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FILTRATIONS OF A LIE ALGEBRA OF VECTOR FIELDS AND NILPOTENT APPROXIMATION OF CONTROLLED SYSTEMS

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1. Let M be an n-dimensional manifold of class C^∞, let $\operatorname{Der} M$ be the Lie algebra of the smooth vector fields on M, and let $q_0 \in M$ be a distinguished point. To each subset $\mathcal{E} \subset \operatorname{Der} M$ and integer $l \geq 0$ there corresponds a degenerate flag $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_l \subseteq T_{q_0}M$, where

$$E_k = \operatorname{span}\{(\operatorname{ad} X_1 \circ \cdots \circ \operatorname{ad} X_{i-1} X_i)(q_0) : X_j \in \mathcal{E},\ 1 \leq j \leq i,\ i \leq k\}. \tag{1}$$

We shall consider only those sets $\mathcal{E}$ for which $E_l = T_{q_0}M$; this immediately implies that $E_1 \neq 0$. Apart from this the spaces E_i are arbitrary. Some of them may coincide, and we do not exclude the case $n < l$.

In §2, given a set $\mathcal{E}$ we construct a filtration of the Lie algebra $\operatorname{Der} M$ and a nilpotent Lie algebra $V^-(\mathcal{E})$ of length l associated with this filtration. We show that there exists a coordinate map $\Phi: O_{q_0} \to R^n$ of a neighborhood O_{q_0} of the point $q_0 \in M$ into R^n that reduces the fields of the set $\mathcal{E}$ to canonical form and induces an isomorphism of the Lie algebra $V^-(\mathcal{E})$ and a certain Lie algebra of the polynomial vector fields on R^n which depends only on the dimensions of the subspaces E_i.

Next, in §3 we consider a controlled system of the form

$$\dot q = f(q) + \sum_{i=1}^{r} g^i(q) u^i, \qquad q \in M,\ u^i \in R,\ q(0) = q_0, \tag{2}$$

where $f(q)$ and the $g^i(q)$, $i = 1, \ldots, r$, are smooth vector fields on M. Let $\tilde u(\tau) = (\tilde u^1(\tau), \ldots, \tilde u^r(\tau)) \in L^r_\infty$ be a fixed control, let the time-dependent field $f(q) + \sum_1^r g^i(q)\tilde u^i(\tau)$ be complete and let $P_\tau: M \to M$, $\tau \in R$, be the flow on M determined by this field, that is,

$$\frac{\partial}{\partial \tau} P_\tau(q) f(P_\tau(q)) + \sum_{i=1}^{r} g^i(P_\tau(q))\tilde u^i(\tau), \qquad P_0(q) = q.$$

A time-dependent change of coordinates $q(\tau) \to P_\tau^{-1}(q(\tau))$ transforms (2) into the system

$$\dot q = \sum_{i=1}^{r} h^i_\tau(q) u^i, \qquad q(0) = q_0,\ h^i_\tau = P^{-1}_{\tau *} g^i. \tag{3}$$

The controlled system (3) which is equivalent to (2) is particularly convenient in that its right-hand side is homogeneous with respect to the control parameters u^i (for details see [1]). Let $t > 0$; and for all controls $u(\cdot) = (u^1(\cdot), \ldots, u^r(\cdot)) \in L^r_\infty[0,t]$ sufficiently near zero let a map $G_t: u(\cdot) \to q(t)$ be defined, where

$$\dot q(\tau) = \sum_{i=1}^{r} h^i_\tau(q(\tau)) u^i(\tau), \qquad \tau \in [0,t],\ q(0) = q_0.$$

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For any $\varepsilon > 0$ the image of an ε-neighborhood of zero in $L_\infty^r[0,t]$ under the map G_t is called the *ε-attainability set* of system (3) within the time t and is denoted by $\mathfrak{A}_t(\varepsilon)$.

Putting

$$(4) \qquad \mathcal{E} = \{h_\tau^i \mid \tau \text{ is a point of density on the curve } \tau \to h_\tau^i,\ 0 \le \tau \le t,\ i = 1, \dots, r\},$$

and applying the results of §2, we get in §3 the canonical "nilpotentization" of system (3), and then we compare the ε-attainability sets of system (3) and its "nilpotentization" for small ε.

The idea of using different nilpotent approximations of the Lie algebra connected with a controlled system and the flag it generates in the tangent space to the manifold M is often met in papers on geometric control theory (see [2]–[4]). In addition, a canonical form for the family of vector fields determining a regular distribution of planes was given in [5] in connection with nonholonomic variational problems, and the flags generated by such families were used systematically.

2. We again consider the subset $\mathcal{E} \subset \operatorname{Der} M$ and the flag (1). For $k = 0, 1, \dots, l$ we put

$$(5) \quad \mathcal{E}_{-k} = \{Y \in \operatorname{Der} M | (\operatorname{ad} X_1 \circ \cdots \circ \operatorname{ad} X_i Y)(q_0) \in E_{i+k},\ \forall X_j \in \mathcal{E},\ 1 \le j \le i,\ 0 \le i \le l - k\}.$$

Obviously $\mathcal{E}_0 \subset \mathcal{E}_{-1} \subset \cdots \subset \mathcal{E}_{-l} = \operatorname{Der} M$ and $\mathcal{E} \subset \mathcal{E}_{-1}$. Our aim is to prove that the filtration of the $C^\infty(M)$-module $\operatorname{Der} M$ thus obtained is compatible with the structure of the Lie algebra, but first we describe some constructions concerning differential operators on R^n.

We put $k_i = \dim(E_i/E_{i-1})$, $i = 1, \dots, l$, and let $R^n = \bigoplus_{i=1}^l R^{k_i}$ be the standard decomposition of R^n into a direct sum such that

$$R^n = \{x = (x_1, \dots, x_l) | x_i = (x_{i1}, \dots, x_{ik_i}) \in R^{k_i},\ i = 1, \dots, l\}.$$

Any differential operator on R^n with smooth coefficients has the form $\sum_\alpha a_\alpha(x) \partial^{|\alpha|}/\partial x^\alpha$, where $a_\alpha \in C^\infty(R^n)$ and α is a multi-index:

$$\alpha = (\alpha_1, \dots, \alpha_l), \quad \alpha_i = (\alpha_{i1}, \dots, \alpha_{ik_i}), \quad |\alpha_i| = \sum_{j=1}^{k_i} \alpha_{ij}, \qquad i = 1, \dots, l;$$

$$|\alpha| = |\alpha_1| + \cdots + |\alpha_l|.$$

The space $D(R^n)$ of all differential operators with smooth coefficients forms an associative algebra with composition of operators as multiplication. The differential operators with polynomial coefficients form a subalgebra in $D(R^n)$ with generators 1, x_{ij}, $\partial/\partial x_{ij}$, $i = 1, \dots, l$, $j = 1, \dots, k_i$. We introduce a $\mathbf{Z}$-grading into this subalgebra by giving the weights ν to the generators by $\nu(1) = 0$, $\nu(x_{ij}) = i$, and $\nu(\partial/\partial x_{ij}) = -i$. Accordingly

$$\nu\left(x^\alpha \frac{\partial^{|\beta|}}{\partial x^{|\beta|}}\right) = \sum_{i=1}^l (|\alpha_i| - |\beta_i|) i,$$

where α and β are multi-indices.

A differential operator with polynomial coefficients is said to be *homogeneous of weight* r if all the monomials occurring in it have weight r. It is easy to see that $\nu(D_1 \circ D_2) = \nu(D_1) + \nu(D_2)$ for any homogeneous differential operators D_1 and D_2 (it is possible to ascribe an arbitrary weight to the zero operator). All that has been said is also true for vector fields which are differential operators of the first order; moreover, of course, $\nu([X_1, X_2]) = \nu(X_1) + \nu(X_2)$ for any homogeneous vector fields X_1 and X_2. We note further that a differential operator of order N has weight at least $-Nl$; in particular, the weight of nonzero vector fields is at least $-l$. We let

$$V^-(k_1,\dots,k_l) = \operatorname{span}\left\{ x^\alpha \frac{\partial}{\partial x_{ij}} \,\middle|\, \nu(x^\alpha) < i \right\}$$

denote the linear span of the set of homogeneous vector fields of negative weight. Clearly $V^-(k_1,\dots,k_l)$ is a nilpotent subalgebra of length l in the Lie algebra $\operatorname{Der} R^n$.

Now let $X = \sum_{i,j} a_{ij}\partial/\partial x_{ij}$ be an arbitrary smooth vector field. Expanding the coefficients a_{ij} in a Taylor series in powers of x_{rs} and grouping the terms with the same weights, we get an expansion $X \sim \sum_{m=-l}^{+\infty} X^{(m)}$, where $X^{(m)}$ is a homogeneous field of weight m. This expansion enables us to introduce a decreasing filtration in the Lie algebra $\operatorname{Der} R^n$ by putting

$$\operatorname{Der}^m(k_1,\dots,k_l) = \{X \in \operatorname{Der} R^n | X^{(i)} = 0 \text{ for } i < m\}, \qquad -l \le m < +\infty.$$

It is easy to see that

$$[\operatorname{Der}^{m_1}(k_1,\dots,k_l),\ \operatorname{Der}^{m_2}(k_1,\dots,k_l)] \subseteq \operatorname{Der}^{m_1+m_2}(k_1,\dots,k_l).$$

There is an obvious isomorphism of graded Lie algebras which is nevertheless important for our purposes

$$\bigoplus_{m=-l}^{-1} (\operatorname{Der}^m(k_1,\dots,k_l)/\operatorname{Der}^{m+1}(k_1,\dots,k_l)) \approx V^-(k_1,\dots,k_l).$$

THEOREM 1. *There exist a neighborhood Q_{q_0} of the point q_0 in M and a coordinate map $\Phi: O_{q_0} \to R^n$, $\Phi(q_0) = 0$, $\Phi_{*q_0}(E_i) = \bigoplus_{j=1}^{i} R^{k_j}$, $1 \le i \le l$, such that $\Phi_*\mathcal{E}_{-i} = \operatorname{Der}^{-i}(k_1,\dots,k_l)$.*

In particular, $\Phi_*\mathcal{E} \subseteq \operatorname{Der}^{-1}(k_1,\dots,k_l)$, and we get

COROLLARY. *Under the conditions of Theorem 1, for any $X_1,\dots,X_i \in \mathcal{E}$, $i = 1,\dots,l$,*

$$\Phi_*(\operatorname{ad} X_1 \circ \cdots \circ \operatorname{ad} X_{i-1} X_i) \in \operatorname{Der}^{-i}(k_1,\dots,k_l).$$

Theorem 1 implies that the filtration (5) is compatible with the structure of the Lie algebra in $\operatorname{Der} M$; that is, $[\mathcal{E}_{-i}, \mathcal{E}_{-j}] \subseteq \mathcal{E}_{-i-j}$. We put $V^-(\mathcal{E}) = \bigoplus_{i=1}^{l}(\mathcal{E}_{-i}/\mathcal{E}_{1-i})$; then the map Φ_* induces an isomorphism of the graded nilpotent Lie algebras: $V^-(\mathcal{E}) \approx V^-(k_1,\dots,k_l)$; this isomorphism also is denoted by Φ_*. The image of the vector field $X \in \mathcal{E} \subseteq \mathcal{E}_{-1}$ in the Lie algebra $V^-(\mathcal{E})$ under the canonical factorization $\mathcal{E}_{-1} \to \mathcal{E}_{-1}/\mathcal{E}_0$ is denoted by X^-.

3. We return to a consideration of the controlled system (3) and the map G_t; everywhere below the set $\mathcal{E}$ is determined by (4) and $t > 0$ is fixed. Recall that we are assuming $E_l = T_{q_0}M$; the Nagano-Sussmann theorem enables us to suppose that this condition is satisfied for any regular controlled system, in particular for any analytic system with fixed initial value $q(0) = q_0$ [6].

Obviously on taking quotients the time-dependent fields $h_\tau^i \in \mathcal{E}$ generate the curves h_τ^{i-}, $\tau \in [0,t]$, in the algebra $V^-(\mathcal{E})$. Let $\Phi: O_{q_0} \to R^n$ be a coordinate map satisfying the conditions of Theorem 1. Then $\hat{h}_\tau^i = \Phi_* h_\tau^{i-} \in V^-(k_1,\dots,k_l)$ are nonstationary vector fields in R^n with polynomial coefficients.([1]) The controlled system

(6) $$\dot{x} = \sum_{i=1}^{r} \hat{h}^i_\tau(x)u^i, \quad x \in R^n, \; x(0) = 0,$$

is the "nilpotentization" of system (3).

For every $u(\cdot) \in L^r_\infty[0,t]$ we put $\hat{G}_t(u(\cdot)) = x(t)$, where

$$\dot{x}(\tau) = \sum_{i=1}^{r} \hat{h}^i_\tau(x(\tau))u^i(\tau), \qquad \tau \in [0,t], \; x(0) = 0.$$

It is not hard to show that

(7) $$\hat{G}_t(u(\cdot)) = \bigoplus_{i=1}^{l} \left(\int_0^t d\tau_1 \cdots \int_0^{\tau_{i-1}} d\tau_i \left(\sum_{i=1}^{r} \hat{h}^j_{\tau_i} u^j(\tau_i) \right) \circ \cdots \circ \left(\sum_{j=1}^{r} \hat{h}^j_{\tau_1} u^j(\tau_1) \right) \right) x_i,$$

where $x_i = (x_{i1}, \ldots, x_{ik_i})^{\mathrm{T}}$ is a coordinate vector-valued function, the components of the vector-valued function x_i have weight i and the differential operator applied to them has weight $-i$; as a result a vector-valued function of weight 0 is obtained; that is, an element of R^{k_i}. The ε-attainability set of the system (6) within time t is denoted by $\hat{\mathfrak{A}}_t(\varepsilon)$.

Let $\Delta_\varepsilon \colon (x_1, \ldots, x_l) \to (\varepsilon x_1, \ldots, \varepsilon^l x_l)$ be a dilation of the space R^n. Then (7) implies that $\forall \varepsilon_1, \varepsilon_2$, $\Delta_{\varepsilon_1}\hat{\mathfrak{A}}_t(\varepsilon_2) = \hat{\mathfrak{A}}_t(\varepsilon_1\varepsilon_2)$.

DEFINITION. The point $q \in \mathfrak{A}_t(\varepsilon)$ ($x \in \hat{\mathfrak{A}}_t(\varepsilon)$) is *normally attainable for system* (3) (system (6)) if the set $G_t^{-1}(q)$ ($\hat{G}_t^{-1}(x)$) contains a regular point of the map G_t (the map $\hat{G}_t$).

PROPOSITION 1. *Assume that the point $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_l) \in \hat{\mathfrak{A}}_t(1)$ is normally attainable for system* (6). *Then for all sufficiently small $\varepsilon > 0$ the point $q(\varepsilon) = \Phi^{-1}(\varepsilon\overline{x}_1, \ldots, \varepsilon^l\overline{x}_l)$ lies in $\mathfrak{A}_t(\varepsilon)$ and is normally attainable for system* (3).

Everywhere below the fields f and g^i (see (2)) and also the set M are assumed to be analytic.

PROPOSITION 2. *There exists $\varepsilon > 0$ such that the set of regular points of the map G_t is open and everywhere dense in an ε-neighborhood of zero in the space $L^r_1[0,t]$.*

COROLLARY. *The set of normally attainable points is open and everywhere dense in $\mathfrak{A}_t(\varepsilon)$ for all sufficiently small $\varepsilon > 0$.*

We define a (nonsmooth) homeomorphism $\Gamma \colon R^n \to R^n$ by the formula

$$\Gamma(x) = \bigoplus_{i=1}^{l} |x_i|^{(1-i)/i} x_i, \qquad \Gamma(0) = 0;$$

clearly $\Gamma(\Delta_\varepsilon(x)) = \varepsilon\Gamma(x)$.

DEFINITION. Let $\mathfrak{M}(\varepsilon)$, $\varepsilon \geq 0$, be a family of sets in R^n, where $\mathfrak{M}(\varepsilon_1) \subseteq \mathfrak{M}(\varepsilon_2)$ when $\varepsilon_1 < \varepsilon_2$ and $\mathfrak{M}(0) = 0$. We call a vector $y \in R^n$ *interior for the family* $\mathfrak{M}(\varepsilon)$ if there is a neighborhood O_y of the point y in R^n such that $\{\alpha x | x \in O_y, \; 0 \leq \alpha \leq \varepsilon\} \subset \mathfrak{M}(\varepsilon)$ for all sufficiently small $\varepsilon > 0$.

Propositions 1 and 2 imply the following result (the words "almost all" mean "all in a certain open everywhere dense subset").

([1])The fields $\hat{h}^i_\tau$, unlike h^{i-}_τ, depend on the choice of the coordinate map Φ. The difference between h^{i-}_τ and $\hat{h}^i_\tau$ is of the same kind as that between the k-jet of a smooth function at a given point and its Taylor polynomial of degree k.

THEOREM 2. *Almost all the interior vectors of the family of sets* $\Gamma(\hat{\mathfrak{A}}_t(\varepsilon))$, $\varepsilon \geq 0$, *are interior vectors of the family* $\Gamma \circ \Phi(\mathfrak{A}_t(\varepsilon))$.

In order to obtain an inverse inclusion one must introduce some restriction on the "speed of oscillation" of the admissible controls. We get

$$U(\varepsilon, c) = \{u(\cdot) \in L_\infty^r[0,t] \mid \|u(\cdot)\|_{L_\infty} < \varepsilon,\ \mathrm{Var}_{[0,t]} u(\cdot) \leq c\|u(\cdot)\|_{L_1}\},$$

and

$$\mathfrak{A}_t(\varepsilon, c) = G_t(U(\varepsilon, c)), \qquad \hat{\mathfrak{A}}_t(\varepsilon, c) = \hat{G}_t(\varepsilon, c)).$$

THEROEM 3. *For any* $c > 0$, *almost all the interior vectors of the family of sets* $\Gamma \circ \Phi(\mathfrak{A}_t(\varepsilon, c))$, $\varepsilon \geq 0$, *are interior vectors of the family* $\Gamma(\hat{\mathfrak{A}}_t(\varepsilon, c))$.

Thus it has been shown that the attainability sets of system (3) and of its "nilpotentization" (6) in a certain definite sense have locally "almost the same" structure.

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