Geometry and Control

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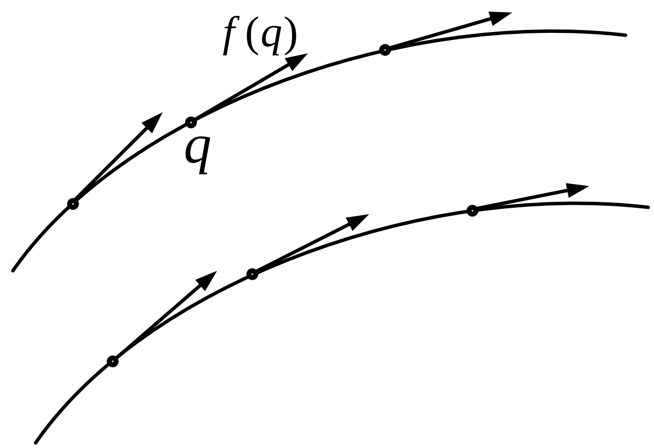
Smooth dynamical system:

$$\dot{q}(t) = f(q(t)), \quad q \in M, \; t \in \mathbb{R},$$



generates a flow

$$P^t : M \to M, \quad P^t : q(0) \mapsto q(t), \quad t \in \mathbb{R}.$$

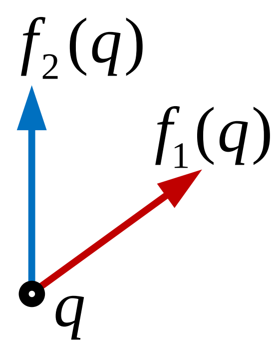
Control system:

$$\dot{q} = f_u(q), \quad u \in U.$$

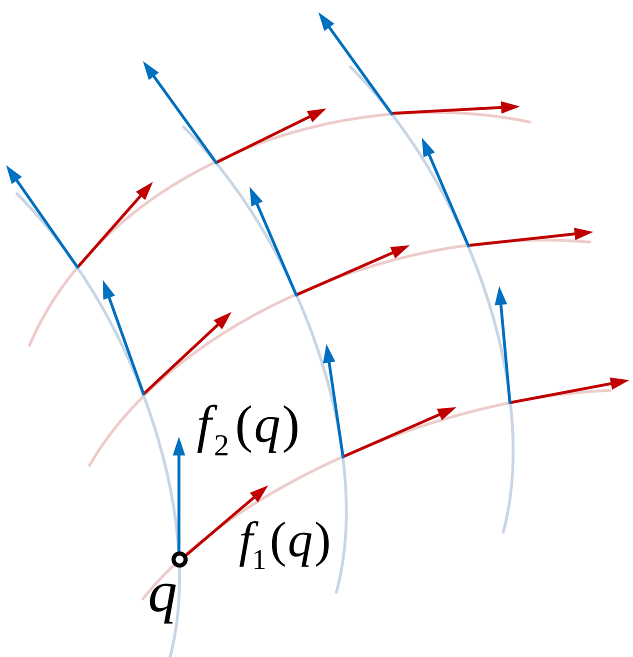
Control: $t \mapsto u(t),\ t \geq 0$.

Trajectory: $t \mapsto q(t)$, where $\dot{q}(t) = f_{u(t)}(q(t))$.

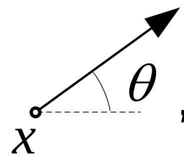
Special case: $U = \{1, 2\}$:



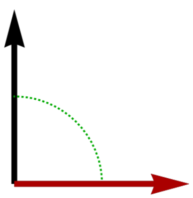
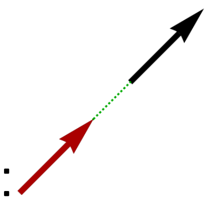
Trajectories:



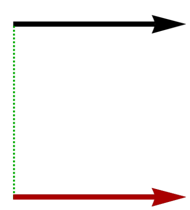
Example. Unicycle:

$q = (x, \theta)$: 

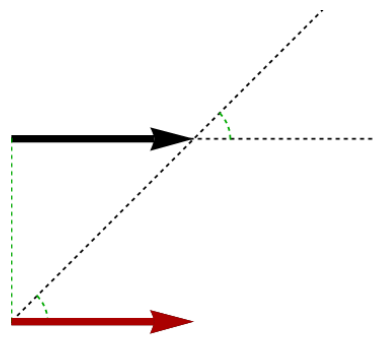
, $M = \mathbb{R}^2 \times S^1$.

P_1^t :  , P_2^t : 

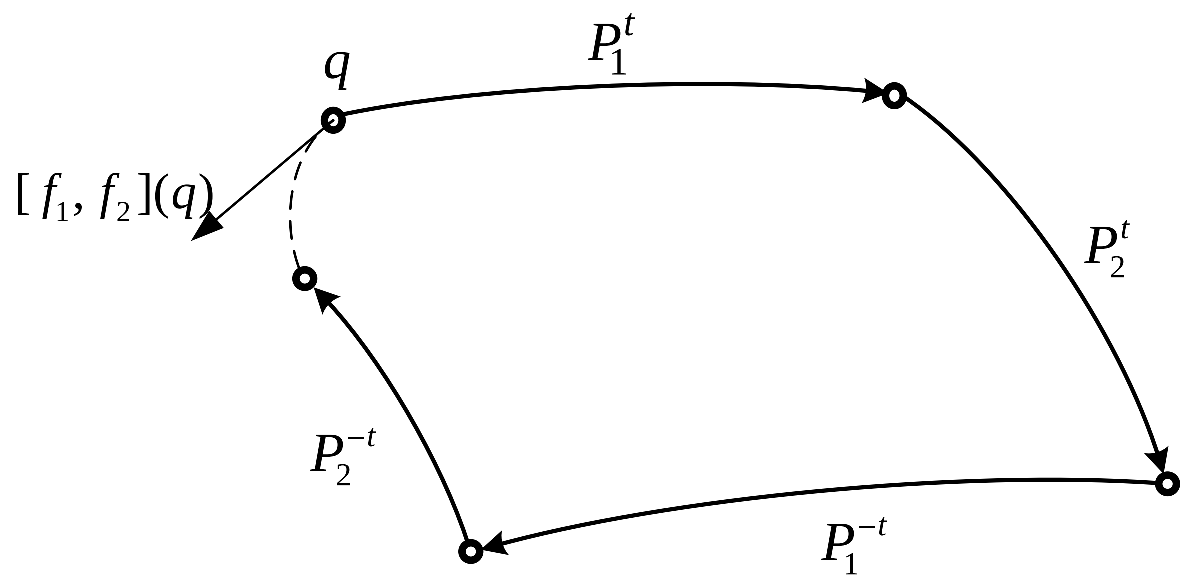
Third directions is the "parallel parking":



We realize it using only $P_1^{\pm t}$ and $P_2^{\pm t}$:



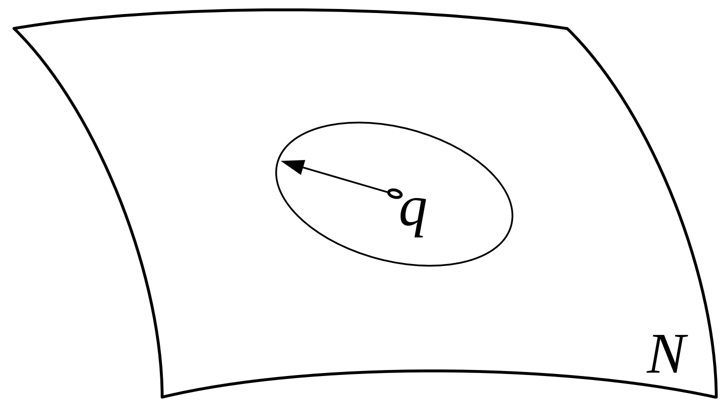
The commutator of flows and vector fields:



$$P_2^{-t} \circ P_1^{-t} \circ P_2^t \circ P_1^t(q) = q + t^2[f_1, f_2](q) + O(t^3).$$

In the example, the field $g = [f_1, f_2]$ generates the parallel parking.

Similar problem on a non flat surface:



M is the "spherical bundle" of all tangent vectors of the length 1 to the surface N.

The flow P_1^t is as before, P_2^t is the geodesic flow:

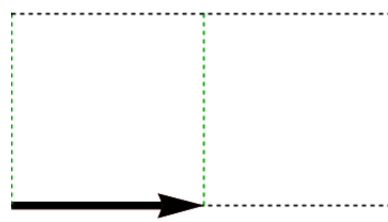
$$P_2^t : (x(0), \dot{x}(0)) \mapsto (x(t), \dot{x}(t)),$$

where $\ddot{x}(t) \perp N$.

Then $g = [f_1, f_2]$ generates the "parallel parking".

The difference between the flat and non flat cases:

"translation" f_2 and "parallel parking" g obviously commute in the flat case.



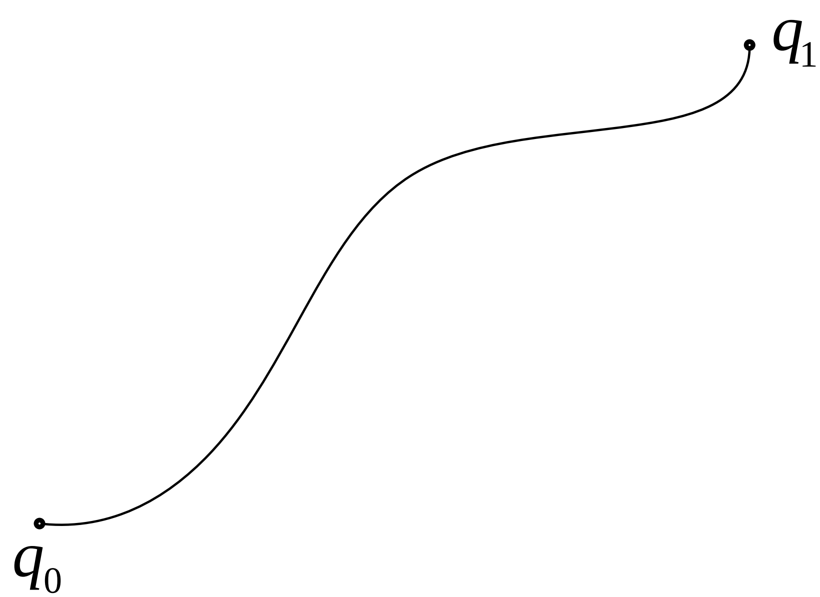
In general, $[g, f_2] = \kappa f_1$, where κ is the curvature!

In higher dimensions, the field $[g, f_2]$ may be linearly independent on $f_1, f_2, [f_1, f_2] = g$.

General case:

$$Lie_q f_U = span\{[f_{u_1}, [\cdots, f_{u_k}] \ldots](q) : u_i \in U,\ k = 1, 2, \ldots\}.$$

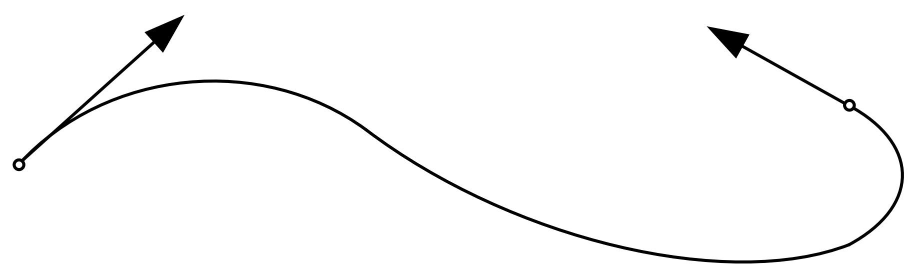
Theorem (Rashevskij–Chow). *If* $Lie_q f = T_q M$, $\forall q \in M$, *then* $\forall q_0, q_1 \in M$ $\exists t \mapsto (u(t), q(t))$ *such that* $\dot{q}(t) = f_{u(t)}(q(t))$, $q(0) = q_0$, $q(t_1) = q_1$.



Optimal control

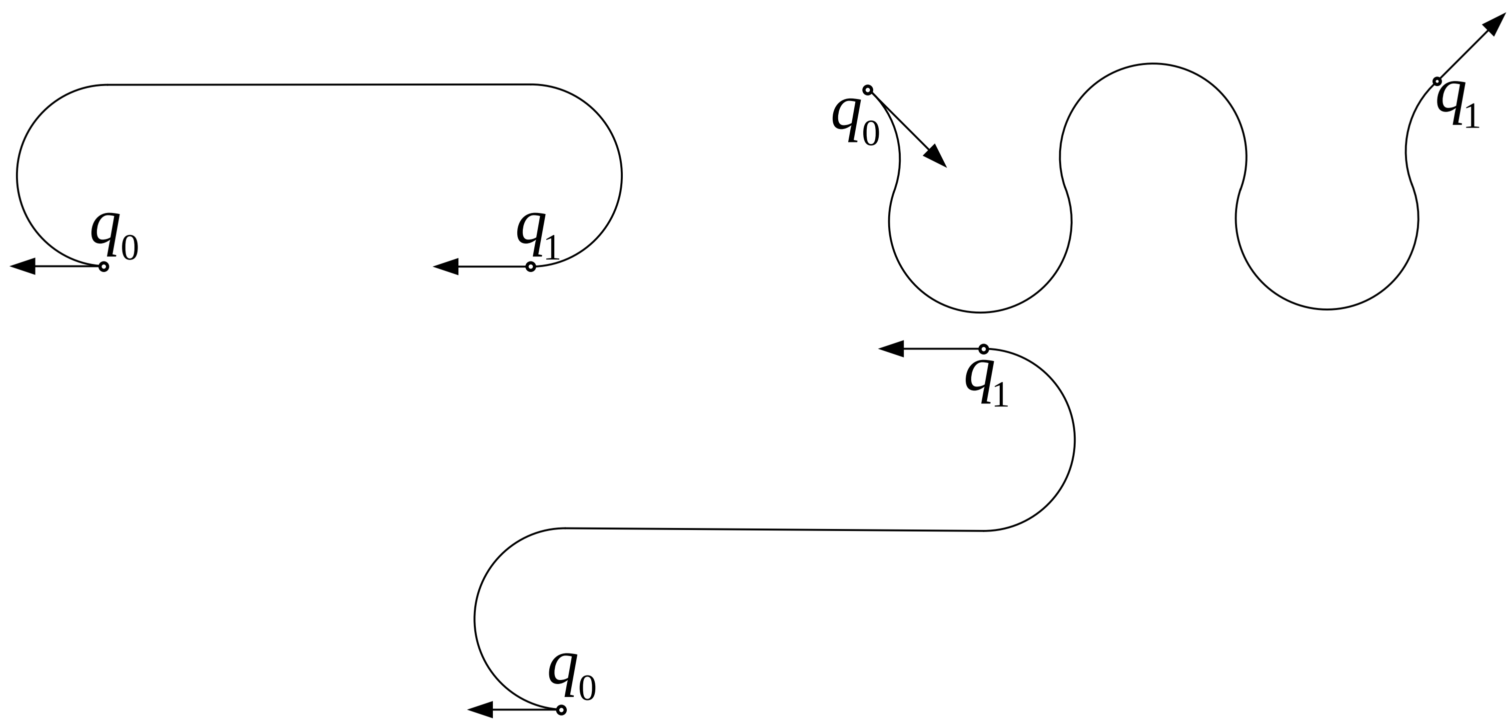
Example. $M = \mathbb{R}^2 \times S^1$, $\dot{q} = u f_1(q) + f_2(q)$.

Admissible trajectories $t \mapsto q(t) = (x(t), \theta(t))$, where $\dot{x} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\dot{\theta} = u$ is the curvature of the curve $x(\cdot)$.



I) Markov–Dubins problem: $|u| \le c$, minimise the length of $x(\cdot)$ (= time t_1).

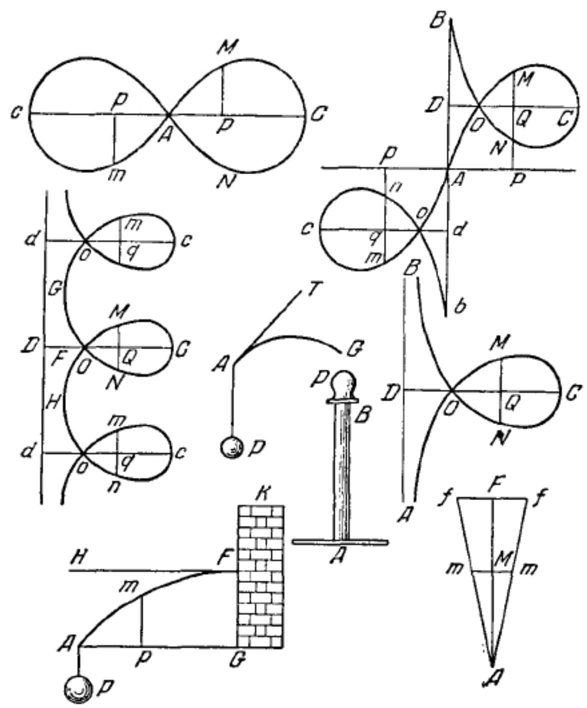
"Geodesics" are special concatenations of circle and linear segments:



Optimal piece has no more than 3 switchings.

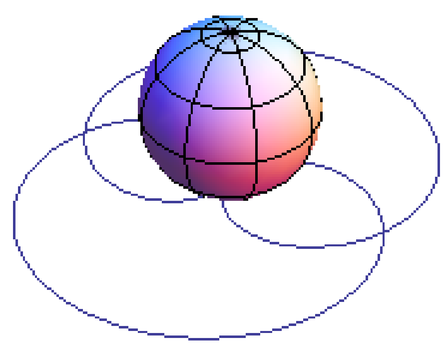
II) Euler elastic problem: $u \in \mathbb{R}$, minimise $\int_0^{t_1} u^2(t)\, dt$.

"Geodesics" or *elasticas*: the curves whose curvature is a linear function of coordinates, $u(t) = \langle a, x(t) \rangle + \alpha$.



Both problems are translation and rotation invariant, where $S^1 = \mathrm{SO}(2)$ is the group of rotations.

Example (rolling without slipping or twisting).
Here $M = \mathbb{R}^2 \times \mathrm{SO}(3)$, $q = (x, X)$, where $x \in \mathbb{R}^2$, $X \in \mathrm{SO}(3)$.



$$u = \binom{u_1}{u_2}, \quad \dot{x} = u, \quad \dot{q} = u_1 F_1(q) + u_2 F_2(q).$$

Minimize $\int_0^{t_1} |u(t)|\, dt$, the length of $x(\cdot)$.

Minimal length:

$$\delta(q_0, q_1) = \inf \left\{ \int_0^{t_1} |u(t)|\, dt : q(0) = q_0, q(t_1) = q_1 \right\}$$

is a metric on M:

$$\delta(q_0, q_1) = \delta(q_1, q_0), \quad \delta(q_0, q_2) \leq \delta(q_0, q_1) + \delta(q_1, q_2).$$

"Rolling geodesics" are again elasticas.

General sub-Riemannian problem:

$$\dot{q} = \sum_{i=1}^{k} u_i F_i(q), \quad q \in M, \quad u = (u_1, \dots, u_k)^T \in \mathbb{R}^k,$$

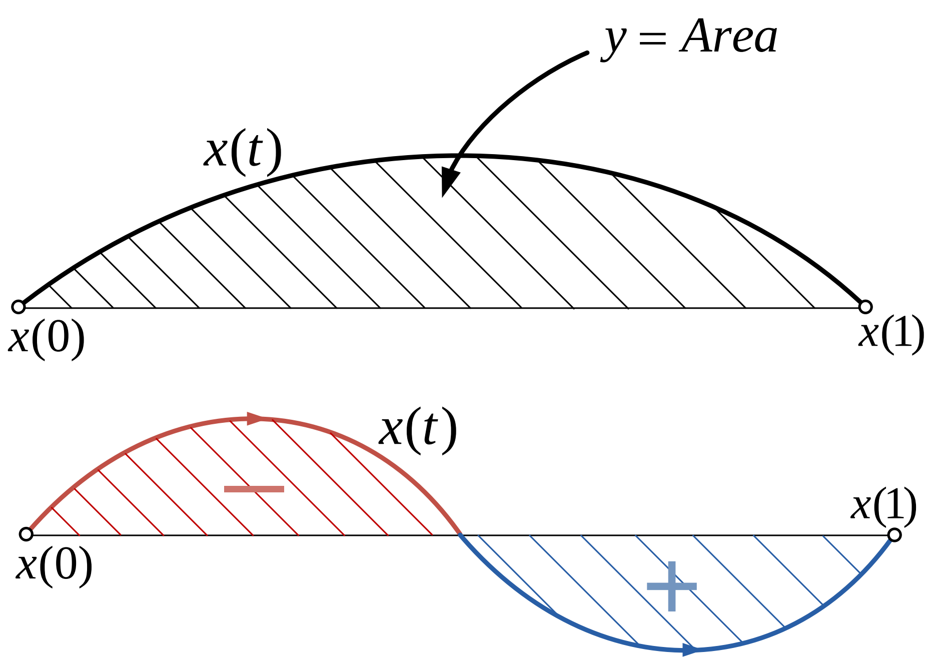
defines a "Carnot–Caratheodory metric" on M:

$$\delta(q_0, q_1) = \inf \left\{ \int_0^{t_1} |u(t)| \, dt : q(0) = q_0, \; q(t_1) = q_1 \right\}$$

3-dim examples:

I) Dido problem. $q = (x, y),\ x \in \mathbb{R}^2,\ y \in \mathbb{R}$,

$$\begin{cases} \dot{x} = u \\ 2\dot{y} = x \wedge u \end{cases}$$

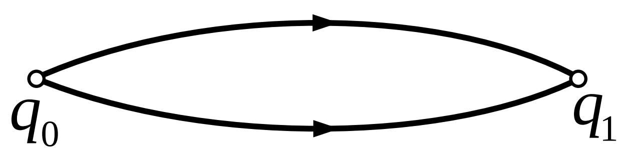


II) Interpolation problem.

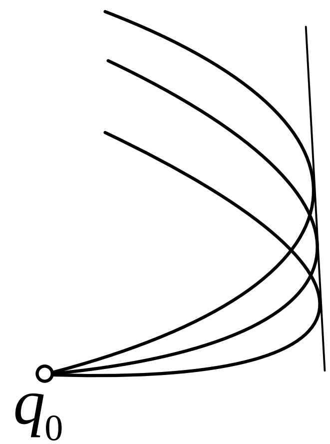
$$M = \mathbb{R}^2 \times S^1, \quad \dot{q} = u_1 f_1 + u_2 f_2.$$

Where to "cut" geodesics?

Maxwell points: at least two geodesics of equal length connect the same points.



Conjugate points: the envelope of the family of geodesics starting from q_0.

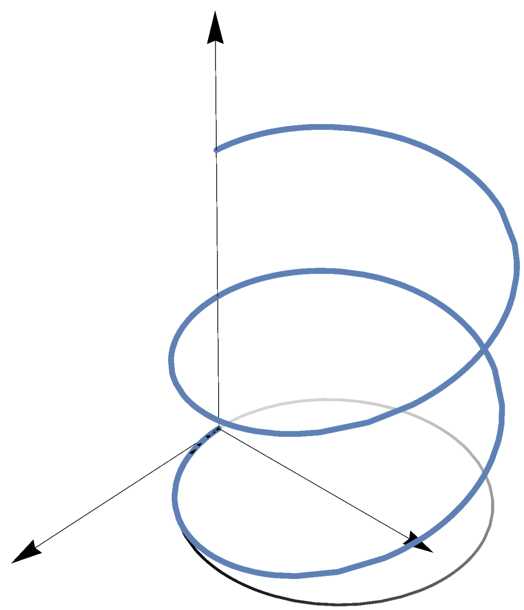


The wave front $W_{q_0}(r)$ is the set of endpoints of the length r geodesics starting from q_0.

The sphere $S_{q_0}(r)$ is a part of the wave front.

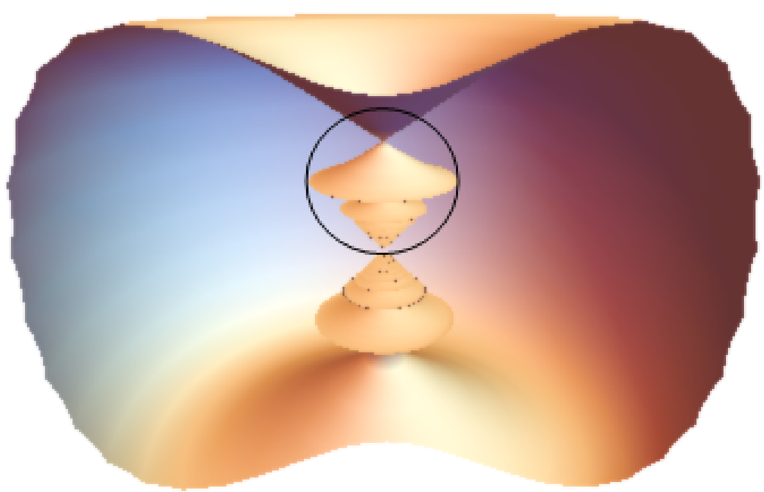
Dido problem.

Geodesics:

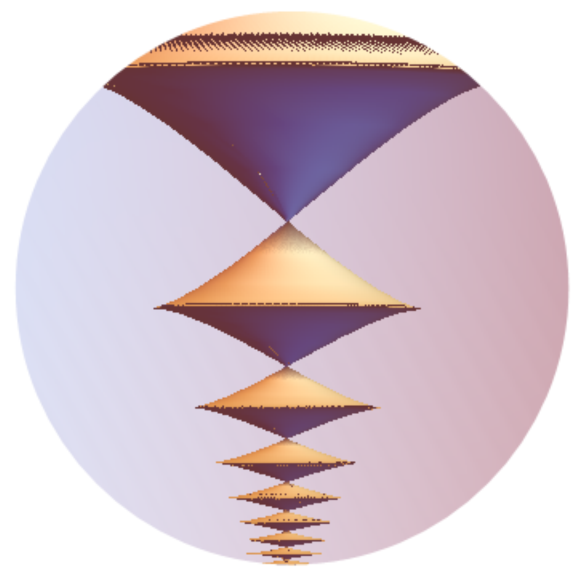


They are characterized by the initial velocity and curvature; greater the curvature, more tough is the spiral.

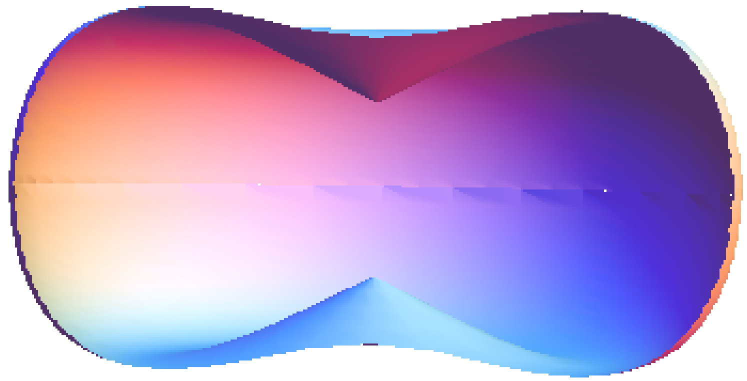
The wave front:



$\Rightarrow$



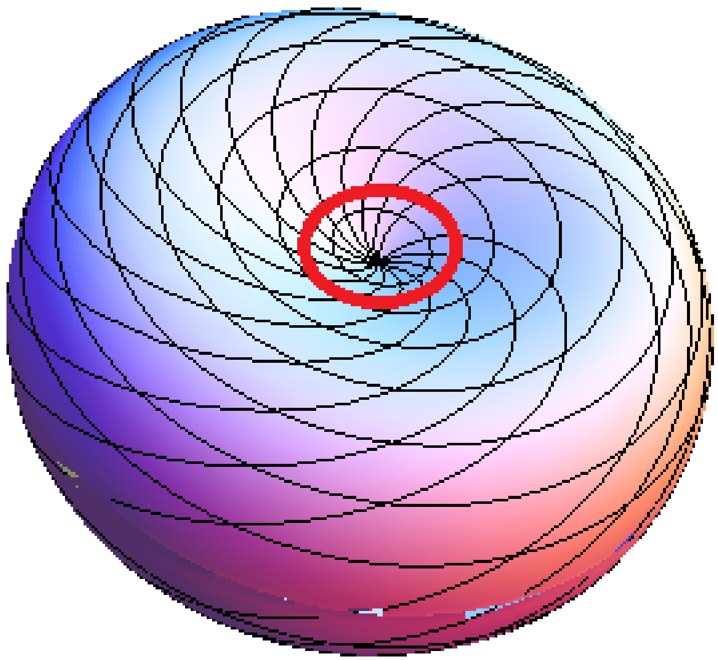
The sphere:



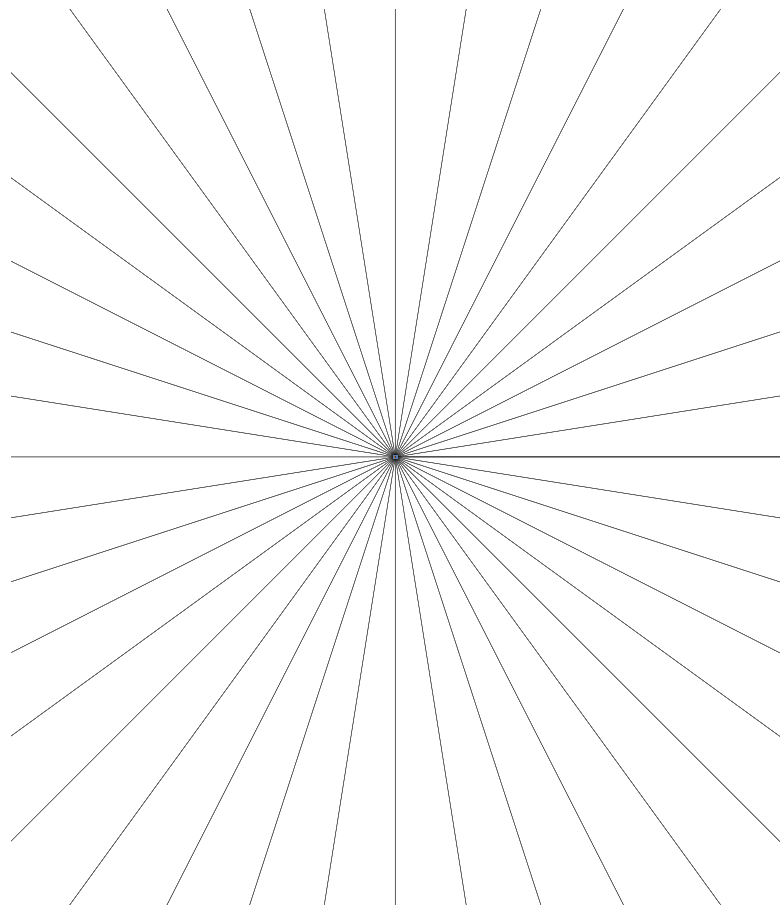
First Maxwell and conjugate points coincide and belong to the vertical line due to the rotational symmetry.

Breaking the symmetry.

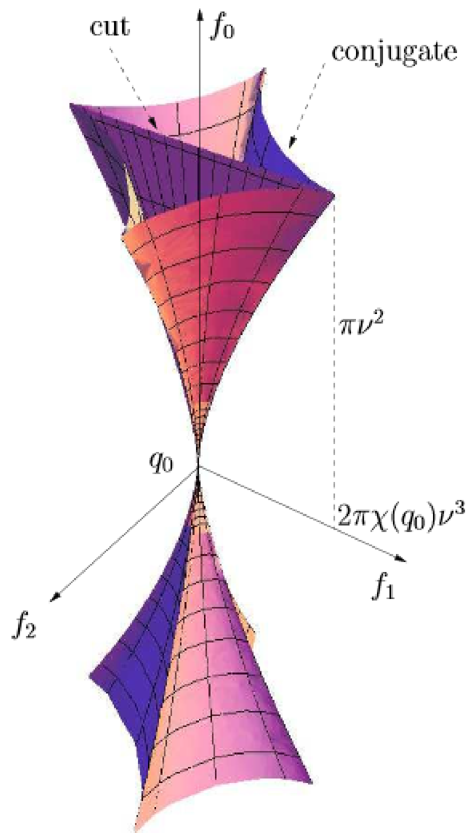
Look under the microscope:



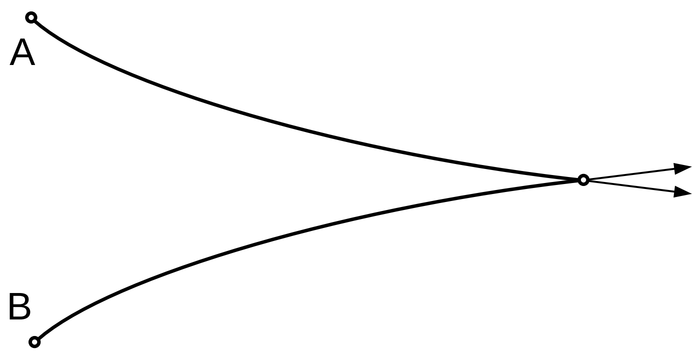
Symmetric $\Rightarrow$ Generic



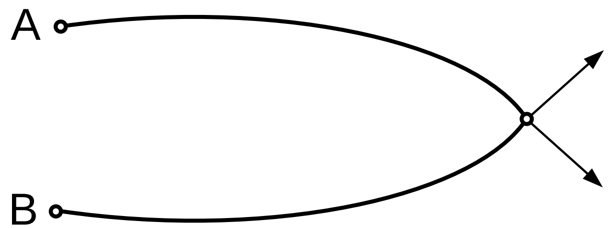
First conjugate and cut loci:



Curvature, basic idea:



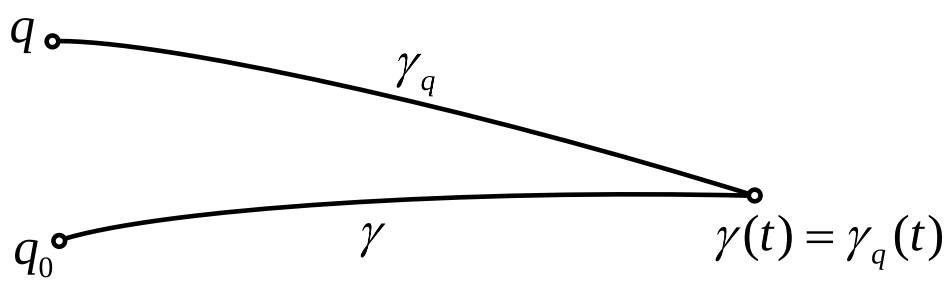
negative curvature



positive curvature

Bigger the curvature – bigger the difference between velocities in the intersection point.

Do it infinitesimaly:



$$b_t(q) \doteq \tfrac{1}{2}|\dot\gamma_q(t) - \dot\gamma(t)|^2.$$

In particular, $b_t(\gamma(s)) = \frac{1}{2}\left|\frac{(t-s)}{t}\dot\gamma(t) - \dot\gamma(t)\right|^2 = \frac{s^2}{2t^2}|\dot\gamma(t)|^2.$

Riemannian case:

$$D^2_{q_0}(v) = \frac{1}{t^2}|v|^2 + \frac{1}{3}\langle R(v,\dot\gamma)\dot\gamma, v\rangle + O(t).$$

General subriemannian case (ample geodesic):

$$D^2_{q_0}(v) = \frac{1}{t^2} Q(v) + \frac{1}{3} R_\gamma(v) + O(t), \quad v \in \Delta_{q_0}.$$

$Q(v) \geq |v|^2$ and R_γ is the *curvature along* γ. We have:

$$Q(v) = |v|^2, \ \forall v \in \Delta_{q_0} \quad \Leftrightarrow \quad \delta_{q_0} = T_{q_0} M.$$

Quadratic form Q measures “nonholonomy orders” and precedes the curvature.

Let $\Phi^\tau : M \to M$ be a horizontal flow s. t. $\Phi^\tau(q_0) = \gamma(\tau),\ \tau \in \mathbb{R}$.

We set $\Delta^t = \Phi_*^{-t} \Delta_{\gamma(t)} \subset T_{q_0} M$.

Geodesic flag $\Delta_{q_0} = \Delta^{(1)} \subset \Delta^{(2)} \subset \cdots \subset \Delta^{(i)} \subset$ is defined as follows:

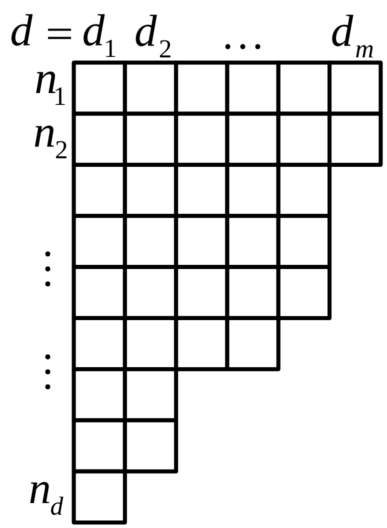
$$\Delta^{(i)} = \frac{d^{i-1}}{dt^{i-1}} \Delta^t \Big|_{t=0}.$$

The flag depends only on γ and not on the choice of Φ^τ.

Geodesic γ is *ample* if $\Delta^{(m)} = T_{q_0} M$ for some $m > 0$.

Let $\dim \Delta_{q_0} = d$, $\dim M = n$, and γ is equiregular. We set: $d_1 = d$, $d_i = \dim \Delta^{(i)} - \dim \Delta^{(i-1)}$.

Young diagram:



$$\operatorname{spec} Q_\gamma = \{n_1^2, \dots, n_d^2\}.$$

If $n = 3$, $d = 2$, then: $\operatorname{spec} Q = \{1, 4\}$, $\quad \operatorname{spec} R_\gamma = \{0, r_\gamma\}$.

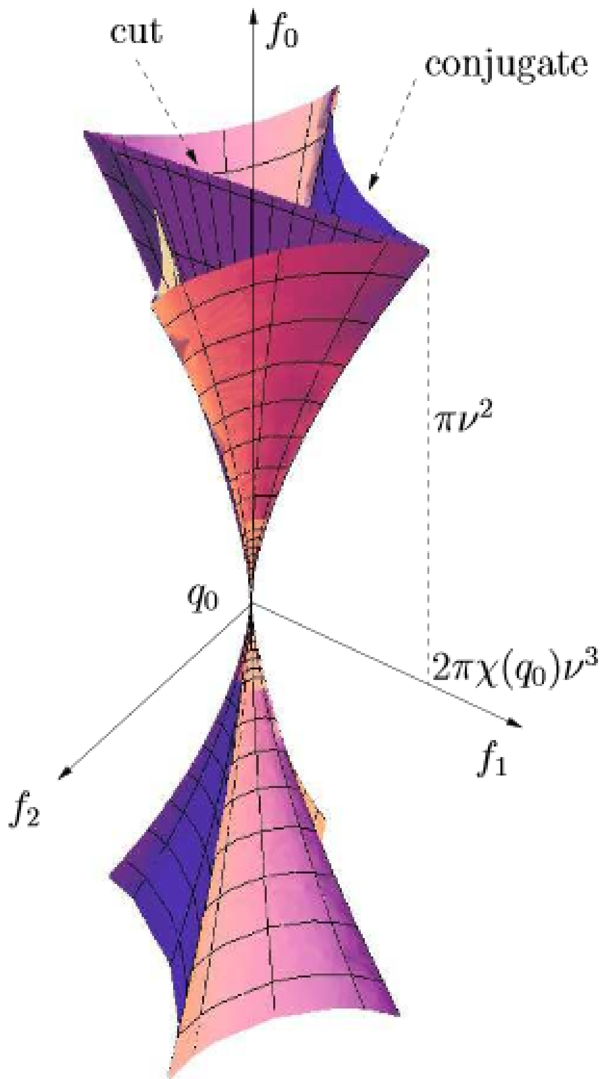
Starting from q_0 geodesics are characterized by their initial "momenta" $p \in T^*_{q_0}M$ Given $p \in T^*_{q_0}M$, let γ_p be the correspondent geodesic, $|\dot\gamma|^2 = \langle p, \dot\gamma\rangle$. We set: $r(p) = \frac{5}{4} r_{\gamma_p}$.

Theorem. $r : T^*_{q_0}M \to \mathbb{R}$ *is a quadratic form,* $r\Big|_{\Delta^\perp_{q_0}} > 0$. *Such a form can be canonically written as follows:*

$$r(p) = \langle p, f_0\rangle^2 + \alpha_1 \langle p, f_1\rangle^2 + \alpha_2 \langle p, f_2\rangle^2,$$

where f_1, f_2 *is an orthonormal frame of* Δ_{q_0} *and* $\alpha_1 \geq \alpha_2$.

Principal invariants: $\kappa = \frac{1}{2}(\alpha_1 + \alpha_2), \quad \chi = \frac{1}{6}(\alpha_1 - \alpha_2).$



$$\nu = 1/|\langle p, f_0 \rangle|$$

$$length_{conj}(\gamma_p) = 2\pi\nu + \pi\kappa\nu^3 + O(\nu^4)$$

A subriemannian structure is locally isometric to the Dido problem on the surfac with Gaussian curvature κ iff $\chi = 0$.

It is is isometric to its own metric tangent space iff $\chi = \kappa = 0$, iff r is a rank 1 quadratic form.