambda_t}(\dot{\lambda}_t, \xi) = 0$$

for every $t \in I$ and every $\xi \in T_{\lambda_t} \Delta^\perp$.

The following result of [14] establishes a clear geometrical characterization of singular curves in terms of abnormal extremals.

Proposition 5. *An admissible curve $\gamma \in \Omega$ is a singular curve if and only if $t \mapsto \gamma_t$, $t \in I$, is the projection of an abnormal extremal $t \mapsto \lambda_t$, $t \in I$. As a matter of terminology, we say that λ_t is an abnormal lift of γ_t .*

Let us equip (M, Δ) with a sub-Riemannian metric $|\cdot|$, that is a scalar product on Δ smoothly depending on the base point. Then the triple $(M, \Delta, |\cdot|)$ defines a rank-two sub-Riemannian structure on M . Once we have fixed a sub-Riemannian structure, we define an energy functional $J : \Omega \rightarrow \mathbb{R}$ by

$$J(\gamma) = \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 dt.$$

The energy functional J is evidently smooth on Ω (and only lower semicontinuous with respect to the weak $W^{1,2}$ -topology).

Definition 6. The extended endpoint map $\Phi : \Omega \rightarrow M \times \mathbb{R}$ denotes the pair

$$\Phi(\gamma) = (F(\gamma), J(\gamma)).$$

The problem of finding admissible curves γ that minimize the energy J , can be reformulated as a constrained minimum problem on Φ . The Lagrange multiplier's rule implies that a curve γ

is a candidate minimizer if there exists a nonzero covector $\bar{\lambda} = (\lambda, \lambda_0) \in (T_{\gamma(1)}^*M \times \mathbb{R})$, defined up to scalar multiples, such that

$$(2.1) \quad \bar{\lambda}\Phi = \lambda d_\gamma F + \lambda_0 d_\gamma J = 0.$$

If $\lambda_0 = 0$, then γ is singular and it is the projection of an abnormal lift starting at λ , in the sense of Proposition 5. If $\lambda_0 = -1$, γ is instead a normal extremal curve, and small pieces of γ are geodesics in the classical sense, i.e. short enough pieces of γ are energy minimizers among all admissible curves connecting the two endpoints. These two possibilities are not mutually exclusive in principle, and an admissible curve may be at the same time both normal and abnormal. This motivates the following definition.

Definition 7. An admissible curve γ is strictly abnormal if it is not normal, that is if it does not admit a normal extremal lift.

Singular curves may be minimizing admissible curves [18], and their appearance is at the core of all the major difficulties in the sub-Riemannian setting. We refer the interested reader to [4, 21, 22] for a more comprehensive discussion of these points.

2.2. Nice singular curves. Let $\gamma \in \Omega$ be a singular curve and $\lambda : I \rightarrow \Delta^\perp$ its abnormal lift. It is well-known [5, 9, 10] that for a rank-two sub-Riemannian distribution

$$\langle \lambda_t, [X, Y](\gamma_t) \rangle = 0$$

for any X, Y local smooth sections of Δ , that is $\lambda_t \in (\Delta_{\gamma_t}^2)^\perp$ for every $t \in I$ (i.e. λ_t satisfies the so-called Goh condition).

Definition 8. A curve $\lambda : I \rightarrow \Delta^\perp \subset T^*M$ is a nice abnormal extremal if, letting $\pi : T^*M \rightarrow M$ be the canonical projection, it holds that

$$\lambda_t \in (\Delta_{\gamma_t}^2)^\perp \setminus (\Delta_{\gamma_t}^3)^\perp, \quad \gamma_t := \pi(\lambda_t)$$

for every $t \in I$.

We remark that nice abnormal extremals coincide with the so-called regular abnormal extremals introduced in [17]. If $\gamma \in \Omega$ is the projection of a nice abnormal extremal, it will be called a nice singular curve.

Remark 3. The property of being a nice singular curve depends just on the curve γ . On the contrary, the property of being rank-two-nice as in Section 1.2 is rather a requirement on the pair (γ, y) , where $y = \gamma(1)$ is a point on γ which is not conjugate along the curve. For the moment we will just investigate properties of nice singular curves, without any further assumption on their final points.

Nice singular curves satisfy the generalized Legendre condition [6, Theorem 4.4] which is a necessary, second-order condition for the optimality of γ . Let us call $\text{He}_\gamma(F)$ the Hessian map at γ of the endpoint map F , that is let us consider the bilinear application

$$\text{He}_\gamma(F) : \ker(d_\gamma F) \times \ker(d_\gamma F) \rightarrow T_{F(\gamma)}M / \text{Im}(d_\gamma F).$$

The projection of $\text{He}_\gamma(F)$ along the abnormal direction $\lambda \in \text{Im}(d_\gamma F)^\perp \subset T_{F(\gamma)}^*M$ induces a well-defined quadratic form

$$\lambda \text{He}_\gamma(F) : \ker(d_\gamma F) \rightarrow \mathbb{R};$$

if both the negative and the positive index of $\lambda\text{He}_\gamma(F)$ are infinite, then γ cannot be optimal by [4, Theorem 20.6]. In particular, since the abnormal covector is defined up to real nonzero multipliers, we may select λ so that

$$(2.2) \quad \text{ind}(\lambda\text{He}_\gamma(F)) < +\infty,$$

and in this setting the generalized Legendre condition becomes a necessary condition for the validity of (2.2).

More is actually true: indeed nice singular curves are smooth [5, Theorem 3.3], and for every such γ , there exists $0 < s \leq 1$ such that $\gamma|_{[0,s]}$ is a strict local minimizer for the $W^{1,2}$ -topology on the space of admissible curves joining x_0 and $\gamma(s)$. This property depends just on the sub-Riemannian manifold (M, Δ) , and not on the metric chosen on it. More details on nice singular curves can be found in [5] and in [2, Chapter 12].

2.3. Adapted coordinates. We briefly present a procedure [1, 3, 16, 22] that permits to pass from admissible curves to their associated controls.

Let (M, Δ) be a rank-two sub-Riemannian structure and $\gamma \in \Omega$ a reference admissible curve. Our study being local in the space of horizontal curves around γ , by possibly lifting both Δ and γ to a covering of the neighborhood $\{\gamma(t) \mid t \in I\}$, it is not restrictive to assume that γ has no self-intersections. Then there exist a neighborhood $O_\gamma \subset M$ of γ , and X_1, X_2 smooth vector fields on M such that:

- (i) γ is an integral curve of X_1 associated with the control $(1, 0)$, satisfying $\dot{\gamma}_t = X_1(\gamma_t)$, for a.e. $t \in I$;
- (ii) $\Delta_x = \text{span}\{X_1(x), X_2(x)\}$, for every $x \in O_\gamma$.

The horizontal curves contained in O_γ are then described by the solutions $t \mapsto x_t$, $t \in I$ of the differential system

$$\dot{x}_t = u_1(t)X_1(x_t) + u_2(t)X_2(x_t) \quad \text{a.e. on } I, \quad x(0) = x_0, \quad u \in \mathcal{U}_1 \subset L^2(I, \mathbb{R}^2),$$

where the open set $\mathcal{U}_1 \subset L^2(I, \mathbb{R}^2)$ is a neighborhood of $(1, 0)$ that consists of all the pairs (u_1, u_2) such that the curve $t \mapsto x_t$ is defined on the whole of I .

Definition 9. A local chart on \mathcal{U}_1 is the choice of a neighborhood $\mathcal{V}_1 \subset L^2(I, \mathbb{R}^2)$ of zero and a system of coordinates

$$(u_1, u_2) \mapsto (1 + v_1, v_2)$$

on \mathcal{U}_1 and centered at $(1, 0)$.

With the choice of a local chart, any admissible curve $t \mapsto x_t$, $t \in I$, can be written as

$$(2.3) \quad \dot{x}_t = (1 + v_1(t))X_1(x_t) + v_2(t)X_2(x_t), \quad x(0) = x_0, \quad \text{a.e. } t \in I.$$

Finally, let us consider the map $A : \mathcal{V}_1 \rightarrow \Omega$ that associates to the pair $(v_1, v_2) \in \mathcal{V}_1$ the only solution (up to zero-measure sets) $\gamma \in \Omega$ to (2.3) (see [1]). In particular A is a submersion, and permits to reinterpret both the endpoint map F and the energy J as defined on \mathcal{V}_1 , that is

$$(2.4) \quad J(A(v_1, v_2)) = \frac{1}{2} \|(1 + v_1, v_2)\|_{L^2(I, \mathbb{R}^2)}^2$$

and $F(v_1, v_2) := F(A(v_1, v_2))$ for every $(v_1, v_2) \in \mathcal{V}_1$.

We can now compute F by the formula

$$F(v_1, v_2) = x_0 \circ \overrightarrow{\text{exp}} \int_0^1 (1 + v_1(t))X_1 + v_2(t)X_2 dt,$$

and with our conventions, a control $v \in \mathcal{V}_1$ is said singular if and only if

$$d_v F = d_{A(v)} F \circ d_v A$$

is not a submersion, and its corank is the corank of $d_v F$.

2.4. The endpoint map near a nice singular curve. Let $\gamma \in \Omega(y)$ be a reference nice singular curve for (M, Δ) , and let us choose local coordinates on M centered at $(1, 0)$, so that γ becomes an integral curve of

$$\dot{\gamma}_t = X_1(\gamma_t), \quad \text{a.e. on } I, \quad \gamma(0) = x_0.$$

From now on we will always assume that γ is a corank-one strictly abnormal singular curve, as anticipated in Section 1.2. By the corank-one assumption on γ , the subspace

$$(d_0 F)^\perp \subset T_{F(0)}^* M$$

has dimension one. On the other hand, the strict abnormality of γ implies that if we choose $\bar{\lambda}$ as in (2.1), then forcedly $\lambda_0 = 0$. In particular γ admits a unique extremal lift up to real multiples, which is necessarily abnormal. By the variation of the constants' formula [4, Chapter 2] we describe, locally around γ , the endpoint map $F(v_1, v_2)$ as a perturbation of $y = F(0)$.

Setting

$$g_t := e_*^{(1-t)X_1} X_2, \quad t \in I,$$

we write

$$\begin{aligned} (2.5) \quad F(v_1, v_2) &= x_0 \circ \overrightarrow{\exp} \int_0^1 (1 + v_1(t)) X_1 + v_2(t) X_2 dt \\ &= x_0 \circ e^{X_1} \circ \overrightarrow{\exp} \int_0^1 v_1(t) X_1 + v_2(t) g_t dt \\ &= y \circ \overrightarrow{\exp} \int_0^1 v_1(t) X_1 + v_2(t) g_t dt. \end{aligned}$$

Then we define $G : \mathcal{V}_1 \rightarrow M$ to be the endpoint map associated to the (non-autonomous) system

$$\dot{z} = v_1 X_1(z) + v_2 g_t(z) \quad \text{a.e. on } I, \quad z_0 = y, \quad z \in M, \quad (v_1, v_2) \in \mathbb{R}^2$$

and we have the identity

$$(2.6) \quad F_{x_0}(v_1, v_2) = G_y(v_1, v_2),$$

where we made explicit the initial datum in each map. Since x_0 and y are fixed in this paper, we can indifferently use F or G to analyze the geometry of (M, Δ) around γ , and we will extensively use this flexibility in the sequel if there is no ambiguity.

2.4.1. First-order conditions. Starting from (2.5), the differential $d_0 G$ (or equivalently $d_0 F$) is computed by [5, Section 4]

$$(2.7) \quad d_0 G(v_1, v_2) = \int_0^1 v_1(t) dt X_1(y) + \int_0^1 v_2(t) g_t(y) dt, \quad \text{for every } (v_1, v_2) \in L^2(I, \mathbb{R}^2).$$

Let us split the space of controls $L^2(I, \mathbb{R}^2)$ as the direct sum

$$L^2(I, \mathbb{R}^2) = \ker(d_0 G) \oplus E$$

where $E \simeq \mathbb{R}^{m-1}$ is a finite-dimensional complement of $\ker(d_0 G)$. For future purposes, we need the following result.

Lemma 10. $\mathcal{J} := \text{proj}_{\ker(d_0G)}(d_0J)$ is nonzero.

Proof. Let us split the space of controls $L^2(I, \mathbb{R}^2)$ as

$$L^2(I, \mathbb{R}^2) = L^2(I, \mathbb{R}) \oplus L^2(I, \mathbb{R}) = (Z_1 \oplus C_1) \oplus (V_2 \oplus W_2),$$

where V_2 denotes the restriction of $\ker(d_0G)$ to the second component of the control, W_2 is its finite-dimensional complement, C_1 are the constants and Z_1 the zero-mean controls. We claim that, since γ is a strictly abnormal singular curve, then

$$(2.8) \quad \text{Im}(d_0G) = \{d_0G(0, w_2) \mid w_2 \in W_2\}.$$

Let us first show how to conclude once we have established the claim. By (2.8), we can find $w_2^0 \in W_2$ different from zero such that:

$$d_0G(0, w_2^0) = \int_0^1 w_2^0(t) g_t(y) dt = X_1(y) \in \text{Im}(d_0G).$$

Setting $\widehat{w} := (1, -w_2^0)$ we have the following orthogonal decomposition of $\ker(d_0G)$, namely

$$\ker(d_0G) = Z_1 \oplus \mathbb{R}\widehat{w} \oplus V_2.$$

Finally, since $d_0J = (1, 0)$ (compare with (2.4)), we deduce that

$$\mathcal{J} = \text{proj}_{\ker(d_0G)}(d_0J) = \frac{\widehat{w}}{\|\widehat{w}\|^2} \neq 0,$$

thus proving the lemma.

Now we prove the claim, and we reason by contradiction assuming

$$\{d_0G(0, w_2) \mid w_2 \in W_2\} \subset \text{Im}(d_0G).$$

In particular, we see from (2.7) that $X_1(y)$ is the only direction in $\text{Im}(d_0G)$ which is not covered by elements of the form $\{d_0G(0, w_2) \mid w_2 \in W_2\}$. Let $\lambda \in \text{Im}(d_0G)^\perp \subset T_y^*M$. Since $\langle \lambda, X_1(y) \rangle = 0$, we deduce that there exists $\xi \neq \lambda \in \mathbb{R}^{m^*}$ such that

$$\langle \xi, d_0G(w_2, 0) \rangle = 0$$

for every $w_2 \in W_2$, and such that $\langle \xi, X_1(y) \rangle \neq 0$. From (2.7) we see that $\{d_0G(v_1, 0) \mid v_1 \in C_1 \oplus Z_1\} = \mathbb{R}(1, 0)$ is one-dimensional and spanned by the constants, with

$$\langle \xi, d_0G(c_1, 0) \rangle = c_1 \langle \xi, X_1(y) \rangle,$$

while from (2.4) we have the identity:

$$d_0J(v_1, v_2) = \langle 1, v_1 \rangle_{L^2(I, \mathbb{R})} = \int_0^1 v_1(t) dt,$$

so that constant controls in C_1 suffice also to span the differential of J . Choosing $\lambda_0 \in \mathbb{R} \setminus \{0\}$ satisfying:

$$\langle \xi, X_1(y) \rangle = \langle \xi, d_0G(1, 0) \rangle = \lambda_0 d_0J(1, 0) = \lambda_0,$$

we see that $(\xi, -\lambda_0) \in \mathbb{R}^{m+1}$ is a normal covector, that is

$$\xi d_0F(v_1, v_2) = \xi d_0G(v_1, v_2) = \lambda_0 d_0J(v_1, v_2)$$

for every $(v_1, v_2) \in L^2(I, \mathbb{R}^2)$. Then we have the absurd, since γ is strictly abnormal by assumption. \square

Corollary 11. *The control space admits the following orthogonal decomposition:*

$$(2.9) \quad L^2(I, \mathbb{R}^2) = \ker(d_0G) \oplus E = Z_1 \oplus \mathbb{R}\mathcal{J} \oplus V_2 \oplus E,$$

where:

- (a) Z_1 is the set of zero-mean controls in $L^2(I, \mathbb{R}) \oplus \{0\}$,
- (b) $\mathcal{J} = \text{proj}_{\ker(d_0G)}(d_0J)$, and
- (c) V_2 denotes the restriction of $\ker(d_0G)$ to the subspace $\{0\} \oplus L^2(I, \mathbb{R})$.

2.4.2. First-order conditions on the extended endpoint map. Let us briefly discuss the kernel $\ker(d_0\Phi)$ of the differential of the extended endpoint map $\Phi = (F, J)$ introduced in Definition 6. In particular, since

$$\text{Im}(d_0J) = \mathbb{R}(1, 0),$$

let us just notice that

$$(2.10) \quad \ker(d_0\Phi) = \ker(d_0F) \cap \ker(d_0J) = (Z_1 \oplus \mathbb{R}\mathcal{J} \oplus V_2) \cap \ker(d_0J) = Z_1 \oplus V_2,$$

is of codimension one in $\ker(d_0F)$.

2.5. Second-order conditions. We analyze in this section the quadratic form

$$q := \lambda \text{He}_0(G),$$

which, we recall, is a well-defined real-valued quadratic form on $\ker(d_0G)$. To begin with, we observe from (2.7) that the relation $\lambda \in \text{Im}(d_0G)^\perp$ translates into the conditions

$$(2.11) \quad \langle \lambda, X_1(y) \rangle = 0 \quad \text{and} \quad \langle \lambda, g_t(y) \rangle \equiv 0, \quad \text{for every } t \in I.$$

Differentiating the second of these equalities and recalling that

$$g_t := e_*^{(1-t)X_1} X_2, \quad t \in I,$$

we obtain

$$(2.12) \quad \frac{d}{dt} \langle \lambda, g_t(y) \rangle = -\langle \lambda, [X_1, g_t](y) \rangle \equiv 0 \quad \text{i.e.} \quad \langle \lambda, [X_1, g_t](y) \rangle \equiv 0, \quad \text{for every } t \in I.$$

Combining (2.11) and (2.12), the Hessian

$$q(v_1, v_2) := \lambda \text{He}_0(G)(v_1, v_2) = \left\langle \lambda, \left(\int_0^1 \int_0^t (v_1(\tau)X_1 + v_2(\tau)g_\tau) \circ (v_1(t)X_1 + v_2(t)g_t) d\tau dt \right) (y) \right\rangle$$

can be rewritten [4, Exercise 20.4] as:

$$(2.13) \quad q(v_1, v_2) = \int_0^1 \left\langle \lambda, \left[\int_0^t v_1(\tau)X_1 + v_2(\tau)g_\tau d\tau, v_1(t)X_1 + v_2(t)g_t \right] (y) \right\rangle dt \\ = \int_0^1 \left\langle \lambda, \left[\int_0^t v_2(\tau)g_\tau d\tau, v_2(t)g_t \right] (y) \right\rangle dt \quad \text{for every } (v_1, v_2) \in \ker(d_0G).$$

Notice that q has no explicit dependence on the first component v_1 . It then follows from (2.7) and Corollary 11 that the kernel of its associated bilinear form b contains the subspace Z_1 of zero-mean controls. The space of controls $L^2(I, \mathbb{R}^2)$ admits then a second orthogonal decomposition

$$(2.14) \quad L^2(I, \mathbb{R}^2) = \ker(d_0G) \oplus E = N \oplus Z \oplus P \oplus E,$$

where N and P are, respectively, the negative and the positive eigenspaces of q , Z is its kernel and we have the inclusion $Z_1 \subset Z$ (compare with (2.9)).

2.5.1. *On the Hessian of the extended endpoint map.* In analogy with Section 2.4.2, we discuss here the Hessian $\text{He}_0(\Phi)$ of the extended endpoint map $\Phi = (F, J)$. We have already observed that γ admits, up to scalar multiples, only one abnormal covector $\bar{\lambda} = (\lambda, 0) \in T_y^*M \times \mathbb{R}$ with $\lambda \in d_0G^\perp$. This implies that

$$\hat{q} := \bar{\lambda} \text{He}_0(\Phi)|_{\ker(d_0\Phi)}$$

has formally the same expression of q in (2.13) (the component λ_0 of the abnormal covector $\bar{\lambda}$ being zero along d_0J), although its domain is strictly smaller than the domain of q .

3. CONJUGATE POINTS

We present in this section the study of conjugate points along the nice singular curve γ . All the results presented in this section apply to the Hessian \hat{q} introduced in Section 2.5.1, with the obvious modifications needed to take care of its smaller domain of definition.

3.1. **Adapted norms and completion spaces.** By virtue of the equality:

$$(3.1) \quad \begin{aligned} F(v_1, v_2) &= x_0 \circ \overrightarrow{\text{exp}} \int_0^1 (1 + v_1(t))X_1 + v_2(t)X_2 dt \\ &= x_0 \circ \overrightarrow{\text{exp}} \int_0^1 (1 + v_1(t))e_*^{-w_2(t)X_2} X_1 dt \circ e^{w_2(1)X_2} = F(v_1, w_2), \quad w_2(t) := \int_0^t v_2(\tau) d\tau, \end{aligned}$$

F can be thought as a map on the space

$$L^2(I, \mathbb{R}) \oplus H^1(I, \mathbb{R}).$$

The map G in (2.6) can also be expressed as a function of the pair (v_1, w_2) , by:

$$(3.2) \quad y \circ \overrightarrow{\text{exp}} \int_0^1 (1 + v_1(t))e_*^{-w_2(t)g_t} X_1 - X_1 dt \circ e^{w_2(1)X_2}.$$

In these coordinates, we have:

$$(3.3) \quad \ker(d_0G) = \left\{ (v_1, w_2) \in L^2(I, \mathbb{R}) \oplus H^1(I, \mathbb{R}) \mid c_1 := \int_0^1 v_1(\tau) d\tau \text{ and} \right. \\ \left. c_1 X_1(y) + w_2(1)X_2(y) - \int_0^1 w_2(t)\dot{g}_t(y) dt = 0 \right\},$$

The last equality in (3.3) implies that there exists a constant $C > 0$, such that for every $(v_1, w_2) \in \ker(d_0G)$ there hold the estimates

$$(3.4) \quad |c_1| \leq C \|w_2\|_{L^2(I, \mathbb{R})}, \quad |w_2(1)| \leq C \|w_2\|_{L^2(I, \mathbb{R})}.$$

On $L^2(I, \mathbb{R}) \oplus H^1(I, \mathbb{R})$, q has the following expression³:

$$(3.5) \quad q(w_2) = \int_0^1 \langle \lambda, [\dot{g}_t, g_t](y) \rangle w_2(t)^2 dt + \int_0^1 \left\langle \lambda, \left[w_2(1)X_2 + \int_0^t w_2(\tau)\dot{g}_\tau d\tau, w_2(t)\dot{g}_t \right] (y) \right\rangle dt.$$

³Recall that q does not explicitly depend on v_1 (compare with (2.13)).

Since $t \mapsto \gamma(t) = x_0 \circ e^{tX_1}$ is by assumption a nice singular curve, there exists a constant $\kappa > 0$ such that for every $t \in I$, there holds the Legendre condition

$$(3.6) \quad \langle \lambda, [\dot{g}_t, g_t](y) \rangle \geq \kappa;$$

in fact, were the above product zero for some $t \in I$, then so would be

$$\langle \lambda, e_*^{(1-t)X_1} [[X_1, X_2], X_2](\gamma(t)) \rangle = \langle \lambda_t, [[X_1, X_2], X_2](\gamma(t)) \rangle, \quad \lambda_t := \left(e^{(1-t)X_1} \right)^* \lambda,$$

and λ_t would annihilate $\Delta_{\gamma_t}^3$, contradicting the assumptions ⁴.

Let us equip $H^1(I, \mathbb{R})$ with the norm $\| \cdot \|_2$, defined by

$$\|w_2\|_2 = |w_2(1)|_{\mathbb{R}} + \|w_2\|_{L^2(I, \mathbb{R})}.$$

The completion of the space $H^1(I, \mathbb{R})$ with respect to the norm $\| \cdot \|_2$ is then isomorphic to

$$\mathbb{R} \oplus L^2(I, \mathbb{R}) \simeq H^{-1}(I, \mathbb{R}).$$

The completion $\overline{\ker(d_0G)}$ of $\ker(d_0G)$ in $L^2(I, \mathbb{R}) \oplus \mathbb{R} \oplus L^2(I, \mathbb{R})$ is the set:

$$(3.7) \quad \overline{\ker(d_0G)} = \left\{ (v_1, c_2, w_2) \in L^2(I, \mathbb{R}) \oplus \mathbb{R} \oplus L^2(I, \mathbb{R}) \mid c_1 := \int_0^1 v_1(\tau) d\tau \text{ and} \right. \\ \left. c_1 X_1(y) + c_2 X_2(y) - \int_0^1 w_2(t) \dot{g}_t(y) dt = 0 \right\};$$

also, we see from (3.4) that for every $(v_1, c_2, w_2) \in \overline{\ker(d_0G)}$, we have

$$(3.8) \quad |c_1| \leq C \|w_2\|_{L^2(I, \mathbb{R})}, \quad |c_2| \leq C \|w_2\|_{L^2(I, \mathbb{R})}.$$

3.2. Conjugate points. Consider again the Hessian map q of (3.5), and extend it to $\overline{\ker(d_0G)}$ by

$$(3.9) \quad q(c_2, w_2) = \int_0^1 \langle \lambda, [\dot{g}_t, g_t](y) \rangle w_2(t)^2 dt + \int_0^1 \left\langle \lambda, \left[c_2 X_2 + \int_0^t w_2(\tau) \dot{g}_\tau d\tau, w_2(t) \dot{g}_t \right] (y) \right\rangle dt.$$

By a slight abuse of notation we don't introduce any new terminology for this extension; however it will be convenient in what follows to denote

$$\overline{\ker(d_0G)}_2 := \overline{\ker(d_0G)} \Big|_{\mathbb{R} \oplus L^2(I, \mathbb{R})},$$

the restriction of $\overline{\ker(d_0G)}$ to the second coordinates of the control.

Definition 12. The point y is a conjugate point along γ if and only if the quadratic form q in (3.9) is degenerate on $\overline{\ker(d_0G)}_2$. Equivalently, denoting with b the bilinear form associated with q , y is a conjugate point if and only if there exists a nonzero $(\bar{c}_2, \bar{w}_2) \in \overline{\ker(d_0G)}_2$, such that the linear operator

$$b((\bar{c}_2, \bar{w}_2), \cdot) : \overline{\ker(d_0G)}_2 \rightarrow \mathbb{R}$$

is zero. The multiplicity of y as a conjugate point is given by the dimension of $\ker(q)$ in $\overline{\ker(d_0G)}_2$.

⁴The sign convention is not important, since we can always consider $-\lambda$ instead of λ .

We summarize some known facts (see [23, Theorem 1], [5, Section 4]) about conjugate points along rank-two-nice singular curves that we will use in the sequel. These results are obtained applying the above arguments to the restrictions $\gamma_s := \gamma|_{[0,s]}$ for every $s \in I$, and noticing that the Hessian rescales accordingly as follows:

$$q^s(c_2, w_2) = \int_0^s \langle \lambda, [\dot{g}_t, g_t](\gamma(s)) \rangle w_2(t)^2 dt + \int_0^s \left\langle \lambda, \left[c_2 X_2 + \int_0^t w_2(\tau) \dot{g}_\tau d\tau, w_2(t) \dot{g}_t \right] (\gamma(s)) \right\rangle dt.$$

Proposition 13. *For a rank-two-nice singular curve γ the following properties hold true.*

- (a) *For sufficiently small times $s \in I$, the Hessian map is positive definite.*
- (b) *Conjugate points are isolated along γ , and every conjugate point has a finite multiplicity.*
- (c) *The negative index of q equals the sum of the multiplicities of all conjugate points along γ . In particular, it is finite.*

The definition of conjugate points motivates our second assumption in Section 1.2, that is we suppose from now on that y is not a conjugate point along γ . In particular, from now on γ will be a rank-two-nice singular curve.

3.3. Analytical properties of q . The goal of this section is to show that the choice of the norm $\|\cdot\|_2$ allows the decomposition of q as the sum of a coercive and a compact operator.

Proposition 14. *Let us consider the linear operator $T : \overline{\ker(d_0 G)}_2 \rightarrow \mathbb{R} \oplus L^2(I, \mathbb{R})$, associated to the bilinear form induced by*

$$\mathcal{Q}(c_2, w_2) := q(c_2, w_2) - \int_0^1 \langle \lambda, [\dot{g}_t, g_t](y) \rangle w_2(t)^2 dt, \quad (c_2, w_2) \in \overline{\ker(d_0 G)}_2.$$

Then T is a compact and self-adjoint operator on $\overline{\ker(d_0 G)}_2$, with respect to the product topology on $\mathbb{R} \oplus L^2(I, \mathbb{R})$.

Proof. Starting from (3.9), the linear operator T is given by:

$$T(c_2, w_2) = \left(c_2 \langle \lambda, [X_2, \dot{g}_t](y) \rangle + \left\langle \lambda, \left[\int_0^1 \langle \lambda, [X_2, \dot{g}_t](y) \rangle w_2(t) dt, \int_0^1 w_2(\tau) \dot{g}_\tau d\tau, \dot{g}_t \right] (y) \right\rangle - \left\langle \lambda, \left[\int_t^1 w_2(\tau) \dot{g}_\tau d\tau, \dot{g}_t \right] (y) \right\rangle \right).$$

Recall indeed that for any $a, b \in L^2(I, \mathbb{R})$ and any two smooth vector fields $X_t, Y_t \in \text{Vec}(M)$, $t \in I$, one has the following identity:

$$\int_0^1 \left\langle \lambda, \left[\int_0^t a(\tau) X_\tau d\tau, b(t) Y_t \right] (y) \right\rangle dt = \int_0^1 \left\langle \lambda, \left[a(t) X_t, \int_t^1 b(\tau) Y_\tau d\tau \right] (y) \right\rangle dt,$$

from which the polarization of \mathcal{Q} follows.

The self-adjointness of T follows directly from the fact that it is a linear operator associated to a bilinear form, so that it remains to prove its compactness. Observe that the last component of T can be expressed as the sum

$$c_2 \langle \lambda, [X_2, g_t](y) \rangle + \int_0^1 K(t, \tau) w_2(\tau) d\tau,$$

with

$$K(t, \tau) := \langle \lambda, [\dot{g}_\tau, \dot{g}_t](y) \rangle \chi_{[0,t]}(\tau) - \langle \lambda, [\dot{g}_\tau, \dot{g}_t](y) \rangle \chi_{[t,1]}(\tau), \quad K(t, \tau) \in L^2(I \times I, \mathbb{R}),$$

and therefore it is a compact operator. In fact, $c_2 \mapsto c_2 \langle \lambda, [X_2, \dot{g}_t](y) \rangle$ is a rank-one operator, while the compactness of

$$w_2 \mapsto \int_0^1 K(t, \tau) w_2(\tau) d\tau$$

is classical, and proved e.g. in [13, Chapter 6]. \square

Let us consider the following operator $R : \overline{\ker(d_0 G)}_2 \rightarrow \mathbb{R} \oplus L^2(I, \mathbb{R})$:

$$R(c_2, w_2) = \begin{pmatrix} 0 \\ \langle \lambda, [\dot{g}_t, g_t](y) \rangle w_2(t) \end{pmatrix}.$$

Following Proposition 14, we have the identity

$$q(c_2, w_2) = \langle (R + T)(c_2, w_2), (c_2, w_2) \rangle, \quad (c_2, w_2) \in \overline{\ker(d_0 G)}_2.$$

The Legendre condition (3.6) and the estimates (3.8) imply that R induces a norm on $\overline{\ker(d_0 G)}_2$, say $\|\cdot\|_R$, which is equivalent to the product norm on $\mathbb{R} \oplus L^2(I, \mathbb{R})$. It is given by the formula:

$$(3.10) \quad \|(c_2, w_2)\|_R^2 = \langle R(c_2, w_2), (c_2, w_2) \rangle = \langle R^{1/2}(c_2, w_2), R^{1/2}(c_2, w_2) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard $\mathbb{R} \oplus L^2(I, \mathbb{R})$ inner product. In fact, notice that

$$\|\langle \lambda, [\dot{g}_t, g_t](y) \rangle w_2\|_{L^2(I, \mathbb{R})} = 0 \Leftrightarrow \|w_2\|_{L^2(I, \mathbb{R})} = 0 \Leftrightarrow \|(c_2, w_2)\|_{\mathbb{R} \oplus L^2(I, \mathbb{R})} = 0.$$

This yields the following result.

Corollary 15. *The linear operator \mathcal{L} induced by the quadratic form q admits on $\overline{\ker(d_0 G)}_2$ a decomposition of the form: $\mathcal{L} = R + T$, where R is a coercive operator and T is compact with respect to the product topology on $\mathbb{R} \oplus L^2(I, \mathbb{R})$.*

Proposition 16. *There exists a constant $K > 0$ such that, if we define⁵ $q_{\overline{P}} := q|_{\overline{P}} : \overline{P} \rightarrow \mathbb{R}$, then*

$$q_{\overline{P}}(c_2, w_2) \geq K \|(c_2, w_2)\|_{\mathbb{R} \oplus L^2(I, \mathbb{R})}^2$$

for every $(c_2, w_2) \in \overline{P}$.

Proof. Let us define $R_{\overline{P}}$ and $T_{\overline{P}}$ to be the restrictions to the subspace \overline{P} of the operators R and T , respectively. If $\|\cdot\|_R$ denotes the norm appearing in (3.10), and provided by the Legendre condition, then $\langle R_{\overline{P}}(c_2, w_2), (c_2, w_2) \rangle \equiv 1$ on the set $\{(c_2, w_2) \in \overline{P}, \|(c_2, w_2)\|_R = 1\}$. Let us consider:

$$\begin{aligned} \alpha &= \inf\{q_{\overline{P}}(c_2, w_2) \mid (c_2, w_2) \in \overline{P}, \|(c_2, w_2)\|_R = 1\} \\ &= 1 + \inf\{\langle T_{\overline{P}}(c_2, w_2), (c_2, w_2) \rangle \mid (c_2, w_2) \in \overline{P}, \|(c_2, w_2)\|_R = 1\}. \end{aligned}$$

Clearly, $\alpha \geq 0$. We claim that, in fact, $\alpha > 0$, and this will conclude the proof, since then as a consequence of (3.6) we would have

$$q_{\overline{P}}(c_2, w_2) \geq \alpha \|(c_2, w_2)\|_R \geq \sqrt{\kappa} \alpha \|w_2\|_{L^2(I, \mathbb{R})} \geq \frac{\sqrt{\kappa} \alpha \min\{1, C^{-1}\}}{2} \|(c_2, w_2)\|_{\mathbb{R} \oplus L^2(I, \mathbb{R})},$$

where C is the constant appearing in (3.8).

Assume then that $\alpha = 0$. Since $T_{\overline{P}}$ is the restriction of a compact, self-adjoint operator by Proposition 14, it is itself compact and self-adjoint. In particular its eigenvalues are bounded, countable, and can only accumulate at zero (see e.g., [15]). Clearly,

$$-1 = \inf\{\langle T_{\overline{P}}(c_2, w_2), (c_2, w_2) \rangle \mid (c_2, w_2) \in \overline{P}, \|(c_2, w_2)\|_R = 1\},$$

⁵Here \overline{P} denotes the completion of the positive eigenspace P of q (see (2.14)).

and therefore -1 coincides with the lowest bound of the spectrum $\sigma(T_{\overline{P}})$, which is actually an eigenvalue as a consequence of the Fredholm alternative. This implies that we can find $(\overline{c}_2, \overline{w}_2) \in \overline{P}$, such that $\|(\overline{c}_2, \overline{w}_2)\|_R = 1$, and such that $q_{\overline{P}}(\overline{c}_2, \overline{w}_2) = 0$. But then, since $q_{\overline{P}}$ is a nonnegative quadratic form, this implies that $q_{\overline{P}}$ is actually degenerate on $\overline{\ker(d_0G)}_2$ ([12, Lemma 6.2]), which is absurd since y is not conjugate, and the proof is concluded. \square

Proposition 16 implies that the eigenvalues of q do not accumulate towards zero on \overline{P} ; since this is clearly true on \overline{N} and $\overline{\ker(d_0G)}_2$ does not intersect the kernel \overline{Z} , Proposition 16 yields the following fundamental Corollary.

Corollary 17. *Let $\mathcal{L} : \overline{\ker(d_0G)}_2 \rightarrow \overline{\ker(d_0G)}_2$ be the linear operator associated to the bilinear form b induced by q , i.e. such that:*

$$\langle \mathcal{L}(c_2, w_2), (c'_2, w'_2) \rangle = b((c_2, w_2), (c'_2, w'_2))$$

for every $(c'_2, w'_2) \in \overline{\ker(d_0G)}_2$. Then \mathcal{L} is bounded from below, hence invertible on $\overline{\ker(d_0G)}_2$.

4. CUTTING THE KERNEL OF THE HESSIAN MAP

4.1. A change of coordinates. We restrict in this section to the Banach space $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$. Let us introduce the Banach subspace

$$L^\infty(I, \mathbb{R})_0 := \left\{ v_1^0 \in L^\infty(I, \mathbb{R}) \mid \int_0^1 v_1^0(t) dt = 0 \right\} \subset L^\infty(I, \mathbb{R}).$$

Every element $v_1 \in L^\infty(I, \mathbb{R})$ can be decomposed uniquely as an orthogonal (with respect to the $L^2(I, \mathbb{R})$ product) sum $v_1 = \overline{v}_1 + v_1^0$, where $v_1^0 \in L^\infty(I, \mathbb{R})_0$ and $\overline{v}_1 \in \mathbb{R}$ is the average over I of v_1 . This implies that $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R}) \simeq \mathbb{R} \oplus L^\infty(I, \mathbb{R})_0 \oplus L^2(I, \mathbb{R})$, and that $(\overline{v}_1, v_1^0, v_2)$ is a coordinate system on $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$.

Given $\alpha > 0$, and define the two open neighborhoods of the origin $\mathcal{V}_2^0, \mathcal{V}_3^0 \subset L^\infty(I, \mathbb{R})$ by:

$$\begin{aligned} \mathcal{V}_2^0 &= \{ v_1^0 \in L^\infty(I, \mathbb{R})_0 \mid 1 + v_1^0(t) > \alpha, \text{ a.e. on } I \}, \\ \mathcal{V}_3^0 &= \left\{ v_1^0 \in L^\infty(I, \mathbb{R})_0 \mid 1 + \frac{v_1^0(t)}{1 + \overline{v}_1} > \alpha, \text{ a.e. on } I \right\}. \end{aligned}$$

We set as well $\mathcal{V}_2 = \mathbb{R} \oplus \mathcal{V}_2^0 \oplus L^2(I, \mathbb{R})$ and $\mathcal{V}_3 = \mathbb{R} \oplus \mathcal{V}_3^0 \oplus L^2(I, \mathbb{R})$.

Definition 18. We define $\rho : \mathcal{V}_2 \rightarrow \mathcal{V}_3$ by:

$$\rho(\overline{v}_1, v_1^0, v_2) = \left(\overline{v}_1, (1 + \overline{v}_1)v_1^0, \dot{\phi}_v(v_2 \circ \phi_v) \right),$$

where

$$\phi_v(t) = \int_0^t 1 + v_1^0(\tau) d\tau.$$

Given any $v \in \mathcal{V}_2$, the time-reparametrization $\phi_v : I \rightarrow I$ is well-defined on the set

$$S_v = \left\{ s \in I \mid s = \phi_v(t) \text{ and } \dot{\phi}_v(t) \text{ exists different from zero} \right\},$$

which is of full measure by the Sard lemma for real-valued absolutely continuous functions (see, e.g. [27, Theorem 16]). Moreover, it is not difficult to see that ρ induces a system of coordinates on \mathcal{V}_3 , whose inverse is given explicitly by:

$$(4.1) \quad (\overline{v}_1, v_1^0, v_2) \mapsto \left(\overline{v}_1, \frac{v_1^0}{1 + \overline{v}_1}, \frac{v_2 \circ \phi_v^{-1}}{\dot{\phi}_v \circ \phi_v^{-1}} \right), \quad \phi_v(t) = \int_0^t 1 + \frac{v_1^0(\tau)}{1 + \overline{v}_1} d\tau.$$

In these coordinates, the endpoint map has then the following expression:

$$\begin{aligned} F(\rho(v_1, v_2)) &= x_0 \circ \overrightarrow{\text{exp}} \int_0^1 (1 + v_1^0(s))(1 + \bar{v}_1)X_1 + \dot{\phi}_v(s)v_2(\phi_v(s))X_2 ds \\ &= x_0 \circ \overrightarrow{\text{exp}} \int_0^1 (1 + \bar{v}_1)X_1 + v_2(t)X_2 dt \\ &= y \circ \overrightarrow{\text{exp}} \int_0^1 \bar{v}_1 X_1 + v_2(t)g_t dt. \end{aligned}$$

where the passage from the first to the second line follows by the change of variable $t = \phi_v(s)$, noticing that $1 + v_1^0(s) = \dot{\phi}_v(s)$. In particular, we see that there is no more explicit dependence on the zero mean part v_1^0 in this new system of coordinates. Thus we may regard ρ as a “functional change of coordinates” on \mathcal{V}_3 , which hides the dependence of F on the zero mean part of the control v_1^0 , within the time reparametrization ϕ_v .

4.2. Regularity properties of ρ . We turn now to prove that ρ is an homeomorphism, and we begin stating a technical lemma which is crucial to show that ρ^{-1} is continuous, and whose proof is postponed in Appendix A for the sake of completeness.

Lemma 19. *Let $v \in \mathcal{V}_3$, and $(v^n)_{n \in \mathbb{N}}$ be a sequence converging to v in $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$. Define, for every $n \in \mathbb{N}$, $\phi_n : I \rightarrow I$ to be the time-reparametrization associated with v^n . Then:*

- (a) $\phi_n \rightarrow \phi_v$ uniformly on I ;
- (b) $\phi_n^{-1} \rightarrow \phi_v^{-1}$ pointwise on S_u .

Proposition 20. *The reparametrization map $\rho : \mathcal{V}_2 \rightarrow \mathcal{V}_3$ is a continuous homeomorphism onto its image, i.e. $\mathcal{V}_2 \simeq \mathcal{V}_3$.*

Proof. The inverse map ρ^{-1} has been explicitly computed in (4.1), thus it only remains to prove that ρ and ρ^{-1} are continuous. Actually we will just prove the continuity of ρ^{-1} , since the conclusion for ρ follows along similar (and somewhat simpler) reasonings. Let $v \in \mathcal{V}_3$ and let $(v^n)_{n \in \mathbb{N}}$ be any sequence converging to v in $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$. Then $v_1^{0,n} \rightarrow v_1^0$ and $\bar{v}_1^n \rightarrow \bar{v}_1$ in $L^\infty(I, \mathbb{R})$. Call $w_1^{0,n} := v_1^{0,n}/(1 + \bar{v}_1^n)$, and let w_1^0 be defined similarly. Trivially, $w_1^{0,n}$ converges to w_1^0 in $L^\infty(I, \mathbb{R})$, whence it suffices to establish:

$$(4.2) \quad \lim_{n \rightarrow \infty} \int_\Sigma \left| \frac{v_2^n(\phi_n^{-1}(s))}{(1 + w_1^{0,n})(\phi_n^{-1}(s))} - \frac{v_2(\phi_v^{-1}(s))}{(1 + w_1^0)(\phi_v^{-1}(s))} \right|^2 ds = 0,$$

where $\Sigma \subset I$ is the following full-measured set:

$$\Sigma := \left\{ s \in I \mid s = \phi_v(t) \text{ and } \exists \dot{\phi}_v(t) \neq 0 \right\} \cap \bigcap_{n \in \mathbb{N}} \left\{ s \in I \mid s = \phi_n(t) \text{ and } \exists \dot{\phi}_n(t) \neq 0 \right\}.$$

By the triangular inequality, (4.2) can be bounded in two steps. Indeed:

$$\begin{aligned} & \int_\Sigma \left| \frac{v_2^n(\phi_n^{-1}(s))}{(1 + w_1^{0,n})(\phi_n^{-1}(s))} - \frac{v_2(\phi_v^{-1}(s))}{(1 + w_1^0)(\phi_v^{-1}(s))} \right|^2 ds \\ & \leq 2 \left(\int_\Sigma \left| \frac{v_2^n(\phi_n^{-1}(s))}{(1 + w_1^{0,n})(\phi_n^{-1}(s))} - \frac{v_2(\phi_n^{-1}(s))}{(1 + w_1^0)(\phi_n^{-1}(s))} \right|^2 ds \right. \\ & \quad \left. + \int_\Sigma \left| \frac{v_2(\phi_n^{-1}(s))}{(1 + w_1^0)(\phi_n^{-1}(s))} - \frac{v_2(\phi_v^{-1}(s))}{(1 + w_1^0)(\phi_v^{-1}(s))} \right|^2 ds \right). \end{aligned}$$

By the change of variables $z = \phi_n^{-1}(s)$, the first summand is bounded, for n big enough, by

$$\begin{aligned} & \int_{\phi_n^{-1}(\Sigma)} \frac{|(1+w_1^0(z))v_2^n(z) - (1+w_1^{0,n}(z))v_2(z)|^2}{(1+w_1^{0,n}(z))(1+w_1^0(z))^2} dz \\ & \leq \frac{2}{\alpha^3} \int_0^1 |1+w_1^0(z)|^2 |v_2^n(z) - v_2(z)|^2 + |w_1^0(z) - w_1^{0,n}(z)|^2 |v_2(z)|^2 dz, \end{aligned}$$

and the convergence to zero follows. The convergence to zero of

$$\int_{\Sigma} \left| \frac{v_2(\phi_n^{-1}(s))}{(1+w_1^0)(\phi_n^{-1}(s))} - \frac{v_2(\phi_v^{-1}(s))}{(1+w_1^0)(\phi_v^{-1}(s))} \right|^2 ds$$

follows instead from Lemma 19 and the Lebesgue's dominated convergence theorem. \square

5. NORMAL FORMS AROUND RANK-TWO-NICE SINGULAR CURVES

5.1. A generalized Morse Lemma. The reparametrization map ρ yields a local system of coordinates on $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$, where the kernel Z of the Hessian map q disappears. To put it in more geometrical terms, notice that $\mathcal{V}_3 = \mathbb{R} \oplus \mathcal{V}_3^0 \oplus H^1(I, \mathbb{R})$ has a natural fiber bundle structure $\mathcal{V}_3 \xrightarrow{\pi} \mathcal{V}_3^0$ over \mathcal{V}_3^0 (which is an open submanifold of the Banach manifold $L^\infty(I, \mathbb{R})_0$), where the fiber is the Hilbert space $H = \mathbb{R} \oplus H^1(I, \mathbb{R})$. The fact that the endpoint map F does not depend on the zero mean part of the first control v_1^0 allows therefore to study F on H , agreeing that we identify H with the fiber $\pi^{-1}(\{0\})$. Thus F has the following expression:

$$\begin{aligned} F(c_1, w_2) &= x_0 \circ \overrightarrow{\text{exp}} \int_0^1 (1+c_1)X_1 + \dot{w}_2(t)X_2 dt \\ &= y \circ \overrightarrow{\text{exp}} \int_0^1 c_1 X_1 + \dot{w}_2(t)g_t dt. \end{aligned}$$

Accordingly, when we pass to the completion space $\overline{H} = \mathbb{R} \oplus \mathbb{R} \oplus H^1(I, \mathbb{R})$, F and G rewrite respectively as (see (3.1) and (3.2)):

$$\begin{aligned} F(c_1, c_2, w_2) &= x_0 \circ \overrightarrow{\text{exp}} \int_0^1 (1+c_1)e_*^{-w_2(t)X_2} X_1 dt \circ e^{c_2 X_2}, \\ G(c_1, c_2, w_2) &= y \circ \overrightarrow{\text{exp}} \int_0^1 (1+c_1)e_*^{-w_2(t)g_t} X_1 - X_1 dt \circ e^{c_2 X_2}. \end{aligned}$$

The hessian q is now non-degenerate on $\overline{\ker(d_0 G)}$, and therefore there holds a version of the ‘‘Generalized Morse Lemma’’, much in the spirit of [24, Lemma 1.2], and whose proof is reported below for the sake of completeness. Before giving the actual proof, let us fix the notation $v := (c_1, c_2, w_2) \in \overline{H}$, and let us write $G(v) = G(0) + d_0 G(v) + d_0^2 G(v) + R(v)$, where R denotes a remainder term whose first and second derivatives at zero vanish. Let us also temporarily consider variables $v = (v^{\ker}, v^E)$ adapted to the splitting

$$H = \ker(d_0 G) \oplus E.$$

Proposition 21 (Generalized Morse Lemma). *There exist neighborhoods $\mathcal{W} \subset \overline{H}$ and $\mathcal{O} \subset M$ of the origin, and origin-preserving diffeomorphisms $\sigma : \mathcal{W} \rightarrow \mathcal{W}$ and $\psi : \mathcal{O} \rightarrow \mathcal{O}$, such that for every $v \in \mathcal{W}$ there holds the identity:*

$$(\psi \circ G \circ \sigma)(v) = y + d_0 G(v^E) + \lambda \text{He}_0 G(v^{\ker}).$$

$$\begin{array}{ccc}
\overline{\mathcal{W}} & \xrightarrow{G(0) + d_0G + d_0^2G + tR} & \mathcal{O} \\
\uparrow P_t & & \downarrow Q_t \\
\overline{\mathcal{W}} & \xrightarrow{G(0) + d_0G + d_0^2G} & \mathcal{O}
\end{array}$$

FIGURE 3.

Proof. We begin by proving the following statement: there exist neighborhoods $\overline{\mathcal{W}} \subset \overline{H}$ and $\mathcal{O} \subset M$, respectively of the origin and of y such that for every $t \in I$, there exist an origin-preserving diffeomorphism $P_t : \overline{\mathcal{W}} \rightarrow \overline{\mathcal{W}}$ and a diffeomorphism $Q_t : \mathcal{O} \rightarrow \mathcal{O}$ that preserves y , for which the diagram in Figure 3 commutes. More specifically, we will look for families $(P_t)_{t \in I}$, $(Q_t)_{t \in I}$ of diffeomorphisms of the form $P_t = \overrightarrow{\exp} \int_0^t X_\tau d\tau$ and $Q_t = \overleftarrow{\exp} \int_0^t Y_\tau d\tau$, for suitable locally Lipschitz time-dependent vector fields ⁶ X_τ and Y_τ on $\overline{\mathcal{W}}$ and \mathcal{O} (compare with the classical Moser's trick [20]).

Let us fix local coordinates on a neighborhood $\mathcal{O} \subset M$ of y , subordinated to the splitting $T_y M \simeq \mathbb{R}^m = \text{coker}(d_0G) \oplus \text{Im}(d_0G)$, and let us suppose, without loss of generality, that $G(0) = 0$. Any function $f : \overline{H} \rightarrow \mathbb{R}^m$ can be decomposed as a sum $f = f^\lambda + f^E$, where $f^\lambda = \langle \lambda, f \rangle$ denotes the projection of f along the abnormal direction.

The commutativity condition reads $P_t \circ (d_0G + d_0^2G + tR) \circ Q_t = d_0G + d_0^2G$. Notice that for $t = 0$ the identity holds. Differentiating this equation we obtain:

$$(5.1) \quad P_t \circ (X_t \cdot (d_0G + d_0^2G + tR) + R + (d_0G + d_0^2G + tR) \circ Y_t) \circ Q_t = 0.$$

We look for solutions to (5.1) of the form $X_t = X_t^{\ker} + X_t^E$ and $Y_t = Y_t^\lambda$, where $X_t^{\ker} \in \overline{\text{ker}(d_0G)}$ and Y_t^λ is parallel to λ in \mathbb{R}^m . Solving (5.1) is thus equivalent to solve the following system of equations:

$$(5.2) \quad \begin{cases} d_0G X_t(v) + 2B^E(v, X_t(v)) + td_v R^E X_t(v) = -R^E(v) \\ 2B^\lambda(v, X_t(v)) + td_v R^\lambda X_t(v) + Y_t^\lambda(d_0G(v) + d_0G^2(v, v) + tR(v)) = -R^\lambda(v), \end{cases}$$

where we may forget about P_t and Q_t since they are diffeomorphisms, and $B : \overline{H} \times \overline{H} \rightarrow \mathbb{R}^m$ denotes the vector-valued bilinear form associated to d_0^2G . Let $d_0G^{-1} : \text{Im}(d_0G) \rightarrow E$ denotes the right pseudo-inverse to d_0G . Then we solve the first equation of (5.2) with respect to X_t^E as follows: let $J_t^E(v) : E \rightarrow E$ be defined as

$$J_t^E(v) := \text{Id}_E + 2d_0G^{-1}B^E(v, \cdot) + td_0G^{-1}d_v R^E.$$

For every $t \in I$, $J_t^E(0) = \text{Id}_E$, therefore there exists a neighborhood $\overline{\mathcal{W}}_1^t \subset \overline{H}$ of the origin such that $J_t^E(v)$ is invertible for every $v \in \overline{\mathcal{W}}_1^t$. By compactness, we find $\overline{\mathcal{W}}_1$ independently on t , and for $v \in \overline{\mathcal{W}}_1$ we have:

$$X_t^E(v) = -J_t^E(v)^{-1}d_0G^{-1}(R^E(v) + 2B^E(v, X_t^{\ker}(v)) + td_v R^E X_t^{\ker}(v)).$$

⁶For the definition of the right and left chronological exponentials we refer to [4, Chapter 2]. For us, it will only be important to recall that $\frac{d}{dt}P_t = P_t \circ X_t$ and $\frac{d}{dt}Q_t = Y_t \circ Q_t$.

It remains to find X_t^{\ker} . Once we know it, we substitute in the equation above to find X_t^{\ker} , which in turn will solve (5.2). We turn to the scalar equation:

$$(5.3) \quad \langle A_t(v), X_t^{\ker}(v) \rangle + Y_t^\lambda (d_0G(v) + d_0^2G(v, v) + tR(v)) = -S_t^\lambda(v),$$

where $\langle \cdot, \cdot \rangle$ denotes the Hilbert product on \overline{H} , and

$$\begin{aligned} \langle A_t(v), \cdot \rangle &= 2\lambda B(v, \cdot) - 2\lambda B(v, J_t^E(v)^{-1}d_0G^{-1}(2B^E(v, \cdot) + td_vR^E(\cdot))) \\ &\quad + t\lambda d_vR(\cdot) - t\lambda d_vR(J_t^E(v)^{-1}d_0G^{-1}(2B^E(v, \cdot) + td_vR^E(\cdot))), \\ S_t^\lambda(v) &= \lambda R(v) + 2\lambda B(v, J_t^E(v)^{-1}d_0G^{-1}R^E(v)) + t\lambda d_vR(J_t^E(v)^{-1}d_0G^{-1}R^E(v)). \end{aligned}$$

Notice that the first and the second derivatives at zero of S_t^λ vanish for every $t \in I$.

Actually, since $X_t^{\ker}(v) \in \overline{\ker(d_0G)}$ for every $v \in \overline{W}_1$, we only need to consider the projection $\pi \circ A_t : \overline{W}_1 \rightarrow \overline{\ker(d_0G)}$ (here π denotes the orthogonal projection onto $\overline{\ker(d_0G)}$). Moreover, the operator

$$\pi \circ d_0A_t|_{\overline{\ker(d_0G)}} : \overline{\ker(d_0G)} \rightarrow \overline{\ker(d_0G)}$$

coincides for every $t \in I$ with the operator \mathcal{L} of Corollary 17, and therefore it is invertible. Let us introduce a new system of coordinates in a neighborhood $\overline{W} \subset \overline{W}_1$ of the origin, namely we consider the map

$$(5.4) \quad \Phi_t(v) = (\pi \circ A_t(v), d_0G^*(d_0G + d_0^2G(v, v) + tR(v))),$$

where $d_0G^* : \mathbb{R}^m \rightarrow E$ denotes an operator adjoint to d_0G . An easy computation shows that for every $t \in I$ there holds:

$$(5.5) \quad d_0\Phi_t = \begin{pmatrix} \mathcal{L} & * \\ 0 & d_0G^*d_0G \end{pmatrix}.$$

Thus we can find a neighborhood \overline{W} of the origin, where (5.5) holds independently on t , and $\Phi_t(v) = (w_t^{\ker}(v), w_t^E(v))$ defines a local diffeomorphism on \overline{W} . By the Hadamard Lemma there exists a smooth function $\tilde{S}_t : \overline{W} \rightarrow \overline{\ker(d_0G)}$ such that

$$(5.6) \quad \begin{aligned} S_t^\lambda(v) &= S_t^\lambda(\Phi_t^{-1}(w_t^{\ker}(v), w_t^E(v))) \\ &= (S_t^\lambda \circ \Phi_t^{-1})(0, w_t^E(v)) + \langle \tilde{S}_t(v), w_t^{\ker}(v) \rangle. \end{aligned}$$

Comparing (5.3), (5.4) and (5.6), it suffices to set

$$X_t^{\ker} := \tilde{S}_t, \quad Y_t^\lambda := S_t^\lambda \circ \Phi_t^{-1} \circ d_0G^*$$

to find X_t . Finally, it is not difficult to see that X_t^{\ker} is Lipschitz, and that its first derivative at the origin vanishes, and that also Y_t^λ and X_t^E are Lipschitz with respect to their arguments, and their first and second order derivatives at the origin vanish.

The previous argument proves that P_1 exists in the completion space \overline{H} . It thus remains to show that P_1 induces a local diffeomorphism σ on some open neighborhood $\mathcal{W} \subset H$ of the origin, and the proposition will follow setting $\psi = Q_1$. We know from [24, Theorem 3.2] that P_1 is explicitly given by a system of nonlinear Urysohn integral equations of the second kind with small kernels, of the form

$$P_1(v)(t) = v(t) - \int_0^1 K(v, \tau, t) d\tau,$$

where the K is differentiable with respect to the t -variable. It follows that $P_1(v) \in H = \mathbb{R} \oplus H^1(I, \mathbb{R})$ if and only if $v \in H$, yielding that the $L^2(I, \mathbb{R})$ component of $P_1(v)$ is differentiable

with respect to time if and only if so is the $L^2(I, \mathbb{R})$ component of v . To complete the proof it is then sufficient to set

$$\mathcal{W} := \overline{\mathcal{W}} \cap H, \quad \text{and} \quad \sigma := P_1|_{\mathcal{W}}.$$

□

5.2. Proof of Theorem 1. Let $\mathcal{V}_3 \xrightarrow{\pi} \mathcal{V}_3^0$ be the fiber bundle consider at the beginning of the previous section. We consider the sets \mathcal{O} and \mathcal{W} as in Proposition 21, so that the set $\pi^{-1}(\mathcal{W}) \subset \mathcal{V}_3$ is an open neighborhood of the origin in $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$. Then we define

$$\mathcal{V} := (\pi \circ \rho)^{-1}(\mathcal{W}) \subset \mathcal{V}_2,$$

which is also open because the map ρ is continuous by Proposition 20, and $\rho(\mathcal{V}) \subset \mathcal{V}_3$. The diffeomorphisms

$$\varphi : \pi^{-1}(\mathcal{W}) \rightarrow \pi^{-1}(\mathcal{W}), \quad \varphi := (\sigma, \text{Id}_{L^\infty(I, \mathbb{R})_0})$$

and $\psi : \mathcal{O} \rightarrow \mathcal{O}$, with σ and ψ as in Proposition 21 provides the desired changes of coordinates on $\rho(\mathcal{V})$ and \mathcal{O} , respectively, and the theorem is completely proved once we define

$$\mu := \varphi \circ \rho, \quad \mu : \mathcal{V} \rightarrow \pi^{-1}(\mathcal{W}) \subset L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$$

(notice that we tacitly assumed the decomposition $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R}) = \mathbb{R} \oplus L^2(I, \mathbb{R}) \oplus L^\infty(I, \mathbb{R})_0$ in the component-wise definition of φ).

5.3. Isolation of rank-two-nice singular curves. We discuss in this section some isolation properties of rank-two-nice singular curves in $\Omega(y)$, both among extremal curves (i.e. critical points of the extended endpoint map (G, J)) and among singular curves (which are critical points for G).

Proposition 22. *There exists a neighborhood $\mathcal{V}_4 \subset \mathcal{V}$, such that the only extremal controls contained in $\mathcal{V}_4 \cap \Omega(y)$ are of the form $(v_1^0, 0)$, with v_1^0 of zero mean. Then γ is isolated (up to reparametrizations) among singular curves in $\Omega(y)$, with respect to the $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$ topology.*

Proof. We begin by showing that there are no extremal controls $u \in \rho(\mathcal{V}) \cap \Omega(y)$ which have nonzero second component. By Theorem 1, there exists an origin-preserving diffeomorphism $\varphi : \rho(\mathcal{V}) \rightarrow \rho(\mathcal{V})$ such that:

$$(5.7) \quad \langle \lambda, G(\varphi(w^{\ker}, 0)) \rangle = \langle \lambda, G(0) \rangle + \underbrace{\lambda \text{He}_0(G)(w^{\ker})}_{q(w^{\ker})}, \quad \text{for every } w = (w^{\ker}, 0) \in \rho(\mathcal{V}).$$

Let $(w^{\ker}, 0) \in \rho(\mathcal{V})$ have nonzero second component. Since q is non-degenerate, differentiating (5.7) we see that

$$\lambda d_{\varphi(w^{\ker}, 0)} G : \ker(d_0 G) \rightarrow \mathbb{R}$$

is not the zero operator. More specifically, there exists $z \in \ker(d_0 G)$ such that $d_{(w^{\ker}, 0)} \varphi(z)$ generates its image.

Let u be contained in $\rho(\mathcal{V}) \cap \Omega(y)$ and have nonzero second component; without loss of generality we suppose that $d_u(G, J)$ is of corank one. Then u has the form $u = \varphi(w^{\ker}, 0)$ for some $(w^{\ker}, 0) \in \rho(\mathcal{V})$, which necessarily has nonzero second component as well, and thus by the above reasoning $d_u G$ is of full rank. We can also suppose, shrinking $\rho(\mathcal{V})$ if necessary, that even $d_u J$ is nonzero (recall that $d_0 J = (1, 0)$), but this would contradict the fact that u is an extremal control. To conclude, let us notice that since ρ is an homeomorphism,

$$\mathcal{V}_4 := \mathcal{V} \cap \rho(\mathcal{V})$$

is an open neighborhood of the origin in $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$. All the extremal controls contained in $\mathcal{V}_4 \cap \Omega(y)$ have zero second component, and satisfy the relation

$$y = F(v_1, 0) = x_0 \circ e^{(1+\int_0^1 v_1(t)dt)X_1} = y \circ e^{\int_0^1 v_1(t)dtX_1},$$

yielding that $v_1 = v_1^0$ has zero mean, whence the proposition follows. \square

Corollary 23. *There exists a weak neighborhood of the origin $\mathcal{V}_5 \subset L^2(I, \mathbb{R}^2)$, such that the only singular controls contained in \mathcal{V}_5 are of the form $(v_1^0, 0)$, with v_1^0 of zero mean. In particular, γ is isolated (up to reparametrizations) among the singular curves in $\Omega(y)$, with respect to the weak $L^2(I, \mathbb{R}^2)$ topology.*

Proof. Assume by contradiction that $(v_n)_{n \in \mathbb{N}} \subset \Omega(y)$ is a sequence of abnormal controls weakly converging to zero in $L^2(I, \mathbb{R})$. By Proposition 2, $\gamma_{v_n} \rightarrow \gamma$ uniformly on I and $d_{v_n}F \rightarrow d_0F$ strongly as operators. In particular, $d_{v_n}F$ becomes eventually of corank-one, and the (unique) abnormal norm-one covector λ_{v_n} tends to λ in \mathbb{R}^m . The two conditions in Section 1.2, defining a rank-two nice singular curve, are open in the set of all singular curves in $\Omega(y)$. Therefore we may assume without loss of generality that all the curves γ_{v_n} are rank-two nice, and in turn that all the controls v_n are smooth. Finally, the relations $\gamma_{v_n} \rightarrow \gamma$ and $d_{v_n}F \rightarrow d_0F$ yield the uniform convergence on I even for the dual extremal trajectories $t \mapsto \lambda_{v_n}(t)$:

$$\lim_{n \rightarrow \infty} \sup_{t \in I} \|\lambda_{v_n}(\cdot) - \lambda(\cdot)\| = 0.$$

All these facts together yield that $v_n \rightarrow 0$ in $L^\infty(I, \mathbb{R}) \oplus L^2(I, \mathbb{R})$, therefore Proposition 22 implies that eventually v_n has to be of the form $(v_{n,1}^0, 0)$, with $v_{n,1}^0$ of zero mean, and the claim follows. \square

6. EXAMPLES

6.1. The general framework. We explain in this last section how to compute conjugate points. Let us consider a rank-two totally nonholonomic distribution $\Delta \subset TM$, and let us suppose for simplicity that $\Delta_x = \text{span}\{X_1(x), X_2(x)\}$ in the domain under consideration. We consider as in Definition 9 a local chart $\mathcal{V}_1 \in L^2(I, \mathbb{R}^2)$ centered at the origin.

Definition 24. For every $s \in I$ we define the subset $\mathcal{V}_1^s \subset L^2([0, s], \mathbb{R}^2)$ by

$$\mathcal{V}_1^s := \{v_{[0,s]} \mid v \in \mathcal{V}_1\}.$$

We consider the time- s endpoint map $F^s : \mathcal{V}_1^s \rightarrow M$ and the time- s energy map $J^s : \mathcal{V}_1^s \rightarrow M$, defined respectively as:

$$F^s(v_1, v_2) := x_0 \circ \overrightarrow{\text{exp}} \int_0^s (1 + v_1(t))X_1 + v_2(t)X_2 dt,$$

$$J^s(v_1, v_2) := \frac{1}{2} \|(1 + v_1(t), v_2(t))\|_{L^2([0,s], \mathbb{R}^2)}^2.$$

The time- s extended endpoint map $\Phi^s : \mathcal{V}_1^s \rightarrow M \times \mathbb{R}$ will denote as usual the pair

$$\Phi^s(v_1, v_2) = (F^s(v_1, v_2), J^s(v_1, v_2)).$$

For any $s \in I$, let us consider the restriction of the nice abnormal $t \mapsto \gamma(t)$ onto $[0, s]$. The extremal lift $t \mapsto \lambda_t$ of γ is then parametrized on $[0, s]$ by

$$\lambda_t = (e^{-tX_1})^* \lambda_0, \quad (e^{-tX_1})^* : T_{x_0}^* M \rightarrow T_{\gamma(t)}^* M,$$

where $(e^{-tX_1})^*$ denotes the adjoint of the differential of the flow map $t \mapsto e^{tX_1}$, and $\lambda_0 \in T_{x_0}^*M$ is the initial covector of each such restriction.

There are two quadratic maps that evolve in time, namely the Hessians:

$$q^s := \lambda \text{He}_0(F^s), \quad \text{and} \quad \widehat{q}^s := \bar{\lambda} \text{He}_0(\Phi^s),$$

and to detect conjugate times we introduce appropriate ‘‘Jacobi equations’’ that naturally live on the cotangent space T^*M . Since q^s and \widehat{q}^s have formally the same expression, we will try as much as possible to treat them at the same time, pointing out when necessary the important differences.

6.1.1. The symplectic setting. Given any Hamiltonian function $a \in C^\infty(T^*M)$, we define its Hamiltonian lift $\vec{a} \in \text{Vec}(T^*M)$ by

$$\sigma_\mu(\cdot, \vec{a}) = d_\mu a(\cdot), \quad \text{for every } \mu \in T^*M,$$

where σ denotes the canonical symplectic form on T^*M .

Definition 25. Let $s \in I$. For any $t \in [0, s]$, let

$$g_t^s := e_*^{(s-t)X_1} X_2.$$

Let us then define the fiber-wise linear Hamiltonians $\xi_1, \eta_t^s : T^*M \rightarrow \mathbb{R}$ by:

$$\xi_1(\mu) = \langle \mu, X_1(\pi(\mu)) \rangle, \quad \eta_t^s(\mu) = \langle \mu, g_t^s(\pi(\mu)) \rangle,$$

where $\pi : T^*M \rightarrow M$ denotes the canonical projection, and let $\vec{\xi}_1, \vec{\eta}_t^s, t \in [0, s]$, be their corresponding Hamiltonian lifts.

By (3.7) and (2.10) we have (notice that there is no c_1 component in $\overline{\ker(d_0\Phi^s)}$):

$$(6.1) \quad \overline{\ker(d_0F^s)} = \left\{ w_2 \in L^2([0, s], \mathbb{R}) \left| \int_0^s w_2(t) \dot{g}_t^s(\gamma(s)) dt \in \text{span}\{X_1(\gamma(s)), X_2(\gamma(s))\} \right. \right\},$$

$$\overline{\ker(d_0\Phi^s)} = \left\{ w_2 \in L^2([0, s], \mathbb{R}) \left| \int_0^s w_2(t) \dot{g}_t^s(\gamma(s)) dt \in \text{span}\{X_2(\gamma(s))\} \right. \right\}.$$

In both cases the Hessian is given by:

$$(6.2) \quad q^s(c_2, w_2) = \int_0^s \langle \lambda_s, [g_t^s, \dot{g}_t^s](\gamma(s)) \rangle w_2(t)^2 dt + \int_0^s \left\langle \lambda_s, \left[c_2 g_s^s + \int_0^t w_2(\tau) \dot{g}_\tau^s d\tau, w_2(t) \dot{g}_t^s \right] (\gamma(s)) \right\rangle dt,$$

therefore for clarity’s sake we avoid the notation \widehat{q}^s when referring to the Hessian $\bar{\lambda} \text{He}_0(\Phi^s)$ in the sequel.

We now translate these informations on the cotangent bundle T^*M . It is convenient to identify in the standard way $T_{\gamma(s)}^*M$ with $T_{\lambda_s}(T_{\gamma(s)}^*M)$, interpreting any covector $\nu \in T_{\gamma(s)}^*M$ as the value at λ_s of the Euler vector field \mathbf{e} on $T(T_{\gamma(s)}^*M)$. This means that in a system (p_1, \dots, p_m) of local coordinates on $T_{\gamma(s)}^*M$ we identify $\nu = (\nu_1, \dots, \nu_m)$ with

$$\mathbf{e}(\nu) = \nu_1 \partial_{p_1} + \dots + \nu_m \partial_{p_m}.$$

We declare $\Sigma = \lambda_s^\perp / \mathbb{R} \lambda_s$ to be the skew-orthogonal complement of λ_s in the symplectic space $T_{\lambda_s}(T^*M)$, and we call $\Pi = T_{\gamma(s)}^*M / \mathbb{R} \lambda_s$. Then Σ is a symplectic subspace of $T_{\lambda_s}(T^*M)$ of dimension $2(m-1)$, and Π is a Lagrangian subspace of Σ , therefore of dimension $m-1$. For every $\mu \in T_{\gamma(s)}^*M$, we have that

$$\sigma_{\lambda_s}(\mu, \vec{\xi}_1) = \langle \mu, X_1 \rangle_{\gamma(s)}, \quad \sigma_{\lambda_s}(\mu, \vec{\eta}_t^s) = \langle \mu, g_t^s \rangle_{\gamma(s)},$$

and thus $\sigma_{\lambda_s}(\lambda_s, \vec{\xi}_1) = \sigma_{\lambda_s}(\lambda_s, \vec{\eta}_t^s) = 0$ for every $t \in [0, s]$. In particular, we can treat both $\vec{\xi}_1$ and $\vec{\eta}_t^s$ as elements of Σ .

Let $l_{t,s}^0 := \sigma_{\lambda_s}(\vec{\eta}_t^s, \vec{\eta}_t^s)$. The quadratic form (6.2) becomes

$$(6.3) \quad q^s(w_2) = \int_0^s l_{t,s}^0 w_2(t)^2 dt + \int_0^s \sigma_{\lambda_s} \left(c_2 \vec{\eta}_s^s + \int_0^s w_2(\tau) \dot{\vec{\eta}}_\tau^s d\tau, w_2(t) \dot{\vec{\eta}}_t^s \right) dt,$$

while the kernel conditions (6.1) read

$$(6.4) \quad \begin{aligned} \overline{\ker(d_0 F^s)} &= \left\{ w_2 \in L^2([0, s], \mathbb{R}) \mid \int_0^s w_2(t) \dot{\vec{\eta}}_t^s dt \in \Pi + \text{span}\{\vec{\xi}_1, \vec{\eta}_s^s\} \right\}, \\ \overline{\ker(d_0 \Phi^s)} &= \left\{ w_2 \in L^2([0, s], \mathbb{R}) \mid \int_0^s w_2(t) \dot{\vec{\eta}}_t^s dt \in \Pi + \text{span}\{\vec{\eta}_s^s\} \right\} \end{aligned}$$

or, equivalently,

$$(6.5) \quad \overline{\ker(d_0 F^s)} = \left\{ w_2 \in L^2([0, s], \mathbb{R}) \mid \int_0^s \sigma_{\lambda_s}(\nu, \dot{\vec{\eta}}_t^s) w_2(t) dt = 0, \text{ for every } \nu \in \Pi \cap \{\vec{\xi}_1, \vec{\eta}_s^s\}^\perp + \mathbb{R}\vec{\xi}_1 \right\},$$

$$\overline{\ker(d_0 \Phi^s)} = \left\{ w_2 \in L^2([0, s], \mathbb{R}) \mid \int_0^s \sigma_{\lambda_s}(\nu, \dot{\vec{\eta}}_t^s) w_2(t) dt = 0, \text{ for every } \nu \in \Pi \cap \{\vec{\eta}_s^s\}^\perp + \mathbb{R}\vec{\xi}_1 \right\}.$$

Remark 4. In (6.5), we can define ν modulo $\mathbb{R}\vec{\xi}_1$ since $\sigma_{\lambda_s}(\vec{\xi}_1, \dot{\vec{\eta}}_t^s) \equiv 0$ for every $t \in [0, s]$. We choose to write it explicitly for later convenience.

6.1.2. *Conjugate points.* Following [5, 6] we begin recalling the symplectic definition of a conjugate point along γ (compare with Definition 12).

Definition 26. Conjugate points along the nice abnormal trajectory γ are time instants $s \in \mathbb{R}$ at which the quadratic form q^s in (6.3), whose domain is determined by either one of the two conditions in (6.5), has a nontrivial kernel. The multiplicity of s as a conjugate point equals the dimension of this kernel.

Starting from the definition of q^s in (6.3), we see that its kernel is composed by all elements $w_2^0 \in \overline{\ker(d_0 F^s)}$ (or to $\overline{\ker(d_0 \Phi^s)}$), such that

$$\int_0^s \left(l_{t,s}^0 w_2^0(t) + \sigma_{\lambda_s} \left(c_2^0 \vec{\eta}_s^s + \int_0^t w_2^0(\tau) \dot{\vec{\eta}}_\tau^s d\tau, \dot{\vec{\eta}}_t^s \right) \right) w_2(t) dt = 0$$

for every $w_2 \in \overline{\ker(d_0 F^s)}$ (resp. for every $w_2 \in \overline{\ker(d_0 \Phi^s)}$). By (6.5), this implies that for all $t \in [0, s]$

$$(6.6) \quad l_{t,s}^0 w_2^0(t) + \sigma_{\lambda_s} \left(c_2^0 \vec{\eta}_s^s + \int_0^t w_2^0(\tau) \dot{\vec{\eta}}_\tau^s d\tau, \dot{\vec{\eta}}_t^s \right) = \sigma_{\lambda_s}(-\nu, \dot{\vec{\eta}}_t^s),$$

for some ν that belongs to $\Pi \cap \{\vec{\xi}_1, \vec{\eta}_s^s\}^\perp + \mathbb{R}\vec{\xi}_1$ (respectively, to $\Pi \cap \{\vec{\eta}_s^s\}^\perp + \mathbb{R}\vec{\xi}_1$).

Let $k(t) := \int_0^t w_2^0(\tau) \dot{\vec{\eta}}_\tau^s d\tau + c_2^0 \vec{\eta}_s^s + \nu$. Multiplying both its sides by $\dot{\vec{\eta}}_t^s$, we rewrite (6.6) as

$$(6.7) \quad l_{t,s}^0 \dot{k}(t) = \sigma_{\lambda_s}(\dot{\vec{\eta}}_t^s, k(t)) \dot{\vec{\eta}}_t^s.$$

The corresponding boundary conditions become, respectively,

$$(6.8) \quad (a) \quad \begin{cases} k(0) & \in \Pi \cap \{\vec{\xi}_1, \vec{\eta}_s^s\}^\perp + \mathbb{R}\vec{\xi}_1 + \mathbb{R}\vec{\eta}_s^s, \\ k(s) & \in \Pi + \mathbb{R}\vec{\xi}_1 + \mathbb{R}\vec{\eta}_s^s, \end{cases} \quad (b) \quad \begin{cases} k(0) & \in \Pi \cap \{\vec{\eta}_s^s\}^\perp + \mathbb{R}\vec{\xi}_1 + \mathbb{R}\vec{\eta}_s^s, \\ k(s) & \in \Pi + \mathbb{R}\vec{\eta}_s^s. \end{cases}$$

Remark 5. Recall that, given a Lagrangian subspace Λ and an isotropic subspace Γ , then $\Lambda^\Gamma = \Lambda \cap \Gamma^\perp + \Gamma = (\Lambda + \Gamma) \cap \Gamma^\perp$ is again a Lagrangian subspace (see, e.g. [4]).

Remark 6. Notice that $\vec{\xi}_1$ and $\vec{\eta}_t^s$ are solutions to (6.7). The point is to verify whether or not they also satisfy the boundary conditions in (6.8).

6.2. Regular distributions. We specify now the Jacobi equations for a particular class of rank-two sub-Riemannian structures, the so-called regular distributions, that have been intensively investigated starting from the seminal works [25, 17]. Assume M is an $(m+2)$ -dimensional manifold, and that $\Delta = \text{span}\{X_1, X_2\}$ in the domain under consideration. Moreover we assume that:

- i) $X_1, X_2, \dots, (\text{ad}X_1)^{m-1}X_2$ are linearly independent.
- ii) There exist smooth functions $\beta, \{\alpha^i, i = 0, \dots, m-1\}$ on M , such that

$$(6.9) \quad (\text{ad}X_1)^m X_2 = \beta X_1 + \sum_{i=0}^{m-1} \alpha^i (\text{ad}X_1)^i X_2.$$

- iii) $[[X_1, X_2], X_2]$ is linearly independent from $V = \text{span}\{X_1, X_2, \dots, (\text{ad}X_1)^{m-1}X_2\}$.

Under these hypotheses it turns out [25, Section 8] that integral curves of the vector field X_1 are indeed corank-one abnormal geodesics for Δ . Moreover, these curves are also strictly abnormal as soon as $\beta \neq 0$ along the trajectory.

Let $i \in \mathbb{N}$, and let us define

$$g_t^{s,(i)} := \partial_t^{(i)} g_t^s = (-1)^i e_*^{(s-t)X_1} (\text{ad}X_1)^i X_2,$$

with g_t^s as in Definition 25, and

$$l_{t,s}^{(i)} := \langle \lambda_s, [g_t^{s,(1)}, g_t^{s,(i)}](\gamma(s)) \rangle = (-1)^{i+1} \langle \lambda_t, [[X_1, X_2], (\text{ad}X_1)^i X_2](\gamma(t)) \rangle.$$

Calling $\beta_t := \beta(\gamma(t))$ and $\alpha_t^i := \alpha^i(\gamma(t))$, it is immediate to deduce from (6.9) its symplectic version along the abnormal curve $t \mapsto \gamma(t)$, that is

$$(6.10) \quad \vec{\eta}_t^{s,(m)} = \beta_t \vec{\xi}_1 + \sum_{i=0}^{m-1} \alpha_t^i \vec{\eta}_t^{s,(i)}.$$

Observe that Σ admits the decomposition $\Sigma = \Pi \oplus \{\vec{\xi}_1, \vec{\eta}_t^s, t \in [0, s]\}$ and that, thanks to (6.10), the set $\{\vec{\xi}_1, \vec{\eta}_\tau^s, \dots, \vec{\eta}_\tau^{s,(m-1)}\}$ is a basis for $Z = \{\vec{\xi}_1, \vec{\eta}_t^s, t \in [0, s]\}$, for every $\tau \in [0, s]$. Notice that Z is not a Lagrangian subspace, nonetheless the symplectic form σ_{λ_s} defines a non-degenerate splitting between Π and Z . Returning to (6.7), we write $k(t) = z_t + \theta_t$, with $z_t \in Z$ and $\theta_t \in \Pi$. Then (6.7) splits as the differential system of equations

$$(6.11) \quad l_{t,s}^0 \dot{z}_t = \sigma_{\lambda_s}(\vec{\eta}_t^{s,(1)}, z_t + \theta_t) \vec{\eta}_t^{s,(1)}, \quad \dot{\theta}_t = 0.$$

The boundary conditions read, in this formulation,

$$z_0 = 0, \quad z_s \in \Pi + \text{span}\{\vec{\xi}_1, \vec{\eta}_s^s\} \quad \text{and} \quad \sigma_{\lambda_s}(\vec{\eta}_s^s, \theta_0) = \sigma_{\lambda_s}(\vec{\xi}_1, \theta_0) = 0$$

in the first case, while we have

$$z_0 = 0, \quad z_s \in \Pi + \text{span}\{\vec{\eta}_s^s\} \quad \text{and} \quad \sigma_{\lambda_s}(\vec{\eta}_s^s, \theta_0) = 0$$

in the second. In both cases the condition on z_s is derived from the kernel conditions (6.4).

We can further present z_t as $z_t = z_t^f \vec{\xi}_1 + \sum_{i=0}^{m-1} z_t^i \vec{\eta}_t^{s,(i)}$, and define $\zeta_t := \sigma_{\lambda_s}(\vec{\eta}_t^s, \theta_0)$. In this way, (6.11) is equivalent to the following system of equations:

$$(6.12) \quad \begin{cases} \dot{z}_t^f &= -\beta_t z_t^{m-1}, \\ \dot{z}_t^0 &= -\alpha_t^0 z_t^{m-1}, \\ l_{t,s}^0 (\dot{z}_t^1 + \alpha_t^1 z_t^{m-1}) &= \sum_{j=2}^{m-1} l_{t,s}^{(j)} z_t^j + \dot{\zeta}_t, \quad z_0^1 = 0, \\ \dot{z}_t^j + \alpha_t^j z_t^{m-1} &= -z_t^{j-1}, \quad z_0^j = 0, \quad j = 2, \dots, m-1, \\ \dot{\zeta}_t^{(m)} &= \beta_t \sigma_{\lambda_s}(\vec{\xi}_1, \theta_0) + \sum_{i=0}^{m-1} \alpha_t^i \zeta_t^{(i)}, \quad \zeta_0 = 0. \end{cases}$$

In particular, s is a conjugate point along the abnormal trajectory $t \mapsto \gamma_t$ if and only if (6.12) admits a nontrivial solution that further satisfies $z_s^i = 0$ for all $i = 1, \dots, m-1$, and also $z_s^f = 0$ in the second case

6.2.1. The four-dimensional case. We specialize the theory of the conjugate points for a rank-two sub-Riemannian structure on a four dimensional manifold. In this case the vector field X_1 , that satisfies all the Lie bracket configurations required in the previous part exists if the structure is of maximal growth, that is if it is of Engel type [11]. We study in this case the Jacobi equations for the Hessian map both of the endpoint F^s and of the extended endpoint map Φ^s .

We begin with the endpoint map F^s . In this case we also have that $\sigma_{\lambda_s}(\vec{\xi}_1, \theta_0) = 0$, and the relevant equations are:

$$(6.13) \quad \begin{cases} l_{t,s}^0 (\dot{z}_t^1 + \alpha_t^1 z_t^1) &= \dot{\zeta}_t, \quad z_0^1 = 0, \\ \dot{\zeta}_t &= \alpha_t^0 \zeta_t + \alpha_t^1 \dot{\zeta}_t, \quad \zeta_0 = 0, \quad \dot{\zeta}_0 = 1. \end{cases}$$

Moreover, from (6.10) we easily calculate that

$$l_{t,s}^0 = \frac{d}{dt} \langle \lambda_s, [\dot{g}_t^s, g_t^s](\gamma(s)) \rangle = \alpha_t^1 l_{t,s}^0, \quad l_{0,s}^0 = 1,$$

that is $l_{s,t}^0 = e^{\int_0^t \alpha_r^1 dr}$. From here it is immediate to compute that $z_t^1 = \zeta_t e^{-\int_0^t \alpha_r^1 dr}$. In particular $z_0^1 = 0$ if, and only if $\zeta_0 = 0$ (notice instead that $\zeta_s = 0$ and, in turn, $z_s^1 = 0$, is granted by construction).

It is also possible to give a totally intrinsic description of a conjugate point in this case. In fact, we have that $\sigma_{\lambda_s}(\vec{\eta}_s^s, \theta_0) = \sigma_{\lambda_s}(\vec{\eta}_0^s, \theta_0) = \sigma_{\lambda_s}(\vec{\xi}_1, \theta_0) = 0$. On the other hand, $\theta_0 \in \Pi = T_{\gamma(s)}^* M / \mathbb{R} \lambda_s$, which is three dimensional. Then the linear map $\sigma_{\lambda_s}(\cdot, \theta_0) = \langle \cdot, \theta_0 \rangle_{\gamma(s)}$ cannot have a three dimensional kernel, for otherwise θ_0 would be zero, and this implies that

$$X_1(\gamma(s)) \wedge X_2(\gamma(s)) \wedge g_0^s(\gamma(s)) = 0.$$

For the extended endpoint map Φ^s , the relevant equations become:

$$(6.14) \quad \begin{cases} \dot{z}_t^f &= -\beta_t z_t^1, \\ l_{t,s}^0 (\dot{z}_t^1 + \alpha_t^1 z_t^1) &= \dot{\zeta}_t, \quad z_0^1 = 0, \\ \dot{\zeta}_t &= \alpha_t^0 \zeta_t + \alpha_t^1 \dot{\zeta}_t + \beta_t \sigma_{\lambda_s}(\vec{\xi}_1, \theta_0), \quad \zeta_0 = 0, \quad \dot{\zeta}_0 = 1. \end{cases}$$

In addition to the equations found before, we also need to guarantee $z_s^f = 0$, that is to say $-\int_0^s \beta_t \zeta_t e^{-\int_0^t \alpha_r^1 dr} dt = 0$. This can as well be nicely reinterpreted in terms of vector fields by saying that, at conjugate points, all the equalities found so far imply that

$$X_2(\gamma(s)) \wedge g_0^s(\gamma(s)) \wedge \int_0^s \beta_t e^{-\int_0^t \alpha_r^1 dr} g_t^s(\gamma(s)) dt = 0.$$

Remark 7. With this discussion we can finally complete the explanation of the four-dimensional example presented in Section 1.4. Indeed we find from the structural equations that $\alpha_t^0 \equiv -2$, $\alpha_t^1 \equiv 0$ and $\beta_t \equiv 1$, whence the computations for $a(s)$ and $\widehat{a}(s)$ easily follow from (6.13) and (6.14).

APPENDIX A. A TECHNICAL PROOF

Proof of Lemma 19. We begin with the proof of (a), and we begin observing that the condition $\lim_{n \rightarrow \infty} \|v_1^n - v_1\|_{L^\infty(I, \mathbb{R})} = 0$, implies both that $\lim_{n \rightarrow \infty} \|\bar{v}_1^n - \bar{v}_1\|_{L^\infty(I, \mathbb{R})} = 0$ and $\lim_{n \rightarrow \infty} \|v_1^{0,n} - v_1^0\|_{L^\infty(I, \mathbb{R})} = 0$. Using the triangular inequality we find then that

$$\lim_{n \rightarrow \infty} \left\| \frac{v_1^{0,n}}{1 + \bar{v}_1^n} - \frac{v_1^0}{1 + \bar{v}_1} \right\|_{L^\infty(I, \mathbb{R})} = 0,$$

which proves the first point. As for point (b), for every $s \in I$, we can define ϕ_v^{-1} (respectively, ϕ_n^{-1}) by:

$$\phi_v^{-1}(s) := \inf\{t \in I \mid \phi_v(t) = s\}.$$

Let $s \in S_v$, $t := \phi_v^{-1}(s)$ and $t_n := \phi_n^{-1}(s)$, and assume that

$$\lim_{n \rightarrow \infty} t_n = \bar{t} > t.$$

We claim that, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$ one has:

$$(A.1) \quad s \leq \phi_v(\bar{t}) \leq \phi_n(t_n) + 3\varepsilon \leq s + 3\varepsilon.$$

Notice that this would imply $\phi_v(\bar{t}) = s$, yielding that $s \in I \setminus S_v$ (that is, either $\phi_v(t)$ does not exist, or it exists and it is equal to zero, as ϕ_v would be constant on $[t, \bar{t}]$), which is a contradiction. For every $n \in \mathbb{N}$, let us make the position $w_1^{0,n} := \frac{v_1^{0,n}}{1 + \bar{v}_1^n}$, and similarly define w_1^0 (notice that $w_1^{0,n}$ converges to w_1^0 in $L^\infty(I, \mathbb{R})$ by point (a)). Thus (A.1) follows from:

$$\begin{aligned} s \leq \phi_v(\bar{t}) &= \int_0^{\bar{t}} 1 + w_1^0(\tau) d\tau - \int_0^{t_n} 1 + w_1^0(\tau) d\tau \\ &+ \int_0^{t_n} 1 + w_1^0(\tau) d\tau - \int_0^{t_n} 1 + w_1^{0,n}(\tau) d\tau \\ &+ \int_0^{t_n} 1 + w_1^{0,n}(\tau) d\tau \leq \phi_n(t_n) + 3\varepsilon = s + 3\varepsilon, \end{aligned}$$

since both the difference terms on the first and the second line go to zero as $n \rightarrow \infty$. On the other hand, let us assume that

$$\lim_{n \rightarrow \infty} t_n = \bar{t} < t.$$

By similar arguments, we find even in this case that for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ big enough such that

$$\phi_v(t) = s > \phi_v(\bar{t}) = \int_0^{\bar{t}} 1 + w_1^0(\tau) d\tau \geq \phi_n(t_n) - 3\varepsilon = s - 3\varepsilon$$

for every $n \geq n_\varepsilon$, yielding again an absurd. \square

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