Soft Construction of Floer-type Homologies

Andrei Agrachev

arXiv:2001.05440

A toy example, the Leray–Schauder degree.

Let B be an infinite-dimensional separable Banach space and $S \subset B$ the unite sphere in B. Let $\mathcal{E} = \{E \subset B : \dim E < \infty\}$ be the ordered by the inclusion directed set of finite-dimensional subspaces of B. We set:

$$G_i(S) = H_{i(\dim E - 1)}(S \cap E)$$

and call $G_i(S)$ the Leray–Schauder homology of S.

Now let $\varphi : S \to B$ be a compact map such that $x+\varphi(x) \neq 0$, $\forall x \in S$. $\forall\, \varepsilon > 0$, $\exists\, \varepsilon$-close to φ finite-dimensional map $\varphi_\varepsilon : S \to E_\varepsilon$. We define a map:

$$\Phi_E^\varepsilon : S \cap E \to S \cap E, \quad \Phi_E^\varepsilon(x) = \frac{x + \varphi_\varepsilon(x)}{|x + \varphi_\varepsilon(x)|},$$

for any $E \supset E_\varepsilon$. The degree of this map $d = \deg(\Phi_E^\varepsilon)$ does not depend on E and is the same for all sufficiently good approximations φ_ε. This is the Leray–Schauder degree.

The degree is defined by the homomorphism:

$$\Phi_{E*}^\varepsilon : H_{\dim E-1}(S \cap E) \to H_{\dim E-1}(S \cap E), \quad \Phi_{E*}^\varepsilon(c) = cd,$$

for any $c \in H_{\dim E-1}(S \cap E) = \mathbb{Z}$. We may interpret it as a homomorphism $\Phi_* : G_1(S) \to G_1(S)$, where $\Phi = \frac{I+\varphi}{|I+\varphi|}$.

Floer Homology

We consider a compact smooth manifold M endowed with a symplectic structure σ. Let $\tilde{M}$ be the universal covering of M and $\tilde{\sigma}$ the pullback of σ to $\tilde{M}$; we assume that $\tilde{\sigma}$ is an exact form: $\tilde{\sigma} = ds$.

We denote by Ω the space of contractible closed curves in M of class H^1. In other words, Ω consists of contractible maps $\gamma : S^1 \to M$, where γ is differentiable almost everywhere with the derivative of class L^2. The lifts of $\gamma \in \Omega$ to $\tilde{M}$ are closed curves and we use the same symbol γ for any lift of this curve to $\tilde{M}$.

Let $h_t : M \to \mathbb{R}$ be a measurable bounded w. r. t. $t \in S^1$ family of smooth functions on M. The functional $\varphi_h : \Omega \to \mathbb{R}$ is defined by the formula:

$$\varphi_h(\gamma) = \int_{S^1} s(\dot{\gamma}(t)) - h_t(\gamma(t))\, dt.$$

Given $c \in \mathbb{R}$, we denote by Ω_h^c the Lebesgue set of φ_h:

$$\Omega_h^c = \{\gamma \in \Omega : \varphi_h(\gamma) \leq c\}$$

We assume that M is equipped with a Riemannian structure $\langle \cdot, \cdot \rangle$ adapted to the symplectic structure, i. e. $\sigma(\xi, \eta) = \langle J\xi, \eta \rangle$, $\xi, \eta \in TM$, where $J : TM \to TM$ is a quasi-complex structure, $J^2 = -I$. Then:

$$\nabla_\gamma \varphi_h = -J\dot{\gamma} - \nabla_\gamma h.$$

Second variation of φ_0 at a "constant curve" q:

$$b_q(\xi,\eta) = \int_{S^1} \sigma(\xi(\theta),\dot\eta(\theta))\,d\theta, \quad \xi,\eta \in H^1(S^1;T_qM).$$

We denote by $\imath : H^1(S^1;T_qM) \to H^1(S^1;T_qM)$ the involution defined by the formula $(\imath\xi)(\theta) = \xi(-\theta)$. Then

$$b_q(\imath\xi,\imath\eta) = -b_q(\xi,\eta), \quad \xi,\eta \in H^1(S^1;T_qM).$$

We fix generators $X_1, \ldots, X_l$ of the $C^\infty(M)$-module Vec$M$ of all smooth vector fields on M and define a linear map $X_q : \mathbb{C}^l \to T_q M$ by the formula

$$X_q u = \sum_{j=1}^{l} v^j X_j(q) + w^j J X_j(q),$$

where $u = (u^1, \ldots, u^l),\ u_j = v_j + i w_j \in \mathbb{C},\ j = 1, \ldots, l$ and

$$\langle \xi, \xi \rangle = \min\{|u|^2 : u \in \mathbb{C}^l,\ \xi = X_q u\}.$$

Let W be the space of all curves in M of class H^1 parameterized by the segment $[0,1]$. We fix a parametrisation of S^1 by $[0,1]$; then $\Omega \subset W$.

We define the map $\phi : M \times L^2([0,1];\mathbb{C}^l) \to W$ as follows. Given $q \in M$ and $u(\cdot) \in L^2([0,1];\mathbb{C}^l)$ the curve $\gamma(\cdot) = \phi(q,u(\cdot))$ is the solution of the ordinary differential equation

$$\dot{\gamma}(t) = X_{\gamma(t)}u(t), \quad 0 \leq t \leq 1,$$

with the initial condition $\gamma(0) = q$. We also set $\phi_t(q,u) = (q,\phi(q,u)(t))$ and thus define the map $\phi_t : M \times L^2([0,1];\mathbb{R}^l) \to M \times M$. It is easy to see that ϕ_t is a smooth map and ϕ_t is a submersion for $0 < t \leq 1$.

Let E be a finite-dimensional subspace of $L^2([0,1];\mathbb{C})$ and $E_0 = \left\{ \upsilon \in E : \int_0^1 \upsilon(t)\,dt = 0 \right\}$. We set:

$$\mathcal{X}_q(E) = \left\{ \theta :\mapsto \xi_0 + \int_0^\theta X_q u(t)\,dt : \xi_0 \in T_q M,\ u(\cdot) \in E_0^l \right\}.$$

We say that E is *well-balanced* if $\imath E = E$ and $\ker b_q|_{\mathcal{X}_q(E)} = \ker b_q$.

Lemma 1. *Any finite-dimensional subspace of* $L^2([0,1];\mathbb{C})$ *is contained in a well-balanced subspace.*

We set:

$$B_r = \left\{ u \in L^2([0,1];\mathbb{C}^l) : \|u\| < r \right\}, \qquad U_r(E) = \phi\left(M \times (B_r \cap E^l)\right).$$

Let $\jmath_i(E;c,r)$ be homology homomorphisms

$$H_i\left(\Omega_h^c \cap U_r(E), \Omega_h^{-c} \cap U_r(E)\right) \to H_i\left(\Omega_h^c \cap U_\infty(E), \Omega_h^{-c} \cap U_\infty(E)\right)$$

induced by the inclusion $U_r(E) \subset U_\infty(E)$. Finally, $\mathcal{E}$ is the directed set of well-balanced subspaces partially ordered by the inclusion.

Theorem 1. *There exist*

$$\lim_{c\to\infty} \lim_{r\to\infty} \mathcal{E}\text{-}\lim \operatorname{rank}\left(\jmath_{i+d_E}(E;c,r)\right) = \operatorname{rank}(H_i(M)),$$

where $d_E = \frac{1}{2}(\dim E - 1)\dim M$.

Let $\beta_j(M)$ be the Betti number of M of the dimension j and C_h be the set of all 1-periodic trajectories of the Hamiltonian system. If all 1-periodic trajectories are non-degenerate, then C_h is a finite set.

Theorem 2 (Morse inequalities)**.** *Assume that all 1-periodic trajectories are non-degenerate. Then, for any $k \in \mathbb{Z}$, the following inequality holds:*

$$\sum_{j \le k} (-1)^{k-j} \beta_j(M) \le \sum_{\{\gamma \in C_h : i_h(\gamma) \le k\}} (-1)^{k-i_h(\gamma)},$$

where

$$i_h(\gamma) = \frac{1}{2} \lfloor sgn(d^2_\gamma \varphi_0) - sgn(d^2_\gamma \varphi_h) \rfloor.$$

Step Two Carnot Lie algebras and groups:

$$\mathfrak{g} = V \oplus W,\ [V,V] = W,\ [\mathfrak{g}, W] = 0, \quad \mathfrak{G} = e^{\mathfrak{g}}.$$

To any $\omega \in W^*$ we associate an operator $A_\omega \in \mathrm{so}(V)$ by the formula:

$$\langle A_\omega \xi, \eta \rangle = \langle \omega, [\xi, \eta] \rangle, \quad \xi, \eta \in V.$$

It is easy to see that $\omega \mapsto A_\omega,\ \omega \in W^*$ is an injective linear map. Moreover, any injective linear map from W^* to $\mathrm{so}(V)$ defines a structure of step two Carnot Lie algebra on the space $V \oplus W$ by the same formula. Hence step two Carnot Lie algebras are in the one-to-one correspondence with linear systems of anti-symmetric operators.

An H^1-curve $\gamma : [0,1] \to \mathfrak{G}$ is called *horizontal* if $\dot{\gamma}() \in V_{\gamma(t)}$ for a. e. $t \in [0,1]$.

The following multiplication in $V \times W$ gives a simple realization of $\mathfrak{G}$ with the origin in $V \times W$ as the unit element:

$$(v_1, w_1) \cdot (v_2, w_2) = \left(v_1 + v_2, w_1 + w_2 + \frac{1}{2}[v_1, v_2]\right).$$

Starting from the origin horizontal curves are determined by their projection to the first level and have a form:

$$\gamma(t) = \left(\xi(t), \frac{1}{2}\int_0^t [\xi(t), \dot{\xi}(t)]\, dt\right), \quad 0 \le t \le 1,$$

where $\xi(\cdot) \in H^1([0,1]; U)$, $\xi(0) = 0$.

We set:

$$\varphi(\xi) = \frac{1}{4\pi}\int_0^1 |\dot{\xi}(t)|^2\, dt.$$

We focus on the horizontal curves corresponding to closed cures ξ; they connect the origin with the second level. Given $w \in W \setminus 0$, let Ω_w be the space of horizontal curves connecting $(0,0)$ with $(0,w)$; then

$$\Omega_w = \left\{\xi \in H^1([0,1];V) : \xi(0) = \xi(1) = 0,\ \frac{1}{2}\int_0^1 [\xi(t),\dot{\xi}(t)]\, dt = w\right\}.$$

For any $s > 0$, we set: $\Omega_w^s = \{\xi \in \Omega_w : \varphi(\xi) \le s\}$. Note that central reflection $\xi \mapsto -\xi$ preserves Ω_w^s.

Let $E \subset H^1([0,1];V)$ be a finite-dimensional subspace and $\bar{E} = \Big(E \setminus 0\Big)/\Big(\xi \sim (-\xi)\Big)$ its projectivization. We set $E^s_w = \Omega^s_w \cap E$ and denote by $\bar{E}^s_w$ the image of E^s_w under the factorization $\xi \sim (-\xi)$.

We consider the homology $H_\cdot(\bar{E}^s_w; \mathbb{Z}_2)$ and its image in $H_\cdot(\bar{E}; \mathbb{Z}_2)$ by the homomorphism induced by the imbedding $\bar{E}^s_w \subset \bar{E}$. We have:

$$\text{rank}\Big(H_i(\bar{E}^s_w; \mathbb{Z}_2)\Big) = \beta_i(\bar{E}^s_w) + \varrho_i(\bar{E}^s_w),$$

where $\beta_i(\bar{E}^s_w)$ is rank of the kernel of the homomorphism from $H_i(\bar{E}^s_w; \mathbb{Z}_2)$ to $H_i(\bar{E}; \mathbb{Z}_2)$ induced by the imbedding $\bar{E}^s_w \subset \bar{E}$ and $\varrho_i(\bar{E}^s_w) \in \{0, 1\}$ is the rank of the image of this homomorphism.

For given w, E, s, we introduce two positive atomic measures on the half-line $\mathbb{R}_+$, the "Betti distributions":

$$\mathfrak{b}(\bar{E}^s_w) \doteq \sum_{i \in \mathbb{Z}_+} \frac{1}{s} \beta_i(\bar{E}^s_q) \delta_{\frac{i}{s}}, \qquad \mathfrak{r}(\bar{E}^s_w) \doteq \sum_{i \in \mathbb{Z}_+} \frac{1}{s} \varrho_i(\bar{E}^s_q) \delta_{\frac{i}{s}}.$$

Assume that $\dim W = 2$ and let $\mathcal{E}$ be the directed set of all finite-dimensional subspaces of the Hilbert space $H^1([0,1]); V)$. It appears that there exist limits of these families of measures

$$\lim_{s \to \infty} \mathcal{E}\text{-}\lim \mathfrak{b}(\bar{E}^s_w), \qquad \lim_{s \to \infty} \mathcal{E}\text{-}\lim \mathfrak{r}(\bar{E}^s_w)$$

in the weak topology. Moreover, the limiting measures are absolutely continuous with explicitly computed densities.

Let $\alpha : \Delta \to \mathbb{R}$ be an absolutely continuous function defined on an interval Δ. We denote by $|d\alpha|$ a positive measure on Δ such that $|d\alpha|(S) = \int_S \left|\frac{d\alpha}{dt}\right| dt,\ S \subset \Delta$.

The operators $A_\omega,\ \omega \in W^*$, have purely imaginary eigenvalues. Let $0 \le \alpha_1(\omega) \le \cdots \le \alpha_m(\omega)$ are such that $\pm i\alpha_j m\ j = 1, \ldots, m$, are all eigenvalues of A_ω counted according the multiplicities.

Let $\bar{W}^* = (W \setminus 0)/\Big(w \sim cw, \forall c \neq 0\Big)$ be the projectivization of W^*, $\bar{W}^* = \mathbb{RP}^1$.

Given $w \in W \setminus 0$, we take the line $w^\perp \in W^*$ and consider the affine line

$$\ell_w = \bar{W}^* \setminus \bar{w}^\perp \subset \bar{W}^*.$$

Moreover, we define functions

$$\lambda_j^w : \ell_w \to \mathbb{R}_+,\ j = 1, \dots, m, \qquad \phi^w : \ell_w \to \mathbb{R}_+$$

by the formulas:

$$\lambda_j^w(\bar{\omega}) = \frac{\alpha_j(\omega)}{\langle \omega, w \rangle}, \qquad \phi^w(\bar{\omega}) = \sum_{j=1}^{m} \lambda_j^w(\omega).$$

Theorem 3. *Assume that there exists* $\omega \in W^*$ *such that the matrix* A_ω *has simple spectrum. Then, for any* $w \in W \setminus 0$, *there exist the following limits in the weak topology of the space of positive measures on* $\mathbb{R}_+$:

$$\mathfrak{b}_w = \lim_{s\to\infty} \mathcal{E}\text{-}\lim \mathfrak{b}(\bar{E}^s_w), \qquad \mathfrak{r}_w = \lim_{s\to\infty} \mathcal{E}\text{-}\lim \mathfrak{r}(\bar{E}^s_w).$$

Moreover,

$$\mathfrak{b}_w = \phi^w_*\Big(\sum_{j=1}^m |d\lambda^w_j|\Big), \qquad \mathfrak{r}_w = \chi_{[0,\min \phi^w]} dt,$$

where dt *is the Euclidean measure.*

General scheme.

The object to study is a Banach manifold Ω equipped with a growing family of closed subsets Ω^s, $s \in \mathbb{R}$.

Auxiliary objects are a Banach space B and a submersion $\Phi : U \to \Omega$, where $U \subset B$ is a finite codimension submanifold of B.

Moreover, U is equipped with an ordered by the inclusion directed and exhausting family $\mathcal{V}$ of open bounded subsets and B is endowed by an ordered by the inclusion directed family $\mathcal{E}$ of finite dimensional subspaces such that $\overline{\bigcup\limits_{E \in \mathcal{E}} E} = B$.

Given $E \in \mathcal{E},\ V \in \mathcal{V},\ s \in \mathbb{R}$, we denote by

$$\jmath_i(E, V, s) : H_i\left(\Omega^s \cap \Phi(E \cap V), \Omega^{-s} \cap \Phi(E \cap V)\right)$$

$$\to H_i\left(\Omega^s \cap \Phi(E \cap U), \Omega^{-s} \cap \Phi(E \cap U)\right)$$

the homology homomorphism induced by the inclusion $V \subset U$.

Finally, we select normalizing quantities $r_i(E,s)$, $\rho_i(E,s) \in \mathbb{R}_+$ and build atomic measures:

$$\mathfrak{b}(E,V,s) = \sum_{i \in \mathbb{Z}_+} \rho_i(E,s) \mathsf{rank}\left(\jmath_i(E,V,s)\right) \delta_{r_i(E,s)}$$

in such a way that their exist a limit:

$$\mathfrak{b} = \lim_{s \to \infty} \mathcal{V}\text{-}\lim \mathcal{E}\text{-}\lim \mathfrak{b}(E,V,s).$$