Soft Construction of Floer-type Homologies

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Abstract

Floer homology is a good example of homological invariants living in the infinite dimension. We suggest a way to construct this kind of invariants using only soft essentially finite-dimensional tools; no hard analysis or PDE is involved. This work is partially inspired by the M. Gromov's survey "Soft and hard symplectic geometry" at ICM-86.

1 Introduction

Floer homology (see [4, 5]) is a good example of homological invariant, which lives in the infinite dimension. It is not completely homotopy invariant[1], but it survives a class of "compact homotopies" that are close enough to finite-dimensional perturbations.

Classical construction of Floer homology and similar invariants involves hard analysis and elliptic systems of PDEs. This is a beautiful theory but it is technically heavy and rather restrictive; we cannot go beyond a limited number of situations where good elliptic systems are available.

In this expository paper, partially motivated by the well-known M. Gromov's survey [3], we suggest a soft and essentially finite-dimensional approach to the Floer-type homological invariants. We use finite-dimensional subspaces of an auxiliary Banach space for finite-dimensional approximations of the original big object. The role of this Banach space is similar to the role of the events space in the Kolmogorov's foundation of Probabilities Theory.

In our study, the infinite dimension and infinite size remain only potential infinities. We deal with very big but still finite-dimensional and compact objects, the dimension and size are just big parameters. The "Infinity" is

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[1]indeed, infinite dimension implies an infinite freedom :-)

treated as a singularity and we study asymptotics of usual finite-dimensional homological invariants at this singularity.

Let me describe a toy example, the Leray–Schauder degree. Let B be an infinite-dimensional separable Banach space and $S \subset B$ the unite sphere in B. Let $\mathcal{E} = \{E \subset B : \dim E < \infty\}$ be the ordered by the inclusion directed set of finite-dimensional subspaces of B. Given $E \in \mathcal{E}$, the intersection $E \cap S$ is a $(\dim E - 1)$-dimensional sphere. In particular, the homology group $H_{i(\dim E - 1)}(S \cap E)$ equals $\mathbb{Z}$ for $i = 0, 1$ and equals 0 for other i; it depends only on i and does not depend on E. We set:

$$G_i(S) = H_{i(\dim E - 1)}(S \cap E)$$

and call $G_i(S)$ the Leray–Schauder homology of S. Recall that S is contractible and $H_i(S) = 0, \ \forall i \neq 0$.

Now let $\varphi : S \to B$ be a compact map such that $x + \varphi(x) \neq 0, \ \forall x \in S$. For any $\varepsilon > 0$, there exist a ε-close to φ finite-dimensional map $\varphi_\varepsilon : S \to E_\varepsilon$, where $E_\varepsilon \in \mathcal{E}$. If ε is small enough, then $x + \varphi_\varepsilon(x) \neq 0, \ \forall x \in S$, that allows us to define a map:

$$\Phi_E^\varepsilon : S \cap E \to S \cap E, \quad \Phi_E^\varepsilon(x) = \frac{x + \varphi_\varepsilon(x)}{|x + \varphi_\varepsilon(x)|},$$

for any $E \supset E_\varepsilon$. The degree of this map $d = \deg(\Phi_E^\varepsilon)$ does not depend on E and is the same for all sufficiently good approximations φ_ε. This is the Leray–Schauder degree.

The degree is defined by the homomorphism

$$\Phi_{E*}^\varepsilon : H_{\dim E - 1}(S \cap E) \to H_{\dim E - 1}(S \cap E), \quad \Phi_{E*}^\varepsilon(c) = cd,$$

for any $c \in H_{\dim E - 1}(S \cap E) = \mathbb{Z}$. Since this homomorphism does not depend on ε and E (for small enough ε and big enough E), we may interpret it as a homomorphism $\Phi_* : G_1(S) \to G_1(S)$, where $\Phi = \frac{I + \varphi}{|I + \varphi|}$. This homomorphism survives homotopies $\varphi_t, \ 0 \leq t \leq 1$, where all φ_t are compact and $x + \varphi_t(x) \neq 0, \ \forall x \in S$.

In the next section we give our soft construction of Floer homology and in Section 3 we explain, why it works. In Section 4 we briefly study another example, inspired by the sub-Riemannian geometry. This example demonstrates a much more interesting asymptotic behavior of the homology than in the usual Floer case. We conclude with a draft of general construction that can be adapted to many different situations.

In what follows, the definitions and statements are precise while the proofs are only sketched.

2 Floer Homology

We consider a compact smooth manifold M endowed with a symplectic structure σ. Let $\tilde{M}$ be the universal covering of M and $\tilde{\sigma}$ the pullback of σ to $\tilde{M}$; we assume that $\tilde{\sigma}$ is an exact form: $\tilde{\sigma} = ds$.

We denote by Ω the space of contractible closed curves in M of class H^1. In other words, Ω consists of contractible maps $\gamma : S^1 \to M$, where γ is differentiable almost everywhere with the derivative of class L^2. The lifts of $\gamma \in \Omega$ to $\tilde{M}$ are closed curves and we use the same symbol γ for any lift of this curve to $\tilde{M}$.

Let $h_t : M \to \mathbb{R}$, $t \in S^1$, be a measurable bounded with respect to $t \in S^1$ family of smooth functions on M. We are going to study the functionals $\varphi_h : \Omega \to \mathbb{R}$ defined by the formula

$$\varphi_h(\gamma) = \int_{S^1} s(\dot{\gamma}(t)) - h_t(\gamma(t))\, dt.$$

Note that $\int_{S^1} \langle s, \dot{\gamma}(t) \rangle\, dt = \int_\Gamma \sigma$ for any film Γ such that $\gamma = \partial \Gamma$ and this integral does not depend on the choice of the lift of γ to $\tilde{M}$. Given $c \in \mathbb{R}$, we denote by Ω_h^c the Lebesgue set of φ_h:

$$\Omega_h^c = \{\gamma \in \Omega : \varphi_h(\gamma) \le c\}$$

We use some auxiliary objects. The homology invariants will be constructed with a help of these objects but do not depend on their choice. Namely, we assume that M is equipped with a Riemannian structure $\langle \cdot, \cdot \rangle$ adapted to the symplectic structure, i.e. $\sigma(\xi, \eta) = \langle J\xi, \eta \rangle$, $\xi, \eta \in TM$, where $J : TM \to TM$ is a quasi-complex structure, $J^2 = -I$.

Moreover, we fix generators $X_1, \ldots, X_l$ of the $C^\infty(M)$-module Vec$M$ of all smooth vector fields on M and define a linear map $X_q : \mathbb{C}^l \to T_q M$ by the formula

$$X_q u = \sum_{j=1}^{l} v^j X_j(q) + w^j J X_j(q),$$

where $u = (u^1, \ldots, u^l)$, $u_j = v_j + iw_j \in \mathbb{C}$, $j = 1, \ldots, l$. We assume that $\langle \cdot, \cdot \rangle|_{T_q M}$, $q \in M$, is the image of the standard Euclidean structure on $\mathbb{C}^l$ by

the linear map X_q; in other words,

$$\langle \xi, \xi \rangle = \min\{|u|^2 : u \in \mathbb{C}^l, \ \xi = X_q u\}.$$

A simple model example is an even-dimensional torus with a constant sympletic structure, flat Riemannian metric and constant vector fields $X_1, \ldots, X_l$. Needless to say that, in general, if M is not parallelizable, number l is greater than $\dim M$. Anyway, the described auxiliary objects always exist and can be easily built.

Let W be the space of all curves in M of class H^1 parameterized by the segment $[0, 1]$. We fix a parametrisation of S^1 by $[0, 1]$; then $\Omega \subset W$.

We define the map $\phi : M \times L^2([0, 1]; \mathbb{C}^l) \to W$ as follows. Given $q \in M$ and $u(\cdot) \in L^2([0, 1]; \mathbb{C}^l)$ the curve $\gamma(\cdot) = \phi(q, u(\cdot))$ is the solution of the ordinary differential equation

$$\dot{\gamma}(t) = X_{\gamma(t)} u(t), \quad 0 \le t \le 1,$$

with the initial condition $\gamma(0) = q$. It is important for us that weak convergence of a sequence $u_n \in L^2([0, 1]; \mathbb{C}^l)$ to u ($u_n \rightharpoonup u$ as $n \to \infty$) and convergence of $q_n \in M$ to q imply uniform convergence of $\phi(q_n, u_n)$ to $\phi(q, u)$.

We also set $\phi_t(q, u) = (q, \phi(q, u)(t))$ and thus define the map $\phi_t : M \times L^2([0, 1]; \mathbb{R}^l) \to M \times M$. It is easy to see that ϕ_t is a smooth map and ϕ_t is a submersion for $0 < t \le 1$.

The differential of ϕ_t is a linear operator $D_{(q,u)}\phi_t : T_q M \times L^2([0, 1]; \mathbb{C}^l) \to T_q M \times T_q M$. It is important for us that convergence $q_n \to q$ and weak convergence $u_n \rightharpoonup u$ as $n \to \infty$ imply convergence $D_{(q_n, u_n)}\phi_t \to D_{(q,u)}\phi_t$ in the operator norm. It follows immediately from the standard ODE's "variations formula" for the differential of ϕ_t and the uniform convergence $\phi(q_n, u_n) \to \phi(q, u)$.

Given a subspace $E \subset L^2([0, 1]; \mathbb{C})$, we set $E^l = E \times \cdots \times E \subset L^2([0, 1]; \mathbb{C}^l)$.

Lemma 1. $0 < t \le 1, r > 0$, and $B_r = \{u \in L^2([0, 1]; \mathbb{C}^l) : \|u\| < r\}$. Then there exists a finite-dimensional subspace $E_r \subset L^2([0, 1]; \mathbb{C})$ such that for any subspace $E \supset E_r$ the map $\phi_t|_{M \times (E^l \cap B_r)}$ has no critical points.

Proof. For any $(q, u) \in M \times L^2([0, 1]; \mathbb{C}^l)$ there exists a finite-dimensional subspace $E_{(q,u)}$ such that the restriction of $D_{(q,u)}\phi_t$ to the subspace $T_q M \times E^l(q, u)$ is surjective; simply because ϕ_t is a submersion and M is finite-dimensional.

Moreover, we can take $E_{(q',u')} = E_{(q,u)}$ for all q' close to q and u' close to u in the weak topology. Hence, due to the compactness of M and weak

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compactness of $\bar{B}_r$, we may assume that $\#\{E_{(q,u)} : q \in M,\ u \in B_r\} < \infty$. Then $E_r = \sum\limits_{(q,u) \in M \times B_r} E_{(q,u)}$ is the desired space. $\qquad\square$

If E satisfies conditions of Lemma 1, then

$$U_r(E) \doteq \{(q, u) \in M \times (B_r \cap E^l) : \phi_t(q, u) = (q, q)\} \qquad (1)$$

is a smooth submanifold of $M \times E^l$ of dimension $\dim E^l$ and $\phi(U_r(E^l)) \subset \Omega$.

In the next construction, we prefer to treat S^1 in the intrinsic way as an Abelian Lie group equipped with the Haar measure $d\theta$ rather than directly parameterize it by a segment. Let $q \in M$, we denote by b_q a symmetric bilinear form on the space $H^1(S^1; T_q M)$ defined by the formula:

$$b_q(\xi, \eta) = \int_{S^1} \sigma(\xi(\theta), \dot{\eta}(\theta))\, d\theta, \quad \xi, \eta \in H^1(S^1; T_q M).$$

A vector-function $\eta \in H^1(S^1; T_q M)$ belongs to the kernel of b_q if and only if η is constant, i. e. $\eta(\theta) \equiv \eta(0)$. Indeed, if $\dot{\eta} \neq 0$, then

$$b_q(J\dot{\eta}, \eta) = -\int_{S^1} \langle \dot{\eta}(\theta), \dot{\eta}(\theta) \rangle\, d\theta < 0. \qquad (2)$$

So, if $\dot{\eta} \in H^1(S^1; T_q M)$, then η is not in the kernel; otherwise, we approximate $\dot{\eta}$ by a smooth function in the norm L^2 and plug-in the approximating function in (2) instead of $\dot{\eta}$.

We denote by $\imath : H^1(S^1; T_q M) \to H^1(S^1; T_q M)$ the involution defined by the formula $(\imath\xi)(\theta) = \xi(-\theta)$. Then

$$b_q(\imath\xi, \imath\eta) = -b_q(\xi, \eta), \quad \xi, \eta \in H^1(S^1; T_q M). \qquad (3)$$

Let E be a finite-dimensional subspace of $L^2([0, 1]; \mathbb{C})$ and $E_0 = \left\{ v \in E : \int_0^1 v(t)\, dt = 0 \right\}$. We set:

$$\mathcal{X}_q(E) = \left\{ \theta :\mapsto \xi_0 + \int_0^\theta X_q u(t)\, dt : \xi_0 \in T_q M,\ u(\cdot) \in E_0^l \right\} \subset H^1(S^1; T_q M).$$

We say that E is *well-balanced* if $\imath E = E$ and $\ker b_q|_{\mathcal{X}_q(E)} = \ker b_q$.

Lemma 2. *Any finite-dimensional subspace of $L^2([0, 1]; \mathbb{C})$ is contained in a well-balanced subspace.*

Indeed, we can always add to E a big enough finite-dimensional space of trigonometric polynomials to guarantee that for the enlarged space $\hat{E}$ and

any nonconstant $\eta \in \mathcal{X}_q(\hat{E})$ there exists $\xi \in \mathcal{X}_q(\hat{E})$ that is not orthogonal to $\dot{\eta}$ in $L^2(S^1; T_q M)$.

Let $\mathcal{E}$ be the directed set of well-balanced subspaces ordered by the inclusion, the $\mathcal{E}$-lim of a generalized sequence indexed by the elements of $\mathcal{E}$ is defined in a usual way.

We are now ready to formulate the main result. Let $E \in \mathcal{E}$, $c > 0$, and $r > 0$. We consider relative homology groups

$$G_i(E; c, r) \doteq H_i\left(\phi(U_r(E)) \cap \Omega_h^c, \phi(U_r(E)) \cap \Omega_h^{-c}\right),$$

$i = 0, 1, 2, \ldots$.

Let $\bar{r} > r$ and $\jmath_r^{\bar{r}} : G_i(E; c, r) \to G_i(E; c, \bar{r})$ be the homology homomorphisms induced by the inclusions

$$\phi(U_r(E)) \cap \Omega_h^{\pm c} \subset \phi(U_{\bar{r}}(E)) \cap \Omega_h^{\pm c}.$$

Theorem 1. *There exist*

$$\lim_{c \to \infty} \lim_{r \to \infty} \lim_{\bar{r} \to \infty} \mathcal{E}\text{-}\lim \jmath_r^{\bar{r}}\left(G_{i+d_E}(E; c, r)\right) \cong H_i(M),$$

where $d_E = \frac{1}{2}(\dim E - 1) \dim M$.

Remark. Actually, the families under the limits stabilize; a little bit more precise statement is as follows. For any big enough $c > 0$ there exist $r(c) > 0$, $\bar{r}(c, r) > r(c)$, and $E(c, r, \bar{r}) \in \mathcal{E}$ such that for any $r > r(c)$, $\bar{r} > \bar{r}(c, r)$ and well-balanced $E \supset E(c, r, \bar{r})$ we have:

$$\jmath_r^{\bar{r}}\left(G_{i+d_E}(E; c, r)\right) \cong H_i(M).$$

The limit in Theorem 1 is a soft construction of Floer homology. It implies Morse inequalities for the functional φ_h, which we are going to describe now.

Let $\gamma \in \Omega$ and ξ a vector field along γ; then

$$d_\gamma \varphi_h(\xi) = \int_0^1 \sigma(\xi(t), \dot{\gamma}(t)) - d_{\gamma(0)} h_t(\xi(t)) \, dt = \int_0^1 \langle -J\dot{\gamma}(t) - \nabla_{\gamma(t)} h_t, \xi(t) \rangle \, dt.$$

To get this formula, it is sufficient to consider a very narrow film obtained by the translation of γ in the direction of ξ and apply the Stokes formula to the integral of s over the boundary of the film.

Hence γ is a critical point of φ_h if and only if $\dot{\gamma} = J\nabla_\gamma h_t$, $0 \le t \le 1$. Note that $J\nabla h_t = \vec{h}_t$, where $\vec{h}_t$ is the Hamiltonian vector field associated

to the function h_t on the symplectic manifold M, i.e. $\sigma(\cdot, \vec{h}_t) = dh_t(\cdot)$. We obtain that critical points of φ_{h_t} are exactly periodic solutions of period 1 of the time-varying Hamiltonian system

$$\dot{\gamma}(t) = \vec{h}_t(\gamma(t)), \quad 0 \leq t \leq 1. \tag{4}$$

A similar trick allows to easily compute the Hessian of φ_h at a critical point γ. We take one more vector field η along γ, move our narrow film a little bit in the direction of η, and apply the Stocks formula to the integral of σ over the boundary of the obtained thin 3-dimensional body. We have:

$$d_\gamma^2 \varphi_h(\xi, \eta) = \int_0^1 \sigma(\eta(t), \dot{\xi}(t)) - d_{\gamma(t)}^2 h_t(\eta(t), \xi(t)) \, dt.$$

It follows that the field ξ belongs to the kernel of $d_\gamma^2 \varphi_h$ if and only if ξ is a periodic solution with period 1 of of the linearization of system (4) along γ.

Let $P^t : M \to M$ be the flow on M generated by system (4),

$$\frac{\partial P^t}{\partial t} = \vec{h}_t(P^t(q)), \quad P^0(q) = q, \quad q \in M.$$

A curve $\gamma(t) = P^t(\gamma(0))$, $0 \leq t \leq 1$, is a critical point of φ_h if and only if $\gamma(0)$ is a fixed point of P^1. The linearization of P^1 at $\gamma(0)$ is a linear operator $P^1_{*\gamma(0)} : T_{\gamma(0)}M \to T_{\gamma(0)}M$.

We say that γ is a non-degenerate 1-periodic solution of system (4) if $\det(P^1_{*\gamma(0)} - I) \neq 0$. According to the implicit function theorem, non-degenerate 1-periodic solutions are isolated. Moreover, as we have seen, a 1-periodic solution γ of system (4) (a critical point of φ_h) is non-degenerate if and only if $\ker D_\gamma^2 \varphi_h = 0$.

The next step is to associate an index $i(\gamma) \in \mathbb{Z}$ to any non-degenerate 1-periodic solution of system (4). To do that we need some general properties of symmetric bilinear forms. Let b be a continuous symmetric bilinear form on a Hilbert space H and $F \subset H$ be a closed subspace. We set

$$F^{\perp_b} = \{x \in H : Q(x, F) = 0\},$$

the orthogonal complement to F with respect to the form b; then $F \cap F^{\perp_b} = \ker(b|_F)$.

Let $F \subset H$ be a finite codimension subspace such that $b|_F$ is nondegenerate[2], then $F^{\perp_b}$ is finite dimensional and the signature $sgn(b|_{F^{\perp_b}}) \in \mathbb{Z}$ is well-defined. Recall that the signature of a symmetric bilinear form is the difference between the positive and negative inertia indices.

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[2]We say that a bilinear form $b : H \to H^*$ is non-degenerate if b is invertible.

Lemma 3. *Let b_0, b_1 be symmetric bilinear forms on H, $b_s = sb_1 + (1-s)b_0$, and $F, \hat{F} \subset H$, finite codimension subspaces such that $b_s|_F$ and $b_s|_{\hat{F}}$ are non-degenerate for any $s \in [0,1]$. Then*

$$sgn(b_1|_{F^{\perp b_1}}) - sgn(b_0|_{F^{\perp b_0}}) = sgn(b_1|_{\hat{F}^{\perp b_1}}) - sgn(b_0|_{\hat{F}^{\perp b_0}}).$$

Proof. Step 1. Let $F_0 \subset F$ be a subspace of finite codimension such that $b_s|_{F_0}$ is non-degenerate for any s from a subsegment $[\alpha, \beta] \subset [0,1]$. Then

$$sgn(b_\beta|_{F_0^{\perp b_\beta}}) - sgn(b_\alpha|_{F_0^{\perp b_\alpha}}) = sgn(b_\beta|_{F^{\perp b_\beta}}) - sgn(b_\alpha|_{F^{\perp b_\alpha}}).$$

Indeed, $F_0^{\perp b_s} = F_0^{\perp b_s} \cap F \oplus F^{\perp b_s}$ and $b_s|_{F_0^{\perp b_s} \cap F}$ is a non-degenerate form for any $s \in [\alpha, \beta]$. Hence $sgn(b_s|_{F_0^{\perp b_s} \cap F})$ does not depend on $s \in [\alpha, \beta]$. On the other hand,

$$sgn(b_s|_{F_0^{\perp b_s}}) = sgn(b_s|_{F_0^{\perp b_s} \cap F}) + sgn\left(b_s|_{F^{\perp b_s}}\right).$$

Step 2. The subspace $F \cap \hat{F}$ has a finite codimension in H. Moreover, $\ker(b_s|_{F \cap \hat{F}})$ are finite dimensional, $0 \le s \le 1$. Let $F_s \subset F \cap \hat{F}$ be a compliment to $\ker(b_s|_{F \cap \hat{F}})$ in $F \cap \hat{F}$; then $b_{s'}|_{F_s}$ is non degenerate for all $s' \in O_s$, where O_s is a neighborhood of s in $[0,1]$. We can cover $[0,1]$ by a finite number of such neighborhoods and apply Step 1. $\square$

Lemma 3 allows us to define

$$\lfloor sgn(b_1) - sgn(b_0) \rfloor \doteq sgn(b_1|_{F^{\perp b_1}}) - sgn(b_0|_{F^{\perp b_0}});$$

the defined quantity depends only on b_1 and b_0 and not on the choice of the subspace F. Note that only the difference of the signatures is well-defined; each of the forms b_0, b_1 may have infinite inertia indices.

Now consider symmetric bilinear form $d_\gamma^2 \varphi_h$. The variables for this form are vector fields along γ. The bundle $TM|_\gamma$ is trivializable since γ is contractible. Hence one can find an orthonormal basis of $V_j, W_j, 1 \le j \le \frac{\dim M}{2}$ of $TM|_\gamma$, where J has a canonical form: $JV_j = W_j$, $JW_j = -V_j$. Let $\dim M = 2n$; written in the good basis, $d^2 \varphi_h$ turns into a symmetric bilinear form on $H^1(S^1; \mathbb{C}^n)$ and it has the following expression:

$$d_\gamma^2 \varphi_h(\xi, \eta) = \int_0^1 \langle i\eta(t), \dot{\xi}(t) \rangle - \langle R_t \xi(t), \eta(t) \rangle \, dt. \tag{5}$$

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Here $n = \frac{\dim M}{2}$ and R_t is a real symmetric operator on $\mathbb{C}^n$ with the $2n \times 2n$-matrix $\left\{ \begin{array}{cc} d^2_{\gamma(t)} h_t(V_j, V_k) & d^2_{\gamma(t)} h_t(V_j, W_k) \\ d^2_{\gamma(t)} h_t(W_j, V_k) & d^2_{\gamma(t)} h_t(W_j, W_j) \end{array} \right\}_{j,k=1}^n$.

Recall that $\frac{d}{dt}$ sends the space $H^s(S^1; \mathbb{C}^n)$ in $H^{s-1}(S^1; \mathbb{C}^n)$, and $\int_0^1 \langle \cdot, \cdot \rangle \, dt$ is a non-degenerate pairing of $H^s(S^1; \mathbb{C}^n)$ and $H^{-s}(S^1; \mathbb{C}^n)$. It follows that form (5) can be continuously extended to the space $H^{\frac{1}{2}}(S^1; \mathbb{C}^n)$.

From now on we assume that $\xi, \eta \in H^{\frac{1}{2}}(S^1; \mathbb{C}^n)$. Note that

$$d^2_\gamma \varphi_0 - d^2_\gamma \varphi_h = \int_0^1 \langle R_t \cdot, \cdot \rangle \, dt$$

is a compact form on $H^{\frac{1}{2}}(S^1; \mathbb{C}^n)$. Now we analyse the form $d^2_\gamma \varphi_0$

Write the Fourier expansions:

$$\xi(t) = \sum_{k=-\infty}^{+\infty} e^{i2\pi k t} \xi_k, \qquad \eta(t) = \sum_{k=-\infty}^{+\infty} e^{i2\pi k t} \eta_k.$$

We set: $\quad \xi_+(t) = \sum_{k=1}^{+\infty} e^{i2\pi k t} \xi_k, \quad \xi_-(t) = \sum_{k=-\infty}^{-1} e^{i2\pi k t} \xi_k \quad$ and similarly for η_+, η_-. Then

$$d^2_\gamma \varphi_0(\xi, \eta) = \sum_{k=1}^{+\infty} 2\pi k (\langle \eta_k, \bar{\xi}_k \rangle - \langle \eta_{-k}, \bar{\xi}_{-k} \rangle) = \langle \xi_+, \eta_+ \rangle_{\frac{1}{2}} - \langle \xi_-, \eta_- \rangle_{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle_{\frac{1}{2}}$ is the Hilbert inner product of $H^{\frac{1}{2}}(S^1; \mathbb{C}^n)$ (the real part of the Hermitian product).

We have: $H^{\frac{1}{2}}(S^1; \mathbb{C}^n) = H_+ \oplus \mathbb{C}^n \oplus H_-$, where $H_\pm = \{\xi_\pm : \xi \in H^{\frac{1}{2}}(S^1; \mathbb{C}^n)\}$. Let $s \in [0, 1]$; the restriction of the form $d^2_\gamma \varphi_{sh}$ to $H_\pm$ can be written as follows:

$$d^2_\gamma \varphi_{sh}(\xi_\pm, \eta_p m) = \pm \langle \xi_\pm + s A_\pm \xi_\pm, \eta_\pm \rangle,$$

where $A_\pm : H_\pm \to H_\pm$ is a compact symmetric operator

Let $F_\pm \subset H_\pm$ be the orthogonal complement to the linear hull of all eigenvectors of $A_\pm$ corresponding to the eigenvalues that are smaller or equal than -1. Then $F = F_+ \oplus F_-$ is a finite codimension subspace of $H^{\frac{1}{2}}(S^1; \mathbb{C}^n)$ and the restriction of the form $d^2_\gamma \varphi_{sh}$ to F is non-degenerate. So we are in the conditions of Lemma 3 and we set:

$$i_h(\gamma) = \frac{1}{2} \lfloor sgn(d^2_\gamma \varphi_0) - sgn(d^2_\gamma \varphi_h) \rfloor. \tag{6}$$

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Let $\beta_j(M)$ be the Betti number of M of the dimension j (rank of the j's homology group of M with coefficients in a preliminary chosen field) and C_h be the set of all periodic trajectories of system (4) with period 1. If all 1-periodic trajectories are non-degenerate, then C_h is a finite set.

Theorem 2 (Morse inequalities). *Assume that all 1-periodic trajectories of system (4) are non-degenerate. Then, for any $k \in \mathbb{Z}$, the following inequality holds:*

$$\sum_{j \leq k} (-1)^{k-j} \beta_j(M) \leq \sum_{\{\gamma \in C_h : i_h(\gamma) \leq k\}} (-1)^{k-i_h(\gamma)}.$$

We have $d^2_{\gamma(t)} h(\eta, \xi) = \langle \eta, R_t \xi \rangle$, $\eta, \xi \in T_{\gamma(t)} M$. The linearization of system (4) has a form $\dot{\xi} = J R_t \xi$. Let $t \mapsto \xi(t)$ be a solution to this equation, then $\xi(t) = P^t_{*\gamma(0)} \xi(0)$, $0 \leq t \leq 1$. In what follows, we assume that vector bundle $TM|_\gamma$ is trivialized and treat this differential equation as a linear Hamiltonian system in a fixed symplectic space $T_{\gamma(0)} M$. We also write $T_\gamma M$ and $P^t_{*\gamma}$ instead of $T_{\gamma(0)} M$ and $P^t_{*\gamma(0)}$ in order to simplify notations a little bit. Our next goal is to give an effective formula for index (6) in terms of the linear symplectic transformations $P^t_{*\gamma} : T_\gamma M \to T_\gamma M$. We do it under a mild regularity assumption.

Let Σ, σ be a symplectic space, $\Pi \subset \Sigma$ a Lagrange subspace, and $L(\Sigma)$ the Lagrange Grassmannian (the manifold of all Lagrange subspaces of Σ). Then

$$\mathcal{M}_\Pi = \{\Lambda \in L(\Sigma) : \Lambda \cap \Pi \neq 0\}$$

is a codimension 1 cycle in $L(\Sigma)$. Let $\Lambda(t) \in L(\Sigma)$, $t_0 \leq t \leq t_1$, be a curve in $L(\Sigma)$ such that $\Lambda(t_0) \cap \Pi = \Lambda(t_0) \cap \Pi = 0$. Recall that the Maslov $\mu_\Pi(\Lambda(\cdot))$ of the curve $\Lambda(\cdot)$ with respect to Π is the intersection number of $\Lambda(\cdot)$ and $\mathcal{M}_\Pi$, see [7] for details.

Now consider symplectic space $\Sigma \times \Sigma$ andowed with the symplectic structure $(-\sigma) \oplus \sigma$. Given a linear symplectic transformation $Q : \Sigma \to \Sigma$, its graph $\Gamma_Q = \{(\xi, Q\xi) : \xi \in \Sigma\}$ is a Lagrange subspace of $\Sigma \times \Sigma$. In particular, Γ_I is the diagonal of $\Sigma \times \Sigma$. Let $Q(t)$, $t_0 \leq t \leq t_1$, be a curve in the symplectic group such that the maps $Q(t_0) - I$ and $Q(t_1) - I$ are non-degenerate. The Maslov index of $Q(\cdot)$ is just the Maslov index of the curve $\Gamma_{Q(\cdot)}$ with respect to the diagonal,

$$\mu(Q(\cdot)) \doteq \mu_{\Gamma_I}(\Gamma_{Q(\cdot)}).$$

Theorem 3. *Assume that* $\det\left(\int_0^1 R_t \, dt\right) \neq 0$. *Take $\varepsilon > 0$ and consider a curve Φ_ε in the symplectic group defined on the segment $[-\varepsilon, 1]$ according to*

the rule:

$$\Phi_\varepsilon(t) = \begin{cases} e^{tJ \int_0^1 R_\tau \, d\tau}, & \text{if } -\varepsilon \le t < 0; \\ P_{*\gamma}^t, & \text{if } 0 \le t \le 1. \end{cases}$$

Then $i_h(\gamma) = \mu(\Phi_\varepsilon)$, for any small enough ε.

We can make this formula even more explicit assuming more regularity. Indeed, μ is an intersection number and we can simply count the intersection points. In particular, we assume that R_t continuously depends on $t \in [0,1]$ and introduce subspaces $K_t = \{\xi \in T_\gamma M : R_t \xi = \xi\} \subset T_\gamma M$. We also denote by $\bar{R}_t$ the quadratic form on $T_\gamma M$ defined by the formula: $\bar{R}(\xi) = \langle R_t \xi, \xi \rangle$, $\xi \in T_\gamma M$.

Main regularity assumption is as follows: for any $t \in [0,1]$, if $K_t \ne 0$, then $\bar{R}_t\big|_{K_t}$ is a non-degenerate quadratic form. I emphasize that we require non-degeneracy of the restriction of the form $\bar{R}_t$ to the subspace K_t and do not care about properties of $\bar{R}_t$ on whole space $T_\gamma M$. In principle, the form $\bar{R}_t$ could be non-degenerate on $T_\gamma M$ and be identical 0 on K_t. Under this assumption, the points $t \in [0,1]$ such that $K_t \ne 0$ are isolated.

Corollary 1. *Under just imposed assumptions, the following equality holds:*

$$i_h(\gamma) = \sum_{0 < t < 1} sgn\left(\bar{R}_t\big|_{K_t}\right) + \frac{1}{2}\left(sgn(R_0) + sgn\left(\int_0^1 R_t \, dt\right)\right).$$

3 Sketch of Proofs

The proof of Theorem 1 is divided in two parts. First we show that for any pair of time-varying Hamiltonians h_t^0, h_t^1 and any big enough $c > 0$ (how big, depends on our Hamiltonians) there exist $r(c) > 0$, $\bar{r}(c, r) > r(c)$, $E(c, r, \bar{r}) \in \mathcal{E}$ such that for any $r > r(c)$, $\bar{r} > \bar{r}(c, r)$ and $E \supset E(c, r, \bar{r})$ the groups $j_r^{\bar{r}}(G_{i+d_E}(E; c, r))$ corresponding to h^0 and h^1 are naturally isomorphic. At this point we do not even need the subspace E to be well-balanced.

In the second part of the proof, we show that $j_r^{\bar{r}}(G_{i+d_E}(E; c, r)) = H_i(M)$, if $h = 0$. Here it is essential that E is well-balanced.

So we start from the choice of c. Let us consider the family of time-varying Hamiltonians $h_t^\tau = \tau h_t^1 + (1-\tau)h_t^0$, $\quad \tau, t \in [0,1]$. The correspondent Hamiltonian fields $\vec{h}_t^\tau$ are uniformly bounded, hence there exists $c > 0$ such that all critical points of φ_{h^τ} belong to the interior of $\varphi_{h^\tau}^{-1}([-c, c])$, $0 \le \tau \le t$.

If Ω would be a compact finite-dimensional manifold, then we could easily demonstrate homotopy equivalence of the pairs $\left(\Omega_{h_0}^c, \Omega_{h_0}^{-c}\right)$ and $\left(\Omega_{h_1}^c, \Omega_{h_1}^{-c}\right)$.

Indeed, the flow on Ω generated by a vector field f_τ such that

$$\frac{\partial \varphi_{h\tau}}{\partial \tau} + d_\gamma \varphi_{h\tau}(f_\tau(\gamma)) \leq 0, \quad \forall \gamma \in \varphi^{-1}(\pm c), \ 0 \leq \tau \leq 1, \qquad (1)$$

provides the desired homotopy equivalence.

Unfortunately, Ω is noncompact and infinite-dimensional. Still Ω and the differential of φ_h are good enough to guarantee the existence of a finite-dimensional substitute that provides the required weaker equivalence. I am going to explain how it works.

First of all, $\frac{\partial \varphi_{h\tau}}{\partial \tau} = \varphi_{h^1} - \varphi_{h^0}$ is uniformly bounded. Moreover, the gradient of φ_h in the L^2-norm has a form: $\nabla_\gamma \varphi_h = -J\dot\gamma - \nabla_\gamma h$. Since any bound on the L^2-norm of $\dot\gamma$ implies a bound on $|\varphi_h|$, we obtain that $\nabla_\gamma \varphi_h$ is separated from 0 on $\varphi_h^{-1}(\pm c)$ if c is sufficiently big. Hence we can find an open set of uniformly bounded in the L^2-norm by a constant $\rho(c)$ vector fields that satisfy inequality (1).

Now we take $r > 0$, $\bar{r} = r + 2\rho(c)$ and a finite-dimensional space $E_{\bar{r}}^l$ guaranteed by Lemma 1. The weak continuity of the differential of $\varphi_h \circ \phi$ and weak compactness of $B_{\bar{r}}$ imply the existence of a finite-dimensional space $E^l \supset E_{\bar{r}}^l$ and a field f_τ on $M \times B_{\bar{r}}$, which takes values in $TM \times E^l$, has the L^2-norm bounded by $2\rho(c)$, and such that

$$\frac{\partial(\varphi_{h\tau} \circ \phi)}{\partial \tau} + d_{(q,u)}(\varphi_{h\tau} \circ \phi)(f_\tau(q,u)) < 0,$$

$$\forall (q,u) \in U_{\bar{r}}(E) \cap (\varphi_{h\tau} \circ \phi)^{-1}(\pm c), \quad 0 \leq \tau \leq 1.$$

The field f_τ is tangent to the finite-dimensional submanifold $U_{\bar{r}}(E^l)$. Moreover, starting in $U_r(E^l)$ trajectories of this vector field stay inside $U_{\bar{r}}(E^l)$ and thus well-defined on the whole segment $[0,1]$. This allows us to control the homology homomorphism induced by the imbedding $U_r(E) \subset U_{\bar{r}}(E)$.

Moreover, $\phi^{-1}(\gamma)$ is an affine subspace of $\gamma(0) \times L^2([0,1]; \mathbb{C}^l)$ for any $\gamma \in \Omega$. Hence $\phi^{-1}(\gamma) \cap U_{\bar{r}}(E)$ and $\phi^{-1}(\gamma) \cap U_r(E)$ are convex subsets of $\gamma(0) \times E^l$ (actually, balls) and ϕ induces the homotopy equivalence of the spaces $U_r(E) \cap \phi^{-1}(\Omega_{h\tau}^{\pm c})$ and $U_{\bar{r}}(E) \cap \phi^{-1}(\Omega_{h\tau}^{\pm c})$ on there images. This is actually all we need to complete the first part of the proof of Theorem 1.

Now consider the case $h = 0$. Critical points of φ_0 are exactly constant curves $\gamma(t) \equiv q$, $q \in M$. In other words, critical points form a smooth manifold M. The Hessian of φ_0 at q is just the quadratic form b_q from Section 2. As we know, the restriction of this form to a complement to the tangent space to M has zero kernel. If Ω would be finite-dimensional and

$\varphi_0^{-1}([-c, c])$ compact we could say that φ_0 is a "Morse–Bott" function and conclude that $H_k(\Omega_0^c, \Omega_0^{-c}) = H_{k-\imath}(M)$, where $\imath$ is the negative inertia index of b_q, see [8].

Unfortunately, our objects are infinite-dimensional and noncompact. Moreover, both positive and negative inertia indices of b_q are infinite. Still, as in the first part of the proof, an appropriate finite-dimensional substitute provides all we need.

Given $r > 0$ and a sufficiently big well-balanced spaced E, let us restrict φ_0 to $\phi(U_r(E^l))$ and first study this restriction in a neighborhood of the manifold M of constant curves. Let $\psi = \varphi_0\big|_{\phi(U_r(E^l))}$ and $q \in M$; the Hessian of ψ at q is nondegenerate on the complement to the tangent space to M and its positive end negative inertia indices are equal. In particular, both inertia indices are equal to $\frac{1}{2}\big(\dim(\phi(U_r(E^l)) - \dim M\big)$ and there exist a neighborhood of M in $\phi(U_r(E^l))$ that does not contain other critical points of ψ.

We need and we have a stronger property: there exists a neighborhood $\mathcal{O}$ of M in Ω such that any finite-dimensional subspace of $L^2([0, 1]; \mathbb{C})$ is contained in a well-balanced subspace E such that the restriction of ϕ_0 to $\phi(U_r(E^l)) \cap \mathcal{O})$ has only constant critical points[3]. Indeed, $\nabla_\gamma \varphi_0 = -J\dot{\gamma}$ almost linearly depends on $\dot{\gamma}$ for γ close to the constants and the estimates used to justify Lemma 2 (see, in particular, formula (2.2)) give also a low bound $|\nabla_\gamma \psi| \geq \epsilon \|\dot{\gamma}\|$ that is uniform with respect to E.

Out of a neighborhood $\mathcal{O}$, the norm of $\nabla_\gamma \varphi_0$ has a uniform low bound, hence the norm of the gradient-like vector field $\frac{\nabla \varphi_0}{\|\nabla \varphi_0\|^2}$ has an upper bound $\hat{\rho}$ and we take $\bar{r} = r + 3c\hat{\rho}$. Then we find a finite-dimensional approximation g of the field $-\frac{\nabla(\varphi_0 \circ \phi)}{\|\nabla(\varphi_0 \circ \phi)\|^2}$ on $(M \times B_{\bar{r}}) \cap \phi^{-1}(\Omega)$ whose norm is bounded by $\frac{3}{2}\hat{\rho}$ and such that $\langle \nabla(\varphi_0 \circ \phi), g \rangle < -\frac{2}{3}$, similarly to what we did in the first part of the proof. Finally, we restrict everything to a finite-dimensional manifold $U_{\bar{r}}(E)$; then any starting in $U_r(E)) \cap \phi^{-1}(\Omega_0^c)$ trajectory of the field g reaches either $\phi^{-1}(\mathcal{O})$ or $\phi^{-1}(\Omega_0^{-c})$, and this is actually all we need to control homology groups $j_r^{\bar{r}}\big(G_i(E; c, r)\big)$.

Now turn to Theorem 2. We have some freedom in the choice of the generators $X_1, \ldots, X_l$ of $\text{Vec} M$ and we are going to adapt them to the periodic trajectories $\gamma \in C_h$. Namely, we may assume that first $\dim M$ generators give a basis of the bundle $TM|_\gamma$ adapted to the quasi-complex structure $J|_\gamma$. In other words, we may assume that $d_\gamma^2 \varphi_h$ has a form (2.5), where only the "compact part" R_t depends on $\gamma \in C_h$, $R_t = R_t^\gamma$.

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[3]Here "constants" means "constant curves".

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The proof proceeds similarly to the proof of the second part of Theorem 1. We take $r > 0$ big enough and assume that the well-balanced space E is such that $\phi(U_r(E^l))$ contains C_h and all kernels of the bilinear forms

$$d^2_\gamma \varphi_h(\xi, \eta) = \int_0^1 \langle i\eta(t), \dot{\xi}(t) \rangle - s\langle R^\gamma_t \xi(t), \eta(t) \rangle \, dt, \quad 0 \le s \le 1, \ \gamma \in C_h. \quad (2)$$

Moreover, we take E sufficiently big and well-balanced to guarantee that $\varphi_h\big|_{\phi(U_{\bar{r}}(E^l))}$ does not have critical points out of C_h and that the Hessian of $\varphi_h\big|_{\phi(U_{\bar{r}}(E^l))}$ at any $\gamma \in C_h$ is non-degenerate. Then negative inertia index of the Hessian equals $\frac{1}{2}\dim \phi(U_r(E^l)) + \imath_h(\gamma)$ and homology groupes $\jmath^{\bar{r}}_r\big(G_i(E; c, r)\big)$ are controlled by these indices and a gradient-like vector field.

Theorem 3. To compute the index $\imath_h(\gamma)$ we have to count, according the multiplicities and signs, the passing through zero eigenvalues of the family of quadratic forms (2). A field $\xi \in T_\gamma M$ belongs to the kernel of the form (2) if and only if it is a 1-periodic solution of the linear Hamiltonian system $\dot{\xi} = sJR^\gamma_t \xi$. We consider linear symplectic transformations

$$Q_t(s) : \xi(0) \mapsto \xi(t), \quad \xi(0) \in T_\gamma M,$$

where $\dot{\xi}(t) = sJR^\gamma_t \xi(t)$, $0 \le t \le 1$, and set $Q(s) \doteq Q_1(s)$.

We see that ξ belongs to the kernel of the form (2) if and only if $Q(s)\xi(0) = \xi(0)$; in other words, if and only if $\xi(0) \in \Gamma_{Q(s)} \cap \Gamma_I$ (notations of Section 2, just before Theorem 3). The contribution of $Q(s)$ to the intersection number $\mu(Q(\cdot))$ is equal to the one-half of the signature of the quadratic form

$$\xi(0) \mapsto \Big\langle J\xi(0), \frac{\partial Q}{\partial s}\xi(0) \Big\rangle, \quad \xi(0) \in \{\xi \in T_\gamma M : \xi = Q(s)\xi\},$$

if this quadratic form is non-degenerate. Let $\xi(t) = Q_t(s)\xi(0)$ We have:

$$\Big\langle J\xi(0), \frac{\partial Q(s)}{\partial s}\xi(0) \Big\rangle = \Big\langle J\xi(1), \frac{\partial Q(s)}{\partial s}\xi(0) \Big\rangle =$$

$$\int_0^1 \Big\langle J\dot{\xi}(t), \frac{\partial Q_t(s)}{\partial s}\xi(0) \Big\rangle + \Big\langle J\xi(t), \frac{\partial^2 Q_t(s)}{\partial t \partial s}\xi(0) \Big\rangle \, dt =$$

$$\int_0^1 -\Big\langle R^\gamma_t \xi(t), \frac{\partial Q_t(s)}{\partial s}\xi(0) \Big\rangle + \Big\langle \xi(t), R^\gamma_t \frac{\partial Q_t(s)}{\partial s}\xi(0) \Big\rangle + \langle \xi(t), R^\gamma_t \xi(t) \rangle \, dt$$

$$= \int_0^1 \langle \xi(t), R^\gamma_t \xi(t) \rangle \, dt.$$

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On the other hand, according to the standard perturbations theory, the derivative with respect to s of the passing through zero eigenvalue of the form (2) is equal to

$$\frac{\partial}{\partial s} \int_0^1 \langle i\xi(t), \dot{\xi}(t)\rangle - s\langle R_t^\gamma \xi(t), \xi(t)\rangle \, dt = -\int_0^1 \langle \xi(t), R_t^\gamma \xi(t)\rangle \, dt.$$

The formulas for $\imath_h(\gamma)$ and $\mu(Q(\cdot))$ match! We actually got these formulas under a transversality assumption but we can drop this assumption due to the homotopy invariance of the intersection number. We also have: $Q_0(s) = Q_t(0) = I$, $s, t \in [0, 1]$; hence the curve $s \mapsto Qs) = Q_1(s)$ in the symplectic group is homotopic to the curve $t \mapsto Q_t(1) = P_{*\gamma}^t$ that is presented in Theorem 3, and the Maslov index is homotopy invariant. Additional "boundary" terms appear because both curves treated as curves in the Lagrange Grassmannian of the symplectic space $\Sigma \times \Sigma$ start at the diagonal and the intersection number has to be adjusted according to that.

4   Step Two Carnot Groups

In this section we describe an example of an asymptotics of Betti numbers that is much more interesting than in the Floer case and is explicitly computed. This calculation was inspired by the joint work with A. Lerario and A. Gentile [1], see also [2, 6]. A motivation, applications and proofs will appear in the forthcoming paper.

A step two Carnot Lie algebra $\mathfrak{g}$ is a graduated nilpotent Lie algebra with two levels generated by the first level and with a fixed Euclidean structure on the first level. The correspondent simply connected Lie group $\mathfrak{G} = e^{\mathfrak{g}}$ is called a step two Carnot group. We have: $\mathfrak{g} = V \oplus W$, $[V, V] = W$, $[\mathfrak{g}, W] = 0$. The Euclidean inner product on V is denoted by $\langle \cdot, \cdot \rangle$ and the norm by $|\cdot|$. To any $\omega \in W^*$ we associate an operator $A_\omega \in \mathrm{so}(V)$ by the formula:

$$\langle A_\omega \xi, \eta \rangle = \langle \omega, [\xi, \eta] \rangle, \quad \xi, \eta \in V. \tag{1}$$

It is easy to see that $\omega \mapsto A_\omega$, $\omega \in W^*$ is an injective linear map. Moreover, any injective linear map from W^* to $\mathrm{so}(V)$ defines a structure of step two Carnot Lie algebra on the space $V \oplus W$ by the same formula read in the opposite direction. We see that step two Carnot Lie algebras are in the one-to-one correspondence with linear systems of anti-symmetric operators.

Any subspace of Lie algebra forms a left-invariant vector distribution on the Lie group. We are interested in the distribution on the group $\mathfrak{G}$ formed by V. A class H^1 curve $\gamma : [0, 1] \to \mathfrak{G}$ is called *horizontal* if $\dot{\gamma}() \in V_{\gamma(t)}$ for

a. e. $t \in [0,1]$. Any two points of $\mathfrak{G}$ can be connected by a horizontal curve; it follows from the fact that V generates Lie algebra $\mathfrak{g}$.

Minimum of the lengths of horizontal curves connecting two given points in $\mathfrak{G}$ is the *sub-Riemannian* or *Carnot–Karatheodory* distance between the points. Step two Carnot groups are rather simple and symmetric models of sub-Riemannian spaces that are not Riemannian.

The following multiplication in $V \times W$ gives a simple realization of $\mathfrak{G}$ with the origin in $V \times W$ as the unit element:

$$(v_1, w_1) \cdot (v_2, w_2) = \left(v_1 + v_2, w_1 + w_2 + \frac{1}{2}[v_1, v_2] \right).$$

The Lie algebra is realized by left-invariant vector fields:

$$(v, w) \mapsto \left(v + \xi, w + \eta + \frac{1}{2}[v, \xi] \right), \quad (v, w) \in \mathfrak{G}, \ \xi \oplus \eta \in V \oplus W = \mathfrak{g},$$

such a field belongs to the horizontal distribution if and only if $\eta = 0$.

Starting from the origin horizontal curves are determined by their projection to the first level and have a form:

$$\gamma(t) = \left(\xi(t), \frac{1}{2} \int_0^t [\xi(t), \dot{\xi}(t)] \, dt \right), \quad 0 \le t \le 1,$$

where $\xi(\cdot) \in H^1([0,1]; U)$, $\xi(0) = 0$. The sub-Riemannian length of γ is equal to $\int_0^1 |\dot{\xi}(t)| \, dt$. For the same reason as in the Riemannian geometry, it is convenient to substitute the length by the essentially equivalent to the length functional, the action $c \int_0^1 |\dot{\xi}(t)|^2 \, dt$, where c is a normalizing constant. For us it is convenient to take $c = \frac{1}{4\pi}$ and we set:

$$\varphi(\xi) = \frac{1}{4\pi} \int_0^1 |\dot{\xi}(t)|^2 \, dt.$$

We focus on the horizontal curves corresponding to closed cures ξ; they connect the origin with the second level. Given $w \in W \setminus 0$, let Ω_w be the space of horizontal curves connecting $(0,0)$ with $(0,w)$; then

$$\Omega_w = \left\{ \xi \in H^1([0,1]; V) : \xi(0) = \xi(1) = 0, \ \frac{1}{2} \int_0^1 [\xi(t), \dot{\xi}(t)] \, dt = w \right\}.$$

For any $s > 0$, we set: $\Omega_w^s = \{ \xi \in \Omega_w : \varphi(\xi) \le s \}$. Note that central reflection $\xi \mapsto -\xi$ preserves Ω_w^s.

Let $E \subset H^1([0,1];V)$ be a finite-dimensional subspace and $\bar{E} = (E \setminus 0)/(\xi \sim (-\xi))$ its projectivization, $\bar{E}$ is homotopy equivalent to $\mathbb{RP}^{\dim E - 1}$. We set $E_w^s = \Omega_w^s \cap E$ and denote by $\bar{E}_w^s$ the image of E_w^s under the factorization $\xi \sim (-\xi)$.

We consider the homology $H.(\bar{E}_w^s; \mathbb{Z}_2)$ and its image in $H.(\bar{E}; \mathbb{Z}_2)$ by the homomorphism induced by the imbedding $\bar{E}_w^s \subset \bar{E}$. We have:

$$\text{rank}\big(H_i(\bar{E}_w^s; \mathbb{Z}_2)\big) = \beta_i(\bar{E}_w^s) + \varrho_i(\bar{E}_w^s),$$

where $\beta_i(\bar{E}_w^s)$ is rank of the kernel of the homomorphism from $H_i(\bar{E}_w^s; \mathbb{Z}_2)$ to $H_i(\bar{E}; \mathbb{Z}_2)$ induced by the imbedding $\bar{E}_w^s \subset \bar{E}$ and $\varrho_i(\bar{E}_w^s)$ is the rank of the image of this homomorphism, $\varrho_i(\bar{E}_w^s) \in \{0,1\}$.

For given w, E, s, we introduce two positive atomic measures on the half-line $\mathbb{R}_+$, the "Betti distributions":

$$\mathfrak{b}(\bar{E}_w^s) \doteq \sum_{i \in \mathbb{Z}_+} \frac{1}{s} \beta_i(\bar{E}_q^s) \delta_{\frac{i}{s}}, \quad \mathfrak{r}(\bar{E}_w^s) \doteq \sum_{i \in \mathbb{Z}_+} \frac{1}{s} \varrho_i(\bar{E}_q^s) \delta_{\frac{i}{s}}.$$

Assume that $\dim W = 2$ and let $\mathcal{E}$ be the directed set of all finite-dimensional subspaces of the Hilbert space $H^1([0,1]); V)$. It appears that there exist limits of these families of measures

$$\lim_{s \to \infty} \mathcal{E}\text{-}\lim \mathfrak{b}(\bar{E}_w^s), \quad \lim_{s \to \infty} \mathcal{E}\text{-}\lim \mathfrak{r}(\bar{E}_w^s)$$

in the weak topology. Moreover, the limiting measures are absolutely continuous with explicitly computed densities.

Some notations. Let $\alpha : \Delta \to \mathbb{R}$ be an absolutely continuous function defined on an interval Δ. We denote by $|d\alpha|$ a positive measure on Δ such that $|d\alpha|(S) = \int_S \left|\frac{d\alpha}{dt}\right| dt$, $S \subset \Delta$. Note that the measure $|d\alpha|$ depends only on the function α and not on the choice of the parameter on the interval.

The introduced in (1) operators A_ω, $\omega \in W^*$, are anti-symmetric and have purely imaginary eigenvalues. Let $0 \leq \alpha_1(\omega) \leq \cdots \leq \alpha_m(\omega)$ are such that $\pm i\alpha_j$, $j = 1, \ldots, m$, are all eigenvalues of A_ω counted according the multiplicities; then $\omega \mapsto \alpha_j(\omega)$ are Lipschitz functions. Let $\bar{W}^* = (W \setminus 0)/(w \sim cw, \forall c \neq 0)$ be the projectivization of W^*, $\bar{W}^* = \mathbb{RP}^1$.

Given $w \in W \setminus 0$, we take the line $w^\perp \in W^*$ and consider the affine line

$$\ell_w = \bar{W}^* \setminus \bar{w}^\perp \subset \bar{W}^*.$$

Moreover, we define functions

$$\lambda_j^w : \ell_w \to \mathbb{R}_+, \ j = 1, \ldots, m, \qquad \phi^w : \ell_w \to \mathbb{R}_+$$

by the formulas:

$$\lambda_j^w(\bar{\omega}) = \frac{\alpha_j(\omega)}{\langle \omega, w \rangle}, \qquad \phi^w(\bar{\omega}) = \sum_{j=1}^{m} \lambda_j^w(\bar{\omega}).$$

In particular, ϕ_*^w transforms measures on ℓ_w into measures on $\mathbb{R}_+$. The Euclidean measure on $\mathbb{R}_+$ is denoted by dt; if $S \subset \mathbb{R}_+$, then $\chi_S dt$ is the product of dt and the characteristic function of S.

For simplicity, we compute the limits of the Betti distributions only under a generic assumption on the Carnot group.

Theorem 4. *Assume that there exists $\omega \in W^*$ such that the matrix A_ω has simple spectrum. Then, for any $w \in W \setminus 0$, there exist the following limits in the weak topology of the space of positive measures on $\mathbb{R}_+$:*

$$\mathfrak{b}_w = \lim_{s \to \infty} \mathcal{E}\text{-}\lim \mathfrak{b}(\bar{E}_w^s), \qquad \mathfrak{r}_w = \lim_{s \to \infty} \mathcal{E}\text{-}\lim \mathfrak{r}(\bar{E}_w^s).$$

Moreover,

$$\mathfrak{b}_w = \phi_*^w \Big(\sum_{j=1}^{m} |d\lambda_j^w| \Big), \qquad \mathfrak{r}_w = \chi_{[0, \min \phi^w]} dt.$$

Remark. The internal sequences actually stabilize: there exists $E(s) \in \mathcal{E}$ such that $\mathfrak{b}(\bar{E}_w^s) = \mathfrak{b}(\bar{E}(s)_w^s)$ and $\mathfrak{r}(\bar{E}_w^s) = \mathfrak{r}(\bar{E}(s)_w^s)$ for any $E \supset E(s)$. An interesting feature of the Betti measures $\mathfrak{b}_w$ and $\mathfrak{r}_w$ is their sensitivity to the endpoint w that reflects well the anisotropy of the space of horizontal curves already at the homological level, even if the spaces Ω_w are all contractible.

5 Conclusion

We conclude with a draft of general scheme for a soft construction of Floer-type asymptotic homologies. The object to study is a Banach manifold Ω equipped with a growing family of closed subsets Ω^s, $s \in \mathbb{R}$. Auxiliary objects are a Banach space B and a submersion $\Phi : U \to \Omega$, where $U \subset B$ is a finite codimension submanifold of B. Moreover, U is equipped with an ordered by the inclusion directed and exhausting family $\mathcal{V}$ of open bounded subsets and B is endowed by an ordered by the inclusion directed family $\mathcal{E}$ of finite dimensional subspaces such that $\overline{\bigcup_{E \in \mathcal{E}} E} = B$.

Given $E \in \mathcal{E}$, $V \in \mathcal{V}$, $s \in \mathbb{R}$, we consider the relative homology groups:

$$G_i(E, V, s) \doteq H_i \left(\Omega^s \cap \Phi(E \cap V), \Omega^{-s} \cap \Phi(E \cap V) \right).$$

Moreover, for $V, W \in \mathcal{V}, \ V \subset W$, we denote by

$$\jmath_V^W : G_i(E, V, s) \to G_i(E, W, s)$$

the homology homomorphism induced by the inclusion $V \subset W$ and denote by $\mathcal{V}_V$ the directed subfamily: $\mathcal{V}_V \doteq \{W \in \mathcal{V} : W \supset V\}$.

Finally, we select normalizing quantities $r_i(E, s), \rho_i(E, s) \in \mathbb{R}_+$ and build atomic measures:

$$\mathfrak{b}(E, V, W, s) = \sum_{i \in \mathbb{Z}_+} \rho_i(E, s) \mathrm{rank}\big(\jmath_V^W G_i(E, V, s)\big) \delta_{r_i(E,s)}$$

in such a way that their exist a limit:

$$\mathfrak{b} = \lim_{s \to \infty} \mathcal{V}\text{-}\lim \mathcal{V}_V\text{-}\lim \mathcal{E}\text{-}\lim \mathfrak{b}(E, V, W, s).$$

We treat this limit as an asymptotic distribution of Betti numbers.

Some of ingredients may be superfluous. In the Leray–Schauder case we take a single limit (variables $s \in \mathbb{R}$ and $V, W \in \mathcal{V}$ are absent) and in the two step Carnot group case we take the double one (variables $V, W \in \mathcal{V}$ are absent). Of course, the equivariant version is also available if there is a natural group action; in the Carnot group case we used the involution $\xi \mapsto (-\xi)$.

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