QUADRATIC MAPPINGS IN GEOMETRIC CONTROL THEORY

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The article is dedicated to local investigation of mappings of type "input-state" of smooth controlled systems. The homological theory of quadratic mappings and the geometry of the Lagrange Grassmannian are used for the study of level sets and images of mappings of type "input-state," including for the obtaining of the necessary and sufficient conditions for local optimality.

1. INTRODUCTION

1. Let M be a differentiable manifold with ~~isolated~~ point $\mu_0 \in M$, and $f_0(\mu)$ a smooth vector field on M. Field f is associated with a curve on M, a solution of the differential equation $d\mu/d\tau = f_0(\mu)$ with initial condition $\mu(0) = \mu_0$. Let $g(\mu)$ be another smooth vector field. Perturbing field f_0 with the help of fields proportional to g, we obtain new curves that are trajectories of equations of the form

$$\frac{d\mu}{d\tau} = f_0(\mu) + u(\tau)\, g(\mu) \tag{1}$$

with the same initial condition μ_0. As $u(\tau)$ we can take an arbitrary local summable function, but as soon as such a function is chosen, Eq. (1) (with the initial condition) uniquely determines trajectory $\mu(t)$, $t \in \mathbf{R}$. Mappings F_t: $u(\cdot) \to \mu(t)$ of a space of functions in manifold M arise. The mappings are fairly complicated: although the velocity of each trajectory passing through point $\mu \in M$ lies on the same affine line $f_0(\mu) + ug(\mu)$, $u \in \mathbf{R}$, the totality of the points of all the trajectories can form a set of an arbitrarily large dimension; as a rule, the image of mapping F_t has a nonempty interior in M.

We can consider a more general situation by replacing $f_0(\mu) + ug(\mu)$ with an arbitrary family $f(\mu, u)$ of vector fields on M that depend smoothly on $u \in U$, where U is some smooth manifold.

Again we define mappings F_t: $u(\cdot) \to \mu(t)$, where

$$\frac{d}{d\tau}\mu(\tau) = f(\mu(\tau), u(\tau)), \quad \mu(0) = \mu_0. \tag{2}$$

A one-parameter family of mappings F_t, $t > 0$, is called a smooth controlled system.

According to the viewpoint accepted in this work, optimal control theory consists of the study of image evolution and sets of the level of mappings F_t with the growth of parameter t. Only local investigations are conducted here, i.e., we study the behavior of mappings F_t near a fixed critical point $\tilde{u}(\cdot)$ (it is easy to see that any point critical for F_t is also critical for all F_τ, $0 < \tau < t$).

The view of a controlled system as a family of mappings $u(\cdot) \to \mu_t$, $t \geq 0$, originates in the foundation-laying works of L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, that are summed up in [14]. Paper [11] explicitly formulates a geometric approach to optimization problems for which the optimality, in some sense or another, of trajectory $\tilde{\mu}_\tau = F_\tau(\tilde{u}(\cdot))$, $0 \leq \tau \leq t$, is interpreted as the belonging of point μ_t to the boundary of the image of F_t or of some modification of this mapping. If the question is of local optimality, then we consider the restriction of F_t to a small neighborhood of $\tilde{u}(\cdot)$.

The key result of classical theory, Pontryagin's maximum principle, provides a necessary optimality condition obtained by linearizing system (2) with respect to variable μ along trajectory $\tilde{\mu}_\tau$. This principle proved to be a very effective tool for solving problems of optimal control. Nevertheless, in many problems, in the first place for systems important in applications having the so-called singular extremals, the application of the maximum principle proves to be insufficient for the finding optimal trajectories (see, for example, [10]). For the study of such systems, various

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additional optimality conditions that take into account approximations of higher order, first of all of the second, were found through the efforts of many experts. Within the scope of the geometric approach, beginning with the middle of the 1970s, the optimality conditions of high order were also studied by Gamkrelidze and the author. In order to attain present understanding, it was necessary to thoroughly reconstruct the theory's language and to significantly expand the formulation of the problem (see [3—7]). An important moment at the beginning of this activity was the familiarity with [18].

It is intuitively clear that the strict local optimality of trajectory $\tilde{\mu}_\tau = F_\tau(\tilde{u}(\cdot))$ is almost the same as the isolation of point $\tilde{u}(\cdot)$ on the level set $F_\tau^{-1}(\tilde{\mu}_\tau)$. In the present work, we undertake to study the local structure of level set $F_t^{-1}(\tilde{\mu}_t)$: the traditional question of whether point $\tilde{\mu}_t$ lies on the boundary of or inside the image of the small neighborhood of $\tilde{u}(\cdot)$ under mapping F_t is inserted into the problem of calculating relative homology groups $H_*(F_t^{-1}(\tilde{\mu}_t), F_t^{-1}(\tilde{\mu}_t)\backslash\tilde{u}(\cdot))$.

Classical examples and models are regular variational problems. A standard variational problem in $\mathbf{R}^n$ with integral functional

$$I=\int_0^T \varphi\left(x(\tau), \frac{dx(\tau)}{d\tau}\right) d\tau, \quad \varphi>0,$$

and boundary conditions $x(0) = x_0$, $x(T) = x_1$ is, essentially, equivalent to the controlled system

$$F_\theta : u(\cdot) \mapsto x(\theta), \text{ where } \frac{dx}{d\theta}=\frac{u(\theta)}{\varphi(x, u(\theta))}.$$

Indeed, the replacement of time $\theta(t)=\int_0^t \varphi\left(x(\tau), \frac{dx(\tau)}{d\tau}\right) d\tau$ leads to a homeomorphism of the level set $F_\theta^{-1}(x_1)$ and $I^{-1}(\theta)$. Under such a homeomorphism the critical points of mapping F_θ turn into the critical points (extremals) of functional I.

Furthermore, under reasonable constraints on φ it turns out that:

the minimum of functional I equal θ $\Leftrightarrow$ x_1 is a boundary point of the image F_θ.

The most important information on the behavior of functional I in the neighborhood of the fixed extremal $\tilde{x}(\cdot)$ is provided by the second variation [the Hessian of I at $\tilde{x}(\cdot)$]. For regular problems the second variation is an integral quadratic form of finite index; moreover, the index of this form completely defines the behavior of functional I near $\tilde{x}(\cdot)$.

The analog of the second variation for a general controlled system is the Hessian of mapping F_t at the critical point $\tilde{u}(\cdot)$. However, in contrast to the regular variational problem, the arising quadratic forms usually prove to be singular and, what is even more significant, the Hessian is, generally speaking, not a scalar quadratic form but a vector quadratic mapping! Moreover, these effects occur not in some pathological cases but for completely natural systems: for example, when $f(\mu, u) = f_0(\mu) + ug(\mu)$, $u \in \mathbf{R}$, $f_0(\mu)$, and $g(\mu)$ are left-invariant fields on a semisimple Lie group. Phenomena of such kind greatly enrich our subject compared to the classical situation and a significant part of the investigation becomes the study of the topology of quadratic mappings. However, before describing the obtained results, it is necessary to refine the initial concepts.

2. Throughout the whole work smoothness means infinite differentiability while piecewise continuous and piecewise smooth functions of a real argument are regarded as continuous on the left. Homology (cohomology) groups are everywhere, unless otherwise stipulated, singular homology (cohomology) groups of a topological space. The assertion that a typical element of a given topological space has a certain property means that this property is fulfilled for all elements of an open everywhere dense subset.

Suppose that M and U are smooth manifolds of dimensions d and r, respectively, $\mu_0 \in M$ and $\mu \mapsto f_t(\mu, u)$ is a family of smooth vector fields on M depending smoothly on $u \in U$ and piecewise smoothly on $t \in \mathbf{R}$. Thus, $f_t(\mu, u) \in T_\mu M$ $\forall \mu \in M$, $u \in U$, $t \in \mathbf{R}$. Manifold M is called a phase space and U a set of controlling parameters.

Remark. As a rule, we consider in control theory sets of controlling parameters of a more general nature than smooth manifolds. The methods of the present work are completely applicable in the case when U is a manifold with boundary and "with corners." They are designed for such applications, but the crux of the matter, in our view, is better seen in a purely smooth situation. There are also such problems in which $f_t(\mu, u)$ depends nondifferentiably on μ; to such problems our methods are not applicable in principle.

By $L_\infty([0, t]; U)$, as usual, we denote the totality of all measurable curves $u(\tau) \in U$, $\tau \in [0, t]$ such that the set $\overline{u([0, t]\backslash\Gamma)} \subset U$ is compact for a subset [depending on $u(\cdot)$] of measure zero $\Gamma \subset [0, t]$. It is easy to see that $L_\infty([0, t], U)$ is a Banach manifold of class C_∞ modeled on $L_\infty^r[0, t]$.

Let $u(\cdot) \in L_\infty([0, t]; U)$. Consider the differential equation

$$\frac{d\mu}{d\tau} = f_\tau(\mu, u(\tau)), \quad \tau \in [0, t] \tag{3}$$

with the initial condition $\mu(0) = \mu_0$.

The solution of such an equation is, generally speaking, not definable on the whole interval [0, t]. In order to avoid similar difficulties, we assume the following conditions to be fulfilled:

Manifold M is imbedded into the Euclidean space $\mathbf{R}^N$ as a smooth manifold and, simultaneously, as a closed subset; in particular, $f_t(\mu, u) \in T_\mu M \subset T_\mu \mathbf{R}^N = \mathbf{R}^N$. For any $K \Subset U$, $t > 0$, there is a constant $c_t(K)$ such that

$$|f_\tau(\mu, u)| \leqslant c_t(K)(1 + |\mu|) \text{ for } u \in K, \ \tau \in [0, t], \ \mu \in M \subset \mathbf{R}^N.$$

The made assumption evidently guarantees the extendability of the solutions of equations of form (3) to the whole interval [0, t]. We define mapping F_t: $L_\infty([0, t]; U) \to M$ by setting $F_t(u(\cdot)) = \mu(t)$, where $\mu(\tau)$, $0 \le \tau \le t$, satisfies differential equation (3), $\mu(0) = \mu_0$.

It is easy to show that F_t is an infinitely differentiable mapping. The critical points of mapping F_t, by analogy with classical calculus of variations, are called extremals.

Fix some point $\tilde{u}(\cdot) \in L_\infty([0, t]; U)$, and let $\tilde{\mu}_t = F_t(\tilde{u}(\cdot))$.

We are interested in the following questions:

I) Is it true that $\tilde{\mu}_t \in \operatorname{int} F_t(\mathcal{O}_{\tilde{u}(\cdot)})$ for any neighborhood $\mathcal{O}_{\tilde{u}(\cdot)}$ of point $\tilde{u}(\cdot)$ in $L_\infty([0, t]; U)$?

II) What can we say about the topology of the intersections of a set of level $F_t^{-1}(\tilde{u}(\cdot))$ and small neighborhoods of point $\tilde{u}(\cdot)$, in the first place, about homology groups

$$H_*(F_t^{-1}(\tilde{\mu}_t), F_t^{-1}(\tilde{\mu}_t)\backslash\tilde{u}(\cdot))\,?$$

If $\tilde{u}(\cdot)$ is a regular point of mapping F_t, then by the implicit function theorem there are local coordinates Φ: $\mathcal{O}_{\tilde{u}(\cdot)} \to L_\infty^r[0, t]$ and φ: $O\tilde{\mu}_t \to \mathbf{R}^d$, defined in some neighborhoods of points $\tilde{u}(\cdot) \in L_\infty([0, t]; U)$ and $\tilde{\mu}_t \in M$, such that $\varphi \cdot F_t \cdot \Phi^{-1}$: $L_\infty^r[0, t] \to \mathbf{R}^d$ is a linear surjective mapping. Consequently, the answer to the first question is affirmative. A set of level $F_t^{-1}(\tilde{\mu}_t)$ near $\tilde{u}(\cdot)$ is arranged quite simply: $F_t^{-1}(\tilde{\mu}_t) \cap \mathcal{O}_{\tilde{u}(\cdot)}$ is diffeomorphic to a subspace of codimension d in $L_\infty^r[0, t]$, in particular, $H_i(F_t^{-1}(\tilde{\mu}_t), F_t^{-1}(\tilde{\mu}_t)\backslash\tilde{u}(\cdot)) = 0$, $i \ge 0$.

Assume now that $\tilde{u}(\cdot)$ is an extremal and, furthermore, $\tilde{u}(\tau)$ depends piecewise smoothly on $\tau \in [0, t]$. Denote by F_t' the differential of mapping F_t at point $\tilde{u}(\cdot)$; then F_t': $T_{\tilde{u}(\cdot)}L_\infty([0, t]; U) \to T_{\tilde{\mu}_t}M$ is a linear mapping. The tangent space $T_{\tilde{u}(\cdot)}L_\infty([0, t]; U)$ consists of all measurable in essence bounded curves $v(\tau)$, $0 \le \tau \le t$ in TU such that $v(\tau) \in T_{\tilde{u}(\tau)}U$ $\forall \tau \in [0, t]$. Each of spaces $T_{\tilde{u}(\tau)}U$, $0 \le \tau \le t$ is isomorphic to $\mathbf{R}^r$; moreover, isomorphism I_τ: $T_{\tilde{u}(\tau)}U \to \mathbf{R}^r$ can be, of course, selected to depend piecewise smoothly on τ. We obtain isomorphism $v(\tau) \to I_\tau v(\tau)$, $\tau \in [0, t]$, of spaces $T_{\tilde{u}(\cdot)}L_\infty([0, t]; U)$ and $L_\infty^r[0, t]$. Since extremal $\tilde{u}(\cdot)$ is fixed in the sequel, it is convenient for the purpose of simplifying notation to fix once and for all an isomorphism of spaces $T_{\tilde{u}(\cdot)}L_\infty([0, t]; U)$ and $L_\infty^r[0, t]$, and in the sequel not to distinguish at all between these spaces. In particular, we shall write

$$F_t': L_\infty^r[0, t] \to T_{\tilde{\mu}_t}M.$$

The fact that $\tilde{u}(\cdot)$ is an extremal (a critical point of mapping F_t) is equivalent to the relation

$$\operatorname{im} F_t' \neq T_{\tilde{\mu}_t}M.$$

Let $L_\infty^r[0, t] \supset \ker F_t'$ be the kernel and $T_{\tilde{u}t}M/\mathrm{im}\, F_t' = \mathrm{coker}\, F_t'$ the cokernel of mapping F_t at point $\tilde{u}(\cdot)$. Denote by F_t the Hessian of mapping F_t at point $\tilde{u}(\cdot)$; then

$$F_t'' : \ker F_t' \times \ker F_t' \to \mathrm{coker}\, F_t'$$

is a symmetric bilinear mapping (see [1, Sec. 1]).

3. We will seek the answers to questions I and II by studying quadratic mapping $v(\cdot) \to F_t''(v(\cdot), v(\cdot))$, $v(\cdot) \in \ker F_t'$. In the case when $\dim \mathrm{coker}\, F_t' = 1$, this quadratic mapping is actually a real quadratic form. If, in addition, the quadratic form is definite on a subspace of a finite codimension in $\ker F_t'$, then to answer the questions interesting us it is enough to find its inertia index (see [1, Sec. 1]). In reality, however, these remarkable properties are not very often fulfilled. Not to be unsupported by evidence, we give a typical example.

Assume that a structure of a semisimple Lie group with Lie algebra $\mathfrak{M}$ is defined on M; moreover, $\mu_0 = e$ is a unit element in M.

Consider the controlled system on M defined by the differential equation

$$\frac{d\mu}{d\tau} = f_0(\mu) + u(\tau)\, g(\mu), \quad \mu(0) = e, \quad u(\tau) \in R, \quad 0 \leqslant \tau \leqslant t,$$

where $f_0(\mu)$ and $g(\mu)$ are left-invariant vector fields on M. Set $\tilde{u}(\tau) = 0$, $0 \le \tau \le t$.

Let $a = f_0(e)$ and $b = g(e)$ be elements of Lie algebra $\mathfrak{M} = T_eM$. Instead of mapping F_t: $u(\cdot) \to \mu(t)$, it is convenient to consider the equivalent mapping

$$G_t : u(\cdot) \mapsto e^{-ta}\mu(t).$$

It is easy to see that $G_t(0) = e$. Let G_t': $L_\infty[0, t] \to \mathfrak{M}$, G_t'': $\mathrm{Ker}\, G_t' \times \ker G_t' \to \mathrm{coker}\, G_t'$ be the differential and the Hessian of mapping G_t at the origin. Somewhat later, in Sec. 2, we will obtain explicit expressions for the derivatives of arbitrarily high order of an arbitrary controlled system. For the time being, however, we give the necessary formulas without justification:

$$G_t' v(\cdot) = \int_0^t e^{\tau\, \mathrm{ad}\, a} b v(\tau)\, d\tau, \quad \forall v(\cdot) \in L_\infty[0, t], \tag{4}$$

$$G_t''(v_1(\cdot), v_2(\cdot)) = \int_0^t \left[\int_0^\tau e^{\theta\, \mathrm{ad}\, a} b v_1(\theta)\, d\theta,\ e^{\tau\, \mathrm{ad}\, a} b v_2(\tau) \right] d\tau + \mathrm{im}\, G_t',$$
$$\forall v_1(\cdot),\ v_2(\cdot) \in \ker G_t';$$

the brackets $[\cdot, \cdot]$ denote, as usual, a commutator in Lie algebra $\mathfrak{M}$ and $(\mathrm{ad}^n a)b = [a, (\mathrm{ad}^{n-1} a)b]$, $n = 1, 2, \ldots$, $(\mathrm{ad}^0 a)b = b$.

Equation (4) implies that $\mathrm{im}\, G_t' = \mathrm{span}\,\{(\mathrm{ad}^n a)b \mid n = 0, 1, \ldots\}$. Assume that a is a regular element of a semisimple Lie algebra $\mathfrak{M}$. Let H_a be a Cartan subalgebra containing a and $\langle \cdot, \cdot \rangle$ the Killing form on $\mathfrak{M}$. Since $\langle H_a, \mathrm{ad}^n ab \rangle = 0\ \forall n > 0$, and the restriction of Killing's form on H_a is nondegenerate, $\dim (H_a \cap \mathrm{im}\, G_t) \le 1$. Consequently, $\dim \mathrm{coker}\, G_t' \ge \mathrm{rank}\, \mathfrak{M} - 1$, where by definition $\mathrm{rank}\, \mathfrak{M} = \dim H_a$.

Thus, for any Lie algebra of rank at least three, G_t'' is certainly a vector (not scalar) bilinear mapping. Furthermore, its scalar projections

$$\psi G_t''(v_1(\cdot), v_2(\cdot)) = \int_0^t \psi \left[\int_0^\tau e^{\theta\, \mathrm{ad}\, a} b v_1(\theta)\, d\theta,\ e^{\tau\, \mathrm{ad}\, a} b v_2(\tau) \right] d\tau,$$

where $\psi \in (\mathrm{im}\, G_t')^\perp$, are completely continuous forms and we do not have to think of any definiteness.

Let us return to the general case. A reasonable nondegeneracy condition of a bilinear mapping F_t'' is the requirement that the origin in space $\mathrm{coker}\, F_t'$ not be a critical value of mapping

$$v(\cdot) \mapsto F_t''(v(\cdot), v(\cdot)), \quad v(\cdot) \in \ker F_t' \setminus 0. \tag{5}$$

If dim coker $F_t' = 1$ and F_t'' is a real bilinear form, then this nondegeneracy condition is equivalent to the regular condition ker $F_t'' = 0$. However, in the case when dim coker $F_t' > 1$ the nondegeneracy of F_t'' does not at all entail the nondegeneracy of all the scalar forms $\varphi F_t'$, $\varphi \in (\text{im } F_t')^{\perp}\backslash 0$ [the latter would have meant that mapping (5) does not always have critical values].

If F_t'' is nondegenerate in the specified sense, then cone $\{v \in \ker F_t' \mid F_t''(v, v) = 0\}$ approximates well the level $F_t^{-1}(\tilde{\mu}_t)$ near point $\tilde{u}(\cdot)$ and a natural "quadratic" analog of question I from subsection 2 turns out to be the question of whether the quadratic mapping $v(\cdot) \rightarrow F_t''(v(\cdot), v(\cdot))$ is essentially surjective (see [1]).

The pairing of arbitrary vector $x \in T_\mu M$ and covector $\xi \in T_\mu{}^*M$ is denoted simply by ξx (as a product of a row by a column). Thus, for any $\psi \in (\text{im } F_t')^{\perp} \subset T_{\mu_t}{}^*M$, expression $\psi F_t''$ is a real quadratic (= symmetric bilinear) form on ker F_t''. The most important invariant of an arbitrary real quadratic form q is its inertia index ind q which is either a nonnegative integer or $+\infty$. For a vector quadratic mapping F_t'' the role of the index is played by function $\psi \mapsto$ ind ψF_t, $\psi \in (\text{im } F_t')^{\perp}\backslash 0$, which takes on nonnegative integer values and $+\infty$. In [1] it is shown how to isolate various properties of mapping F_t'' with the help of this function. However, in order to successfully apply the methods of the mentioned work, it is necessary to have flexible explicit formulas for ind $\psi F_t''$. Section 3 is devoted to the description of such formulas. Before that, in Sec. 2 we give one special representation of the Taylor expansion of mapping F_t, from which, in particular, we obtain an invariant expression for F_t''. A general study of quadratic mapping $v(\cdot) \rightarrow F_t''(v(\cdot), v(\cdot))$ is conducted in Sec. 4, while Sec. 5 is devoted to concrete calculations for certain special classes of systems.

In conclusion, I express my gratitude to my teacher R. V. Gamkrelidze for his constant support and attention to this work.

2. VARIATION OF A CONTROLLED SYSTEM

1. When working with smooth controlled systems (as in many other cases), it is convenient to use the following operator notation.

Point $\mu \in M$ is identified with homomorphism $\varphi \rightarrow \varphi(\mu)$ of algebra $C_\infty(M)$ in **R**. Automorphism $\Phi^*: \varphi(\cdot) \rightarrow \varphi(\Phi(\cdot))$ of algebra $C_\infty(M)$ corresponds to diffeomorphism $\Phi: M \rightarrow M$; value $\Phi(\mu)$ of diffeomorphism Φ at point μ is written in operator language as $\mu.\Phi^*$, a composition of an automorphism and a multiplicative functional (which is in turn a multiplicative functional). At the same time, the composition operation $(\Phi_1, \Phi_2) \rightarrow \Phi_2.\Phi_1$ converts the totality of all diffeomorphisms into a group denoted by Diff M. Note that $(\Phi_2.\Phi_1)^* = \Phi_1^*.\Phi_2^*$. It is not hard to show that any automorphism of algebra $C_\infty(M)$ has form Φ^* for some $\Phi \in$ Diff M, so that relation $\Phi \rightarrow \Phi^*$ establishes an isomorphism of group Diff M and a group of automorphisms of algebra $C_\infty(M)$. Smooth vector fields on M are identified with derivations of algebra $C_\infty(M)$, i.e., with **R**-linear mappings $X: C_\infty(M) \rightarrow C_\infty(M)$, that satisfy Leibnitz's rule:

$$X(\varphi_1\varphi_2) = (X\varphi_1)\varphi_2 + \varphi_1(X\varphi_2) \; \forall \varphi_1\varphi_2 \in C_\infty(M).$$

Commutator $[X, Y] = X{\cdot}Y - Y{\cdot}X$ converts the space of all smooth vector fields into a Lie algebra denoted by Der M. The value of vector field X at point $\mu \in M$ (tangent vector to manifold M at point μ) is written as $\mu.X$. By symbol $T_\mu M$, as usual, we denote a tangent space to manifold M at point μ; for us this is the space of all **R**-linear functionals ξ on $C_\infty(M)$ satisfying the condition $\xi(\varphi_1\varphi_2) = (\xi\varphi_1)\varphi_2(\mu) + \varphi_1(\mu)(\xi\varphi_2)$. By symbol Ad Φ, where $\Phi \in$ Diff M we denote the inner automorphism Ad $\Phi: X \rightarrow \Phi^*.X.\Phi^{*-1}$ of algebra Der M and by symbol ad Y, where $Y \in$ Der M, the inner derivation ad $Y: X \rightarrow [Y, X]$ of the same Lie algebra, $X \in$ Der M.

Let N be one more manifold of class C_∞ and $\Phi: M \rightarrow N$ a diffeomorphism. By symbol Φ_*: Der M $\rightarrow$ Der N we denote the differential of mapping Φ, and by symbol $\Phi_{*\mu}: T_\mu M \rightarrow T_{\Phi(\mu)}N$, where $\mu \in M$, the corresponding linear mapping of tangent spaces. This differential is defined for any smooth mapping and not just for diffeomorphisms; if $\Phi \in$ Diff M, then $\Phi_* = \text{Ad } \Phi^{*-1}$.

We introduce in algebra $C_\infty(M)$ Whitney's topology, the topology of the uniform convergence of derivatives of any order on compacta. Whitney's topology can be specified with the help of the family of seminorms $\|\cdot\|_{k,K}$, $k \geq 0$, $K \Subset M$, where seminorm $\|\cdot\|_{k,K}$ defines the topology of the uniform convergence of all derivatives up to order k on compactum K. Seminorms $\|\cdot\|_{k,K}$ in contrast to the topology given by them are not uniquely defined by manifold M and can be chosen by various methods. In what follows it is assumed that such a choice was made and the seminorms are fixed. For an arbitrary vector field $X \in$ Der M we set

$$\| X \|_{k,K} = \sup \{ \| X\varphi \|_{k,K} \mid \| \varphi \|_{k+1,K} = 1 \}, \quad k=0, \quad K \Subset M.$$

Diffeomorphisms and vector fields define continuous linear operators in Fréchet space $C_\infty(M)$ while points and tangent vectors define continuous linear functionals. In the space $\mathscr{L}(C_\infty(M))$ of all continuous linear operators in M and the space $C_\infty(M)^*$ of all continuous linear functionals in M we introduce a topology of pointwise convergence: sequence $A_i \in \mathscr{L}(C_\infty(M))$, $i = 1, 2, \ldots$ [$a_i \in C_\infty(M)^*$, respectively], converges to $A \in \mathscr{L}(C_\infty(M))$ [$a \in C_\infty(M)^*$] if and only if $A_i\varphi \to A\varphi$ ($a_i(\varphi) \to a(\varphi)$) $\forall \varphi \in C_\infty(M)$.

Suppose that φ_t, $t \in \mathbf{R}$ is a one-parameter family of elements from $C_\infty(M)$. It is said to be measurable if $\forall \mu \in M$ scalar function $t \to \varphi_t(\mu)$ is measurable; a measurable family is defined to be locally integrable if

$$\int_{t_1}^{t_2} \| \varphi_\tau \|_{k,K} d\tau < +\infty \quad \forall t_1, \quad t_2 \in \mathrm{R}, \quad k > 0, \quad K \Subset M.$$

It is easy to see that for locally integrable families φ_t, functions $\int_{t_1}^{t_2} \varphi_\tau d\tau : \mu \to \int_{t_1}^{t_2} \varphi_\tau(\mu) d\tau$, belong to $C_\infty(M)$, $t_1, t_2 \in \mathbf{R}$.

A family φ_t, $t \in \mathbf{R}$ is called absolutely continuous if there exists a local integrable family Ψ_t such that $\varphi_t = \varphi_{t_0} + \int_{t_0}^{t} \Psi_\tau d\tau$. Using the separability of $C_\infty(M)$ we can prove, as we do for scalar functions, that for almost all t

$$\frac{d}{dt}\varphi_t = \frac{d}{dt}\int_0^t \Psi_\tau d\tau = \Psi_t.$$

For one-parameter families A_t, $t \in \mathbf{R}$, of operators from $\mathscr{L}(C_\infty(M))$ the concepts of measurability, continuity, differentiability, local integrability, and absolute convergence are defined by the requirement that $\forall \varphi \in C_\infty(M)$ family $A_t\varphi$ have the corresponding property. The fact that a local integrable family can be integrated and a differentiable one differentiated as well as the validity of Leibnitz's form

$$\frac{d}{dt}(A_t \circ B_t) = \left(\frac{d}{dt} A_t\right) \circ B_t + A_t \circ \left(\frac{d}{dt} B_t\right)$$

is proved with the help of the Banach—Steinhaus theorem (for a more detailed proof of basic analysis operations for such families, see [4, 5]). The same is analogous for one-parameter families of linear functionals from $C_\infty(M)$.

Family A_t, $t \in \mathbf{R}$ is called absolutely continuous if it is representable in the form $A_t = A_{t_0} + \int_0^t B_\tau d\tau$. We have $(d/dt)A_t = B_t$ for almost all t.

Locally summable families $X_t \in \mathrm{Der}\, M$, $t \in \mathbf{R}$, are called nonstationary local vector fields on M and absolutely continuous families Φ_t^*, where $\Phi_t \in \mathrm{Diff}\, M$, $t \in \mathbf{R}$, satisfying the condition $\Phi_0 = \mathrm{id}$ are called nonstationary fluxes on M or, simply, fluxes (symbol id denotes the identity mapping). A nonstationary field X_t, $t \in \mathbf{R}$, defines an ordinary differential equation $(d/dt)\mu = \mu . X_t$ on M. A nonstationary field X_t is called complete if for $\forall \mu_0 \in M$ there is an absolutely continuous solution $\mu(t)$ of this equation satisfying the condition $\mu(0) = \mu_0$ for all $t \in \mathbf{R}$. A complete field defines flux Φ_t^*, the uniquely absolutely continuous solution of operator equation $(d/dt)A_t = A_t . X_t$ with initial condition $A_0 = \mathrm{id}$; here $\mu(t) = \mu_0 . \Phi_t^*$. For flux Φ_t^* the following notation is used, which reflects well its origin and is convenient for calculations:

$$\Phi_t^* = \overrightarrow{\exp} \int_0^t X_\tau d\tau.$$

The asymptotic expansion

$$\overrightarrow{\exp}\int_0^t X_\tau d\tau \approx \mathrm{id} + \sum_{m=1}^{\infty}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{m-1}} (X_{\tau_m}\circ\ldots\circ X_{\tau_1})\, d\tau_m,$$

the exact sense of which consists of inequalities

$$\left\| \overrightarrow{\exp}\int_0^t X_\tau\, d\tau - \mathrm{id} - \sum_{\alpha=1}^{m}\int_0^t d\tau_1 \ldots \int_0^{\tau_{\alpha-1}} (X_{\tau_\alpha}\circ\ldots\circ X_{\tau_1})\, d\tau_\alpha)\,\varphi \right\|_{k,K} \leqslant$$

$$\leqslant c_1 e^{c_2\int_0^t \|X_\tau\|_{k,\tilde K} d\tau} \left(\int_0^t \| X_\tau \|_{k+m,\tilde K} d\tau\right)^{m+1} \|\varphi\|_{k+m+1,\tilde K}, \tag{1}$$

$$\forall\varphi\in C_\infty(M),\quad k,\ m \geqslant 0,\quad K \Subset M,$$

where c_1 and c_2 depend only on k and m, holds (for details including estimates of the constants and $\tilde K \Subset M$ see [4]).

Next, family of operators Ad Φ_t^*, $t \in \mathbf{R}$, acting in space Der M, is the unique absolutely continuous solution of the operator equation $(d/dt)\mathscr{A}_t = \mathscr{A}_t \cdot \mathrm{ad}\ X_t$ with initial condition $\mathscr{A}_0 = \mathrm{id}$, implying the asymptotic expansion

$$\mathrm{Ad}\,\Phi_t^* Y \approx Y + \sum_{m=1}^{\infty}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{m-1}} (\mathrm{ad}\, X_{\tau_m}\circ\ldots\circ \mathrm{ad}\, X_{\tau_1} Y)\, d\tau_m,$$

$Y \in \mathrm{Der}\ M$ with estimates of the remainder analogous to (1) (see [4]). These facts are reflected in the notation used below:

$$\mathrm{Ad}\,\overrightarrow{\exp}\int_0^t X_\tau d\tau = \overrightarrow{\exp}\int_0^t \mathrm{ad}\, X_\tau d\tau.$$

When X_t is independent of t, $X_t \equiv X$, $t \in \mathbf{R}$, the traditional notation is used:

$$\overrightarrow{\exp}\int_0^t X d\tau = e^{tX},\quad \overrightarrow{\exp}\int_0^t \mathrm{ad}\, X d\tau = e^{t\,\mathrm{ad}\, X}.$$

For complete nonstationary fields X_t and $X_t + Y_t$ nonstationary field $\overrightarrow{\exp}\int_0^t \mathrm{ad}\, X_\tau d\tau Y_t$ is also complete and the "constant variation formula"

$$\overrightarrow{\exp}\int_0^t (X_\tau + Y_\tau)\, d\tau = \overrightarrow{\exp}\int_0^t \left(\overrightarrow{\exp}\int_0^\tau \mathrm{ad}\, X_\theta d\theta Y_\tau\right) d\tau \circ \overrightarrow{\exp}\int_0^t X_\tau d\tau, \tag{2}$$

which is checked by direct differentiation of the right- and left-hand sides with respect to t, is valid.

2. Suppose that X_t, Y_t, $t \in \mathbf{R}$ are nonstationary fields on M such that nonstationary fields of form $X_t + \varepsilon Y_t$, $\varepsilon \in \mathbf{R}$ are complete. Consider the one-parameter family of fluxes

$$\Phi_t^*(\varepsilon) = \overrightarrow{\exp}\int_0^t (X_\tau + \varepsilon Y_\tau)\, d\tau,\quad \varepsilon,\ t\in\mathbf{R}.$$

Formula (2) and expansion (1) imply that

$$\frac{\partial}{\partial\varepsilon}\Phi_t^*(\varepsilon) = \int_0^t \mathrm{Ad}\,\Phi_\tau^*(\varepsilon)\, Y_\tau d\tau \circ \Phi_t^*(\varepsilon). \tag{3}$$

Set

$$\Xi_t(\varepsilon)=\int_0^t \operatorname{Ad}\Phi_\tau^*(\varepsilon)Y_\tau d\tau.$$

LEMMA 1. Assume that for a given $t \in \mathbf{R}$, $k \geq 1$, equalities

$$\frac{\partial^i}{\partial\varepsilon^i}\Big|_{\varepsilon=0}\Xi_t(\varepsilon)=0,\quad 0\leqslant i<k-1,\quad \frac{\partial^{k-1}}{\partial\varepsilon^{k-1}}\Big|_{\varepsilon=0}\Xi_t(\varepsilon)=\xi_t^k\in T_{\mu_0}M.$$

hold. Then

$$\mu_0\circ\Phi_t^*(\varepsilon)=\left(\mu_0+\frac{\varepsilon^k}{k}\xi_t^k\right)\circ\Phi_t^*(0)+O(\varepsilon^{k+1}).$$

The statement of the lemma follows from (1) and (3).

In the hypotheses of Lemma 1, vector

$$\xi_t^k\circ\Phi_t^*(0)=(\Phi_t(0)_*\,\xi_k)\in T_{\Phi_t(0)(\mu_0)}M$$

is tangent to curve $\varepsilon \mapsto \Phi_t(\varepsilon)(\mu_0)$ at point $\Phi_t(0)(\mu_0)$. In other words, the principal term in the power series expansion in ε of curve $\varepsilon \mapsto \mu_t(\varepsilon) = \Phi_t(\varepsilon)(\mu_0)$ in M coincides to within a positive factor with the principal term of the power series expansion in ε of curve $\varepsilon \mapsto \mu_t(0)\cdot\Phi_t(0)_*\Xi_t(\varepsilon)$ in $T_{\mu_t(0)}M$. At the same time, if for curve $\mu_t(\varepsilon)$ only the principal term of the expansion is a tangent vector; then the whole expansion of curve $\mu_t(0)\cdot\Phi_t(0)_*\Xi_t(\varepsilon)$, by definition, consists of tangent vectors. Since the aforementioned relates to the arbitrary fields X_τ and Y_τ, the power series expansion in ε of fields $\Xi_t(\varepsilon)$ must, in principle, give a universal expression for the differential, the Hessian, and all the invariant information about higher derivatives of an arbitrary controlled system.

Using formula (3) we get

$$\frac{\partial}{\partial\varepsilon}\Xi_t(\varepsilon)=\int_0^t\frac{\partial}{\partial\varepsilon}\left(\Phi_\tau^*(\varepsilon)Y_\tau\left(\Phi_\tau^*(\varepsilon)\right)^{-1}\right)d\tau=$$

$$=\int_0^t\left[\operatorname{Ad}\Phi_\vartheta^*(\varepsilon)Y_\vartheta d\vartheta,\ \operatorname{Ad}\Phi_\tau^*(\varepsilon)Y_\tau\right]d\tau=\int_0^t\left[\Xi_\tau(\varepsilon),\ \frac{\partial}{\partial\tau}\Xi_\tau(\varepsilon)\right]d\tau.$$

Thus,

$$\frac{\partial}{\partial\varepsilon}\Xi_t(\varepsilon)=\int_0^t\left[\Xi_\tau(\varepsilon),\ \frac{\partial}{\partial\tau}\Xi_\tau(\varepsilon)\right]d\tau,\quad \Xi_t(0)=\int_0^t\operatorname{Ad}\Phi_\tau^*(0)Y_\tau d\tau.$$

For arbitrary absolutely continuous in τ stationary fields $\mathscr{A}_\tau$, $\mathscr{B}_\tau$ we set

$$(\nabla_{\mathscr{A}_\cdot}\mathscr{B}_\cdot)_t=\int_0^t\left[\mathscr{A}_\tau,\ \frac{d}{d\tau}\mathscr{B}_\tau\right]d\tau.$$

We have

$$\frac{\partial}{\partial\varepsilon}\Xi_\cdot=\nabla_\Xi\Xi_\cdot.$$

Differentiating this relation with respect to ε, we get

$$\frac{\partial^2}{\partial\varepsilon^2}\Xi_\cdot=\nabla_{\nabla_\Xi\Xi}\Xi_\cdot+\nabla_\Xi\nabla_\Xi\Xi_\cdot.$$

Differentiating once again, we obtain an expression of $(\partial^3/\partial\varepsilon^3)\Xi$ through Ξ with the help of operation ∇, etc. Thus, we can write out the whole power series expansion in ε of $\Xi_t(\varepsilon)$ through $\Xi_t(0)$, using only operation ∇. We can, however, act differently.

Set

$$Z_t = \mathrm{Ad}\,\Phi_t^*(0)\,Y_t = \overrightarrow{\exp}\int_0^t \mathrm{ad}\,X_\tau d\tau Y_\tau.$$

Equation (2) implies that

$$\Xi_t(\varepsilon) = \int_0^t \mathrm{Ad}\left(\overrightarrow{\exp}\int_0^\tau X_\vartheta + \varepsilon Y_\vartheta d\vartheta\right) Y_\tau d\tau =$$

$$= \int_0^t \left(\mathrm{Ad}\,\overrightarrow{\exp}\int_0^\tau \varepsilon Z_\vartheta d\vartheta\right)\left(\mathrm{Ad}\,\overrightarrow{\exp}\int_0^\tau X_\vartheta d\vartheta\right) Y_\tau d\tau =$$

$$= \int_0^t \overrightarrow{\exp}\int_0^\tau \varepsilon\,\mathrm{ad}\,Z_\vartheta d\vartheta Z_\tau d\tau.$$

Consequently,

$$\Xi_t(\varepsilon) \approx \sum_{m=0}^{\infty} \varepsilon^m \int_0^t d\tau_0 \int_0^{\tau_0} d\tau_1 \ldots \int_0^{\tau_{m-1}} (\mathrm{ad}\,Z_{\tau_m} \ldots \mathrm{ad}\,Z_{\tau_1} Z_{\tau_0})\,d\tau_m. \tag{4}$$

Consider a more general situation by replacing the family of fields $X_t + \varepsilon Y_t$ with an arbitrary family of complete nonstationary fields $X_t(\varepsilon)$ that depends smoothly on $\varepsilon \in \mathbf{R}$. The "constant variation formula," taking into account (1), implies equality

$$\frac{\partial}{\partial\varepsilon}\overrightarrow{\exp}\int_0^t X_\tau(\varepsilon)\,d\tau = \int_0^t \overrightarrow{\exp}\int_0^\tau \mathrm{ad}\,X_\vartheta(\varepsilon)\,d\vartheta \frac{\partial}{\partial\varepsilon} X_\tau(\varepsilon)\,d\tau \circ$$

$$\circ \overrightarrow{\exp}\int_0^t X_\tau(\varepsilon)\,d\tau.$$

So the assertion of Lemma 1 remains true if we set

$$\Phi_t^*(\varepsilon) = \overrightarrow{\exp}\int_0^t X_\tau(\varepsilon)\,d\tau,$$

$$\Xi_t(\varepsilon) = \int_0^t \mathrm{Ad}\,\Phi_\tau^*(\varepsilon)\frac{\partial}{\partial\varepsilon} X_\tau(\varepsilon)\,d\tau =$$

$$= \int_0^t \left(\overrightarrow{\exp}\int_0^\tau \mathrm{ad}\,X_\vartheta(\varepsilon)\,d\vartheta \frac{\partial}{\partial\varepsilon} X_\tau(\varepsilon)\right) d\tau.$$

For k = 1, 2, ..., we set

$$Z_t^{(k)} = \overrightarrow{\exp}\int_0^t \mathrm{ad}\,X_\tau(0)\,d\tau \frac{\partial^k}{\partial\varepsilon^k}\Big|_{\varepsilon=0} X_t(\varepsilon).$$

It is easy to show that all the coefficients of the power series expansion in ε of the family of fields $\int_0^t (\overrightarrow{\exp}\int_0^\tau \times$ $\operatorname{ad} X_\vartheta(\varepsilon)\, d\vartheta \frac{\partial}{\partial\varepsilon} X_\tau(\varepsilon))\, d\tau$ is expressed through nonstationary fields $Z_t^{(k)}$, k = 1, 2, ..., with the help of ∇. We shall not dwell on this and will write out the expansion that generalizes (4). We have

$$\overrightarrow{\exp}\int_0^t \operatorname{ad} X_\tau(\varepsilon)\, d\tau = \overrightarrow{\exp}\int_0^t \operatorname{ad}\left(\overrightarrow{\exp}\int_0^\tau \operatorname{ad} X_\vartheta(0)\, d\vartheta\, (X_\tau(\varepsilon) - X_\tau(0))\right)\circ$$

$$\circ\overrightarrow{\exp}\int_0^t \operatorname{ad} X_\tau(0)\, d\tau \approx \overrightarrow{\exp}\int_0^t \operatorname{ad}\left(\sum_{k=1}^\infty \frac{\varepsilon^k}{k!} Z_\tau^{(k)}\right) d\tau \circ \overrightarrow{\exp}\int_0^t \operatorname{ad} X_\tau(0)\, d\tau.$$

Consequently,

$$\int_0^t \left(\overrightarrow{\exp}\int_0^\tau \operatorname{ad} X_\vartheta(\varepsilon)\, d\vartheta\right) \frac{\partial}{\partial\varepsilon} X_\tau(\varepsilon)\, d\tau \approx \sum_{m=0}^\infty \int_0^t d\tau_0 \int_0^{\tau_0} d\tau_1 \ldots$$

$$\ldots \int_0^{\tau_{m-1}} \left(\left(\sum_{k_m=1}^\infty \frac{\varepsilon^{k_m}}{k_m!} \operatorname{ad} Z_{\tau_m}^{(k_m)}\right)\circ\ldots\circ\left(\sum_{k_1=1}^\infty \frac{\varepsilon^{k_1}}{k_1!} \operatorname{ad} Z_{\tau_1}^{k_1}\right) \sum_{k_0=0}^\infty \frac{\varepsilon^{k_0}}{k_0!} Z_{\tau_0}^{k_0+1}\right) d\tau. \tag{5}$$

Now everything is ready for the definition of the variations of system (1.2) at point $\tilde{u}(\cdot)$. We will give this definition under the assumption that U is an open subset in R^r. The general case is reduced to this by an introduction of local coordinates $L_\infty([0, t]; U)$ (the variation, generally speaking, depends on the choice of local coordinates in the space of controls $L_\infty([0, t], U)$, but it does not depend on the local coordinates in phase space M). A complete variation of system (1.2) is defined to be the one-parameter family of mappings $\mathscr{V}_t$: $L_\infty([0, t]; V) \times L_\infty^r[0, t] \to T_{\mu_0}M$, $t > 0$, which are defined by the rule

$$\mathscr{V}_t(u(\cdot);\ v(\cdot)) =$$

$$= \mu_0\circ\int_0^t \left(\overrightarrow{\exp}\int_0^\tau \operatorname{ad} f_\vartheta(\cdot, u(\vartheta))\ d\vartheta \frac{\partial f_\tau}{\partial u}(\cdot, u(\tau))\, v(\tau)\right) d\tau.$$

For k = 1, 2, ..., a k-order variation at point $(\tilde{u}(\cdot), 0)$ is defined to be the (k − 1)-st derivative of mapping $\mathscr{V}_t$ at the point $(\tilde{u}(\cdot), 0)$. A k-order variation is denoted by

$$\mathscr{V}_t^{(k)}(\tilde{u}(\cdot)): L_\infty^r[0, t] \times \ldots \times L_\infty^r[0, t] \to T_{\mu_0}M$$

and is a symmetric multilinear mapping.

The motivation for such a definition of variation is provided by Lemma 1, and the method of calculation by formula (5). Let

$$v(\cdot) \in L_\infty^r[0, t],$$

setting

$$Z_t^{(k)} = \left(\overrightarrow{\exp}\int_0^t \operatorname{ad} f_\tau(\cdot, \tilde{u}(\tau))\, d\tau\right) \frac{\partial^k}{\partial u^k} f_\tau(\cdot, \tilde{u}(\tau))\, (v(\tau), \ldots, v(\tau)),$$

$$k = 1, 2, \ldots,$$

we get

$$\mathscr{V}_t^{(1)}(\tilde{u}(\cdot))\,v(\cdot)=\mu_0\circ\int_0^t Z_\tau^{(1)}d\tau,$$

$$\mathscr{V}_t^{(2)}(\tilde{u}(\cdot))(v(\cdot),v(\cdot))=\mu_0\circ\int_0^t \left(Z_\tau^{(2)}+\left[\int_0^\tau Z_\vartheta^{(1)}d\vartheta,\ Z_\tau^{(1)}\right]\right)d\tau$$

$$\mathscr{V}_t^{(3)}(\tilde{u}(\cdot))(v(\cdot),v(\cdot),v(\cdot))=\mu_0\circ\int_0^t \left(Z_\tau^{(3)}+\int_0^\tau\left([Z_\vartheta^{(2)},Z_\tau^{(1)}]+\right.\right.$$

$$\left.\left.+2[Z_\vartheta^{(1)},Z_\tau^{(2)}]+2\left[\int_0^\vartheta Z_{\vartheta'}^{(1)}d\vartheta',\ [Z_\vartheta^{(1)},Z_\tau^{(1)}]\right]\right)d\vartheta\right)d\tau \tag{6}$$

etc.

We turn, finally, to mapping F_t. Suppose that $\tilde{\Phi}_t{}^*=\overrightarrow{\exp}\int_0^t f_\tau(\cdot,\tilde{u}(\tau))\,d\tau$, $t\in\mathbf{R}$, Lemma 1 implies the following relations for the differential and the Hessian of F_t at point $\tilde{u}(\cdot)$:

$$F_t'v(\cdot)=\tilde{\Phi}_{t*}\mathscr{V}_t^{(1)}(\tilde{u}(\cdot))\,v(\cdot),$$
$$\forall v(\cdot)\in L_\infty^r[0,t];$$
$$F_t''(v_1(\cdot),v_2(\cdot))=\Phi_{t*}\mathscr{V}_t^{(2)}(\tilde{u}(\cdot))(v_1(\cdot),v_2(\cdot))+\operatorname{im}F_t',$$
$$\forall v_1(\cdot),v_2(\cdot)\in\ker F_t'.$$

It is convenient to make a change in M and consider instead of F_t mapping $G_t=\tilde{\Phi}_t{}^{-1}.F_t$, for which equality $G_t(\tilde{u}(\cdot))=\mu_0$ is fulfilled. Since $\tilde{\Phi}_t$ is a diffeomorphism, from the standpoint of the questions interesting us, mappings F_t and G_t are completely equivalent, for example, $F_t^{-1}(\tilde{\mu}_t)=G_t^{-1}(\mu_0)$. Let G_t' be the differential and G_t'' the Hessian of mapping G_t at point $\tilde{u}(\cdot)$; then $F_t'+\tilde{\Phi}_{t*}G_t'$, $F_t''=\tilde{\Phi}_{t*}G_t''$, so that

$$G_t'v(\cdot)=\mathscr{V}_t^{(1)}(u(\cdot))\,v(\cdot)$$
$$G_t''(v_1(\cdot),v_2(\cdot))=\mathscr{V}_t^{(2)}(u(\cdot))(v_1(\cdot),v_2(\cdot))+\operatorname{im}G_t', \tag{7}$$
$$v(\cdot)\in L_\infty^r[0,t],\ v_1(\cdot),\ v_2(\cdot)\in\ker G_t'.$$

3. SECOND VARIATION AND SYMPLECTIC GEOMETRY

1. We begin by writing out the explicit expressions for G_t' and G_t''. Let

$$Z_t=\tilde{\Phi}_{t*}^{-1}\frac{\partial}{\partial u}f_t(\cdot,\tilde{u}(t)),\quad H_t=\tilde{\Phi}_{t*}^{-1}\frac{\partial^2}{\partial u^2}f_t(\cdot,\tilde{u}(t)),\ t>0$$

[recall that $\tilde{\Phi}_{t*}^{-1}=\overrightarrow{\exp}\int_0^t \operatorname{ad}f_\tau(\cdot,\tilde{u}(\tau))\,d\tau$, and extremal $\tilde{u}(\tau)$ depends piecewise smoothly on τ]. Then Z_t: $\mathbf{R}^r\to$ Der M is a linear mapping and H_t: $\mathbf{R}^r\times\mathbf{R}^r\to$ Der M is a symmetric bilinear depending piecewise smoothly on $t\geq 0$. Formulas (2.6) and (2.7) imply that

$$G_t'v(\cdot)=\mu_0\circ\int_1^t Z_\tau v(\tau)\,d\tau,\quad \operatorname{im}G_t'=\sum_{0<\tau<t}(Z_\tau\mathbf{R}^r)$$

$$G_t''(v_1(\cdot),v_2(\cdot))=\mu_0\circ\int_0^t\left(H_\tau(v_1(\tau),v_2(\tau))+\right.$$

$$\left.+\left[\int_0^\tau Z_\theta v_1(\theta)\,d\theta,\ Z_\tau v_2(\tau)\right]\right)d\tau+\operatorname{im}G_t',$$

$$v(\cdot)\in L^r_\infty[0,t],\ v_1(\cdot),\ v_2(\cdot)\in\ker G_t'.$$

Fix time $t > 0$ and set

$$\Pi = \operatorname{im} G_t' = \sum_{0<\tau<t} Z_\tau R^r$$

Suppose that $\mathcal{E}_\Pi$ is a submodule in Der M consisting of all vector fields, the value of which at point μ_0 lies in Π, and norm $(\mathcal{E}_\Pi)$ is the normalizer of subspace $\mathcal{E}_\Pi$ in Lie algebra Der M, i.e.,

$$\operatorname{norm}(\mathcal{E}_\Pi) = \{X\in \operatorname{Der} M \mid [X, \mathcal{E}_\Pi]\subset\mathcal{E}_\Pi\}\subset\mathcal{E}_\Pi.$$

Set $E_\Pi = \mathcal{E}_\Pi/\operatorname{norm}\mathcal{E}_\Pi$. The operation of vector field commutation defines in the following way the structure of a nilpotent Lie algebra on space $E_\Pi \oplus T_{\mu_0}M/\Pi$.

Let $x, y \in E_\Pi$, $\xi, \eta \in T_{\mu_0}M/\Pi$; moreover, $x = X + \operatorname{norm}\mathcal{E}_\Pi$, $y = Y + \operatorname{norm}\mathcal{E}_\Pi$, where $X, Y \in \mathcal{E}_\Pi$; then

$$[x+\xi, y+\eta] \overset{\text{def}}{=} (\mu_0\circ[X, Y]+\Pi)\in T_{\mu_0}M/\Pi.$$

For any $\tau \in [0, t]$ we denote by $h_\tau(v_1, v_2)$ the image of vector $\mu_0\cdot H_\tau(v_1, v_2)$ under factorization $T_{\mu_0}M \to T_{\mu_0}M/\Pi$, $v_1, v_2 \in R^r$, while by $\mathfrak{z}_\tau v$ we denote the image of vector field $Z_\tau v$ under factorization $\mathcal{E}_\Pi \to \mathcal{E}_\Pi/\operatorname{norm}\mathcal{E}_\Pi = E_\Pi$. Then

$$G_t''(v_1(\cdot), v_2(\cdot)) = \int_0^t (h_\tau(v_1(\tau), v_2(\tau)) + \left[\int_0^\tau \mathfrak{z}_\theta v_1(\theta)\,d\theta,\ \mathfrak{z}_\tau v_2(\tau)\right] d\tau. \tag{1}$$

We turn to the description of space E_Π.

Suppose that I_{μ_0} is the maximal ideal $C_\infty(M)$ consisting of functions that vanish at point μ_0. The following chain of inclusions is checked directly:

$$I_{\mu_0}^2 \operatorname{Der} M\subset I_{\mu_0}\mathcal{E}_\Pi\subset\operatorname{norm}\mathcal{E}_\Pi\subset I_{\mu_0}\operatorname{Der} M\subset\mathcal{E}_\Pi. \tag{2}$$

Using the penultimate inclusion in chain (2), we get that mapping $Z \to \mu_0\cdot Z$, which associates to each field $Z \in E_\Pi$ its value at point μ_0, vanishes on norm $\mathcal{E}_\Pi$, and, consequently, induces some linear mapping $\overline{\mu}_0$: $E_\Pi \to \Pi$. Here, as it is not hard to see, im $\overline{\mu}_0 = \Pi$, ker $\overline{\mu}_0 = I_{\mu_0}$ Der $M/\operatorname{norm}\mathcal{E}_\Pi$. Next, taking into account the first two inclusions in chain (2), we get that mapping $(\varphi, X) \to \varphi X$, which associates to function $\varphi \in I_{\mu_0}$ and to vector field $X \in$ Der M the field $\varphi X \in I_{\mu_0}$ Der M, induces some linear mapping $\overline{j}$: $T_{\mu_0}{}^*M \otimes (T_{\mu_0}M/\Pi) \to E_\Pi$. It is easy to see that

$$\operatorname{im}\overline{j} = I_{\mu_0}\operatorname{Der} M/\operatorname{norm}\mathcal{E}_\Pi,\quad \ker\overline{j} = \Pi^\perp\oplus(T_{\mu_0}M/\Pi).$$

Since $T_{\mu_0}{}^*M/\Pi^\perp = \Pi^*$, we obtain a (natural) exact sequence

$$0\to\Pi^*\oplus(T_{\mu_0}M/\Pi)\xrightarrow{j} E_\Pi\xrightarrow{\overline{\mu}_0}\Pi\to 0, \tag{3}$$

where inclusion j induces mapping $\overline{j}$.

Let codim $\Pi = k > 0$. The exact sequence implies that dim $E_\Pi = (d - k)(k + 1)$. The commutation operation in Lie algebra $E_\Pi \oplus T_{\mu_0}M/\Pi$ is connected to exact sequence (3) in the following way:

$$[z, j(\omega\oplus\nu)] = (\omega\overline{\mu}_0 z)\,\nu,\quad \forall z\in E_\Pi,\quad \omega\oplus\nu\in\Pi^*\oplus(T_{\mu_0}M/\Pi).$$

The assignment of local coordinates $q = (q_1, ..., q_d)^T$: $O \to \mathbf{R}^d$ in some neighborhood O of point μ_0 in M leads to the identification of a space of vector fields in O with the space of smooth mappings $q \to X(q)$ from $\mathbf{R}^d$ into $\mathbf{R}^d$. It also automatically defines coordinates in space E_Π. Let coordinates q_α be such that $q_\alpha(\mu) = 0$, $\alpha = 1, ..., d$, and subspace $\Pi \subset T_{\mu_0}M$ are identified with plane $q_1 = ... = q_k = 0$ in $\mathbf{R}^d$. Then module $\mathscr{E}_\Pi$ is identified with submodule $\{X = (X_1, ..., X_d)^T \in C_\infty{}^d(\mathbf{R}^d) \mid X_i(0) = 0, i = 1, ..., k\}$ in $C_\infty{}^d(\mathbf{R}^d)$ and the space norm $\mathscr{E}_\Pi$ with subspace

$$\{X = (X_1, \ldots, X_d)^T \in C^n_\infty(\mathbf{R}^d) \mid X(0) = 0,\ \frac{\partial X_i}{\partial q_j}(0) = 0,\ i = 1, \ldots, k;\ j = k+1, \ldots, d\}.$$

Consequently, values $X_j(0)$, $\partial X_i/\partial q_j(0)$, $i = 1, ..., k$; $j = k + 1, ..., d$ give coordinates in space $E_\Pi = \mathscr{E}_\Pi/\text{norm}\ \mathscr{E}_\Pi$. Let $x = (X_{k+1}(0), ..., X_d(0), \partial X_1/\partial q_{k+1}(0), ..., \partial X_k/\partial q_d(0))$ be some element of space E_Π expressed in these coordinates. Clearly, $\overline{\mu}_0 x = (X_{k+1}(0), ..., X_d(0))^T$. Thus, our coordinates split the exact sequence (3). Besides that, for any $x, y \in E_\Pi$,

$$[x, y] = \left(\sum_{j=k+1}^{d} \left(\frac{\partial Y_1}{\partial q_j} X_j - \frac{\partial X_1}{\partial q_j} Y_j \right), \ldots \sum_{j=k+1}^{d} \left(\frac{\partial Y_k}{\partial q_j} X_j - \frac{\partial X_k}{\partial q_j} Y_j \right) \right)^T.$$

Note that in the important special case $k = \text{codim}\ \Pi = 1$, mapping $(x, y) \to [x, y]$ from $E_\Pi \times E_\Pi$ into $T_{\mu_0}M/\Pi = \mathbf{R}$ is a symplectic form on E_Π; moreover, the coordinates introduced above are canonic for this form. In this case, Lie algebra $E_\Pi \oplus T_{\mu_0}M/\Pi = E_\Pi \oplus \mathbf{R}$ is isomorphic to the generalized Heisenberg algebra (see [13]), and exact sequence (3) takes on the form

$$0 \to \Pi^* \xrightarrow{j} E_\Pi \xrightarrow{\overline{\mu}_0} \Pi \to 0.$$

Suppose now that k is arbitrary and $\psi \in \Pi^\perp \setminus 0$. Mapping $(x, y) \to \psi[x, y]$ defines a skew-symmetric bilinear form on E_Π. Identity $\psi[z, j(\omega \otimes \nu)] = (\omega\overline{\mu}_0 z)(\psi\nu)$ implies that the kernel of this form coincides with $j(\Pi^* \oplus (\psi^\perp))$. Consequently, form $(x, y) \to \psi[x, y]$ defines a symplectic structure on space

$$E_{\Pi,\psi} = E_\Pi / j\,(\Pi^* \oplus (\psi^\perp)).$$

Since, clearly, $(T_{\mu_0}M/\Pi)/(\psi^\perp) \approx \mathbf{R}$, and j is an injective mapping, we have exact sequence

$$0 \to \Pi^* \to E_{\Pi,\psi} \to \Pi \to 0, \tag{4}$$

that is induced by sequence (3). As in the case when $k = 1$, the introduction of local coordinates in a neighborhood of point μ_0 splits exact sequence (4) and defines canonic coordinates.

Here and below in this section, vector $\psi \in \Pi^\perp \setminus 0$ is fixed; set $\Sigma = \mathscr{E}_{\Pi,\psi}$ and denote by σ a symplectic structure on Σ. According to the aforementioned, Σ is a quotient-space of $\mathscr{E}_\Pi$ by a subspace of codimension $2 \dim \Pi = 2(d - k)$ in $\mathscr{E}_\Pi$. Symplectic structure σ is induced by the skew-symmetric form $(X, Y) \to \psi(\mu_0 \cdot [X, Y])$ on $\mathscr{E}_\Pi$. Denote by $\Pi_0 = \ker \overline{\mu}_0$ the image of subspace $I_{\mu_0}\mathscr{E}_\Pi$ in Σ. It is clear that Π_0 is a Lagrangian subspace in Σ. In coordinates $(q_1, ... q_d)$ canonic factorization $\mathscr{E}_\Pi \to \Sigma$ takes the form

$$X = (X_1, \ldots, X_d) \mapsto \left(X_{k+1}(0), \ldots, X_d(0),\ \sum_{i=1}^{k} \psi_i \frac{\partial X_i}{\partial q_{k+1}}(0), \ldots, \sum_{i=1}^{k} \psi_i \frac{\partial X_i}{\partial q_d}(0)\right),$$

where $\psi = (\psi_1, ..., \psi_k, 0, ..., 0)$.

Let $z_\tau(\psi)v$ be the image of field $Z_\tau v$ under factorization $\mathscr{E}_\Pi \to \Sigma$; then

$$\psi G_t''(v_1(\cdot), v_2(\cdot)) = \int_0^t (\psi h_\tau(v_1(\tau), v_2(\tau)) + \sigma\left(\int_0^\tau z_\theta(\psi) v_1(\theta)\, d\theta,\ z_\tau(\psi) v_2(\tau)\right) d\tau. \tag{5}$$

Condition $v_t(\cdot) \in \ker G_t'$ is equivalent to relation $\int_0^t z_\tau(\psi) v_i(\tau)\, d\tau \in \Pi_0$.

In the sequel, argument ψ in notation $z_\tau(\psi)$ and $z_\tau^{(i)}(\psi) = (d^i/d\tau^i) z_\tau(\psi)$, $i \geq 0$ will be, as a rule, dropped. Since covector ψ is fixed, this does not cause any misunderstanding. Finally we set $m = d - k = \dim \Pi$.

2. The principal goal of this section is to obtain explicit and possibly more flexible expressions for the inertia index of quadratic form $\psi G_t''$. The problem of calculating the inertia index of an integral quadratic form has a long history; for the second variation of the regular variation problem it was solved by Morse [19] in terms of the so-called conjugate points. In a number of works (see [15, 17]) Morse's formula was generalized to certain degenerate situations. The interpretation of Morse's results in terms of symplectic geometry is also sufficiently well known (see, for example, [9]). Our problem is to obtain an explicit expression for the index that is stable for practically any degeneration. Here it is reasonable to give up altogether the taking of conjugate points.

In the first place we should establish the conditions for the finiteness of ind $\psi G_t''$ or (which is very close) the conditions for the nonnegativity of $\psi G_t''(v(\cdot), v(\cdot))$ for any $v(\cdot)$ different from zero by only a subset of a sufficiently small diameter in $[0, t]$.

LEMMA 1. Assume that for some $l \geq 0$ and half-interval $(\overline{\tau}, \tau] \subset (0, t]$ identities $\sigma(z_\theta^{(i)} v_1, z_\theta v_2) = 0$, $\forall \theta \in (\overline{\tau}, \tau]$, $\forall v_1, v_2 \in \mathbf{R}^r$, $\forall i < l$ are fulfilled. Then

1) $\sigma(z_\theta^{(l-i)} v_1, z_\theta^{(i)} v_2) = (-1)^i \sigma(z_\theta^{(l)} v_1, z_\theta v_2)$, $\overline{\tau} < \theta \leq \tau$, $v_1, v_2 \in \mathbf{R}^r$, $i \leq l$.

2) If $l \geq 2m$, then $\sigma(z_\theta^{(i)} v_1, z_\theta^{(j)} v_2) = 0$, $\forall i, j \geq 0$, $\overline{\tau} < \theta \leq \tau$, $v_1, v_2 \in \mathbf{R}^r$.

Proof. 1) Let us use induction on value l. For $l = 0$ there is nothing to prove. The step of the induction is:

$$\sigma\left(z_\theta^{(l)} v_1, z_\theta v_2\right) = \frac{d}{d\theta}\, \sigma\left(z_\theta^{(l-1)} v_1, z_\theta, v_2\right) - \sigma\left(z_\theta^{(l-1)} v_1, z_\theta^{(1)} v_2\right) =$$

$$= -\sigma\left(z_\theta^{(l-1)} v_1, z_\theta^{(1)} v_2\right) = -\frac{d}{d\theta}\, \sigma\left(z_\theta^{(l-2)} v_1, z_\theta^{(1)} v_2\right) + \sigma\left(z_\theta^{(l-2)} v_1, z_\theta^{(2)} v_2\right) = \sigma\left(z_\theta^{(l-2)} v_1, z_\theta^{(2)} v_2\right) = \ldots$$

$$\ldots = (-1)^i \sigma\left(z_\theta^{(l-i)} v_1, z_\theta^{(i)} v_2\right).$$

2) Since $\dim \Sigma = 2m$, for all τ from some open everywhere dense in $[0, t]$ subset the following decomposition is valid: $z_\tau^{(2m)} v = \sum_{i=0}^{2m-1} \alpha_i(\tau, v)\, z_\tau^{(i)} v$, where $\alpha_i(\tau, v)$ depends smoothly on τ. Differentiating this identity with respect to τ, we can represent a derivative of an arbitrarily high order from $z_\tau v$ in the form of a linear combination of the first $2r - 1$ derivatives. ∎

We can associate to each $\tau \in (0, t]$ an integer $k_\tau \geq 0$ and a quadratic form γ_τ on $\mathbf{R}^r$ in the following way: if form ψh_θ does not equal identically zero on any interval $\overline{\tau} < \theta < \tau$; then we set $k_\tau = 0$, $\gamma_\tau(v) = \psi h_\tau(v, v)$; otherwise let k_τ be a maximal number k such that $\sigma(z_\theta^{(i)} v_1, z_\theta v_2) \equiv 0$ for $i < 2(k - 1)$, $v_1, v_2 \in \mathbf{R}^r$ on some interval $\overline{\tau} < \theta < \tau$, $\gamma_\tau(v) = \sigma(z_\tau^{(k_\tau)} v, z_\tau^{(k_\tau - 1)} v)$, $v \in \mathbf{R}^r$; if, however, maximal k does not exist [that is, $\sigma(z_\theta^{(i)} v_1, z_\theta v_2) \equiv 0$ for $i \leq 2m$), then we set $k_\tau = m + 1$, $\gamma_\tau = 0$.

Proposition 1. If ind $\psi G_t'' < +\infty$, then:

a) $\sigma(z_\tau^{(k_\tau - 1)} v_1, z_\tau^{(k_\tau - 1)} v_2) = 0$, $\forall v_1, v_2 \in \mathbf{R}^r$, $\tau \in (0, t]$;

b) $\gamma_\tau(v) \geq 0$, $\tau \in (0, t]$, $v \in \mathbf{R}^r$.

Conversely, if condition a) is fulfilled and $\gamma_\tau(v) \geq \varepsilon |v|^2$ for any $v \in \mathbf{R}^r$, $\tau \in (0, t]$ and some $\varepsilon > 0$, then ind $\psi G_t'' < +\infty$.

Proof. Assume that $\psi h_\tau = 0$ for $v \in \mathbf{R}^r$, $\tau \in (t_1, t_2)$ and z_τ is smooth on this interval. Then for every function $v(\cdot)$ different from zero only on segment $[t_1, t_2]$ we have

$$\psi G_t''(v(\cdot), v(\cdot)) = \int_{t_1}^{t_2} \sigma\left(\int_{t_1}^{\tau} z_\theta v(\theta)\, d\theta,\ z_\tau v(\tau)\right) d\tau.$$

Integrating by parts [whereby $v(\tau)$ is integrated and $\int_{t_1}^{\tau} z_\theta v(\theta)\, d\theta$ and z_τ are differentiated] and setting $w_1(\tau) = \int_{t_1}^{\tau} v(\theta)\, d\theta$, we get

$$\psi G_t''(v(\cdot), v(\cdot)) = -\int_{t_1}^{t_2} \sigma(z_\tau v(\tau),\ z_\tau w_1(\tau))\, d\tau -$$

$$-\int_{t_1}^{t_2} \sigma\left(\int_{t_1}^{\tau} z_\theta v(\theta)\, d\theta,\ z_\tau^{(1)} w_1(\tau)\right) d\tau + \sigma\left(\int_{t_1}^{t_2} z_\tau v(\tau)\, d\tau,\ z_{t_2} w_1(t_2)\right) =$$

$$= \int_{t_1}^{t_2} \sigma(z_\tau w_1(\tau),\ z_\tau v(\tau))\, d\tau + \int_{t_1}^{t_2} \sigma(z_\tau^{(1)} w_1(\tau),\ z_\tau w_1(\tau))\, d\tau +$$

$$+ \int_{t_1}^{t_2} \sigma\left(\int_{t_1}^{\tau} z_\theta^{(1)} w_1(\theta)\, d\theta,\ z_\tau^{(1)} w_1(\tau)\right) d\tau -$$

$$- \sigma\left(\int_{t_1}^{t_2} z_\tau w_1(\tau)\, d\tau,\ z_{t_2} w_1(t_2)\right).$$

If $k_\tau > 1$ for $\tau \in (t_1, t_2]$, then the first two summands on the right-hand side equal zero; in this case the third term is integrated by parts again, and so on. If, in addition, $k_\tau = \text{const}$ for $\tau \in (t_1, t_2)$, then finally we get

$$\begin{aligned}\psi G_t'' = &\int_{t_1}^{t_2} \sigma\left(z_\tau^{(k_\tau-1)} w_{k_\tau}(\tau),\ z_\tau^{(k_\tau-1)} w_{k_\tau-1}(\tau)\right) d\tau + \\ &+ \int_{t_1}^{t_2} \gamma_\tau(w_{k_\tau}(\tau))\, d\tau + \int_{t_1}^{t_2} \sigma\left(\int_{t_1}^{\tau} z^{(k_\tau)} w_{k_\tau}(\theta)\, d\theta,\ z_\tau^{(k_\tau)} w_{k_\tau}(\tau)\right) d\tau - \\ &- \sum_{j=0}^{k_\tau-1} (-1)^{k_\tau+j} \sigma\left(\int_{t_1}^{t_2} z_\theta^{(k_\tau)} w_{k_\tau}(\theta)\, d\theta,\ z_{t_2}^{(j)} w_{j+1}(t_2)\right),\end{aligned} \tag{6}$$

where $w_0(\tau) = v(\tau)$, $w_i(\tau) = \int_{t_1}^{\tau} w_{i-1}(\theta)\, d\theta$ for $i > 0$.

Note that $\| w_i(\cdot) \|_\infty \le |t_2 - t_1|\, \| w_{i-1}(\cdot) \|_\infty$. Clearly, $(v_1, v_2) \to \sigma(z_\tau^{(k_\tau-1)} v_1, z_\tau^{(k_\tau-1)} v_2)$ is a skew-symmetric form on $\mathbf{R}^r$. Assume that this form differs from zero for some $\tau \in [t_1, t_2]$, without loss of generality we can assume that $\tau = t_2$. It is not hard to see that on a subspace of codimension rk_τ in $L_\infty^r[t_1, t_2]$, defined by conditions $w_j(t_2) = 0$, $j = 1, \ldots, k_\tau$, the following relation is true:

$$\begin{aligned}\psi G_t''(v(\cdot), v(\cdot)) = &\int_{t_1}^{t_2} \sigma\left(z_{t_2}^{(k_\tau-1)} w_{k_\tau}(\tau),\ z_{t_2}^{(k_\tau-1)} w_{k_\tau-1}(\tau)\right) d\tau + \\ &+ O(|t_2 - t_1|\, \| w_{k_\tau}(\cdot) \|^2).\end{aligned}$$

If form $\psi G_t''$ has a finite index, then we can easily deduce from $\| w_{k_\tau}(\cdot) \|_\infty \le |t_2 - t_1|\, \| w_{k_\tau-1}(\cdot) \|_\infty$ that form

$$\int_{t_1}^{t_2} \sigma\left(z_{t_2}^{(k_\tau-1)} w_{k_\tau}(\tau),\ z_{t_2}^{(k_\tau-1)} w_{k_\tau-1}(\tau)\right) d\tau \tag{7}$$

must also have a finite index. At the same time there exists involution $J: \mathbf{R}^r \to \mathbf{R}^r$, $J^2 = \text{id}$ of space $\mathbf{R}^r$ such that

$$\sigma\left(z_{t_2}^{(k_\tau-1)}Jv_1,\ z_{t_2}^{(k_\tau-1)}Jv_2\right)=-\sigma\left(z_{t_2}^{(k_\tau-1)}v_1,\ z_{t_2}^{(k_\tau-1)}v_2\right)\ \forall v_1,\ v_2\in R^r$$

(the corresponding involution obviously exists for each skew-symmetric form on R^r). Consequently, each subspace in $L_2{}^r[t_1, t_2]$ on which form (7) is positive is put in correspondence to a subspace of the same dimension on which form (7) is negative. Since the kernel of form (7) evidently has an infinite codimension, we arrive at a contradiction.

Thus, the finiteness of the index of form $\psi G_t''$ implies the identity

$$\sigma\left(z_\tau^{(k_\tau-1)}v_1,\ z_\tau^{(k_\tau-1)}v_2\right)=0,\quad \forall v_1,\ v_2\in R^r.$$

In this case, equality (6) implies that on a subspace of codimension rk_τ in $L_\infty{}^r[t_1, t_2]$ defined by conditions $w_j(t_2) = 0$, $j = 1, \ldots, k_\tau$, the following relation is fulfilled:

$$\psi G_t''(v(\cdot),\ v(\cdot))=\int_{t_1}^{t_2}\gamma_\tau(w_{k_\tau}(\tau))\,d\tau+O\left(|t_2-t_1|^2\,\|w_{k_\tau}(\cdot)\|_\infty^2\right).$$

If the index of the form is finite, then the index of form $\int_{t_1}^{t_2}\gamma_\tau(w_{k_\tau}(\tau))\,d\tau$, must also be finite, which is evidently possible only for $\gamma_\tau \geq 0$.

Assume now that hypothesis a) from the formulation of Proposition 1 is fulfilled and, in addition, $\gamma_\tau(v) \geq \varepsilon|v|^2$, $\forall v \in R^r$. The definition of numbers k_τ and forms γ_τ implies that in this case k_τ has discontinuities only at the points where the smoothness of curves $\tau \to z_\tau$ and $\tau \to \psi h_\tau$ is violated.

Let $0 = \tau_0 < \tau_1 < \ldots < \tau_l = t$ be a partition of interval $[0, t]$; moreover, all the points where the smoothness of z_τ and ψh_τ is violated are among the points τ_i.

Every $v(\cdot) \in L_\infty{}^r[0, t]$ is uniquely representable in the form $v(\cdot)=\sum_{i=1}^{l}v_i(\cdot)$, where $v_i(\tau)$ differ from zero only for $\tau_{i-1} < \tau \leq \tau_i$. Let $v(\cdot) \in L_\infty{}^r[0, t]$ be such that

$$\int_{\tau_{i-1}}^{\tau_i} z_\tau v(\tau)\,d\tau=0,\quad i=1,\ \ldots,\ l. \tag{8}$$

Then $\psi G_t''(v(\cdot),\ v(\cdot))=\sum_{i=1}^{l}\psi G_t''(v_i(\cdot),\ v_i(\cdot))$.

Conditions (8) isolate in $L_\infty{}^r[0, t]$ a subspace with a finite codimension. Consequently, to prove the finiteness of the index of form $\psi G_t''$ it is sufficient to prove the finiteness of the index of each of the forms $v(\cdot) \to \psi G_t''(v_i(\cdot), v_i(\cdot))$. In particular, it is enough to consider only such $v(\cdot)$ which vanish outside some segment $(\tau_{i-1}, \tau_i]$. Since k_τ = const for $\tau \in (\tau_{i-1}, \tau_i]$, representation (6), where $t_1 = \tau_{i-1}$, $t_2 = \tau_i$, is valid. Taking into account hypothesis a), we obtain that on a subspace of finite codimension in $L_2{}^r[\tau_{i-1}, \tau_i]$ our form coincides with the form

$$\int_{\tau_{i-1}}^{\tau_i}\gamma_\tau(w_{k_\tau}(\tau))\,d\tau+\int_{\tau_{i-1}}^{\tau_i}\sigma\Big(\int_{\tau_{i-1}}^{\tau}z_\theta^{(k_\tau)}\,w_{k_\tau}(\theta)\,d\theta,\ z_\tau^{(k_\tau)}\,w_{k_\tau}(\tau)\,d\tau\Big)\,\tau\,,$$

that is, with the restriction on some subspace of form

$$Q_i(w)=\int_{\tau_{i-1}}^{\tau_i}\gamma_\tau(w(\tau))\,d\tau+\int_{\tau_{i-1}}^{\tau_i}\sigma\int_{\tau_{i-1}}^{\tau}z_\theta^{(k_\tau)}w(\theta)\,d\theta,\ z_\tau^{(k_\tau)}\,w(\tau)\,d\tau,$$

$$w(\cdot)\in L_2^r[\tau_{i-1},\ \tau_i].$$

However, according to the classical Hilbert—Schmidt theorem, the quadratic form Q_i is positively definite on some subspace of finite codimension in $L_r{}^2[\tau_{i-1}, \tau_i]$.

As we already noted, if $\gamma_\tau(v) \geq \varepsilon|v|^2\ \forall\tau \in [0, t]$, $v \in R^r$, then γ_τ and k_τ are piecewise smooth continuous on the left functions on $[0, t]$, smooth at any point where curve $t \to z_\tau$ is smooth.

First we will learn how to compute the index of form $\psi G_t''$ when $\gamma_\tau(v) \geq \varepsilon |v|^2$, and then in subsection 6 we note what must be changed if γ_τ degenerates. Thus, everywhere below, if otherwise not stipulated, it is assumed that $\gamma_\tau(v) \geq \varepsilon |v|^2$ $\forall \tau \in [0, t]$, $v \in \mathbf{R}^r$, and some $\varepsilon > 0$.

3. In this and the following subsections we widely use notation and results from symplectic geometry that are collected in the Appendix to this section.

Consider the families of subspaces

$$\Gamma_\tau = \sum_{i=0}^{k_\tau - 1} z_\tau^{(i)} \mathbf{R}^r, \quad \overline{\Gamma}_\tau = \sum_{j=k_\tau}^{2k_\tau - 1} z_\tau^{(j)} \mathbf{R}^r.$$

LEMMA 2. a) $\overline{\Gamma}_\tau \cap \Gamma_\tau^\angle = 0$, $\Gamma_\tau \cap \overline{\Gamma}_\tau^\angle = 0$; b) $\dim \Gamma_\tau = \dim \overline{\Gamma}_\tau = rk_\tau$; c) families of subspaces Γ_τ and $\overline{\Gamma}_\tau$ depend piecewise smoothly on $\tau \in [0, t]$; moreover, this dependence is smooth at any point where curves $\tau \to z_\tau$, $\tau \to \psi h_\tau$ are smooth.

Proof. a) Identity $\sigma(z_\tau^{(i)}, z_\tau^{j}) = 0$, $i, j < k_\tau$, equivalent to inclusion $\Gamma_\tau \subset \Gamma_\tau^\angle$ follows from Lemma 1. Assume that $\left(\sum_{j=k_\tau}^{n} z_\tau^{(j)} v_j\right) \angle \Gamma$, moreover, $v_n \neq 0$, $k_\tau \leq n \leq 2k_\tau - 1$. Then

$$0 = \sigma\left(z_\tau^{(2k_\tau - n - 1)} v_n, \sum_{j=k_\tau}^{n} z_\tau^{(j)} v_j\right) = \sigma\left(z_\tau^{(2k_\tau - n - 1)} v_n, z_\tau^{(n)} v_n\right) =$$
$$= (-1)^{n + k_\tau - 1} \gamma_\tau(v_n),$$

which contradicts the positive definiteness of form γ_τ. Consequently, $\overline{\Gamma}_\tau \cap \Gamma_\tau^\angle = 0$.

Analogous arguments prove relation $\overline{\Gamma}_\tau \cap \Gamma_\tau^\angle = 0$ and also assertion b). Assertion c) follows from b) and the constancy of k_τ near any point where curves z_τ, $\psi h(\tau)$ are smooth.

Form γ_τ, like every quadratic form on $\mathbf{R}^r$, is defined by some self-adjoint mapping $\hat{\gamma}_\tau$: $\mathbf{R}^r \to \mathbf{R}^{r*}$, so that $\gamma_\tau(v) = (\hat{\gamma}_\tau v)v$. Denote by $\hat{\gamma}_\tau^{-1}$ a quadratic form on $\mathbf{R}^{r*}$ definable by mapping $\hat{\gamma}_\tau^{-1}$, that is, $\hat{\gamma}_\tau^{-1}(v^*) = v^*(\hat{\gamma}^{-1} v^*)$. Next, for every $x \in \Sigma$ mapping $v \to \sigma(z_\tau^{(k_\tau)} v, x)$ is a linear form on $\mathbf{R}^r$; thus, $\sigma(z_\tau^{(k_\tau)}\cdot, x) \in \mathbf{R}^r$, and correspondence

$$x \mapsto \frac{1}{2} \gamma_\tau^{-1}\left(\sigma\left(z_\tau^{(k_\tau)}\cdot, x\right)\right), \quad x \in \Sigma, \tag{9}$$

defines a quadratic form on symplectic space Σ.

Assume that $v_\tau^1, \ldots, v_\tau^r$ is a basis in $\mathbf{R}^r$ such that $(\hat{\gamma}_\tau v_\tau^i) v_\tau^j = 0$ for $i \neq j$; then

$$\frac{1}{2} \gamma_\tau^{-1}\left(\sigma\left(z_\tau^{(k_\tau)}\cdot, x\right)\right) = \sum_{i=1}^{r} \frac{\sigma\left(z_\tau^{(k_\tau)} v_\tau^i, x\right)^2}{2\gamma_\tau\left(v_\tau^i\right)}.$$

Every smooth function on a symplectic space is a Hamiltonian; it defines a Hamiltonian system of differential equations on this space; if the Hamiltonian depends on time τ, then the system is nonstationary; if the Hamiltonian is a quadratic form, then the Hamiltonian system is linear. Nonstationary quadratic Hamiltonian (9) is associated with the Hamiltonian system

$$\dot{x} = \sum_{i=1}^{r} \frac{\sigma\left(z_\tau^{(k_\tau)} v_\tau^i, x\right)}{\gamma_\tau\left(v_\tau^i\right)} z_\tau^{(k_\tau)} v_i; \quad x \in \Sigma, \quad \tau \in [0, t], \tag{10}$$

which is called a Jacobi system for the extremal of the controlled system under consideration. A flux on Σ defined by the Jacobi system is a one-parameter family of linear symplectic transformations of space Σ. Since symplectic transformations carry Lagrangian planes into Lagrangian planes, system (10) also defines a flux on the Lagrange Grassmannian $L(\Sigma)$. According to the notation given in the Appendix, this flux on $L(\Sigma)$ is generated by the nonstationary vector field $(1/2)\gamma_\tau^{-1}(\sigma(z_\tau^{(k_\tau)}\cdot, \Lambda))$, $\Lambda \in L(\Sigma)$. The differential equation

$$\dot{\Lambda} = \frac{1}{2} \gamma_\tau^{-1}\left(\sigma\left(z_\tau^{(k_\tau)}\cdot, \Lambda\right)\right), \quad \Lambda \in L(\Sigma), \quad \tau \in [0, t] \tag{11}$$

is also called the Jacobi equation.

In the sequel, solutions of Eq. (11) are defined to be not only continuous but also piecewise continuous curves with the derivative with respect to τ at a point of discontinuity taken to be the limit on the left of the corresponding derivatives. Thus, to uniquely determine a solution of the Jacobi equation it is necessary to give the jumps at points of discontinuity in addition to the initial value.

The following concept is key for what follows.

Definition. A Jacobi curve is defined to be a piecewise smooth curve Λ_τ on $L(\Sigma)$ satisfying the Jacobi equation and the conditions $\Lambda_0 = \Pi_0$, $\Lambda_{\tau+0} = \Lambda_\tau^{\Gamma_{\tau+0}}$, $\forall\tau \in [0, t]$.

Direct calculation shows that $\sigma(\Lambda_\tau, \Gamma_\tau) = 0$; consequently, $\Gamma_\tau \subset \Lambda_\tau$ $\forall\tau \in [0, t]$; therefore, the curve is continuous at any point of continuity of Γ_τ.

Suppose that the collection of points $0 = \tau_0 < \tau_1 < \ldots < \tau_l = t$ contains all the points at which the smoothness of curves [illegible] is violated. Curve Λ_τ is described more explicitly in the following way:

$$\Lambda_0 = \Pi_0, \quad \Lambda_{\tau_i+0} = (\Lambda_{\tau_i} + \Gamma_{\tau_i+0})^{\perp} \cap \Gamma_{\tau_i+0}, \quad i = 0, 1, \ldots, l-1,$$

$$\Lambda_\tau = \left\{ x_\tau \in \Sigma \mid \dot{x}_\theta = \sum_{i=1}^{\tau} \frac{\sigma\left(z_\theta^{(k_\theta)} v_\theta^i, x_\theta\right)}{\gamma_\theta\left(v_\theta^i\right)} z_\theta^{(k_\theta)} v_\theta^i, \ \tau_i < \theta \leqslant \tau, \ x_{\tau_i+0} \in \Gamma_{\tau_i+0} \right\}$$

for $\tau_i < \tau \leq \tau_{i+1}$.

To compute the index of quadratic form $\psi G_t''$ we need one symplectic invariant of the triple of Lagrangian planes. Let $\Lambda_1, \Lambda_2, \Lambda_3 \subset \Sigma$ be Lagrangian planes and $\hat{\lambda} \in (\Lambda_1 + \Lambda_3) \cap \Lambda_2 / \bigcap_{i=1}^{3} \Lambda_i$. Then $\hat{\lambda}$ is represented in the form $\hat{\lambda} = \hat{\lambda}_1 + \hat{\lambda}_3$, where $\hat{\lambda}_j \in \Lambda_j / \bigcap_{i=1}^{3} \Lambda_i$, $j = 1, 3$. Assume that $\hat{\lambda}_j = \lambda_j + \bigcap_{i=1}^{3} \Lambda_i$, $j = 1, 3$, and set $q(\hat{\lambda}) = \sigma(\lambda_1, \lambda_3)$. It is easy to see that value $q(\hat{\lambda})$ is well defined, i.e., expression $\sigma(\lambda_1, \lambda_3)$ is independent of the choice of the representation of the corresponding coclasses. Thus, correspondence $\hat{\lambda} \to q(\hat{\lambda})$ defines a quadratic form on $(\Lambda_1 + \Lambda_3) \cap \Lambda_2 / \bigcap_{i=1}^{3} \Lambda_i$. We introduce the notation

$$\operatorname{ind}_{\Lambda_2}(\Lambda_1, \Lambda_3) = \operatorname{ind} q + \frac{1}{2} \dim \ker q = \operatorname{ind} q + \frac{1}{2} (\dim(\Lambda_1 \cap \Lambda_2) + \dim(\Lambda_3 \cap \Lambda_2)) - \dim\left(\bigcap_{i=1}^{3} \Lambda_i\right),$$

where ind q is the inertia index of quadratic form $q(\hat{\lambda})$ and ker q is the kernel of this form.

Clearly, $0 \leq \operatorname{ind}_{\Lambda_2}(\Lambda_1, \Lambda_3) \leq m$; moreover, $\operatorname{ind}_{\Lambda_2}(\Lambda_1, \Lambda_1) = 0$,

$$\operatorname{ind}_{\Lambda_1}(\Lambda_1, \Lambda_2) = \operatorname{ind}_{\Lambda_1}(\Lambda_1, \Lambda_2) = \frac{1}{2}(m - \dim(\Lambda_1 \cap \Lambda_2)),$$

$$\operatorname{ind}_{\Lambda_2}(\Lambda_1, \Lambda_3) + \operatorname{ind}_{\Lambda_2}(\Lambda_3, \Lambda_1) + \dim(\Lambda_1 \cap \Lambda_3) = m.$$

We will consider other formal properties of this index (and, in particular, its expression through the Maslov index of the triple of Lagrangian planes) in Sec. 4, while now we give one more definition.

Definition. Let Λ_t, $t \in [t_0, t_1]$ be some piecewise smooth curve on the manifold of Lagrangian planes $L(\Sigma)$. Curve $\Lambda.$ is called simple if there exists a Lagrangian plane $\mathscr{P} \subset \Sigma$ such that

$$\Lambda_t \cap \mathscr{P} = 0, \quad \operatorname{ind}_{\mathscr{P}}(\Lambda_t, \Lambda_{t+0}) = 0 \quad \forall t \in [t_0, t_1].$$

LEMMA 3. Let Λ_t, $t \in [t_0, t_1]$, be a piecewise smooth curve on $L(\Sigma)$. Then for every $t \in [t_0, t_1]$ there is a neighborhood O_t of point t in $[t_0, t_1]$ such that curve $\Lambda.|_{O_t}$ is simple.

Proof. Proposition A5 of the Appendix to this section implies the existence of a Lagrangian plane $\mathscr{P}$ such that quadratic form $\lambda \to \sigma(\lambda_t, \lambda_{t+0})$ on $(\Lambda_t + \Lambda_{t+0}) \cap \mathscr{P}$, where $\lambda = \lambda_t + \lambda_{t+0}$, $\lambda_t \in \Lambda_t$, $\lambda_{t+0} \in \Lambda_{t+0}$, is nonnegative and $\Lambda_t \cap \mathscr{P} = \Lambda_{t+0} \cap \mathscr{P} = 0$. In this case, $\operatorname{ind}_{\mathscr{P}}(\Lambda_t, \Lambda_{t+0}) = 0$.

THEOREM 1. Let Λ_τ, $0 \leq \tau \leq t$ be a Jacobian curve and $\tau_{l+1} = 0 = \tau_0 < \tau_1 < \ldots < \tau_l = t$ an arbitrary partition of interval $[0, t]$. Then

$$\sum_{i=0}^{l} \operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}}) \leqslant \operatorname{ind} \psi G_t'' + m. \tag{12}$$

If, however, the given partition of interval [0, t] is such that all pieces $\Lambda.|_{[\tau_i,\tau_{i+1}]}$ of the Jacobian curve are simple, $0 \le i < l$, then the inequality in formula (12) becomes an equality.

Before proving the theorem, we describe several additional methods for calculating ind $\psi G_t''$; they are all closely connected with formula (12).

Consider the family of quadratic forms that depends on $s \in [0, t]$

$$\psi G_s''(v(\cdot), v(\cdot)) = \int_0^s \psi h_\tau(v(\tau))\,d\tau +$$
$$+ \int_0^s \sigma\left(\int_0^\tau z_\theta v(\theta)\,d\theta,\ z_\tau v(\tau)\right) d\tau,$$
$$v(\cdot) \in L_\infty^r[0, s],\quad \int_0^s z_\tau v(\tau)\,d\tau \in \Pi_0.$$

It is easy to see that ind $\psi G_s''$ is a nondecreasing continuous on the left function of parameter s. This function is completely characterized by its jumps at the points of discontinuity, i.e., by the values ind $\psi G_{s+0}''$ − ind $\psi G_s''$. If in Theorem 1 ind $\psi G_t''$ is expressed in the form of an "integral sum," now we want to compute explicitly the "derivative" of this integral." To do that we will have to introduce one more family of subspaces in Σ, but to construct this family it is not even necessary to solve differential equations.

Let $0 \le \alpha \le \beta \le t$, and set

$$\Delta_\beta^\alpha = \Pi_0 \cap \{z_\tau v \mid \alpha < \tau \leqslant \beta,\ v \in R^r\}^{\angle},\quad \Delta_{\beta+0}^\alpha = \bigcup_{\varepsilon>0} \Delta_{\beta+\varepsilon}^\alpha.$$

It is easy to see that

$$\dim \Delta_\beta^\alpha = \operatorname{codim}\Big(\sum_{\alpha<\tau<\beta} \bar{\mu}_0 z_\tau R^r\Big).$$

Indeed, this is implied by the obvious relations

$$\Delta_\beta^{\alpha\angle} = \Pi_0 + \sum_{\alpha<\tau<\beta} z_\tau R^r,\quad \ker \bar{\mu}_0 = \Pi_0.$$

In particular, $\Delta_t^0 = 0$, since $\sum_{0<\tau<t} \bar{\mu}_0 z_\tau R^r = \Pi$. Note also that $\Delta_{\bar{\beta}}^{\bar{\alpha}} \subset \Delta_{+0}^\alpha \subset \Delta_\beta^\alpha$ if $\bar{\alpha} \le \alpha < \beta < \bar{\beta}$. If, however, $\bar{\mu}_0 z_\tau$ depends analytically on $\tau \in [0, t]$, then $\Delta_{\alpha+0}^\alpha = 0$ $\forall \alpha \in [0, t]$.

THEOREM 2. For any $s \in [0, t)$ the relation

$$\operatorname{ind} \psi G_{s+0}'' - \operatorname{ind} \psi G_s'' = \dim(\Lambda_{s+0} \cap \Pi_0 / \Lambda_{s+0} \cap \Delta_{s+0}^s) - \dim(\Delta_s^0/\Delta_{s+0}^0) +$$
$$+ \operatorname{ind}_{\Pi_0}(\Lambda_s, \Lambda_{s+0}) + \frac{1}{2}(\dim(\Lambda_s \cap \Pi_0) - \dim(\Lambda_{s+0} \cap \Pi_0))$$

is true.

The given expression looks outwardly rather cumbersome; however, the last three terms contribute only when s is a point of discontinuity of the Jacobi curve and the second term is almost trivial.

The point $s \in (0, t)$ is called the conjugate point to zero for the Jacobi equation if $\Lambda_s \cap \Pi_0 \neq 0$, and the value dim $(\Lambda_s \cap \Pi_0)$ is called the multiplicity of the conjugate point s. The following assertion specifies the exact limits within which Morse's classic formula for the index of the second variation is true.

COROLLARY. Assume that $\Delta_s^0 = \overset{s}{\Delta}_{s+0}$, and s is a point of continuity of Jacobi curve Δ. Then

$$\operatorname{ind} \psi G^{*}_{s+0} - \operatorname{ind} \psi G^{*}_{s} = \dim (\Lambda_s \cap \Pi_0).$$

4. The proofs of Theorems 1 and 2 will be given in Sec. 5, while in this section we give a homotopy interpretation of formula (12).

Let $\Lambda \in L(\Sigma)$. Below, without special stipulations, we constantly use the identification of the tangent space $T_\Lambda L(\Sigma)$ with the space of quadratic forms on Λ described in the Appendix to this section.

Definition. A smooth curve $\mathcal{A}_\tau$, $\tau \in [t_0, t_1]$ on $L(\Sigma)$ is called nondecreasing if velocity $(d/d\tau)\mathcal{A}_\tau \in T_{\mathcal{A}_\tau}L(\Sigma)$ is a nonnegative quadratic form on $\mathcal{A}_\tau$.

In other words, $\mathcal{A}_\tau$ is a nondecreasing curve if $(d/d\tau)\mathcal{A}_\tau \geq 0$ $\forall\tau[t_0, t_1]$. In particular, Jacobi curve Λ_τ is nondecreasing since

$$\frac{d}{d\tau} \Lambda_\tau = \frac{1}{2} \gamma_\tau^{-1} (\sigma (z_\tau^{(k_\tau)} \cdot, \Lambda_\tau)) > 0.$$

LEMMA 4. Any two Lagrangian planes $\mathcal{A}_0$, $\mathcal{A}_1 \in L(\Sigma)$ can be joined by a simple smooth nondecreasing curve $\mathcal{A}_\tau$, $0 \leq \tau \leq 1$.

Proof. Lemma 3 implies the existence of a Lagrangian plane $\mathcal{B}$ such that $\mathcal{A}_0, \mathcal{A}_1 \in \mathcal{B}^\pitchfork$ and $\operatorname{ind}_{\mathcal{B}}(\mathcal{A}_0, \Lambda_1) = 0$. In Proposition $\mathcal{A}$1 and its corollary a structure of an affine space is defined on the set $\mathcal{B}^\pitchfork$. A rectilinear segment joining $\mathcal{A}_0$ to $\mathcal{A}_1$ in this affine space is, as is not hard to see, a simple nondecreasing curve in $L(\Sigma)$.

In Proposition A6 a formula is given that allows us to calculate the Maslov index of a given continuous closed curve in terms of the Maslov index of the approximate triple Lagrangian planes. If the curve is nondecreasing, then it is convenient to use the index that was involved in the formulation of the assertion of Theorem 1. To begin with, we express one index through the other.

LEMMA 5. Let $\Lambda_i \in L(\Sigma)$, i = 1, 2, 3; then

$$\mu (\Lambda_1, \Lambda_2, \Lambda_3) + 2\operatorname{ind}_{\Lambda_2} (\Lambda_1, \Lambda_3) + \dim (\Lambda_1 \cap \Lambda_3) = m.$$

Proof. Recall that

$$\operatorname{ind}_{\Lambda_2} (\Lambda_1, \Lambda_3) = \operatorname{ind} q + \frac{1}{2} \dim \ker q,$$

where q is a quadratic form given on space $\Lambda_2 \cap (\Lambda_1 + \Lambda_3) / \bigcap_{i=1}^{3} \Lambda_i$. Here, $q(\hat{\lambda}) = \sigma(\lambda_1, \lambda_3)$, where $\lambda_j \in \Lambda_j$, j = 1, 3, $\hat{\lambda} = \lambda_1 + \lambda_3 + \bigcap_{i=1}^{3} \Lambda_i$. At the same time, $\mu(\Lambda_1, \Lambda_2, \Lambda_3)$ coincides with the signature of form q. The assertion of the lemma follows now from the equality

$$\dim (\Lambda_2 \cap (\Lambda_1 + \Lambda_3) / \bigcap_{i=1}^{3} \Lambda_i) = m - \dim (\Lambda_1 \cap \Lambda_3).$$

COROLLARY. Let $\Lambda_1 \cap \Lambda_3 \supset S$ be an isotopic subspace in Σ. Then

$$\operatorname{ind}_{\Lambda_2} (\Lambda_1, \Lambda_3) = \operatorname{ind}_{\Lambda_2^S} (\Lambda_1, \Lambda_3).$$

Indeed, $\Lambda_1^S = \Lambda_1$, $\Lambda_3^S = \Lambda_3$, $\mu(\Lambda_1^S, \Lambda_2^S, \Lambda_3^S) = \mu(\Lambda_1, \Lambda_2, \Lambda_3)$.

Proposition 2. Suppose that $\mathcal{A}_\tau$, $t_0 \leq \tau \leq t_1$ is a continuous closed nondecreasing curve on $L(\Sigma)$ and $t_0 = \tau_0 < \tau_1 < \ldots < \tau_N < \tau_{N+1} = t_1$ is some partition of segment $[t_0, t_1]$, $\mathcal{B} \in L(\Sigma)$. Then

$$\sum_{i=0}^{N} \operatorname{ind}_{\mathscr{B}} (\mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}}) \leqslant \operatorname{Ind} \mathscr{A}. \tag{13}$$

If, however, the partition of segment $[t_0, t_1]$ is such that curves $\mathscr{A}|_{[\tau_i,\tau_{i+1}]}$ are simple, then the inequality in formula (13) becomes an equality.

Proof. Consider first the case when curves $\mathscr{A}|_{[\tau_i,\tau_{i+1}]}$ are simple.

Let T_i be a Lagrangian plane transversal to curve $\mathscr{A}|_{[\tau_i,\tau_{i+1}]}$, $i = 0, \ldots, N$. Then, according to Proposition A6,

$$2 \operatorname{Ind} \mathscr{A}. = \sum_{i=0}^{N} (\mu (T_i, \mathscr{B}, \mathscr{A}_{\tau_i}) - \mu (T_i, \mathscr{B}, \mathscr{A}_{\tau_{i+1}})).$$

At the same time the chain rule for the Maslov index implies

$$\mu (T_i, \mathscr{B}, \mathscr{A}_{\tau_i}) - \mu (T_i, \mathscr{B}, \mathscr{A}_{\tau_{i+1}}) = \mu (\mathscr{B}, \mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}}) -$$
$$- \mu (T_i, \mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}}) = - \mu (\mathscr{A}_{\tau_i}, \mathscr{B}, \mathscr{A}_{\tau_{i+1}}) - \mu (T_i, \mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}}),$$

Let $\tau \in [\tau_i, \tau_{i+1}]$; we denote by $p_\tau : \Sigma \to \mathscr{A}_\tau$ the projection operator of space Σ parallel to T_i onto $\mathscr{A}_\tau$. Then $-\mu(T_i, \mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}})$ coincides with the signature of quadratic form $\alpha \to \sigma(p_{\tau_{i+1}}\alpha, \alpha)$ given on space $\mathscr{A}_{\tau_i}$. Since $\mathscr{A}_\tau$ is a nondecreasing curve, quadratic form $\alpha \to \sigma(\dot{p}_\tau \alpha, \alpha)$ is nonnegative on $\mathscr{A}_\tau$, $\forall \tau \in [\tau_i, \tau_{i+1}]$. At the same time, identity $\sigma(p_\tau x, \xi) = \sigma(x, \xi)$, valid for any $\tau \in [\tau_i, \tau_{i+1}]$, $x \in \Sigma$, $\xi \in T_i$ (see A1) implies that $\sigma(\dot{p}_\tau x, \xi) = 0$, $\forall x \in \Sigma$, $\xi \in T_i$. Therefore, form $\alpha \to \sigma(\dot{p}_\tau \alpha, \alpha)$, being nonnegative on subspace $\mathscr{A}_\tau$, a direct complement to T_1 in Σ, is also nonnegative on the whole Σ. Furthermore, for any $\alpha \in \mathscr{A}_{\tau_i}$ we have

$$\sigma (p_{\tau_{i+1}} \alpha, \alpha) = \int_{\tau_i}^{\tau_{i+1}} \sigma (\dot{p}_\tau \alpha, \alpha) \, d\tau \geqslant 0.$$

The kernel of form $\alpha \to \sigma(p_{\tau_{i+1}}\alpha, \alpha)$ on $\mathscr{A}_{\tau_i}$ coincides with $\mathscr{A}_{\tau_i} \cap \mathscr{A}_{\tau_{i+1}}$. Consequently,

$$- \mu (T_i, \mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}}) = m - \dim (\mathscr{A}_{\tau_i} \cap \mathscr{A}_{\tau_{i+1}}).$$

According to Lemma 5,

$$- \mu (\mathscr{A}_{\tau_i}, \mathscr{B}, \mathscr{A}_{\tau_{i+1}}) = 2 \operatorname{ind}_{\mathscr{B}} (\mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}}) - m.$$

Adding everything, we get

$$\operatorname{Ind} \mathscr{A}. = \sum_{i=0}^{N} \operatorname{ind}_{\mathscr{B}} (\mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}}).$$

Inequality (13) is now the "triangle inequality" for ind, which is important in itself.

LEMMA 6. For any $\Lambda_1, \Lambda_2, \Lambda_3, \mathscr{B} \in L(\Sigma)$ the following inequality is valid:

$$\dim (\Lambda_1 \cap \Lambda_3) + \operatorname{ind}_{\mathscr{B}} (\Lambda_1, \Lambda_3) \leqslant \operatorname{ind}_{\mathscr{B}} (\Lambda_1, \Lambda_2) +$$
$$+ \operatorname{ind}_{\mathscr{B}} (\Lambda_2, \Lambda_3) + \dim \left(\bigcap_{i=1}^{3} \Lambda_i \right).$$

Proof. By virtue of the corollary to Lemma 5, it suffices to consider the case $\bigcap_{i=1}^{3} \Lambda_i \subset \mathscr{B}$. We join sequentially by simple continuous nondecreasing curves Λ_1 to Λ_2, Λ_2 to Λ_3, and Λ_3 to Λ_1. According to the already proven part of

The point the Maslov index d of this curve can be computed from the formula
$(\Lambda_s \cap \Pi_0)$ is
which Mor
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$$d=\operatorname{ind}_{\mathscr{B}}(\Lambda_1, \Lambda_2)+\operatorname{ind}_{\mathscr{B}}(\Lambda_2, \Lambda_3)+\operatorname{ind}_{\mathscr{B}}(\Lambda_3, \Lambda_1)=$$
$$=\operatorname{ind}_{\mathscr{B}}(\Lambda_1, \Lambda_2)+\operatorname{ind}_{\mathscr{B}}(\Lambda_2, \Lambda_3)+m-\dim(\Lambda_1\cap\Lambda_3)-\operatorname{ind}_{\mathscr{B}}(\Lambda_1, \Lambda_3).$$

Let us now use the fact that the equality is valid for any $\mathscr{B}$. Substituting Lagrangian plane Λ_1 for $\mathscr{B}$, we obtain

$$d=\operatorname{ind}_{\Lambda_1}(\Lambda_1, \Lambda_2)+\operatorname{ind}_{\Lambda_1}(\Lambda_2, \Lambda_3)+\operatorname{ind}_{\Lambda_1}(\Lambda_3, \Lambda_1)=$$
$$=\frac{1}{2}(m-\dim(\Lambda_1\cap\Lambda_2))+\frac{1}{2}(m-\dim(\Lambda_1\cap\Lambda_3))+$$
$$+\operatorname{ind}_{\Lambda_1}(\Lambda_2, \Lambda_3)\geqslant m-\dim\left(\bigcap_{i=1}^{3}\Lambda_i\right).$$

Comparing the two expressions for d, we arrive at the inequality

$$\operatorname{ind}_{\mathscr{B}}(\Lambda_1, \Lambda_2)+\operatorname{ind}_{\mathscr{B}}(\Lambda_2, \Lambda_3)-\operatorname{ind}_{\mathscr{B}}(\Lambda_1, \Lambda_3)\geqslant\dim(\Lambda_1\cap\Lambda_3)-$$
$$-\dim\left(\bigcap_{i=1}^{3}\Lambda_i\right).$$

LEMMA 7. Let $\mathscr{A}_\tau$, $t_0 \le \tau \le t_1$ be a simple nondecreasing curve in $L(\Sigma)$. Then $\mathscr{A}_{t_0}\cap\mathscr{A}_{t_1}=\bigcap_{t_0<\tau<t_1}\mathscr{A}_\tau$.

Proof. Proposition 2 implies the equality

$$\operatorname{ind}_{\mathscr{B}}(\mathscr{A}_{t_0}, \mathscr{A}_{t_1})=\operatorname{ind}_{\mathscr{B}}(\mathscr{A}_{t_0}, \mathscr{A}_\tau)+\operatorname{ind}_{\mathscr{B}}(\mathscr{A}_\tau, \mathscr{A}_{t_1}),$$

$\forall\tau\in[t_0, t_1]$, $\mathscr{B}\in L(\Sigma)$. Therefore, according to Lemma 6,

$$\dim(\Lambda_{t_0}\cap\Lambda_{t_1})\leqslant\dim(\Lambda_{t_0}\cap\Lambda_\tau\cap\Lambda_{t_1}).$$

The last is possible only when $\Lambda_{t_0}\cap\Lambda_{t_1}\subset\Lambda_\tau$.

COROLLARY. Let $\mathscr{A}_\tau$, $t_0 \le \tau \le t_1$ be a continuous nondecreasing closed curve in $L(\Sigma)$. Then

$$\operatorname{Ind}\mathscr{A}_.\geqslant m-\dim\left(\bigcap_{t_0<\tau<t_1}\mathscr{A}_\tau\right).$$

Proof. Denote $\mathscr{B}=\mathscr{A}_{t_0}=\mathscr{A}_{t_1}$, and let $t_0=\tau_0<\tau_1<\ldots<\tau_N<\tau_{N+1}=t_1$ be a partition of segment $[t_0, t_1]$ such that all the curves $\mathscr{A}|_{[\tau_i,\tau_{i+1}]}$ are simple. Then

$$\operatorname{Ind}\mathscr{A}_.=m-\frac{1}{2}(\dim(\mathscr{B}\cap\mathscr{A}_{\tau_1})+\dim(\mathscr{B}\cap\mathscr{A}_{\tau_N}))+$$
$$+\sum_{i=1}^{N-1}\operatorname{ind}_{\mathscr{B}}(\mathscr{A}_{\tau_i}, \mathscr{A}_{\tau_{i+1}})\geqslant m+\sum_{j=2}^{N-1}\dim(\mathscr{B}\cap\mathscr{A}_{\tau_j})-$$
$$-\sum_{i=1}^{N-1}\dim(\mathscr{A}_{\tau_i}\cap\mathscr{B}\cap\mathscr{A}_{\tau_{i+1}})\geqslant m-\dim\left(\mathscr{B}\bigcap_{i=1}^{N}\mathscr{A}_{\tau_i}\right).$$

Using Lemma 7 we get $\mathscr{B}\bigcap_{i=1}^{N}\mathscr{A}_{\tau_i}=\bigcap_{t_0<\tau<t_1}\mathscr{A}_\tau$.

THEOREM 3. Let Λ_τ, $\tau\in[0, t]$ be a Jacobian curve and $\tau_1, \ldots, \tau_N$ all its points of discontinuity. Join in the order specified below by simple continuous nondecreasing curves Π_0 to Λ_{+0}, Λ_{τ_i} to Λ_{τ_i+0}, $i = 1, \ldots, N$, Λ_t to Π_0. Jacobian curve Λ_τ, together with the pieces added this way, forms a continuous nondecreasing closed curve in $L(\Sigma)$, which we denote by $\bar{\Lambda}$. Then

$$\operatorname{ind}\psi G_t^{\cdot}=\operatorname{Ind}\bar{\Lambda}_.-m.$$

The proof immediately follows from Theorem 1 and Proposition 3.

5. Proof of Theorem 1. The triangle inequality (Lemma 6) implies that it is enough to prove the theorem for the case when all the curves $\Lambda\,|_{[\tau_i,\tau_{i+1}]}$ are simple. At the same time, Proposition 2 implies that for nondecreasing simple curve $\Lambda\,|_{[\tau_i,\tau_{i+1}]}$ and arbitrary $\tau \in [\tau_i, \tau_{i+1}]$ the equality

$$\mathrm{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_\tau)+\mathrm{ind}_{\Pi_0}(\Lambda_\tau, \Lambda_{\tau_{i+1}})=\mathrm{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}})$$

is true, i.e., the triangle inequality becomes an equality in this case.

Thus, it is enough to prove Theorem 1 for a single, arbitrary, sufficiently small partition of the interval [0, t]. In particular, we shall assume that among the points $\tau_1, \ldots, \tau_\ell$ there are points at which the smoothness of the curve Λ_τ, $\tau\in[0, t]$ is violated. When k_τ is constant on the half-intervals $[\tau_i, \tau_{i+1}]$, we set $k_\tau = k_i$ for $\tau\in[\tau_i, \tau_{i+1}]$, $i = 0, 1, \ldots, \ell-1$.

In the sequel we will use special distribution spaces with supports on a given segment of the real line, Sobolev spaces with negative numbers. Recall the definition of these spaces.

Let $-\infty < t_0 < t_1 < +\infty$ and $k \geq 0$ is an integer. In the space $C_\infty^r[t_0, t_1]$ we introduce the inner product

$$(a, b)_{k,[t_0,t_1]}=\int_{t_0}^{t_1}\sum_{i=0}^{k}(a_\tau^{(i)}, b_\tau^{(i)})\,d\tau, \quad \forall a_\tau,\ b_\tau\in C_\infty^r\ [t_0, t_1];$$

here $a_\tau^{(i)} = (d^i/d\tau^i)a_\tau$, $b_\tau^{(i)} = (d^i/d\tau^i)b_\tau$, and $(\cdot,\cdot)$ is the standard inner product in $\mathbf{R}^r$.

The completion of space $C_\infty^r[t_0, t_1]$ in norm $\|a\|_{k,[t_0,t_1]} = \sqrt{(a, a)_{k,[t_0,t_1]}}$ is, obviously, a Hilbert space. It is called Sobolev space of order k and is denoted by $H_k^r[t_0, t_1]$. It is easy to see that $H_k^r[t_0, t_1]$ consists of all $k - 1$ times continuously differentiable functions a_τ such that $a_\tau^{(k-1)}$ is absolutely continuous and $a_\tau^{(k)} \in L_2^r[t_0, t_1]$.

Next, let $u(\tau) \in L_2^r[t_0, t_1] = H_0^r[t_0, t_1]$. Mapping $a_\tau\mapsto \int_{t_0}^{t_1}(a_\tau, u(\tau))\,d\tau$, where $a_\tau \in H_k^r[t_0, t_1]$, determines a linear continuous functional on $H_k^r[t_0, t_1]$. Set $\|u(\cdot)\|_{k,[t_0,t_1]} = \sup_{\|a\|_{k,[t_0,t_1]}=1}\int_{t_0}^{t_1}(a_\tau, u(\tau))\,d\tau$ to be a regular operator norm. The completion of space $L_2^r[t_0, t_1]$ in norm $\|\cdot\|_{-k,[t_0,t_1]}$ is called a Sobolev space of order $(-k)$ and is denoted by $H_{-k}^r[t_0, t_1]$. Space $H_{-k}^r[t_0, t_1]$ is obviously isomorphic to the dual space to $H_k^r[t_0, t_1]$; consequently, it is a Hilbert space. In addition, since any functional on $C_\infty^r[t_0, t_1]$ is continuous in norm $\|\cdot\|_{k,[t_0,t_1]}$, and, moreover, continuous in standard topology of space $C_\infty^r[t_0, t_1]$, then $H_k^r[t_0, t_1]$ is contained in the space of all r-dimensional vector distributions with support in $[t_0, t_1]$. Distributions lying in $H_{-k}^r[t_0, t_1]$ can be characterized in the following way. Let $i = 0, 1, 2, \ldots$, $\chi^0(\tau) = \chi(\tau)$ (Heaviside function) and the asterisk denote the operation of convolution for distributions: vector distribution $\bar{u}(\tau) = (u^1(\tau), \ldots, u^r(\tau))^T$ with support in $[t_0, t_1]$ lies in $H_{-k}^r[t_0, t_1]$ if and only if distribution $\chi^{k-1}(\cdot)*u(\cdot) = (\chi^{k-1}(\cdot)*u^1(\cdot), \ldots, \chi^{k-1}(\cdot)*u^r(\cdot))^T$ is a locally summable vector function; moreover,

$$\chi^{k-1}(\cdot)*u(\cdot)\,|_{[t_0,t_1]}\in L_2^r[t_0, t_1]=H_0^r[t_0, t_1].$$

Note that no matter what distribution v with support in $[t_0, t_1]$ is, the result of its contraction with χ^{k-1} has support in $[t_0, +\infty)$; moreover, on the half-closed interval $[t_1, +\infty)$ it coincides with some polynomial of degree at most $k - 1$. The coefficients of this polynomial together with the restriction of $\chi^{k-1}*v$ to $[t_0, t_1]$ contain the complete information on distribution v. When $v \in H_{-k}^r[t_0, t_1]$, this enables us to define, with the help of the norm of space L_2^r, the norm equivalent to $\|\cdot\|_{-k,[t_0,t_1]}$. Since for our purposes it makes no sense to distinguish between the equivalent norms, we save notation $\|\cdot\|_{-k,[t_0,t_1]}$ for a new norm. The exact definition is as follows: for every $u \in H_{-k,[t_0,t_1]}$ set

$$\|u\|_{-k,[t_0,t_1]}=\left(\int_{t_0}^{t_1}|\chi^{k-1}*u(\tau)|^2\,d\tau+\sum_{i=0}^{k-1}|\langle u, \chi^i\rangle|^2\right)^{\frac{1}{2}}$$

(the angled brackets $\langle u, \chi^i \rangle$ denote the result of applying vector distribution u to function τ^i; if

$$u \in L_1^r[t_0, t_1], \text{ then } \langle u, \chi^i \rangle = \int_{t_0}^{t_1} \tau^i u(\tau)\, d\tau \in R^r.$$

Let us return to consideration of partition $0 = \tau_0 < \tau_1 < \ldots < \tau_l < \tau_{l+1} = t$ of interval [0, t]. Since any function from $L_\infty^r[0, t]$, being restricted to interval $[\tau_i, \tau_{i+1}]$, certainly belongs to space $H_{-k_i}^r[\tau_i, \tau_{i+1}] \supset H_0^r[\tau_i, \tau_{i+1}] \supset L_\infty^r[\tau_i, \tau_{i+1}]$, norm

$$\|\cdot\|_{-k.,t} = \left(\sum_{i=0}^{l-1} \|\cdot\|^2_{-k_i,[\tau_i,\tau_{i+1}]}\right)^{\frac{1}{2}}$$

is defined on space $L_\infty^r[0, t]$.

LEMMA 8. Bilinear from $\psi G_t''$ is continuous in norm $\|\cdot\|_{-k\cdot,t}$.

Proof. Obviously it is enough to prove that for every i satisfying the condition $k_i > 0$, the quadratic form

$$v(\cdot) \mapsto \int_{\tau_i}^{\tau_{i+1}} \sigma\left(\int_{\tau_i}^{\tau} z_\theta v(\theta),\ z_\tau v(\tau)\right) d\tau$$

is continuous in norm $\|\cdot\|_{-k_i,[\tau_i,\tau_{i+1}]}$. But this immediately follows from equality (6) derived in the proof of Proposition 1 and the obvious inequality

$$\int_{\tau_i}^{\tau_{i+1}} |w(\tau)|^2 d\tau \leqslant \|v(\cdot)\|^2_{-k_i,[\tau_i,\tau_{i+1}]},$$

where $w(\tau) = \int_0^\tau \frac{(\tau-\theta)^{k_i-1}}{(k_i-1)!} d\theta$ is k_i-fold indefinite integral of $v(\cdot)$.

The complement of space $L_\infty^r[0, t]$ in norm $\|\cdot\|_{-k.,t}$ is Hilbert space $\bigoplus_{i=0}^{l} H^r_{-k_i}[\tau_i, \tau_{i+1}]$. We introduce the notation $H^r_{-k.}[0, t] = \bigoplus_{i=0}^{l} H^r_{-k_i}[\tau_i, \tau_{i+1}]$; element

$$\left(\begin{pmatrix} u_0^1 \\ \vdots \\ u_0^r \end{pmatrix}, \ldots, \begin{pmatrix} u_{l-1}^1 \\ \vdots \\ u_{l-1}^r \end{pmatrix}\right) = (u_0, \ldots, u_{l-1}) \in H^r_{-k.}[0, t]$$

will be denoted by the single symbol u. Note that $u = (u_0, \ldots, u_{l-1})$ is not, generally speaking, a distribution on interval [0, t] since this element assumes "two values" at points τ_j, $j = 1, \ldots, l$: for example, if $u_{i-1} = \delta_{\tau_i} \in H_{-1}[\tau_{i-1}, \tau_i]$ and $u_i = \delta_{\tau_i}' \in H_{-2}[\tau_i, \tau_{i+1}]$. Space $H_{-k.}^r[0, t]$ is dual to space $H^r_{k.}[0, t] \overset{\text{def}}{=} \bigoplus_{i=0}^{l-1} H^r_{k_i}[\tau_i, \tau_{i+1}]$. In its turn, any piecewise smooth function on [0, t] smooth on each interval (τ_i, τ_{i+1}), $i = 0, 1, \ldots, l$, is identified in an obvious way with an element of space $H_k[0, t]$; the elements of space $H_{-k_0}^r[0, t]$ act on these functions like usual vector distributions do on smooth functions $\langle u, a \rangle = \sum_{i=0}^{l} \langle u_i, a|_{[\tau_i,\tau_{i+1}]} \rangle$ (precisely the "bifurcation" of $u \in H_{-k_i}^r[0, t]$ at points τ_i enables u to act on functions that have discontinuities at these points but are smooth on the right and on the left). Elements of $H_{-k.}^r[0, t]$ can also be multiplied by piecewise smooth functions that are smooth on intervals (τ_i, τ_{i+1}) and by matrices the elements of which are such functions. In particular, for $v \in H_{-k.}^r[0, t]$ we define $z \cdot v$ and $\overline{\mu}_0 z \cdot v$, which, when the bases in spaces Σ and Π are fixed, become elements from $H_{-k.}^{2m}[0, t]$ and $H_{-k.}^{m}[0, t]$, respectively.

The complement of the domain of form $\psi G_t''$ in norm $\|\cdot\|_{-k.t}$ is the subspace

$$\{v \in H^r_{-k.}[0, t] \mid \langle \bar{\mu}_0 z.v, 1\rangle = 0\} \stackrel{\text{def}}{=} \{\bar{\mu}_0 z.\}^{\perp}_{-k}.$$

which is a Hilbert subspace of dimension m in $H_{-k.}^r[0, t]$.

The bilinear form $\psi G_t''$ is uniquely extended with respect to continuity to form Q given on space $\{\bar{\mu}_0 z\}_{-k.}^{\perp}$. Here ind Q = ind $\psi G_t''$.

To write form Q in an explicit form we need to give an exact meaning to the expression $\chi_*(z\cdot v)$, where $v \in H_{-k.}^r[0, t]$, and χ, as before, is the Heaviside function (double meaning may arise again because of the "bifurcation" of v at points τ_i). Let $u = (u_0, \ldots, u_{l-1})$, $u_i \in H_{-k_i}^r[\tau_i, \tau_{i+1}]$; then distribution $\chi_* u_i$ has a support in $[\tau_i, +\infty)$; moreover, on the interval $(\tau_{i+1}, +\infty)$ it coincides with the vector $\langle u_i, 1\rangle \in R^r$. In turn, distribution $((\chi_* u_i)(\tau) - \langle u_i, 1\rangle \chi(\tau - \tau_{i+1})) \in H_{1-k_i}[\tau_i, \tau_{i+1}] \subset H_{-k_i}[\tau_i, \tau_{i+1}]$. Set $\chi_* u = (w_0, \ldots, w_{l-1})$, where

$$w_i(\tau) = \left(\sum_{j=0}^{i-1} \langle u_j, 1\rangle\right)(\chi(\tau - \tau_i) - \chi(\tau - \tau_{i+1})) + (\chi_* u_i)(\tau) - \\ - \langle u_i, 1\rangle\, \chi(\tau - \tau_{i+1}).$$

Let $\bar{h}_\tau$ be a symmetric r × r matrix corresponding to form ψh_τ, i.e., $\psi h_\tau(v_1, v_2) = (\bar{h}_\tau v_1, v_2)$.

LEMMA 9. Whatever $v \in \{\bar{\mu}_0 z.\}_{-k.}^{\perp}$ is, vector distribution $\sigma(\chi_*(z\cdot v), z.\cdot)^T + \bar{h}\cdot v$ lies in the space $H_{k.}^r[0, t] \subset H_{-k.}^r[0, t]$, here

$$Q(v, v) = \langle v^T, \sigma(\chi_*(zv), z\cdot)^T + \bar{h}v\rangle.$$

Proof. It suffices to prove the existence of constant c such that

$$\left\| \sigma\left(\int_0^\tau z_\theta v(\theta)\, d\theta,\ z_\tau\cdot\right)^T + \bar{h}_\tau v(\tau)\right\|_{k.,t} \leqslant c\, \|v\|_{-k.,t},\quad \forall v \in C^r_\infty[0, t],$$

from this inequality the assertions of the lemma are derived with respect to continuity. To prove the last inequality it is enough, in turn, to establish inequalities

$$\left\| \sigma\left(\int_{\tau_i}^\tau z_\theta v(\theta)\, d\theta,\ z_\tau\cdot\right)\right\|_{k_i,[\tau_i,\tau_{i+1}]} \leqslant c_i \|v\|_{-k_i,[\tau_i,\tau_{i+1}]}\quad \forall v \in C^r_\infty[\tau_i \tau_{i+1}], \\ i = 0, 1, \ldots, l-1$$

for some constants c_i.

Let $0 \le n \le k_i$; then taking into account Lemma 1, we get

$$\frac{d^n}{d\tau^n}\sigma\left(\int_{\tau_i}^\tau z_\theta v(\theta)\, d\theta,\ z_\tau\cdot\right) = \sigma\left(\int_{\tau_i}^\tau z_\theta v(\theta)\, d\theta,\ z_\tau^{(n)}\cdot\right) = \\ = (-1)^{k_i-1}\sigma(z_\tau^{(k_i-1)} w(\tau), z_\tau^{(n)}\cdot) + (-1)^{k_i}\sigma\left(\int_{\tau_i}^\tau z_\theta^{(k_i)} w(\theta)\, d\theta,\ z_\tau^{(n)}\cdot\right),$$

where

$$w(\tau)=\int_{\tau_l}^{\tau}\frac{(\tau-\theta)^{k_l-1}}{(k_l-1)!}v(\theta)\,d\theta.$$

The required inequality now follows immediately from the definition of norms $\|\cdot\|_{k_i,[\tau_i,\tau_{i+1}]}$ and $\|\cdot\|_{-k_i,[\tau_i,\tau_{i+1}]}$.

LEMMA 10. Quadratic form Q is positive definite on a subspace of finite dimension in $(\overline{\mu}_0 z)_{-k.}^{\perp}$.

Proof. For i = 0, 1, ..., $l-1$ we have

$$H^r_{-k_l}[\tau_l,\tau_{l+1}]\subset H^r_{-k.}[0,t]=\bigoplus_{l=0}^{l-1}H^r_{-k_l}[\tau_l,\tau_{l+1}];$$

here

$$H^r_{-k_l}[\tau_l,\tau_{l+1}]\cap\{\overline{\mu}_0 z\}^{\perp}_{-k.}=$$
$$=\{v\in H^r_{-k_l}[\tau_l,\tau_{l+1}]\mid\langle\overline{\mu}_0 zv,1\rangle=0\}\overset{\text{def}}{=}\{\overline{\mu}_0 z\}^{\perp}_{-k_l}.$$

Subspace $\bigoplus_{l=0}^{l-1}\{\overline{\mu}_0 z\}^{\perp}_{-k_l}$ evidently has a finite codimension in $(\overline{\mu}_0 z)_{-k.}^{\perp}$. Quadratic form Q, being restricted to subspace $\bigoplus_{l=0}^{l-1}\{\overline{\mu}_0 z\}^{\perp}_{-k_l}$, decomposes into the direct sum of forms

$$Q(v,v)=\sum_{l=0}^{l-1}\langle v_l^T,\sigma(\chi*(zv_l),z\cdot)^T+\bar{h}v_l\rangle,$$

if

$$v=(v_0,\ldots,v_{l-1}),\quad v_l\in\{\overline{\mu}_0 z\cdot\}^{\perp}_{-k_l},\quad l=0,1,\ldots,l-1.$$

Therefore, to prove Lemma 10, it suffices to establish the positive definiteness of each of the forms

$$Q^l(v,v)\overset{\text{def}}{=}\langle v_l^T,\sigma(\chi*(zv_l),z\cdot)^T+\bar{h}v_l\rangle$$

on the subspace of finite codimension in $(\overline{\mu}_0 z)_{-k_i}^{\perp}$, i = 0, 1, ..., $l-1$. Denote $w_i=[\chi^{k_i-1}/(k_i-1)!]*v_i$. If $\int_{\tau_l}^{\tau_{l+1}}z_\tau^{(k_l)}w_l(\tau)d\tau=0$, then

$$Q^l(v_l,v_l)=\int_{\tau_l}^{\tau_{l+1}}\gamma_\tau(w_l(\tau))\,d\tau+$$
$$+\int_{\tau_l}^{\tau_{l+1}}\sigma\left(\int_{\tau_l}^{\tau}z_\theta^{(k_l)}w_l(\theta)\,d\theta,\;z_\tau^{(k_l)}w_l(\tau)\right)d\tau. \tag{14}$$

Indeed, when $v_i\in L_\infty{}^r[\tau_i,\tau_{i+1}]$, equality (14) follows from (6) and the general case is obtained with respect to continuity. In turn, since $\gamma_\tau(w)\geq\varepsilon|w|^2$, the quadratic form found on the right-hand side of equality (14) is positive definite on some subspace of finite codimension in $H_{-k_i}{}^r[\tau_i,\tau_{i+1}]$. This is the corollary to the Hilbert–Schmidt theorem on the spectrum of a compact self-adjoint operator. This implies that form Q is positive definite on some subspace $H_{-k.}{}^r[0,t]$.

Remark. It is very essential that form Q, unlike $\psi G_t''$, is not simply positive on a subspace of finite codimension but is positive definite.

LEMMA 11. Suppose that p is a continuous quadratic form on Hilbert space E, which is positive definite on some subspace of finite codimension in E. For the arbitrary closed subspace V in E symbol $p \mid V$ denotes the restriction of the quadratic form p to subspace V and

$$V_p^{\perp} = \{e \in E \mid p(e, v) = 0, \quad \forall v \in V\}.$$

Then

$$\operatorname{ind} p = \operatorname{ind}(p \mid V) + \operatorname{ind}(p \mid V_p^{\perp}) + \dim(V \cap V_p^{\perp}) - \\ - \dim(V \cap \ker p). \tag{15}$$

Proof. Passing, if it is necessary, to the space E/ker p, we can assume that ker p = 0. If in addition space E is finite-dimensional, then the assertion of the lemma turns into a standard fact of linear algebra. The proof of the general case repeats verbatim that of the finite-dimensional one since the hypotheses of the lemma guarantee the finiteness of all values occurring in equality (15) and also (for ker p = 0) the fulfillment of the identity $\dim W_p^{\perp} = \operatorname{codim} W$ for any closed subspace W of finite codimension.

Let $V_i = \{v = (v_0, \ldots, v_{l-1}) \in (\overline{\mu}_0 z)_{-k}^{\perp} \mid v_j = 0 \text{ for } i \le j \le l-1\}$. In particular, $V_0 = 0$, $V_l = (\overline{\mu}_0 z)_{-k}^{\perp}$. Introduce the notation $Q_i = Q \mid V_i$. Then, by virtue of Lemma 1,

$$\operatorname{ind} Q_{i+1} = \operatorname{ind} Q_i + \operatorname{ind}(Q_{i+1} \mid V_{iQ_{i+1}}^{\perp}) + \dim(V_i \cap V_{iQ_{i+1}}^{\perp}) - \\ - \dim(V_i \cap \ker Q_{i+1}). \tag{16}$$

LEMMA 12. Let

$$\mathcal{R}_i = \{(\lambda, v_i) \in \Lambda_{\tau_i} \oplus H_{-k_i}[\tau_i, \tau_{i+1}] \mid \lambda + \langle z v_i, 1 \rangle \in \Pi_0\} \subset \Lambda_{\tau_i} \oplus \\ \oplus H_{-k_i}^{r}[\tau_i, \tau_{i+1}]$$

and

$$R_i : (\lambda, v_i) \mapsto \langle v_i^T, \sigma(\lambda + \chi_*(z v_i), z \cdot)^T + \bar{h} v_i \rangle, \quad (\lambda, v_i) \in \mathcal{R}_i$$

is a quadratic form on $\mathcal{R}_i$. Then

$$1)\ \operatorname{ind}(Q_{i+1} \mid V_{iQ_{i+1}}^{\perp}) = \operatorname{ind} R_i$$

$$2)\ \dim(V_i \cap V_{iQ_{i+1}}^{\perp}) - \dim(V_i \cap \ker Q_{i+1}) = \\ = \dim(\Pi_0 \cap \Lambda_{\tau_i} / \Pi_0 \cap \Lambda_{\tau_i} \cap \Lambda_{\tau_{i+1}}) - \dim\Big(\bigcap_{0<\tau<\tau_i} \Lambda_\tau \Big/ \bigcap_{0<\tau<\tau_{i+1}} \Lambda_\tau\Big).$$

Proof. It is easy to see that

$$V_{iQ_{i+1}}^{\perp} = \{v \in V_{i+1} \mid \bar{h}_\tau v(\tau) + \sigma((\chi_* z v)(\tau) + \eta, z_\tau \cdot)^T = 0\}$$

for $0 \le \tau \le \tau_i$ for some $\eta \in \Pi_0$.

We present $v \in V_{iQ_{i+1}}^{\perp}$ in the form $v = (u, v_i)$, where

$$u \in \bigoplus_{j=0}^{i-1} H_{-k_j}^{r}[\tau_j, \tau_{j+1}], \quad v_i \in H_{-k_i}^{r}[\tau_i, \tau_{i+1}],$$

and set $\hat{y} = \eta + \chi_*(zu)$. Let $\tau \in (0, \tau_i) \setminus \{\tau_1, \ldots, \tau_{i-1}\}$; differentiating relation

$$\bar{h}_\tau v(\tau) + \sigma(\hat{y}(\tau), z_\tau \cdot)^T = 0$$

$2k_\tau$ times with respect to τ, and taking into account Lemma 1, we obtain

$$\overline{\gamma}_\tau u(\tau)+(-1)^{k_\tau}\sigma\left(\hat{y}_\tau,\ z_\tau^{(2k_\tau)}\cdot\right)^T=0,$$

where $\overline{\gamma}_\tau$ is a symmetric $r\times r$ matrix corresponding to the quadratic form γ_τ. Consequently,

$$\frac{d}{d\tau}\hat{y}=(-1)^{k_\tau-1}z_\tau\ \ \overline{\gamma}_\tau^{-1}\left(\sigma\left(\hat{y}_\tau,\ z_\tau^{(2k_\tau)}\cdot^T\right)\right).$$

Therefore, distribution $\hat{y}$ (and, consequently, u, too) is a smooth function near τ; moreover, its derivatives are uniformly bounded with respect to τ. Thus, if $u=(u_0, u_1, \dots, u_{i-1})$, then each of the u_j is representable in the form of a sum of a smooth function on $[\tau_j, \tau_{j+1}]$ and a distribution of order k_j-1 that is concentrated at points τ_j and τ_{j+1} (recall that, a priori, $u_j\in H_{-k_j}{}^r[\tau_j, \tau_{j+1}]$).

Consequently, $\hat{y}$ is also representable in the form of a sum of some piecewise smooth function y_τ and distributions concentrated at the points τ_j, $j=0, \dots, i$. Clearly, $(y_{\tau_j+0}-y_{\tau_j})\in\Gamma_{\tau_j}+\Gamma_{\tau_j+0}$; moreover, any vector from $\Gamma_{\tau_j}+\Gamma_{\tau_j+0}$ can be represented in the form of $y_{\tau_j+0}-y_{\tau_j}$ at the expense of the appropriate choice of the "pointwise part" of distributions u_{j-1} and u_j. Furthermore, $y_0=\eta\in\Gamma_0$, $y_\tau=\langle zu, 1\rangle+\eta$ for $\tau\geq\tau_i$.

For $v=(u, v_i)\in V_{iQ_{i+1}}{}^\perp$ we have

$$\begin{aligned}Q_{i+1}(v, v)&=\langle v^T(\tau),\ \sigma(y_\tau-\eta,\ z_\tau\cdot)^T+\overline{h}_\tau v\rangle=\\&=\langle v^T(\tau),\ \sigma(y_\tau, z_\tau\cdot)^T+\overline{h}_\tau v\rangle-\sigma(\eta,\ \langle zv,\ 1\rangle)=\\&=\langle v_i^T,\ \sigma(y_{\tau_i}+\chi_* zv_i,\ z\cdot)^T+\overline{h}v_i\rangle-0\end{aligned}$$

(In this calculation we used the condition that $\langle zv, 1\rangle\in\Pi_0$ $\forall v\in V_{i+1}$ and also the identity $\sigma(y_\tau, z_\tau\cdot)^T+\overline{h}_\tau u(\tau)=0$ for $0\leq\tau\leq\tau_i$).

Assertion 1 of Lemma 12 now follows from the following representation of the Jacobian curve.

LEMMA 13. Let

$$\Delta_\tau=\left\{y_\tau\in\Sigma\ \middle|\ \begin{array}{l}0\mapsto y_\theta\ (0\leqslant\theta\leqslant\tau)\ \text{is a piece wise continuous curve}\\ \dot{y}_\theta=z_\theta u_\theta\quad\text{for}\quad\theta\neq\tau_j;\ \ \sigma(y_\theta,\ z_\theta)^T+\overline{h}_\theta u_\theta\equiv0,\ \ y_0\in\Pi_0,\\ (y_{\tau_j+0}-y_{\tau_j})\in\Gamma_{\tau_j}+\Gamma_{\tau_j+0}.\end{array}\right\}.$$

Then $\Lambda_\tau=\Delta_\tau\oplus\Gamma_\tau$.

Proof. Differentiating relation $\sigma(y_\theta, z_\theta\cdot)^T+h_\theta u_\theta\equiv0$ $2k_\theta$ times with respect to θ, we obtain, by virtue of equation $\dot{y}_\theta=z_\theta u_\theta$,

$$\sigma(y_\theta, z_\theta^{(i)})\equiv0,\ \ i=0, 1, \dots, 2k_\theta-1,\ \ \ u_\theta=(-1)^{k_\theta-1}\overline{\gamma}_\theta^{\,-1}\sigma(y_\theta,\ z_\theta)^T.$$

From this we conclude that $\Delta_\tau\subset(\Gamma_\tau+\overline{\Gamma}_\tau)^\angle=\Gamma_\tau^\angle\cap\overline{\Gamma}_\tau^\angle$.*

In addition, $\dim\Delta_\tau$ is locally constant for $\tau\neq\tau_j$.

Suppose now that $x_\theta\in\Lambda_\theta$; moreover, x_θ satisfies the Jacobi equation for $0\leq\theta\leq\tau$; then

$$\frac{d}{d\theta}\sigma(x_\theta, y_\theta)\equiv\sigma\left(\overline{\gamma}_\theta^{-1}z_\theta^{(k_\theta)}\sigma\left(z_\theta^{(k_\theta)},\ x\right)^T,\ y_\theta\right)+\sigma(x_\theta,\ z_\theta u_\theta)\equiv0,$$
$$\sigma(x_{\tau_j}, y_{\tau_j})=0,\ \tau_j<\tau.$$

Consequently, $\Delta_\tau\subset\Lambda_\tau{}^\angle=\Lambda_\tau$. Thus, $\Delta_\tau\subset\Lambda_\tau\cap\overline{\Gamma}^\angle$.

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*For the definition and properties of $\overline{\Gamma}_\tau$, see the beginning of Section 3.

Furthermore,

$$\Delta_{+0} = (\Pi_0 + \Gamma_{+0}) \cap \Gamma_{+0}^{\angle} \cap \overline{\Gamma}_{+}^{\angle} = \Lambda_{+0} \cap \overline{\Gamma}^{\angle}.$$

From dimension considerations we obtain that $\Delta_\tau = \Lambda_\tau \cap \overline{\Gamma}_\tau^{\angle}$ for $0 < \tau \le \tau_1$. Since $\overline{\Gamma}_\tau^{\angle} \cap \Gamma_\tau = 0$ and $\Gamma_\tau \subset \Lambda_\tau$, we have $\Lambda_\tau = \Delta_\tau \oplus \Gamma_\tau$ for $0 < \tau \le \tau_1$. Consequently,

$$\Delta_{\tau_1+0} = (\Delta_{\tau_1} + \Gamma_{\tau_1} + \Gamma_{\tau_1+0}) \cap \Gamma^{\angle}_{\tau_1+0} \cap \overline{\Gamma}^{\angle}_{\tau_1+0} = \Lambda_{\tau_1}^{\Gamma_{\tau_1+0}} \cap \overline{\Gamma}^{\angle}_{\tau_1+0} = \\ = \Lambda_{\tau_1+0} \cap \overline{\Gamma}^{\angle}_{\tau_1+0},$$

$\Lambda_{\tau+0} = \Delta_{\tau_1+0} \oplus \Gamma_{\tau_1+0}$, etc. For any τ we get the equality $\Lambda_\tau = \Delta_\tau \oplus \Gamma_\tau$.

We move to the proof of assertion 2 of Lemma 12.

Element $(u, v_i) \in V_{iQ_{i+1}}{}^{\perp}$ lies in $V_i \cap V_{iQ_{i+1}}{}^{\perp} = \ker Q_i$ if and only if $v_i = 0$; element $(u, 0) \in V_{iQ_{i+1}}{}^{\perp}$ lies in V_i if and only if $\eta + \langle zu, 1\rangle \in \Pi_0 \cap \Lambda_{\tau_i}$ for some $\eta \in \Pi_0$. Consequently,

$$\dim(V_i \cap V_{iQ_{i+1}}^{\perp}) = \dim\left(\Pi_0 \cap \Lambda_{\tau_i} \Big/ \bigcap_{0<\tau<\tau_i} \Lambda_\tau\right).$$

Element $(u, 0) \in \ker Q_i$ lies in $V_i \cap \ker Q_{i+1}$ if and only if $\eta + \langle zu, 1\rangle \in \Pi_0 \bigcap_{\tau_i<\tau<\tau_{i+1}} \Lambda_\tau$ for some $\eta \in \Pi_0$. Therefore,

$$\dim(V_i \cap \ker Q_{i+1}) = \dim\left(\Pi_0 \bigcap_{\tau_i<\tau<\tau_{i+1}} \Lambda_\tau \Big/ \bigcap_{0<\tau<\tau_{i+1}} \Lambda_\tau\right).$$

Recall that $\Lambda\,|_{[\tau_i,\tau_{i+1}]}$ is a simple nondecreasing curve and, according to Lemma 7, $\bigcap_{\tau_i<\tau<\tau_{i+1}} \Lambda_\tau = \Lambda_{\tau_i} \cap \Lambda_{\tau_{i+1}}$.

Lemma 12 is finally proven.

So then the application of Lemma 11 enabled us to reduce everything to the calculation of the index of form R_i; we do this calculation also by repeatedly applying Lemma 11. First we overcome the possible discontinuity at point τ_i.

Space $H_{-k_i}{}^{r}[\tau_i, \tau_{i+1}]$ contains the rk_i-dimensional space $H_{-k_i}{}^{r}(\tau_i + 0)$, which consists of vector distributions concentrated at point τ_i. Set

$$I = \mathcal{R} \cap \{\Lambda_{\tau_i} \oplus H^r_{-k_i}(\tau_i + 0)\} = \\ = \left\{\left(\lambda, \sum_{j=0}^{k_i-1} a_j \frac{\partial^j}{\partial \tau^j}\Big|_{\tau_i+0}\right) \Bigg| \lambda \in \Lambda_{\tau_i}, \left(\lambda + \sum_{j=0}^{k_i-1} z^{(j)}_{\tau_i+0} a_j\right) \in \Pi_0\right\}.$$

Elementary calculations show that

$$R_i\left(\left(\lambda, \sum_{j=0}^{k_i-1} a_j \frac{\partial^j}{\partial \tau^j}\Big|_{\tau_i+0}\right)\right) = \sigma\left(\lambda, \sum_{j=0}^{k_i-1} z^{(j)}_{\tau_i+0} a_j\right), \tag{17}$$

while the index of form $R_i\,|\,I_{R_i}{}^{\perp}$ coincides with the index of form

$$R_i^{+}: (\lambda, v) \mapsto \langle v^{T}, \sigma(\lambda + \chi_*(zv) + \bar{h}v, z\cdot)^{T}\rangle,$$

given on the space

$$\mathcal{R}_i^{+} = \{(\lambda, v) \in \Lambda_{\tau_i+0} \oplus H_{-k_i}[\tau_i, \tau_{i+1}] \,|\, (\lambda + \langle zv, 1\rangle) \in \Pi_0\}$$

(the expressions for forms R_i and R_i^+ coincide, while the domains differ somewhat; recall that $\Lambda_{\tau_i+0} = (\Lambda_{\tau_i} + \Gamma_{\tau_i+0}) \cap \Gamma_{\tau_i+0}^{\angle})$. The following equality is a direct consequence of the definitions and relation (17):

$$\operatorname{ind}(R_i \mid I) = \operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_i+0}) - \frac{1}{2}(\dim(\Lambda_{\tau_i} \cap \Pi_0) + \dim(\Lambda_{\tau_i+0} \cap \Pi_0)) + \dim(\Lambda_{\tau_i} \cap \Lambda_{\tau_i+0} \cap \Pi_0).$$

Next element $\left(\lambda, \sum_j a_j \frac{\partial^j}{\partial \tau^j}\Big|_{\tau_i+0}\right)$ from I lies in $\ker(R_i| I)$ if and only if $\left(\nu + \lambda + \sum_j z^{(j)}_{\tau_i+0}\right) \in \Lambda_{\tau_i+0} \cap \Pi_0$ for some $\nu \in \Lambda_{\tau_i} \cap \Pi_0$. This element lies in $I \cap \ker R_i$ if and only if

$$\left(\nu + \lambda + \sum_j z^{(j)}_{\tau_i} a_j\right) \in \Pi_0 \bigcap_{\tau_i < \tau < \tau_{i+1}} \Lambda_\tau = \Pi_0 \cap \Lambda_{\tau_i+0} \cap \Lambda_{\tau_{i+1}}.$$

Consequently,

$$\operatorname{ind} R_i = \operatorname{ind}(R_i \mid I) + \operatorname{ind} R_i^+ + \dim\ker(R_i \mid I) - \dim(\ker R_i \cap I) =$$
$$= \operatorname{ind} R_i^+ + \operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_i+0}) + \frac{1}{2}\dim(\Lambda_{\tau_i} \cap \Pi_0) - \frac{1}{2}\dim(\Lambda_{\tau_i} \cap \Pi_0) -$$
$$- \dim(\Lambda_{\tau_i} \cap \Lambda_{\tau_{i+1}} \cap \Pi_0) + \dim(\Lambda_{\tau_i} \cap \Lambda_{\tau_{i+1}} \cap \Pi_0).$$

Taking into account equality (16) and Lemma 12, we get

$$\operatorname{ind} Q_{i+1} = \operatorname{ind} Q_i + \operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_i+0}) + \operatorname{ind} \mathcal{R}_i^+ + \frac{1}{2}\dim(\Lambda_{\tau_i} \cap \Pi_0) + \frac{1}{2}\dim(\Lambda_{\tau_i+0} \cap \Pi_0) - \dim(\Lambda_{\tau_i+0} \cap \Lambda_{\tau_{i+1}} \cap \Pi_0) - \dim\left(\bigcap_{0<\tau<\tau_i} \Lambda_\tau \Big/ \bigcap_{0<\tau<\tau_{i+1}} \Lambda_\tau\right). \tag{18}$$

It remains to compute $\operatorname{ind} R_i^+$. As is evident from the previous examinations, the use of our principal tool, Lemma 11, is contingent on rather cumbersome calculations. To simplify these calculations at least outwardly we will assume in the sequel that $k_i > 0$ and, consequently, $h_\tau = 0$ for $\tau_i < \tau \le \tau_{i+1}$. In fact, the case when $k_i = 0$ is the most simple one, but some expressions are more symmetric for $k_i > 0$.

Consider the space

$$W_i = \{(\lambda, v) \in \mathcal{R}_i^+ \mid \lambda + \langle zv, 1 \rangle = 0\} \subset \mathcal{R}_i^+.$$

LEMMA 14. If partition $0 = \tau_0 < \tau_1 < \dots < \tau_l = t$ is sufficiently fine, then form $R_i^+ \mid W_i$ is nonnegative, i.e.,

$$\operatorname{ind}(R_i \mid W_i) = 0, \quad i = 0, \dots, l-1.$$

Proof. Subspace $\hat{W}_i = \{(\lambda, v) \in W_i \mid v \in L_2^{\,r}[\tau_i, \tau_{i+1}]\}$ is everywhere compact in W_i; therefore, it is enough to prove the nonnegativity of form R_i^+ on $\hat{W}_i$. Set $w(\tau) = \int_{\tau_i}^{\tau} \frac{(\theta-\tau)^{k_i-1}}{(k_i-1)!} v(\theta)\, d\theta$, the usual process of integration by parts leads to the inclusion

$$\left(\int_{\tau_i}^{\tau_{i+1}} z_\tau v(\tau)\, d\tau + \int_{\tau_i}^{\tau_{i+1}} z_\tau^{(k_i)} w(\tau)\, d\tau\right) \in \Gamma_{\tau_{i+1}}.$$

Therefore, for $(\lambda, v) \in \hat{W}_i$ we have $\left(\lambda - \int_{\tau_l}^{\tau_{l+1}} z_\theta^{(k_l)} w(\theta)\, d\theta\right) \in \Gamma_{\tau_{l+1}}$ and

$$R_l^+((\lambda, v)) = \int_{\tau_l}^{\tau_{l+1}} \sigma\left(\lambda + \int_{\tau_l}^{\tau} z_\theta v(\theta)\, d\theta,\ z_\tau v(\tau)\right) d\tau =$$
$$= \int_{\tau_l}^{\tau_{l+1}} \sigma\left(\lambda - \int_{\tau_l}^{\tau} z_\theta^{(k_l)} w(\theta)\, d\theta,\ z_\tau v(\tau)\right) d\tau.$$

Integrating the last expression by parts repeatedly [$v(\tau)$ is integrated while the rest is differentiated] we obtain

$$R_l^+((\lambda, v)) = \int_{\tau_l}^{\tau_{l+1}} \sigma\left(\int_{\tau_l}^{\tau} z_\theta^{(k_l)} w(\theta)\, d\theta - \lambda,\ z_\tau^{(k_l)} w(\tau)\right) + \gamma_\tau(w(\tau))\, d\tau.$$

It is enough to establish the nonnegativity of the quadratic form (λ, w) occurring on the right-hand side of the last equality in the space

$$\mathscr{L} = \left\{(\lambda, w) \in \Lambda_{\tau_l+0} \oplus L_2^r[\tau_l, \tau_{l+1}] \,\middle|\, \left(\int_{\tau_l}^{\tau_{l+1}} z_\tau^{(k_l)} w(\tau)\, d\tau - \lambda\right) \in \Gamma_{\tau_{l+1}}\right\},$$

if $\tau_{i+1} - \tau_i$ is sufficiently small.

Set $x_\tau = \int_{\tau_l}^{\tau} z_\theta^{(k_l)} w(\theta)\, d\theta - \lambda,\ \tau_l < \tau \leqslant \tau_{l+1}$, and let

$$\mathscr{L}_0 = \{(\lambda, w) \in \mathscr{L} \mid x_{\tau_{l+1}} = 0\},$$
$$\mathscr{L}_1 = \{(\lambda, w) \in \mathscr{L} \mid \sigma(x_\tau, z_\tau^{(k_l)} \cdot)^T + \gamma_\tau w(\tau) \equiv 0,\ \tau_l < \tau \leqslant \tau_{l+1}\}$$

The pair (λ, w), evidently, lies in $\mathscr{L}_1$ if and only if curve x_τ is a solution of Jacobi system (10) with boundary conditions $x_{\tau_{i+0}} \in \Lambda_{\tau_{i+0}}$, $x_{\tau_{i+1}} \in \Gamma_{\tau_{i+1}}$. Since $\Gamma_{\tau_{i+1}} \subset \Lambda_{\tau_{i+1}}$, by the definition of the Jacobian curve Λ_τ any solution of the Jacobian system x_τ satisfying the condition $x_{\tau_{i+1}} \in \Gamma_{\tau_{i+1}}$ also satisfies the condition $x_{\tau_{i+0}} \in \Lambda_{\tau_{i+0}}$. Consequently,

$$\dim \mathscr{L}_1 = \dim \Gamma_{\tau_{l+1}}, \quad \mathscr{L} = \mathscr{L}_0 \oplus \mathscr{L}_1.$$

In addition, space $\mathscr{L}_1$ is obviously contained in the kernel of our quadratic form and for $(\lambda, w) \in \mathscr{L}_0$ this form is

$$\int_{\tau_l}^{\tau_{l+1}} \left(\int_{\tau_l}^{\tau} z^{(k_l)} w(\theta)\, d\theta,\ z_\tau^{(k_l)} w(\tau)\right) d\tau + \gamma_\tau(w(\tau))\, d\tau. \tag{19}$$

Since $\gamma_\tau(w(\tau)) \geq \varepsilon \mid w(\tau) \mid^2$, form (19) is positive for $\tau_{i+1} - \tau_i$ sufficiently small. ■

We are justified to choose from the very beginning an arbitrarily fine partition $0 = \tau_0 < \tau_1 < \ldots < \tau_l = t$; therefore, according to Lemma 14,

$$\operatorname{ind} R_l^+ = \operatorname{ind}(R_l^+ \mid W_{R_l}^\perp) + \dim\left(W \cap W_{R_l^+}^\perp\right) - \dim(W \cap \ker R_l^+).$$

The pair (λ, v) from R_i^+ lies in $W_{R_i^+}^\perp$ if and only if

$$\sigma(\lambda+(\chi_*(zv))(\tau), z_\tau\cdot)\equiv\sigma(\lambda_0, z_\tau\cdot), \quad \tau_i<\tau\leqslant\tau_{i+1},$$

for some $\lambda_0 \in \Lambda_{\tau_i+0}$.

Set $\hat{y} = \lambda - \lambda_0 + \chi_*(zv)$. The same arguments as in the proof of Lemma 12 show that distribution $\hat{y}$ is representable in the form of a sum of a vector-valued function y_τ smooth on $(\tau_i, \tau_{i+1}]$ and a distribution of order $k_i - 1$ concentrated at the points τ_i and τ_{i+1}. Lemma 13 implies that $y_{\tau_{i+1}} \in \Lambda_{\tau_{i+1}}$. Here

$$R_i^+((\lambda, w)) = \langle v^\top, \sigma(\lambda+\chi_*(zv), z\cdot)^\top\rangle = \langle v^\top, \sigma(\lambda_0, z\cdot)^\top\rangle = \\ =\sigma(\lambda_0, \langle zv, 1\rangle)=\sigma(\lambda_0, y_{\tau_{i+1}}).$$

We recall that $\lambda_0 + y_{i+1} = (\lambda + \langle zv, 1\rangle) \in \Pi_0$.

The relations imply that the index of the quadratic form $R_i^+|W^\perp_{R_i^+}$ coincides with the index of the quadratic form given on $(\Lambda_{\tau_i+0}+\Lambda_{\tau_{i+1}})\cap\Pi_0$ that puts in correspondence to each

$$\alpha=(\lambda_1+\lambda_2)\in(\Lambda_{\tau_i+0}+\Lambda_{\tau_{i+1}})\cap\Pi_0 \text{ (where } \lambda_1\in\Lambda_{\tau_i+0}, \lambda_2\in\Lambda_{\tau_{i+1}})$$

the value $\sigma(\lambda_1, \lambda_2)$. Thus,

$$\operatorname{ind}\left(R_i^+|W^\perp_{R_i^+}\right)=\operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i+0}, \Lambda_{\tau_{i+1}})-\frac{1}{2}\dim(\Lambda_{\tau_i+0}\cap\Pi_0)- \\ -\frac{1}{2}\dim(\Lambda_{\tau_{i+1}}\cap\Pi_0)+\dim(\Pi_0\cap\Lambda_{\tau_i+0}\cap\Lambda_{\tau_{i+1}}).$$

Next, let $(\lambda, v) \in W \cap W_{R_i^+}^{\perp}$; then

$$\lambda+\langle zv, 1\rangle-\lambda_0=-\lambda_0\in\Lambda_{\tau_{i+1}}\cap\Lambda_{\tau_i+0}=\bigcap_{\tau_i<\tau<\tau_{i+1}}\Lambda_\tau.$$

In this case, $y_\tau \equiv -\lambda_0$ is a constant; so $v(\tau) \equiv 0$ and $\lambda = -\langle zv, 1\rangle = 0$. Consequently,

$$W\cap W^\perp_{R_i^+}=0, \quad \operatorname{ind}R_i^+=\operatorname{ind}\left(R_i^+|W^\perp_{R_i^+}\right).$$

Substituting an expression for $\operatorname{ind}(R_i^+ | W_{R_i^+}^{\perp})$ in (18), we get

$$\operatorname{ind}Q_{i+1}-\operatorname{ind}Q_i=\operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_i+0})+\operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i+0}, \Lambda_{\tau_{i+1}})+ \\ +\frac{1}{2}\dim(\Lambda_{\tau_i}\cap\Pi_0)-\frac{1}{2}\dim(\Lambda_{\tau_{i+1}}\cap\Pi_0)+ \\ +\dim\left(\bigcap_{0<\tau<\tau_{i+1}}\Lambda_\tau\right)-\dim\left(\bigcap_{0<\tau<\tau_i}\Lambda_\tau\right), \quad i=0, 1, \ldots, l-1. \tag{20}$$

Proposition 2 and Lemma 4 imply that

$$\operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_i+0})+\operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i+0}, \Lambda_{\tau_{i+1}})=\operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}}).$$

In addition, $Q_l = Q$ and

$$\operatorname{ind}_{\Pi_0}(\Lambda_t, \Lambda_0)=\operatorname{ind}_{\Pi_0}(\Lambda_t, \Pi_0)=\frac{1}{2}(m-\dim(\Lambda_t\cap\Pi_0)).$$

Therefore, adding equalities (20), we obtain

$$\operatorname{ind}Q=\sum_{i=0}^{l}\operatorname{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}})+\dim\left(\bigcap_{0<\tau<t}\Lambda_\tau\right)-m. \tag{21}$$

LEMMA 15. For any $t_1, t_2 \in [0, t]$, $t_1 < t_2$, equality

$$\bigcap_{t_1<\tau<t_2}\Lambda_\tau=\Lambda_{t_1}\cap\Big(\sum_{t_1<\tau<t_2} z_\tau R^r\Big)^{\angle}$$

is true.

Proof. 1) If z_τ is smooth on the half-interval $t_1 < \tau \le t_2$, then

$$\bigcap_{t_1<\tau<t_2}\Lambda_\tau=\Lambda_{t_1+0}\cap\Big(\sum_{t_1<\tau<t_2} z_\tau^{(k_\tau)}R^r\Big)^{\angle}.$$

Indeed, if $\lambda\angle\Big(\sum_{t_1<\tau<t_2} z^{(k_\tau)}R^r\Big)$ then λ is a fixed point of Jacobi system (10) on $[t_1, t_2]$, that is, the constant vector $x_\tau \equiv \lambda$, $t_1 < \tau \le t_2$ is a solution of system (10). Conversely, let $\lambda\in\bigcap_{t_1<\tau<t_2}\Lambda_\tau$ then $\lambda\angle\bigcup_{t_1<\tau<t_2}\Lambda_\tau$. Consequently, for every solution x_τ of system (10) satisfying the condition $x_{t_1+0}\in\Lambda_{t_1+0}$, identity $\sigma(\lambda, x_\tau) = 0$, $t_1 < \tau \le t_2$ is fulfilled. Differentiating with respect to τ, we get

$$0=\sigma\left(\lambda,\sum_{i=1}^{r}\sigma(z_\tau^{(k_\tau)}v_\tau^i,x_\tau)\,z_\tau^{(k_\tau)}v_\tau^i\right)=\sum_{i=1}^{r}\sigma(z_\tau^{(k_\tau)}v_\tau^i,\ x_\tau)\,\sigma(\lambda,\ z_\tau^{(k_\tau)}v_\tau^i),=$$
$$=\sigma\left(\sum_{i=1}^{n}\sigma(\lambda,\ z_\tau^{(k_\tau)}v_\tau^i)\,z_\tau^{(k_\tau)}v_\tau^i,\ x_\tau\right).$$

Consequently, $\sum_{i=1}^{r}\sigma(\lambda,\ z^{(k_\tau)}v_\tau^i)\,z_\tau^{(k_\tau)}v_\tau^i\in\Lambda_\tau$. Since λ also lies on the Lagrangian plane Λ_τ, we have

$$0=\sigma\left(\lambda,\sum_{i=1}^{r}\sigma(\lambda,z_\tau^{(k_\tau)}v_i)z_\tau^{(k_\tau)}v_i\right)=\sum_{i=1}^{r}\left(\sigma(\lambda,\ z_\tau^{(k_\tau)}v_i)\right)^2.$$

So $\lambda\in(z_\tau^{(k_\tau)}v_i)^{\angle}$, $i = 1, \ldots, r$; $t_1 < \tau \le t_2$.

2) Under the same conditions we have

$$\Lambda_{t_1+0}\cap\Big(\sum_{t_1<\tau<t_2} z_\tau^{(k_\tau)}R^r\Big)^{\angle}=\Lambda_{t_1+0}\cap\Big(\sum_{t_1<\tau<t_2} z_\tau R^r\Big)^{\angle}.$$

Indeed, the obvious inclusion

$$z_\tau^j v\in\sum_{t_1<\tau<t_2} z_\tau R^r\quad\forall\tau\in(t_1,t_2),\quad v\in R^r$$

implies

$$\Big(\sum_{t_1<\tau<t_2} z_\tau R^r\Big)^{\angle}\subset\Big(\sum_{t_1<\tau<t_2} z_\tau^{(k_\tau)}R^r\Big)^{\angle}.$$

On the other hand, $z_\tau v\in\Lambda_\tau=\Lambda_\tau^{\angle}$; consequently,

$$\left(\sum_{t_1<\tau<t_2} z_\tau R\right)^{\angle} \supset \left(\bigcap_{t_1<\tau<t_2} \Lambda_\tau\right) = \Lambda_{t_1+0} \cap \left(\sum_{t_1<\tau<t_2} z^{(k_\tau)} R^r\right)^{\angle}.$$

3) Let $\tau \in [t_1, t_2]$; moreover, τ does not have to be a point where curve z is smooth. We have

$$\Lambda_\tau \cap \Lambda_{\tau+0} = \Lambda_\tau \cap \Gamma^{\angle}_{\tau+0}.$$

Since

$$\Gamma_{\tau+0} = \text{span}\{z^i_{\tau+0} v \mid 0 \leqslant i < k_{\tau+0},\ v \in R^r\} \subset$$
$$\subset \text{span}\{z_\theta v \mid \tau < \theta \leqslant \tau + \varepsilon,\ v \in R^r\},$$

by virtue of 1) and 2)

$$\bigcap_{\tau<\theta<\tau+\varepsilon} \Lambda_\theta \supset \Lambda_\tau \cap \left(\sum_{\tau<\theta<\tau+\varepsilon} z_\theta R^r\right)^{\angle}. \quad \forall \varepsilon > 0.$$

The inverse inclusion immediately follows from 1) and 2).

COROLLARY. $\Pi_0 \bigcap_{t_1<\tau<t_2} \Lambda_\tau = \Lambda_{t_1} \cap \Delta^{t_2}_{t_1}$.* In particular,

$$\bigcap_{0<\tau<t} \Lambda_\tau = \Delta^0_t = 0.$$

To conclude the proof of Theorem 1, it remains to use equality (21) and recall that

$$\text{ind}\, \psi G''_t = \text{ind}\, Q.$$

Proof of Theorem 2. Applying Theorem 1 to forms $\psi G_s''$ and $\psi G_{s+\varepsilon}''$, we get that for a sufficiently small $\varepsilon >$ 0

$$\text{ind}\, \psi G''_{s+\varepsilon} - \text{ind}\, \psi G''_s = \text{ind}_{\Pi_0}(\Lambda_s, \Lambda_{s+\varepsilon}) + \text{ind}_{\Pi_0}(\Lambda_{s+\varepsilon}, \Pi_0) -$$
$$- \text{ind}_{\Pi_0}(\Lambda_s, \Pi_0) - \dim(\Delta^0_s/\Delta^0_{s+\varepsilon}) = \text{ind}_{\Pi_0}(\Lambda_s, \Lambda_{s+\varepsilon}) +$$
$$+ \frac{1}{2}(\dim(\Lambda_s \cap \Pi_0) - \dim(\Lambda_{s+\varepsilon} \cap \Pi_0)) - \dim(\Delta^0_s/\Delta^0_{s+\varepsilon})$$

Next, $\text{ind}_{\Pi_0}(\Lambda_s, \Lambda_{s+\varepsilon}) = \text{ind}_{\Pi_0}(\Lambda_s, \Lambda_{s+0}) + \text{ind}_{\Pi_0}(\Lambda_{s+0}, \Lambda_{s+\varepsilon})$. It is easy to see that mapping $\tau \to \Lambda_\tau \cap \Pi_0$ is semicontinuous from above (relative to the inclusion of subspaces) at any point of continuity of curve Λ_τ. Therefore, for ε sufficiently close to zero, subspaces $\Pi_0 \cap \Lambda_{s+\varepsilon}$ do not increase monotonically with the growth of ε. In addition, if Λ_τ is a nondecreasing curve in $L(\Sigma)$, then $\text{ind}_{\Pi_0}(\Lambda_{s+0}, \Lambda_{s+\varepsilon})$ does not decrease with the growth of ε, we have $\text{ind}_{\Pi_0}(\Lambda_{s+0}, \Lambda_{s+0}) = \Lambda$. From this it is easy to conclude that for ε sufficiently close to zero

$$\text{ind}_{\Pi_0}(\Lambda_{s+0}, \Lambda_{s+\varepsilon}) = \dim(\Lambda_{s+0} \cap \Pi_0) - \dim(\Lambda_{s+\varepsilon} \cap \Pi_0) =$$
$$= \dim(\Lambda_{s+0} \cap \Pi_0) - \dim(\Pi_0 \bigcap_{s<\tau<s+\varepsilon} \Lambda_\tau).$$

The assertion of Theorem 2 now follows from the corollary to Lemma 15.

6. The Jacobi equation and the formula for calculating the index of the quadratic form $\psi G_t''$ were obtained under the assumption that $\gamma_\tau(v) \geq \varepsilon |v|^2$. However, according to Proposition 1, a necessary condition for $\text{ind}\, \psi G_t''$ to be finite is only the nonnegativity of form γ_τ. In this subsection we describe briefly how the Jacobi equation should be modified if forms γ_τ degenerate.

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*For the definition of space $\Delta_{t_1}^{t_0}$ see p. 2685 (before the statement of Theorem 2).

Let $0 \le k \le m$, $\tau \in (0, t]$. Denote by V_τ^k a subspace in $\mathbf{R}^r$ consisting of all points $v \in \mathbf{R}^r$ for which there is a smooth curve v_θ defined on some interval $\overline{\tau} \le \theta \le \tau$ and satisfying the conditions

$$v_\tau = v,\ h_\theta(w, v(\theta)) \equiv \sigma(z_\theta^{(i)} w, z_\theta v(\theta)) \equiv 0,\ \overline{\tau} \le \theta \le \tau,\ 0 \le i \le 2k,$$

for any $w \in \mathbf{R}^r$.

Clearly, $V_\tau^{k_2} \subset V_\tau^{k_1}$ for $k_1 < k_2$; in addition, $v_\tau^k = \mathbf{R}^r$ for $k < k_\tau$, $V_\tau^{k_\tau} = \ker \gamma_\tau$.

For $0 < k \le m$ we set

$$\gamma_\tau^k : (v + V_\tau^k) \mapsto (-1)^{k-1} \sigma(z_\tau^{2k} v, z_\tau v),\ v \in V_\tau^{k-1}$$

to be a quadratic form on v_τ^{k-1}/V_τ^k.

Suppose that, in addition, $V_\tau^{-1} \overset{\text{def}}{=} \mathbf{R}^r$, $\gamma_\tau^0 \overset{\text{def}}{=} h_\tau$.

We have $\gamma_\tau^k = 0$ for $k < k_\tau$, $\gamma_\tau^{k_\tau} = \gamma_\tau$.

The following assertion is an essential strengthening of Proposition 1 even though its proof differs from the proof of Proposition 1 only by the need to use a larger number of indices.

Proposition 3. If $\operatorname{ind} \psi G_t'' < +\infty$, then

a) $\sigma(z_\tau^{(2k)} v_1, z_\tau v_2) = 0\ \forall v_1, v_2 \in V_\tau^k$, $0 \le k < m$, $\tau \in (0, t]$;

b) $\gamma_\tau^k \ge 0$, $0 \le k \le m$, $\tau \in (0, t]$. Conversely, if condition a) is fulfilled and $\gamma_\tau^k(\overline{v}) \ge \varepsilon |\overline{v}|^2$ for any $\overline{v} \in V_\tau^{k-1}/V_\tau^k$, $0 \le k \le m$, $\tau \in (0, t]$, and some $\varepsilon > 0$, then $\operatorname{ind} \psi G_t'' < +\infty$.

It is easy to see that

$$\lim_{\theta \uparrow \tau} V_\theta^k \supset V_\tau^k,\ 0 \le k \le m,\ \tau \in (0, t];$$

if, however, τ is a point where z_θ is smooth and $\dim V_\theta^k = \text{const}$ for θ close to τ, then V_θ^k depends smoothly on θ near τ. Assume that the sufficient condition for the finiteness of $\operatorname{ind} \psi G_t''$ given in Proposition 3 is fulfilled; then subspaces V_τ^k, $0 \le k \le m$ depend piecewise smoothly on $\tau \in (0, t]$ and smoothly at any point where z_τ, ψh_τ are smooth. It turns out that with the fulfillment of this finite condition for $\operatorname{ind} \psi G_t''$ we can define a (generalized) Jacobi equation and a Jacobi curve Λ_τ so that the assertion of Theorem 1 (and understandably all its corollaries) are fulfilled.

Set

$$\Gamma_\tau^k = \operatorname{span}\left\{\frac{d^l}{d\tau^l}(z_\tau v_\tau) \mid v_\theta \in V_\theta^{k-1},\ 0 \le l \le k\right\},\ 0 \le k \le m,$$

$$\hat{\Gamma}_\tau = \sum_{k=0}^{\infty} \Gamma_\tau^k,\ \tau \in (0, t]$$

to be a piecewise smooth family of isotropic subspaces in Σ.

Suppose that $\tau \in (0, t]$ and $V_\theta^{k-1} \ni v_\theta$ is a smooth curve defined for θ near τ such that $v_\tau \in V_\tau^k$. Then, obviously, $(d^k/d\tau^k)(z_\tau v_\tau) \in \Gamma_\tau^k + \Gamma_\tau^{k+1}$. Therefore, whatever $x \angle (\Gamma_\tau^k + \Gamma_\tau^{k+1})$ is, correspondence

$$\zeta_\tau^k(x) : \overline{v} \to \sigma\left(\frac{d^k}{d_\tau^k}(z_\tau v_\tau), x\right).$$

where

$$\overline{v} = (v_\tau + V_k) \in V_\tau^{k-1}/V_\tau^R \text{ и } v_\theta \in V_\theta^{k-1}\ \forall \theta,$$

Further ... optional).

Let $0 \leq k \leq m$, $\tau \in (0, t]$. Denote by V_τ^k a subspace in $\mathbf{R}^r$ consisting of all points $v \in \mathbf{R}^r$ for which there is a smooth curve v_θ defined on some interval $\bar{\tau} \leq \theta \leq \tau$ and satisfying the conditions

$$v_\tau = v,\ h_\theta(w, v(\theta)) \equiv \sigma(z_\theta^{(i)} w, z_\theta v(\theta)) \equiv 0,\ \bar{\tau} \leqslant \theta \leqslant \tau,\ 0 \leqslant i < 2k,$$

for any $w \in \mathbf{R}^r$.

Clearly, $V_\tau^{k_2} \subset V_\tau^{k_1}$ for $k_1 < k_2$; in addition, $v_\tau^k = \mathbf{R}^r$ for $k < k_\tau$, $V_\tau^{k_\tau} = \ker \gamma_\tau$.

For $0 < k \leq m$ we set

$$\gamma_\tau^k : (v + V_\tau^k) \mapsto (-1)^{k-1} \sigma(z_\tau^{2k} v, z_\tau v),\ v \in V_\tau^{k-1}$$

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Suppose that, in addition, $V_\tau^{-1} \overset{\text{def}}{=} \mathbf{R}^r$, $\gamma_\tau^0 \overset{\text{def}}{=} h_\tau$.

We have $\gamma_\tau^k = 0$ for $k < k_\tau$, $\gamma_\tau^{k_\tau} = \gamma_\tau$.

The following assertion is an essential strengthening of Proposition 1 even though its proof differs from the proof of Proposition 1 only by the need to use a larger number of indices.

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b) $\gamma_\tau^k \geq 0$, $0 \leq k \leq m$, $\tau \in (0, t]$. Conversely, if condition a) is fulfilled and $\gamma_\tau^k(\bar{v}) \geq \varepsilon |\bar{v}|^2$ for any $\bar{v} \in V_\tau^{k-1}/V_\tau^k$, $0 \leq k \leq m$, $\tau \in (0, t]$, and some $\varepsilon > 0$, then $\operatorname{ind} \psi G_t'' < +\infty$.

It is easy to see that

$$\lim_{\theta \uparrow \tau} V_\theta^k \supset V_\tau^k,\ 0 \leqslant k \leqslant m,\ \tau \in (0, t];$$

if, however, τ is a point where z_θ is smooth and $\dim V_\theta^k = \text{const}$ for θ close to τ, then V_θ^k depends smoothly on θ near τ. Assume that the sufficient condition for the finiteness of $\operatorname{ind} \psi G_t''$ given in Proposition 3 is fulfilled; then subspaces V_τ^k, $0 \leq k \leq m$ depend piecewise smoothly on $\tau \in (0, t]$ and smoothly at any point where z_τ, ψh_τ are smooth. It turns out that with the fulfillment of this finite condition for $\operatorname{ind} \psi G_t''$ we can define a (generalized) Jacobi equation and a Jacobi curve Λ_τ so that the assertion of Theorem 1 (and understandably all its corollaries) are fulfilled.

Set

$$\Gamma_\tau^k = \operatorname{span}\left\{\frac{d^i}{d\tau^i}(z_\tau v_\tau) \mid v_\theta \in V_\theta^{k-1},\ 0 \leqslant i \leqslant k\right\},\ 0 \leqslant k \leqslant m,$$

$$\hat{\Gamma}_\tau = \sum_{k=0}^{\infty} \Gamma_\tau^k,\ \tau \in (0, t]$$

to be a piecewise smooth family of isotropic subspaces in Σ.

Suppose that $\tau \in (0, t]$ and $V_\theta^{k-1} \ni v_\theta$ is a smooth curve defined for θ near τ such that $v_\tau \in V_\tau^k$. Then, obviously, $(d^k/d\tau^k)(z_\tau v_\tau) \in \Gamma_\tau^k + \Gamma_\tau^{k+1}$. Therefore, whatever $x \angle (\Gamma_\tau^k + \Gamma_\tau^{k+1})$ is, correspondence

$$\zeta_\tau^k(x) : \bar{v} \to \sigma\left(\frac{d^k}{d_\tau^k}(z_\tau v_\tau), x\right).$$

where

$$\bar{v} = (v_\tau + V_k) \in V_\tau^{k-1}/V_\tau^R \text{ и } v_\theta \in V_\theta^{k-1}\ \forall \theta,$$

unambiguously defines a linear form $\eta_\tau^k(x)$ on V_τ^{k-1}/V_τ^k. Consequently, mapping

$$x \mapsto \frac{1}{2}(\gamma_\tau^k)^{-1}(\eta_\tau^k(x)), \quad x \angle (\Gamma_\tau^k + \Gamma_\tau^{k+1})$$

defines a quadratic form on $(\Gamma_\tau^k + \Gamma_\tau^{k+1})^\angle$

$$x \to \frac{1}{2}\sum_{k=0}^{m}(\gamma_\tau^k)^{-1}(\eta_\tau^k(x)), \quad x \angle (\hat{\Gamma}_\tau^k + \Gamma_\tau^{k+1}),$$

and mapping

$$x \mapsto \frac{1}{2}\sum_{k=0}^{m}(\gamma_\tau^k)^{-1}(\eta_\tau^k(x)), \quad x + \hat{\Gamma}_\tau, \tag{22}$$

defines a quadratic form on $\hat{\Gamma}_\tau^\angle \subset \Sigma$.

We extend form (22) in an arbitrary way to some quadratic form J_τ on the whole space Σ. Form J_τ defines vector field $\Lambda \to J_\tau(\Lambda)$ on $L(\Sigma)$. The differential equation

$$\dot{\Lambda} = J_\tau(\Lambda), \quad \Lambda \in L(\Sigma), \quad \tau \in [0, t], \tag{23}$$

is called the (generalized) Jacobi equation. Observe that Eq. (23) is uniquely defined only for such τ and Λ for which $\Lambda \supset \hat{\Gamma}_\tau$.

As for Eq. (11), the solutions of Eq. (23) are defined to be not only continuous but also piecewise continuous curves. Jacobian curves are called solutions Λ_τ of Eq. (23) satisfying the conditions

$$\Lambda_0 = \Pi_0, \quad \Lambda_{\tau+0} = \Lambda_\tau^{\hat{\Gamma}_{\tau+0}}, \quad \forall \tau \in [0, t).$$

It is not hard to show that $\hat{\Gamma}_\tau \subset \Lambda_\tau$, $\forall \tau \in (0, t]$. Consequently, the Jacobian curve (in contrast to the generalized Jacobi equation) is uniquely defined by form (22) and does not depend on the choice of the extension of this form to the whole space Σ. If the sufficient condition for the finiteness of $\operatorname{ind} \psi G_t''$ from Proposition 1 is fulfilled, then curve Λ_τ, obviously, coincides with the Jacobian curve defined for this case in subsection 3.

Proposition 4. Assume that the sufficient condition for the finiteness of $\operatorname{ind} \psi G_t''$ formulated in Proposition 3 is fulfilled and Λ_τ, $\tau \in [0, t]$ is a Jacobian curve. Then for curve Λ_τ the assertion of Theorem 1 is true.

Proposition 4, which generalizes Theorem 1, is proved in the same way as the theorem, but even more cumbersomely because of the need to use a larger number of indices.

We have learned to construct a Jacobian curve and compute $\operatorname{ind} \psi G_t''$ in a more general situation than in Theorem 1; nevertheless, between the necessary and sufficient conditions for the finiteness of $\operatorname{ind} \psi G_t''$ a gap still remains.

Assume that hypotheses a) and b) of Proposition 3 are fulfilled, and let

$$T = \bigcap_{\varepsilon > 0} \overline{([0, t] \setminus \mathscr{K}_\varepsilon)},$$

where

$$\mathscr{K}_\varepsilon = \{\tau \in (0, t] \mid \gamma_\tau^k(\bar{v}) \geqslant \varepsilon |\bar{v}|^2, \ \forall \bar{v} \in V_\tau^{k-1}/V_\tau^k, \ 0 \leqslant k \leqslant m\}.$$

It is not hard to see that set T is nowhere compact on [0, t]; if, however, z_τ depends in a piecewise analytic manner on τ, then T consists of a finite number of points.

Further considerations are conducted under the assumption that T is a finite set (piecewise analyticity is optional).

APPENDIX TO SEC. 3. THE LAGRANGE GRASSMANIAN

Here we provide information from the symplectic geometry that was used in Sec. 3 in the form we need. The proofs, as a rule, are replaced by references to the bibliography. Everywhere below Σ is a symplectic space of dimension 2m with a symplectic form σ. The expression $S_1 \angle S_2$ for subsets $S_1, S_2 \subset \Sigma$ means that $\sigma(s_1, s_2) = 0$, $\forall s_1 \in S_1, s_2 \in S_2$. We denote by $S^{\angle}$ the skew orthogonal complement of the subset $S \subset \Sigma$, $S^{\angle} = \{x \in \Sigma \mid \sigma(x, s) = 0 \ \forall s \in S\}$.

Subspace $\Gamma \subset S$ is called isotropic if $\Gamma \subset \Gamma^{\angle}$, and Lagrangian (a Lagrangian plane) if $\Gamma = \Gamma^{\angle}$. The set of all Lagrangian subspaces forms a closed submanifold in the Grassmanian $G_m(\Sigma)$, this submanifold is denoted by $L(\Sigma)$ and is called the Lagrange Grassmanian. Thus,

$$L(\Sigma) = \{\Lambda \subset \Sigma \mid \Lambda^{\angle} = \Lambda\}.$$

I. Natural Atlas in L(Σ). Let $\Delta \in L(\Sigma)$; denote by $\Delta^{\pitchfork}$ the set of all Lagrangian planes in $L(\Sigma)$ transversal to Δ and for arbitrary $\Lambda \in \Delta^{\pitchfork}$ let $P_\Lambda: \Sigma \to \Lambda$ be the projection operator of Σ onto Λ parallel to Δ.

Proposition A1. For every $\Delta \in L(\Sigma)$ and $\Lambda \in \Delta^{\pitchfork}$ the following identities are true:

i) $\delta(P_\Lambda x_1, x_2) + \sigma(x_1, P_\Lambda x_2) = \sigma(x_1, x_2) \ \forall x_1, x_2 \in \Sigma$;

ii) $P_\Lambda \Delta = \{0\}$.

Conversely, if some linear operator $\Delta\Psi$ satisfies conditions i) and ii), then $P = P_\Lambda$ for some $\Lambda \in \Delta\Psi$. For the proof see [12, pp. 136—137].

COROLLARY. Set $\Delta\Psi$ has the structure of an affine space; moreover, the linear space associated with it is naturally isomorphic to the space $\mathscr{P}(\Sigma/\Delta)$ of all symmetric bilinear forms on Σ/Δ.

Indeed, the fact the collection of linear operators satisfying conditions i) and ii) forms an affine space is obvious. These conditions, furthermore, imply that $\forall\ \Lambda_1, \Lambda_2 \in \Delta\Psi$ the expression $\sigma((P_{\Lambda_1} - P_{\Lambda_2})x_1, x_2)$, $x_1, x_2 \in \Sigma$ defines a symmetric bilinear form on $\mathscr{P}(\Sigma/\Delta)$. It is easy to show that any form from $\mathscr{P}(\Sigma/\Delta)$ is realized in this way.

The set of affine spaces $\Delta\Psi$ is a natural atlas in $L(\Sigma)$, $\dim L(\Sigma) = m(m + 1)/2$ for all possible Δ.

II. Tangent Space $T_\Lambda L(\Sigma)$. Let $\Lambda \in L(\Sigma)$ and Λ_ε, $\varepsilon \in R$ be a smooth curve in $L(\Sigma)$ satisfying the condition $\Lambda_0 = \Lambda$. In such a case, $\xi = (d/d\varepsilon)\Lambda_\varepsilon |_{\varepsilon=0}$ is a tangent vector to $L(\Sigma)$ at the point Λ. Let $\lambda \in \Lambda$ and λ_ε be a smooth curve in Σ satisfying the conditions $\lambda_\varepsilon \in \Lambda_\varepsilon\ \forall\varepsilon$, $\lambda_0 = \lambda$. It is easy to see that coset $(d/d\varepsilon)\lambda_\varepsilon |_{\varepsilon=0} + \Lambda$ depends only on $\lambda \in \Lambda$ and $(d/d\varepsilon)\Lambda_\varepsilon |_{\varepsilon=0}$, and not on the choice of curve λ_ε. Consequently, correspondence $\lambda \to (d/d\varepsilon)\lambda_\varepsilon |_{\varepsilon=0}$ defines the linear transformation D_ξ: $\Lambda \to \Sigma/\Lambda$. Clearly, mapping $\xi \to D_\xi$ is linear; moreover, $D_\xi = 0$ if and only if $\xi = 0$.

So far we have nowhere taken advantage of the fact that we are dealing with Lagrangian spaces; everything we have said is true for any tangent vector to a Grassmanian manifold. The fact of being Lagrangian implies that bilinear form $1/2\sigma(D_\xi\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \in \Lambda$ is unambiguously defined. Moreover, the identity

$$0=\frac{d}{d\varepsilon}\sigma(\lambda^1, \lambda^2_\varepsilon)|_{\varepsilon=0}=\sigma(D_\xi\lambda_1, \lambda_2)+\sigma(\lambda_1, D_\xi\lambda_2)$$

implies the symmetricity of this form. The corresponding quadratic form on Λ is denoted by the symbol $1/2\sigma(D_\xi\Lambda, \Lambda)$. Comparing dimensions we obtain the following:

Proposition A2. Correspondence $\xi \to 1/2\sigma(D_\xi\Lambda, \Lambda)$ establishes a natural isomorphism of the space $T_\Lambda L(\Sigma)$ tangent to $L(\Sigma)$ at "point" Λ and of the space $\mathscr{P}(\Lambda)$ of the quadratic forms on Λ.

For us it is particularly important that the isomorphism described in Proposition A2 defines the relation of a partial ordering in the space $T_\Lambda L(\Sigma)$: tangent vector $\xi \in T_\Lambda L(\Sigma)$ is called nonnegative if the quadratic form corresponding to it is nonnegative.

Let $h_t \in \mathscr{P}(\Sigma)$, $t \in \mathbf{R}$ be a family of quadratic forms on Σ (a nonstationary quadratic Hamiltonian). A linear Hamiltonian system in Σ corresponds to the Hamiltonian h_t, $t \in R$; we denote by H_t: $\Sigma \to \Sigma$, $t \in \mathbf{R}$ the fundamental matrix of this system, $H_0 = \mathrm{id}$. Clearly, H_t is a linear symplectic transformation. $H_t \in Sp(\Sigma)$. Since symplectic transformations carry Lagrangian planes into Lagrangian planes, flux $H_t{}^*$ in Σ defines the corresponding flux $\mathscr{H}_t{}^*$, $t \in \mathbf{R}$ in $L(\Sigma)$,

$$\mathscr{H}_t(\Lambda)\overset{\text{def}}{=}H_t\Lambda\ \ \forall\Lambda\in L(\Sigma).$$

On the other hand, a family of quadratic forms h_t, $t \in \mathbf{R}$ generates in an obvious way a nonstationary vector field $\vec{h}_t$ on $L(\Sigma)$: It is sufficient to restrict quadratic form h_t to the plane $\Lambda \in L(\Sigma)$ in order to obtain the following tangent vector at "point" Λ:

$$\vec{h}_t|\{\Lambda\}\overset{\text{def}}{=}h_t|\Lambda\ \ \forall\Lambda\in L(\Sigma),$$

or, using notation introduced above,

$$\vec{h}_t|\{\Lambda\}=h_t(\Lambda, \Lambda).$$

The following identity is established by direct calculation:

$$\mathscr{H}_t^*=\overrightarrow{\exp}\int_0^t\vec{h}_\tau d\tau,\quad t\in\mathrm{R}.$$

III. Imbeddings and Projections. Let $\Gamma \subset \Sigma$ be an isotropic subspace, i.e., $\Gamma \subset \Gamma^{\angle}$, $\dim \Gamma = k \leq m$. It is easy to see that the skew inner product $\sigma(\cdot, \cdot)$ induces a symplectic structure in the space $\Gamma^{\angle}/\Gamma$; in addition, $\dim(\Gamma^{\angle}/\Gamma) = 2(m - k)$. Suppose that $\mathcal{A} \in L(\Gamma^{\angle}/\Gamma)$; then the complete preimage of subspace $\mathcal{A}$ under the factorization of $\Gamma^{\angle}/\Gamma$ is a Lagrangian subspace in Σ. The described correspondence defines the imbedding of manifold $L(\Gamma^{\angle}/\Gamma)$ into manifold $L(\Sigma)$. Under this imbedding manifold $L(\Gamma^{\angle}/\Gamma)$ passes into a submanifold in $L(\Sigma)$ consisting of all Lagrangian planes containing Γ (or, which is equivalent, contained in $\Gamma^{\angle}$). Henceforth we will identify manifold $L(\Gamma^{\angle}/\Gamma)$ with the corresponding to it submanifold in $L(\Sigma)$.

Conversely, for any $\Lambda \in L(\Sigma)$ we set $\Lambda^{\Gamma} = \Lambda \cap \Gamma^{\angle} + \Gamma$. It is easy to see that $\Lambda^{\angle} \in L(\Gamma^{\angle}/\Gamma)$, and mapping $\Lambda \to \Lambda^{\Gamma}$ defines the projection of $L(\Sigma)$ onto submanifold $L(\Gamma^{\angle}/\Gamma)$. This projection is discontinuous on $L(\Sigma)$ but is smooth and surjective on submanifolds $\{\Lambda \in L(\Sigma) \mid \dim(\Lambda \cap \Gamma) = \text{const}\}$. If $\Lambda \in L(\Gamma^{\angle}/\Gamma) \subset L(\Sigma)$, then the tangent space $T_{\Lambda}L(\Gamma^{\angle}/\Gamma) \subset T_{\Lambda}L(\Sigma) \approx \mathcal{P}(\Lambda)$ consists of quadratic forms q on Λ such that $\Gamma \subset \ker q$.

Proposition A3. Imbedding $L(\Gamma^{\angle}/\Gamma) \subset L(\Sigma)$ induces an isomorphism of fundamental groups.

For the proof, see [12, pp. 152—154].

COROLLARY. $\pi_1(L(\Sigma)) = \mathbf{Z}$. Indeed, let $\dim \Gamma = n - 1$; then $\dim(\Gamma^{\angle}/\Gamma) = 2$. At the same time, $L(\mathbf{R}^2) = \mathbf{RP}^1 = S^1$.

In fact, imbedding $L(\Gamma^{\angle}/\Gamma) \subset L(\Sigma)$ for $\dim \Gamma = n - 1$ not only induces an isomorphism of fundamental groups of manifolds $L(\Sigma)$ and S^1 but also canonically defines the generator $\gamma \in \pi_1(L(\Sigma))$. Indeed, nonzero tangent vectors to submanifold $L(\Gamma^{\angle}/\Gamma) \subset L(\Sigma)$ are quadratic forms of rank one. Any such form is either nonnegative or nonpositive. Denote by γ the generator of group $\pi_1(L(\Sigma))$, which is expressible as a curve in $L(\Gamma^{\angle}/\Gamma)$ with nonnegative velocity.

We can show that this definition is unambiguous, i.e., it does not depend on the choice of Γ. Indeed, the symplectic group $Sp(\Sigma)$ acts transitively on the set of isotropic subsets of fixed dimension. Since, in addition, group Sp (Σ) is connected, for any isotropic $\Gamma_1, \Gamma_2 \subset \Sigma$ of the same dimension there exists a diffeomorphism, isotropic to the identity one, of manifold $L(\Sigma)$ onto itself that carries $L(\Gamma_1^{\angle}/\Gamma_1)$ into $L(\Gamma_2^{\angle}/\Gamma_2)$.

Definition. Let Λ_θ, $\theta \in S^1$ be a continuous closed curve in $L(\Sigma)$. Curve Λ_θ represents some element d of the fundamental group of manifold $L(\Sigma)$, $d \in \mathbf{Z}$. The number d is called the Maslov index of curve Λ_θ and is denoted by $d = \operatorname{Ind} \Lambda_\theta$.

IV. Maslov Index. Definition. Let $\Lambda_1, \Lambda_2, \Lambda_3 \in L(\Sigma)$, Maslov index $\mu(\Lambda_1, \Lambda_2, \Lambda_3)$ is defined to be the signature of the quadratic form q on vector space $\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$ defined by the formula

$$q(\lambda_1, \lambda_2, \lambda_3) = \sigma(\lambda_1, \lambda_2) + \sigma(\lambda_2, \lambda_3) + \sigma(\lambda_3, \lambda_1), \ \lambda_i \in \Lambda_i, \quad i = 1, 2, 3.$$

Proposition A4 (the most important properties of the Maslov index).

1) Index $\mu(\Lambda_1, \Lambda_2, \Lambda_3)$ is antisymmetric with respect to all arguments.

2) The following identity is true (chain rule):

$$\mu(\Lambda_2, \Lambda_3, \Lambda_4) - \mu(\Lambda_1, \Lambda_3 \Lambda_4) + \mu(\Lambda_1, \Lambda_2, \Lambda_4) - \mu(\Lambda_1, \Lambda_2, \Lambda_3) = 0$$
$$\forall \Lambda_i \in L(\Sigma), \quad i = 1, \ldots, 4.^*$$

3) For any $\Lambda_i \in L(\Sigma)$, i = 1, 2, 3 and any subspace $\Gamma \subset \Lambda_1 \cap \Lambda_2 + \Lambda_2 \cap \Lambda_3 + \Lambda_3 \cap \Lambda_1$ the identity

$$\mu(\Lambda_1^{\Gamma}, \Lambda_2^{\Gamma}, \Lambda_3^{\Gamma}) = \mu(\Lambda_1, \Lambda_2, \Lambda_3)$$

is true.

For the proof of properties 2) and 3) see [13, pp. 32—34]; property 1) is obvious.

We give another definition of the Maslov index, precisely this one is used in the main text.

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*Property 1) means that μ is a "2-cochain" on $L(\Sigma)$ and property 2) means that μ is a "cocycle" on $L(\Sigma)$.

Definition. Let $\Lambda_i \in L(\Sigma)$, $i = 1, 2, 3$. Arbitrary vector $\lambda \in \Lambda_2 \cap (\Lambda_1 + \Lambda_3)$ can be represented in the form of $\lambda = \lambda_1 + \lambda_3$, where $\lambda_1 \in \Lambda_1$, $\lambda_3 \in \Lambda_3$. Set $q(\lambda) = \sigma(\lambda_1, \lambda_3)$. It is easy to see that correspondence $\lambda \to q(\lambda)$ unambiguously defines a quadratic form on $\Lambda_2 \cap (\Lambda_1 + \Lambda_3)$, i.e., expression $\sigma(\lambda_1, \lambda_3)$ depends only on the sum $\lambda_1 + \lambda_3$ whenever λ_1 and λ_3 lie in fixed Lagrangian planes. Maslov index $\mu(\Lambda_1, \Lambda_2, \Lambda_3)$ is defined to be the signature of quadratic form q on $\Lambda_2 \cap (\Lambda_1 + \Lambda_3)$. Note that $\ker q = \Lambda_1 \cap \Lambda_2 + \Lambda_2 \cap \Lambda_3$.

The equivalence of the two definitions of the Maslov index when $\Lambda_1 \cap \Lambda_3 = 0$ is proved in [13, p. 31]; the general case is reduced to this one with the help of assertion 3) of Proposition A4. The second definition of the Maslov index implies the estimate

$$|\mu(\Lambda_1, \Lambda_2, \Lambda_3)| \leqslant m - \dim(\Lambda_1 \cap \Lambda_2 + \Lambda_2 \cap \Lambda_3)\ \forall \Lambda_i \in L(\Sigma).$$

Proposition A5. Let $\Lambda_1, \Lambda_2 \in L(\Sigma)$. Then for any integer k satisfying the inequality $|k| \leq m - \dim(\Lambda_1 \cap \Lambda_2)$ there exists a Lagrangian plane Δ transversal to Λ_1 and Λ_2 such that $\mu(\Lambda_1, \Delta, \Lambda_2) = k$.

The proof uses the description of a set of Lagrangian planes transversal to a given one which is given in subsection 1) as well as the antisymmetry of the Maslov index.

Proposition A6. Let Λ_t, $t \in [t_1, t_2]$ be a continuous closed curve in $L(\Sigma)$, $\Lambda_{t_2} = \Lambda_{t_1}$. Suppose that, in addition, we are given points $t_1 = \tau_0 < \tau_1 < \ldots < \tau_N < \tau_{N+1} = t_2$ and Lagrangian planes $\Delta_0, \Delta_1, \ldots, \Delta_N$ such that $\Delta_i \cap \Lambda_\tau = 0$ for $\tau_i \leq \tau \leq \tau_{i+1}$, $i = 0, \ldots, N$.

Then $\forall \Pi \in \Lambda(\Sigma)$ equality

$$2 \operatorname{Ind} \Delta_\cdot = \sum_{i=0}^{N} \mu(\Pi, \Delta_i, \Lambda_{\tau_{i+1}}) - \mu(\Pi, \Delta_i, \Lambda_{\tau_i})$$

is true.

The proof follows from the results of Sec. 1.9 in [13] (see, in particular, Proposition 1.9.5).

V. Unitary Model. Consider space C^m with the Hermitian product $\langle v, w \rangle = \sum_{i=1}^{m} v_i \bar{w}_i$. The bilinear form $\operatorname{Im} \langle v, w \rangle$ specifies a symplectic structure on C^m while $\operatorname{Re} \langle v, w \rangle$ specifies a Euclidean structure. Denote by Λ_R the set of all real vectors in C^m; clearly, Λ_R and also $i\Lambda_R$ are Lagrangian subspaces.

Any unitary transformation preserving the Hermitian structure also preserves the symplectic structure and, consequently, carries Lagrangian planes into Lagrangian planes. Thus, group U(n) acts on the manifold of Lagrangian planes $L(C^m)$. In fact, this action is transitive. Indeed, suppose that $\Lambda \in L(C^m)$, and let $e_1, \ldots, e_m$ be a basis in Λ orthonormal in the sense of the Euclidean structure $\operatorname{Re}\langle \cdot, \cdot \rangle$, i.e., $\operatorname{Re}\langle e_i, e_j \rangle = \delta_{ij}$, $i, j = 1, \ldots, m$.

Since $\operatorname{Im}\langle e_i, e_j \rangle = 0$, $i, j = 1, \ldots, m$ (Λ is an isotropic plane), basis $e_1, \ldots, e_m$ is orthonormal also in the sense of the Hermitian structure, $\langle e_i, e_j \rangle = \delta_{ij}$, $i, j = 1, \ldots, m$. Therefore, there exists a unitary transformation U that carries the standard basis into the basis $e_1, \ldots, e_m$. Clearly, $U\Lambda_R = \Lambda$. Furthermore, the unitary transformation $U: C^m \to C^m$ carries Λ_R into itself if and only if all the elements of matrix U are real. Real matrices in U(m) form the subgroup O(m). Thus we have established the isomorphism $L(C^m) \approx U(m)/O(m)$.

Let $U \in U(m)$; it is clear that magnitude $\det^2(U)$ depends only on coset U in U(m)/O(m) and, consequently, is unambiguously defined on $L(C^m)$. Here $\det^2(U) \in \{v \in C: |v| = 1\} = S^1$.

Proposition A7. Let Λ_ϑ, $\vartheta \in S^1$ be a continuous closed curve in $L(C^m)$, $\Lambda_\vartheta = U_\vartheta \cdot O(m)$. Then $\operatorname{Ind} \Lambda$ equals the degree of mapping $\vartheta \to \det(U_\vartheta)$ from S^1 into S^1.

Suppose that $\Lambda \in L(\Sigma)$, $\xi \in T_\Lambda L(\Sigma)$, as above, tangent vector ξ is identified with the quadratic form $1/2\sigma(D_\xi \Lambda, \Lambda)$ on Λ (see Proposition A2). The existence of a Euclidean structure allows us to associate with a quadratic form the symmetric operator $s(\xi): \Lambda \to \Lambda$, where

$$\operatorname{Re}\langle s(\xi)\lambda_1, \lambda_2 \rangle = \frac{1}{2}\sigma(D_\xi \lambda_1, \lambda_2)\ \forall \lambda_1, \lambda_2 \in \Lambda.$$

From Proposition A7 we can derive

Proposition A8. Let Λ_θ, $\theta \in S1$ be an absolutely continuous closed curve in $L(C^m)$. Then

$$\operatorname{Ind} \Lambda_{\cdot} = \frac{2}{\pi} \int_{S^1} \operatorname{tr} s\left(\frac{d\Lambda_\theta}{d\theta}\right) d\theta.$$

COROLLARY. Suppose that h_τ, $\tau \in [0, t]$ is a nonstationary quadratic Hamiltonian on Σ and $H_\tau\colon \Sigma \to \Sigma$ is a Hamiltonian flux. Let $\Lambda_0 \in L(\Sigma)$ be such that curve $\Lambda_\tau = H_\tau \Lambda_0$, $\tau \in [0, t)$ in $L(\Sigma)$ satisfies the condition $\Lambda_t = \Lambda_0$. Then

$$\operatorname{Ind} \Lambda_{\cdot} = \frac{2}{\pi} \int_0^t \operatorname{tr} s\,(h_\tau \mid \Lambda_\tau)\, d\tau.$$

4. HOMOLOGY INVARIANTS OF THE SECOND VARIATION

In the present section we describe some invariants of a family of curves on a Lagrange Grassmanian and with their help we study the second variation of a controlled system.

1. In this subsection we use notation from subsection 1 of Sec. 3. Recall that G_t'' is the quadratic mapping of space $\ker G_t'$ into the k-dimensional vector space $\operatorname{coker} G_t' = (\Pi^\perp)^*$. Set $\Psi = \{\psi \in \Pi^\perp \mid \operatorname{ind} \psi G_t'' < +\infty\}$. Set Ψ is obviously a convex but, generally speaking, not a closed cone.

The next rule associates with each $\tau \in (0, t]$ an integer nonnegative $l_\tau \le m$: if quadratic mapping $v \to h_\theta(v, v)$, $v \in R^r$ does not equal identically zero on any interval $\bar{\tau} < \theta < \tau$, then we set $l_\tau = 0$; otherwise, let l_τ be a maximal (among the numbers 1, 2, ..., m) number l such that $[\mathfrak{z}_\theta^{(i)} v_1, \mathfrak{z}_\theta v_2] \equiv 0$ for $i < 2(l - 1)$, $v_1, v_2 \in R^r$ on some interval $\bar{\tau} < \theta < \tau$.

Proposition 3.1 directly implies the following

Proposition 1. Set $\Psi \subset \Pi^\perp$ is contained in the intersection of subspace

$$\left\{\left[\mathfrak{z}_\tau^{(l_\tau-1)} v_1, \mathfrak{z}_\tau^{(l_\tau-1)} v_2\right] \mid 0 < \tau \leqslant t,\ v_1, v_2 \in R^r\right\}^\perp \tag{1}$$

and the convex closed cone

$$\left\{\left[\mathfrak{z}_\tau^{(l_\tau-1)} v, \mathfrak{z}_\tau^{(l_\tau)} v\right] \mid 0 < \tau \leqslant t,\ v \in R^r\right\}^\circ. \tag{2}$$

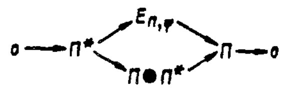
If, in addition, the closure of set $\{[\mathfrak{z}_\tau^{(l_\tau)} v, \mathfrak{z}_\tau^{(l_\tau-1)}] \mid 0 < \tau < t, |v| = 1\}$ does not intersect with subspace (1), then the interior of set Ψ relative to set (1) coincides with the interior of the intersection of (1) and (2) relative to (1).

Remark. Proposition 3.3 enables us to substantially refine the description of cone Ψ given in Proposition 1; moreover, we can isolate not only the interior but also the boundary points of Ψ relative to (1). We shall not, however, dwell on this.

In subsection 1 of Sec. 3 we put in correspondence to each $\psi \in \Pi^\perp \setminus 0$ a symplectic space $E_{\Pi,\psi}$ and the (natural) exact sequence

$$0 \to \Pi^* \to E_{\Pi,\psi} \to \Pi \to 0.$$

The introduction of local coordinates in the neighborhood of point μ_0 leads to an isomorphism of $E_{\Pi,\psi}$ and space $\Pi \oplus \Pi^*$ with the standard symplectic structure $\sigma(x_1 \oplus \xi_1, x_2 \oplus \xi_2) = \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle$, $x_i \in \Pi$, $\xi_i \in \Pi^*$; moreover, diagram



is commutative (the lower arrows denote the imbedding of Π^* into $\Pi \oplus \Pi^*$ as a second term and a coordinate projection of $\Pi \oplus \Pi^*$ onto the first term). This isomorphism is not natural and depends on the choice of local coordinates. However, to construct such an isomorphism we need much less than the introduction of local coordinates. The following assertion describes the situation exactly. (Recall $\Pi^* = T_{\mu_0}{}^*M/\Pi^\perp$.)

LEMMA 1. Let $\Phi: \Pi^\perp \to \mathrm{Der}^* M$ be a linear mapping such that for every $\psi \in \Pi^\perp$ the relations $\mu_0 \cdot \Phi\psi = \Psi$, $\mu_0 \cdot d(\Phi\psi) \perp \Pi \wedge \Pi$ are fulfilled. Then for every $\psi \in \Pi^\perp \backslash 0$ mapping

$$X \mapsto \mu_0 \circ X \oplus (\mu_0 \circ L_X \Phi\psi + \Pi^\perp), \quad X \in \mathcal{F}_\Pi$$

induces an isomorphism of the symplectic spaces

$$S_\psi : E_{\Pi,\psi} \to \Pi \oplus \Pi^*.$$

The proof is a direct calculation.

Now fix once and for all mapping Φ that satisfies the hypothesis of Lemma 1.

Let $\psi \in \Psi\backslash 0$ be such that the Jacobian curve $\Lambda_\tau \in L(E_{\Pi,\psi})$, $\tau \in [0, t]$ corresponding to form $\psi G_t''$ is defined (the most general conditions for the existence of the Jacobian curve on the whole interval [0, t] are given in Proposition 3.3). Set $\Lambda_\tau(\psi) = S_\psi \Lambda_\tau \subset \Pi \oplus \Pi^*$, $\tau \in [0, t]$. Then $\Lambda_\tau(\psi)$ is a nondecreasing curve in $L(\Pi \oplus \Pi^*)$, $\Lambda_0(\psi) = \Pi^* \in L(\Pi \oplus \Pi^*)$. The results of Sec. 3 enable us to calculate $\mathrm{ind}\, \psi G_t''$ in terms of curve $\Lambda_\tau(\psi)$, $0 \le \tau \le t$.

2. The investigation of quadratic mappings on R^{N+1} conducted in [1] relied on objects connected with a space of quadratic forms, namely the index of a form and classes γ_n. Now it is evident that during the investigation of an integral quadratic mapping G_t'' the role analogous to the space of quadratic forms is played by the set of all nondecreasing curves on $L(\Pi \oplus \Pi^*)$ beginning in Π^*. In this subsection we give a definition of the index for these curves and introduce analogs to classes γ_n. Really, the definition of the index is suggested by the results of Sec. 3.

Definition. Suppose that Λ_τ, $0 \le \tau \le t$ is a nondecreasing piecewise smooth curve in $\Pi \oplus \Pi^*$, $\Lambda_0 = \Pi^*$, and $\tau_{l+1} = 0 = \tau_0 < \ldots < \tau_l = t$ is a partition of interval [0, t] such that curves $\Lambda.\big|_{[\tau_i, \tau_{i+1}]}$, $i = 0, 1, \ldots, l-1$ are simple. Set

$$\mathrm{ind}\, \Lambda. = \sum_{i=0}^{l} \mathrm{ind}_{\Lambda_0}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}}) + \dim\Big(\bigcap_{0<\tau<t} \Lambda_\tau\Big) - m.$$

Value $\mathrm{ind}\, \Lambda.$ is a nonnegative integer and, according to the results of Sec. 3, does not depend on the choice of the partition of interval [0, t]. Moreover, if at each point of discontinuity $\overline{\tau}$ of curve $\Lambda.$ we introduce a simple nondecreasing curve joining $\Lambda_{\overline{\tau}}$ with $\Lambda_{\overline{\tau}+0}$ and, in addition, we join by a simple nondecreasing curve Λ_t to Λ_0, then for the nondecreasing continuous closed curve $\overline{\Lambda}$ obtained as a result, the following identity holds:

$$\mathrm{ind}\, \Lambda. = \mathrm{ind}\, \overline{\Lambda}. = \mathrm{Ind}\, \Lambda. + \dim\Big(\bigcap_{0<\tau<t} \Lambda_\tau\Big) - m. \tag{3}$$

Recall that Ind denotes the Maslov index of a closed curve. Identity (3) follows from Proposition 3.2 (see also Theorem 3.3).

In the sequel we assume that some complex structure and Hermitian form $\langle\cdot,\cdot\rangle$ are fixed in the space $\sigma(x_1 \oplus \xi_1, x_2 \oplus \xi_2) = \mathrm{Im}\langle x_1 \oplus \xi_1, x_2 \oplus \xi_2\rangle$. In the Appendix to Sec. 3 we have described an identification of the tangent space $T_\Lambda L(\Pi \oplus \Pi^*)$ with space $\mathcal{P}(\Lambda)$ of quadratic (= symmetric bilinear) forms on Λ. Below, this identification is used without special stipulations. To each form $q \in \mathcal{P}(\Lambda) = T_\Lambda L(\Pi \oplus \Pi^*)$ there corresponds the symmetric linear operator $s(q): \Lambda \to \Lambda$, where $q(\lambda_1, \lambda_2) = \mathrm{Re}\langle s(q)\lambda_1, \lambda_2\rangle$ $\forall \lambda_1, \lambda_2 \in \Lambda$. We give a Riemannian structure on $L(\Pi \oplus \Pi^*)$ by defining the inner product of a pair of tangent vectors $q_1, q_2 \in T_\Lambda L(\Pi \oplus \Pi^*)$ by the formula $(q_1, q_2) \to \mathrm{tr}(s(q_1)s(q_2))$. The length of the arbitrary piecewise smooth curve $\Lambda_\tau \in L(\Pi \oplus \Pi^*)$, $0 \le \tau \le t$ is denoted by the symbol $\rho(\Lambda.)$,

$$\rho(\Lambda_{\cdot})=\int_0^t \sqrt{\operatorname{tr}\left(s\left(\frac{d\Lambda}{d\tau}\right)^2\right)}\,d\tau$$

Proposition 2. Let Λ_τ, $0 \le \tau \le t$ be a nondecreasing piecewise smooth curve in $\Pi \oplus \Pi^*$, $\Lambda_0 = \Pi^*$, having at most N points of discontinuity. Set $\nu = m - \dim(\bigcap_{0<\tau<t} \Lambda_\tau)$. Then

$$\frac{2}{\pi}\rho(\Lambda_{\cdot}) - \nu \leqslant (\operatorname{ind}\Lambda_{\cdot}) \leqslant \frac{2}{\pi}\sqrt{\nu}\,\rho(\Lambda_{\cdot}) + \nu N.$$

Proof. Since Λ_τ is a nondecreasing curve, its velocity $\dot{\Lambda}_\tau \in T_\Lambda L(\Pi \oplus \Pi^*)$ is a nonnegative quadratic form and $s(\dot{\Lambda}_\tau)$ is a nonnegative symmetric operator; moreover, $\operatorname{rank} s(\dot{\Lambda}_\tau) \le \nu$. Therefore, $\operatorname{tr}(s(\dot{\Lambda}_\tau)^2) \le (\operatorname{tr} s(\dot{\Lambda}_\tau))^2 \le \nu \operatorname{tr}(s(\dot{\Lambda}_\tau))^2$. If $\Lambda_{\cdot}$ is a continuous closed curve, then the inequalities being proved below follow from proposition A8 of the Appendix to Sec. 3 and from equality (3). In the general case we need to use also the fact that any pair of Lagrangian planes Δ_1, Δ_2 lies on some nondecreasing continuous closed curve Δ_θ, $\theta \in S^1$ satisfying the condition $\operatorname{Ind}\Delta_\theta = m - \dim(\Delta_1 \cap \Delta_2)$. This fact is established very simply: that at least some pair of Lagrangian curves lies on such a curve is obvious; at the same time, a symplectic group acts transitively on a pair of Lagrangian planes with a fixed intersection dimension.

Denote by $\mathfrak{L}[0, t]$ the space of all piecewise smooth monotone curves $\Lambda_\tau \in \mathfrak{L}(\Pi \oplus \Pi^*)$, $\Lambda_0 = \Pi^*$, *continuous on the half-interval* $(0, t]$ with topology of uniform convergence on each segment $[\varepsilon, t]$ where $0 < \varepsilon \le t$. Discontinuity at zero moment is admissible, i.e., $\Pi^* = \Lambda_0 \ne \Lambda_{+0}$.

By $\mathfrak{L}^s[0, t]$ we denote a subset in $\mathfrak{L}[0, t]$ consisting of all such curves $\Lambda_{\cdot} \in \mathfrak{L}[0, t]$, for which there is a partition $0 = \tau_0 < \tau_1 < \ldots < \tau_{l+1} = t$ of interval $[0, t]$ with the following properties: a) curves $\Lambda_{\cdot}|_{[\tau_i,\tau_{i+1}]}$ are simple for $i = 0, \ldots, l+1$; b) $\Lambda_{\tau_i} \cup \Pi^* = 0$, $i = 1, \ldots, l$, $\Lambda_\theta \cup \Pi^* = 0$ on some interval $\tau < \theta < t$.

Remark. Subset $\mathfrak{L}[0, t]\setminus\mathfrak{L}^s[0, t]$ has an infinite codimension in $\mathfrak{L}[0, t]$, i.e., for an arbitrary finite-dimensional manifold U and space $C(U, \mathfrak{L}[0, t])$ of all continuous mappings from U to $\mathfrak{L}[0, t]$, subspace $C(U, \mathfrak{L}^s[0, t])$ is an everywhere compact subset $C(U, \mathfrak{L}^s[0, t]) = C(U, \mathfrak{L}[0, t])$.

Partitions of interval $[0, t]$ satisfying conditions a) and b) are said to be compatible with a given curve Λ.

Let $\Lambda_{\cdot} \in \mathfrak{L}^s[0, t]$ and $0 = \tau_0 < \tau_1 < \ldots < \tau_{l+1} = t$ be a partition compatible with this curve. Denote $D_i = [\tau_i, \tau_{i+1}]$, $D = \{\tau_1, \ldots, \tau_l\}$ and set

$$K_{D_i}(\Lambda) = \sum_{\tau_i<\tau<\tau_{i+1}} (\Pi^* \cap \Lambda_\tau) \subset \Pi^*, \quad i = 1, \ldots, l,$$

$$K_D(\Lambda) = \bigoplus_{i=1}^{l} K_{D_i}(\Lambda).$$

Suppose, finally, that $\mathfrak{L}^s(D)$ is a subset in $\mathfrak{L}^s[0, t]$ consisting of all curves with which a given partition D is compatible. It is easy to see that $\mathfrak{L}^s(D)$ is an open subset in $\mathfrak{L}^s[0, t]$.

Sets of the level of the integer-valued function ind determine a partition of space $\mathfrak{L}^s[0, t]$: set $\mathfrak{L}^s_n[0, t] = \{\Lambda_{\cdot} \in \mathfrak{L}^s[0, t] \mid \operatorname{ind}\Lambda_{\cdot} = n\}$. Correspondingly,

$$\mathfrak{L}^s_n(D) \overset{\text{def}}{=} \mathfrak{L}^s_n[0, t] \cap \mathfrak{L}^s(D).$$

LEMMA 2. Let $n \ge 0$, $1 \le i \le l$. Then

a) $\dim K_{D_i}(\Lambda) = \operatorname{ind}_{\Pi^*}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}}) - 1/2 \dim(\Lambda_{\tau_i} \cap \Pi^*)$,†

b) subspace $K_{D_i}(\Lambda_{\cdot})$ depends continuously on $\Lambda \in L_n^s(D)$,

c) if some subpartition $\tau_i = \tau_{i_0} < \tau_{i_1} < \ldots < \tau_{i(l_i+1)} = \tau_{i+1}$ of segment D_i is such that $\Pi^* \cap \Lambda_{\tau_{ij}} = 0$ for $j = 0, 1, \ldots, l_i$, then $K_{D_i} = \bigoplus_{j=0}^{l_i} K_{D_{ij}}$, where $D_{ij} = [\tau_{ij}, \tau_{i(j+1)}]$.

†The space $\Lambda_{\tau_i} \cap \Pi^*$ can differ from zero only for $i = l$.

Proof. a) Let T be a Lagrangian plane such that $T \cup \Pi^* = T \cup \Lambda_\tau = 0$ for $\tau_i \le \tau \le \tau_{i+1}$. When proving Proposition 3.2, we have established that

$$2\,\mathrm{ind}_{\Pi^*}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}}) = \mu(\Lambda_{\tau_{i+1}}, \Pi^*, T) - \mu(\Lambda_{\tau_i}, \Pi^*, T),$$

where $\mu(\cdot,\cdot,\cdot)$ is the Maslov index of the triple of Lagrangian planes. Suppose that $p_\tau: \Pi \oplus \Pi^* \to \Lambda_\tau$ is the projection operator of space $\Pi \oplus \Pi^*$ parallel to T onto Λ_τ (that is, $\ker p_\tau = T$). Then $\mu(\Lambda_\tau, \Pi^*, T)$ coincides with the signature of the quadratic form $\xi \to \sigma(p_\tau\xi, \xi)$, $\xi \in \Pi^*$. The kernel of this quadratic form coincides with $\Pi^* \cap \Lambda_\tau$. At the same time, for every $\xi \in \Lambda_\tau \cap \Pi^*$ we have $(d/d\tau)\sigma(p_\tau\xi, \xi) \ge 0$ since Λ_τ is a nondecreasing curve. Consequently,

$$K_{D_i}(\Lambda) = \mathrm{span}\{\xi \in \Pi^* \mid \sigma(p_{\tau_i}\xi, \xi) < 0,\ \sigma(p_{\tau_{i+1}}\xi, \xi) > 0\}$$

and

$$\dim K_{D_i}(\Lambda) = \frac{1}{2}(\mu(\Lambda_{\tau_{i+1}}, \Pi^*, T) - \mu(\Lambda_{\tau_i}, \Pi^*, T) - \dim(\Lambda_{\tau_{i+1}} \cap \Pi^*))$$

(Recall that certainly $\Lambda_{\tau_i} \cap \Pi^* = 0$.)

b) In a) it was implied that $\dim K_{D_i}(\Lambda)$ is locally constant in $\mathfrak{L}^n_s(D)$. Since $K_{D_i}(\Lambda)$ obviously depends on Λ in an upper semicontinuous manner, the fact that the dimension is locally constant implies continuous dependency.

c) By definition $K_{D_i}(\Lambda) = \sum_{j=0}^{l_i} K_{D_{ij}}(\Lambda)$, on the other hand, Proposition 3.2 implies that

$$\mathrm{ind}_{\Pi^*}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}}) = \sum_{j=0}^{l} \mathrm{ind}_{\Pi^*}(\Lambda_{\tau_{ij}}, \Lambda_{\tau_{ij+1}})$$

and, according to assertion a), $\dim K_{D_i}(\Lambda) = \sum_{j=0}^{l_i} \dim K_{D_{ij}}(\Lambda)$. Consequently, the sum $\sum_{j=0}^{l_i} K_{D_{ij}}$ is direct. ■

Let $\Lambda. \in \mathfrak{L}^s(D)$, and set $K_D(\Lambda) = \bigoplus_{i=1}^{l} K_{D_i}(\Lambda)$. According to assertion a) of Lemma 2, $\dim K_D(\Lambda) = \sum_{i=1}^{p} \mathrm{ind}_{\Pi^*}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}})$. Furthermore, since $\Lambda_{\tau_i} \cap \Pi^* = 0$, we have $\mathrm{ind}_{\Pi^*}(\Pi^*, \Lambda_{\tau_1}) = \frac{1}{2}m$, $\bigcap_{0<\tau<t} \Lambda_\tau = 0$; $\mathrm{ind}_{\Pi^*}(\Lambda_t, \Pi^*) = \frac{1}{2}(m - \dim(\Lambda_t \cap \Pi^*))$. Consequently, $\dim K_D(\Lambda) = \mathrm{ind}\,\Lambda$.

Assertion b) of Lemma 2 implies that the family of linear spaces $K_D(\Lambda.)$, $\Lambda. \in \mathfrak{L}_n^s(D)$ forms an n-dimensional vector bundle over $\mathfrak{L}_s^n(D)$ which we denote by $\mathscr{K}_n(D)$. At the same time, for a variable D sets $\mathfrak{L}_s^n(D)$ form an open covering of space $\mathfrak{L}_n^s[0, t]$. For arbitrary $D_i' = \{\tau_1', \dots, \tau_{l'}'\}$ and $D_i'' = \{\tau_1'', \dots, \tau_{l''}''\}$ we have

$$\mathfrak{L}^s_n(D') \cap \mathfrak{L}^s_n(D''),\quad \mathfrak{L}^s_n(D' \cup D'').$$

Suppose that $D_i' = [\tau_i', \tau_{i+1}']$, $D_i'' = [\tau_i'', \tau_{i+1}'']$, and D_j is the j-th segment of bundle $D' \cup D''$. In such a case, each segment D_j is contained in some segment $D_{\alpha j}'$ and in some segment $D_{\beta j}''$. Assertion c) of Lemma 2 implies that inclusions

$$K_{D_j}(\Lambda.) \subset K_{D'_{\alpha_j}}(\Lambda.),\quad K_{D_j}(\Lambda.) \subset K_{D''_{\beta_j}}(\Lambda.)$$

define the canonical isomorphisms

$$K_D(\Lambda.) \approx K_{D'}(\Lambda.),\quad K_D(\Lambda.) \approx K_{D''}(\Lambda.),\quad \forall \Lambda. \in \mathfrak{L}^s_n(D') \cap \mathfrak{L}^s_n(D'').$$

As a result, for various D bundles, $\mathscr{K}_n(D)$ turn out to be coalesced into one vector bundle $\mathscr{K}_n$ over $\mathfrak{L}_n^s[0, t]$.

Let $\kappa_n \in \check{H}^1(\mathfrak{L}_n^s[0, t])$ be a one-dimensional Stiefel—Whitney class of bundle $\mathcal{K}_n$, i.e., $\kappa_n = w_1(\mathcal{K}_n)$.

3. Let us return to the investigation of the quadratic mapping G_t''. In subsection 1 we defined cone $\Psi \subset \Pi^\perp$. Denote by Ψ^s a subset in Ψ consisting of all ψ for which the Jacobian curve $\Lambda.(\psi) \in \mathfrak{L}^s[0, t]$ is defined. In addition, set

$$\Psi_n = \{\psi \in \Psi \setminus 0 \mid \operatorname{ind} \psi G_t'' = n\}, \quad \Psi_n^s = \{\psi \in \Psi^s \mid \operatorname{ind} \Lambda.(\psi) = n\}.$$

Theorem 3.1 implies that

$$\operatorname{ind} \psi G_t'' = \operatorname{ind} \Lambda.(\psi), \quad \forall \psi \in \Psi^s,$$

therefore $\Psi_n^s \subset \Psi$.

Quadratic mapping $\mathfrak{g}_t''$ is defined on the subspace $\overline{\ker}\ \mathfrak{g}_t'$ of the Banach space $L_\infty^r[0, t]$. At the same time, $L_\infty^r[0, t]$ is an everywhere compact linear subspace of the Hilbert space $L_2^r[0, t]$. It is easy to see that G_t'' is continuous in the topology of space $L_2^r[0, t]$; this follows directly from formula (3.1). Denote by $\overline{\ker} G_t'$ the closure of space ker G_t' in $L_2^r[0, t]$,

$$\overline{\ker}\, G_t' = \left\{ v(\cdot) \in L_2^r[0, t] \mid \mu_0 \circ \int_0^t Z_\tau v(\tau)\, d\tau = 0 \right\},$$

and by $\mathfrak{g}_t$ the extension of mapping G_t'' to $\overline{\ker}\, G_t'$ with respect to continuity. Mapping $\mathfrak{g}_t$, like G_t'', is defined by formula (3.1):

$$\mathfrak{g}_t(v(\cdot), v(\cdot)) = \int_0^t \left(h_\tau(v(\tau), v(\tau)) + \left[\int_0^\tau \mathfrak{z}_\theta v(\theta)\, d\theta,\ \mathfrak{z}_\tau v(\tau) \right] \right) d\tau,$$
$$v(\cdot) \in \overline{\ker}\, G_t'.$$

Clearly, $\operatorname{ind} \psi\mathfrak{g} = \operatorname{ind} \psi\, \mathfrak{g}_t''\ \forall \psi \in \Pi^\perp$.

Thus, $\mathfrak{g}_t$ is a quadratic mapping of an infinite-dimensional separable Hilbert space $\overline{\ker} G_t'$ into the k-dimensional space $\operatorname{coker} G_t' = T_{\mu_0}M/\Pi$. Subsection 7 in [1, Sec. 2] is devoted to such mappings. To each $\psi \in \Psi_n$ an n-dimensional vector space $L_\psi \subset \overline{\ker}\, G_t'$ corresponds — an n-dimensional invariant subspace of a self-adjoint operator definable by the quadratic form $\psi\mathfrak{g}$ such that $\psi G_t'' \big|_{L_\psi} < 0$. The family of spaces L_ψ, $\psi \in \Psi_n$ forms the vector bundle $\mathcal{L}_n$ over Ψ_n. By $\pi_n \in H^1(\Psi_n)$ we denote the one-dimensional Stiefel—Whitney class of bundle $\mathcal{L}_n$, i.e., $\pi_n = w_1(\mathcal{L}_n)$.

Denote by $J: \Psi^s \to \mathfrak{L}^s[0, t]$ the mapping $\psi \to \Lambda.(\psi)$, $\psi \in \Psi^s$, and by $J_n: \Psi_n^s \to \mathfrak{L}_n^s[0, t]$ the restriction of mapping J to Ψ_n^s, i.e., $J_n = J \mid \Psi_n^s$.

Proposition 3. Mapping J_n continuously restricts J to Ψ_n^s for every $n \geq 0$.

Proof. Let $\psi \in \Psi_n^s$; according to definitions from Sec. 3, the Jacobian curve $\Lambda_\tau(\psi)$, $0 \leq \tau \leq t$ is an integral curve of a nonstationary vector field on $L(\Pi \oplus \Pi^*)$. Recall the definition of this field; here, for simplicity's sake, we limit ourselves to the case of the piecewise smooth function and one parameter (r = 1). In subsection 1 of Sec. 3 the piecewise smooth function ψh_τ and the piecewise smooth curve $z_\tau(\psi) \in E_{\Pi,\psi}$, $0 \leq \tau \leq t$ were derived. Since throughout this section isomorphism $S_\psi: E_{\Pi,\psi} \to \Pi \oplus \Pi^*$ is fixed, we will identify curve $z_\tau(\psi)$ with the image of this curve under mapping S_ψ.

With each $\tau \in (0, t)$ we associate an integer $k_\tau(\psi) \geq 0$ and a real number $\gamma_\tau(\psi\_$: if $\psi h_\tau \neq 0$, then $k_\tau(\psi) = 0$, $\gamma_\tau(\psi) = \psi h_\tau$; otherwise, $k_\tau(\psi)$ is minimal among numbers k satisfying the condition

$$\sigma(z_\tau^{(k)}(\psi), z_\tau^{(k-1)}(\psi)) \neq 0, \quad \gamma_\tau(\psi) = \sigma(z_\tau^{(k_\tau(\psi))}(\psi), z_\tau^{(k_\tau(\psi)-1)}(\psi)).$$

The condition $\psi \in \Psi^s$ (which guarantees the absence of singularities in the Jacobi equation) implies that $k_\tau \leq m$ and $k_\tau(\psi)$ is locally constant at any point where function ψh_τ and curve $z_\tau(\psi)$, $0 < \tau \leq t$ are smooth; $\gamma_\tau \geq \varepsilon$ for some $\varepsilon > 0$ and all $\tau \in (0, t]$. We set $\Gamma_\tau(\psi) = \operatorname{span}\{z_\tau^{(k)}(\psi) \mid 0 \leq i \leq k(\psi) - 1\}$ to be a k_τ-dimensional isotropic subspace in $\Pi \oplus \Pi^*$, depending piecewise smoothly on τ. Let $\Lambda \in L(\Pi \oplus \Pi^*)$, as in Sec. 3 (see, in particular, the Appendix to that section); the quadratic form $\lambda \to \sigma(z_\tau^{(i)}(\psi), \lambda)^2$, $\lambda \in \Lambda$, and also the element of space $T_\Lambda L(\Pi \oplus \Pi^*)$ corresponding to this form are denoted by the symbol $\sigma(z_\tau^{(i)}(\psi), \Lambda)^2$. The Jacobian curve $\Lambda_\tau(\psi)$ is a continuous on (0, t] solution of the differential equation

$$\frac{d}{d\tau}\Lambda = \frac{1}{2\gamma_\tau(\psi)}\sigma(z_\tau^{(k_\tau(\psi))}(\psi), \Lambda)^2$$

with the initial condition $\Lambda_{+0}(\psi) = \Pi^{*\Gamma_{+0}(\psi)}$.

It is easy to see that $k_\tau(\psi)$ depends upper semicontinuously on ψ. If ψ is such that $k_\tau(\psi') = k_\tau(\psi)$ for all $\tau \in [0, t]$ and $\psi' \in \Psi^S$ sufficiently close to ψ, then the continuity of mapping J at point ψ is the consequence of the standard theorem on the continuous dependence of solutions of differential equations on parameters. If, however, for some sequence $\psi_i \in \Psi^s$, $i = 1, 2, \dots$, $\psi_i \to \psi$ $(i \to \infty)$ inequalities $k_\tau(\psi_i) < k_\tau(\psi)$ are fulfilled, then $\gamma_\tau(\psi_i) \to 0$ $(i \to \infty)$, and the question of the continuous dependence of solutions on a parameter becomes far from trivial. Furthermore, mapping J can turn out to be discontinuous at point ψ; only mappings $J_n = J \mid \Psi_n^s$ are continuous.

Thus, let $\psi \in \Psi_n^s$, $\psi_i \to \psi$ $(i \to \infty)$ and $k_\tau(\psi_i) < k_\tau(\psi)$ for some $\tau \in (0, t]$. The definition of vectors $z_\tau(\psi)$ implies the existence of partition $0 = \tau_0 < \tau_1 < \dots < \tau_{N+1} = t$ of segment [0, t] such that all integer-valued functions $\tau \to k_\tau(\psi_i)$, $i = 1, 2, \dots$ are locally constant on the set $(0, t]\setminus\{\tau_1, \dots, \tau_N\}$. Since all the considerations can be fully conducted separately on each segment $[\tau_j, \tau_{j+1}]$, $j = 0, 1, \dots, N$, without loss of generality (but simplifying the notation) we can assume that $k_\tau(\psi_i)$ and $k_\tau(\psi)$ do not depend on τ, i.e., $k_\tau(\psi_i) = k(\psi_i)$, $0 < \tau \le t$, $i = 1, 2, \dots$, $k_\tau(\psi) = k(\psi)$. Furthermore, passing, if need be, to subsequences, we can attain the fact that $K_\tau(\psi_i)$ should also not depend on i, $k_\tau(\psi_i) = k$ for $i = 1, 2, \dots$.

Thus, we pass to the following situation. There is a sequence ψ_i, $i = 1, 2, \dots$, $\psi_i \to \psi$ $(i \to \infty)$, such that

$$\frac{d}{d\tau}\Lambda_\tau(\psi_i) = \frac{1}{2\gamma_\tau(\psi_i)}\sigma\left(z_\tau^{(k)}(\psi_i), \Lambda\right)^2, \quad \Lambda_{+0}(\psi_i) = \Pi^{*\Gamma_{+0}(\psi_i)},$$
$$\tau \in (0, t]; \quad \operatorname{ind}\Lambda_.(\psi_i) = n; \quad k < k(\psi). \tag{4}$$

We need to prove that $\Lambda_\tau(\psi_i) \to \Lambda_\tau(\psi)$ $(i \to \infty)$ is uniform on every interval $\tau \in [\overline{\tau}, t]$, $0 < \overline{\tau} \le t$. Proposition 2 implies that the lengths of curves $\Lambda_\tau(\psi)$, $\tau \in [0, t]$ are uniformly bounded in i. We make a change of parameter on curves $\Lambda_\tau(\psi_i)$ by parametrizing these curves with an arc length: let functions $\theta \to \tau_i(\theta)$, $\tau_i(0) = 0$ be such that the velocity of curve $\theta \to \Lambda_{\tau(\theta)}(\psi_i)$ has for any θ a unit length, $i = 1, 2, \dots$.

For integer α, $k \le \alpha$, $\tau \in (0, t]$ we set

$$L_\tau^{(\alpha)} = \{\Lambda \in L(\Pi \oplus \Pi^*) \mid z_\tau^{(j)}(\psi) \in \Lambda,\ k \leqslant j < \alpha\}$$

to be a smooth submanifold in $L(\Pi \oplus \Pi^*)$, diffeomorphic to $L(R^{2(m+k-\alpha)})$ for $k \le \alpha \le k(\psi)$,

$$\varnothing = L_\tau^{(k(\psi)+1)} \subset L_\tau^{(k(\psi))} \subset \dots \subset L_\tau^{(k)} = L(\Pi \oplus \Pi^*).$$

Consider the nonstationary vector field $\sigma(z_\tau^{(k)}(\psi), \Lambda)^2$, $\tau \in (0, t]$, $\Lambda \in L(\Pi \oplus \Pi^*)$. For $\Lambda \in L_\tau^{(\alpha)} \mid L^{(\alpha+1)}$ we have

$$\frac{\partial^j}{\partial\tau^j}\sigma\left(z_\tau^{(k)}(\psi), \Lambda\right)^2 = \begin{cases} 0, & \text{for } j < 2(\alpha - k) \\ 2^{\alpha-k}\sigma\left(z_\tau^{(\alpha)}(\psi), \Lambda\right)^2 \ne 0 & \text{for } j = 2(\alpha-k),\ k \leqslant \alpha. \end{cases} \tag{5}$$

Since $z_\tau^{(j)}(\psi_i)$ converges uniformly to $z_\tau^{(j)}(\psi)$ as $i \to \infty$ for any $j \ge 0$ and functions $\gamma_\tau(\psi_i)$ are uniformly bounded in λ and i [in fact, $\gamma_\tau(\psi_i) \to 0$ as $i \to \infty$], (4) and (5) imply that functions $\tau_i(\theta)$ are uniformly Hölder

$$|\tau_i(\theta_1) - \tau_i(\theta_2)| \leqslant c\,|\theta_1 - \theta_2|^{\frac{1}{(2(k(\psi)-k)+1)}},$$

where c is independent of i and θ_1, θ_2.

Thus, the sequences of curves $\theta \to \Lambda_{\tau_i(\theta)}(\psi_i)$ and scalar functions $\theta \to \tau_i(\theta)$ contain uniformly convergent subsequences. Without loss of generality we can assume that sequences $\Lambda_{\tau_i(\theta)}(\psi_i)$ and $\tau_i(\psi)$ themselves uniformly converge to some curve Λ_θ and scalar functions $\tau(\theta)$, $0 < \theta \leq \hat{\theta}$, $\tau(\hat{\theta}) = t$. Since $\Lambda_{\tau_i(\theta)}(\psi_i)$, $\tau_i(\theta)$ are a nondecreasing curve and function, the same is also true for $\hat{\Lambda}_\theta$, $\tau(\theta)$; in addition, ind $\hat{\Lambda}. \leq n$. We will show that $\Lambda_{\tau(\theta)}(\psi) = \Lambda_\theta$, $0 < \theta \leq \hat{\theta}$; moreover, there is a function $\theta_0 \geq 0$ such that function $\tau(\theta)$ continuously and bijectively maps half-interval $[\theta_0, \hat{\theta}]$ into $(0, t]$.

In the first place, we can derive from (4) and (5) the following fact: if on some interval $(\theta_1, \theta_2) \subset (0, \hat{\theta})$, relations $\hat{\Lambda}_\theta \in L_{\tau(\theta)}^{(\alpha)} \setminus L_{\tau(\theta)}^{(\alpha+1)}$, $\theta_1 < \theta < \theta_2$ are fulfilled, then $d\hat{\Lambda}/d\theta$ is positively proportional to $\sigma(z_{\tau(\theta)}^{(\alpha)}(\psi), \hat{\Lambda}_\theta)^2$, i.e.,

$$\frac{d\hat{\Lambda}_\theta}{d\theta} = u(\theta)\,\sigma\left(z_{\tau(\theta)}^{(\alpha)}(\psi), \hat{\Lambda}_\theta\right)^2, \quad u(\theta) > 0, \quad \theta_1 < \theta < \theta_2.$$

Relation $\hat{\Lambda}_\theta \in L_{\tau(\theta)}^{(\alpha)}$ is equivalent to equality

$$\sigma\left(z_{\tau(\theta)}^{(\alpha-1)}(\psi), \lambda_\theta\right) = 0, \quad \forall \lambda_\theta \in \hat{\Lambda}_\theta, \quad \theta_1 < \theta < \theta_2;$$

the differentiation of this equality with respect to θ (at the points where the derivative exists) leads to the identity

$$\sigma\left(z_{\tau(\theta)}^{(\alpha)}(\psi), \lambda_0\right)\left(\frac{dt}{d\theta} + 2u(\theta)\,\sigma\left(z_{\tau(\theta)}^{(\alpha-1)}(\psi), z_{\tau(\theta)}^{(\alpha)}(\psi)\right)\right) = 0,$$
$$\lambda_\theta \in \hat{\Lambda}_\theta, \quad \theta \in (\theta_1, \theta_2),$$

and the difference analog of this identity (true already at all the points) establishes the absolute continuity of function $\tau(\theta)$ on (θ_1, θ_2). Thus, $d\tau/d\theta = 0$ for $\alpha < k(\psi)$ and $dt/d\theta = u(\theta)\gamma_{\tau(\theta)}(\psi) > 0$ for $\alpha = k(\psi)$.

We see that on any interval satisfying condition $\hat{\Lambda}_\theta \in L_{\tau(\theta)}^{k(\psi)}$, $\theta_1 < \theta < \theta_2$, function $\tau(\theta)$ is invertible and curve $\hat{\Lambda}$ is subject to the equation

$$\frac{d\hat{\Lambda}}{d\tau} = \frac{1}{2\gamma_\tau(\psi)}\,\sigma(z_\tau^{(k(\psi))}, \hat{\Lambda})^2, \quad \tau(\theta_1) < \tau < \tau(\theta_2),$$

the same equation as curve $\Lambda_\tau(\psi)$.

Therefore,

$$\hat{\Lambda}_\theta \in L_{\tau(\theta)}^{(k(\psi))}\,(\theta_1 < \theta < \theta_2)\ \&\ \Lambda_{\theta_1+0} = \Lambda_{\tau(\theta_1)} \Rightarrow \hat{\Lambda}_\theta = \Lambda_{\tau(\theta)}(\psi)\,(\theta_1 < \theta \leqslant \theta_2). \tag{6}$$

At the same time, on interval $\theta \in (\theta_1, \theta_2)$ satisfying condition $\hat{\Lambda}_\theta \in L_{\tau(\theta)}^{(\alpha)} \setminus L_{\tau(\theta)}^{(\alpha+1)}$ with $\alpha < k(\psi)$, function $\tau(\theta)$ is constant, $\tau(\theta) \equiv \tau(\theta_1)$, and curve $\hat{\Lambda}$ is subject to the "almost autonomous" equation

$$\frac{d\hat{\Lambda}}{d\theta} = u(\theta)\,\sigma(z_{\tau(\theta_1)}^{(\alpha)}, \hat{\Lambda})^2, \quad u(\theta) > 0, \quad \theta_1 < \theta < \theta_2. \tag{7}$$

Let $\Delta_{\theta_1}^{\alpha} = \hat{\Lambda}_{\theta_1+0} \cap \{z_{\tau(\theta_1)}^{(\alpha)}\}^{\angle}$ be an $(m-1)$-dimensional isotropic subspace in $L(\Pi \oplus \Pi^*)$. Equation (7) leaves manifold

$$L\left(\Delta_{\theta_1}^{\alpha\angle}/\Delta_{\theta_1}^{\alpha}\right) \approx L(\mathrm{R}^2) = S^1$$

invariant.

Note that

$$L\left(\Delta_{\theta_1}^{\alpha\angle}/\Delta_{\theta_1}^{\alpha}\right) \cap L_{\tau(\theta_1)}^{(\alpha+1)} = \Delta_{\theta_1}^{\alpha} + \operatorname{span}\{z_{\tau(\theta_1)}^{(\alpha)}\} = \hat{\Lambda}_{\theta_1+0}^{\operatorname{span}\{z_{\tau(\theta_1)}^{(\alpha)}\}}$$

is the only fixed point of Eq. (7) on $L(\Delta^{\angle}/\Delta)$.

The proposition being proved now follows from (6) and the following assertion.

LEMMA 3. Let $\alpha(\theta) = \max\{\alpha \mid \hat{\Lambda}_\theta \in L_{\tau(\theta)}{}^{(\alpha)}\}$, $0 < \theta \le \hat{\theta}$. Integer-valued function $\alpha(\theta)$ does not decrease with the growth of θ, $0 < \theta < \hat{\theta}$.

Proof. Let $\theta_1 \in (0, \hat{\theta})$, $\tau(\theta_1) < t$. Assume that $\underline{\alpha} = \lim_{\theta \to \theta_1 + 0} \alpha(\theta) < \alpha(\theta_1)$. Then there exists a $\theta_2 > \theta_1$ satisfying conditions $\tau(\theta_2) = \tau(\theta_1)$, $\hat{\Lambda}_{\theta_2} = \hat{\Lambda}_{\theta_1}$, such that the nondecreasing curve $\hat{\Lambda} \mid_{[\theta_1, \theta_2]}$ contains the whole circle $L(\Delta_{\bar{\theta}_1+0}^{\underline{\alpha}\angle}/\Delta_{\bar{\theta}_1+0}^{\underline{\alpha}})$ where $\Delta_{\bar{\theta}_1+0}^{\underline{\alpha}} = \lim_{\theta \to \theta_1+0} \Delta_{\bar{\theta}}^{\underline{\alpha}}$. Indeed, finding ourselves on this circle on the "positive side" of the fixed point of the equation $d\hat{\Lambda}/d\theta = u(\theta)\sigma(z_{\tau(\theta_1)}{}^{(\underline{\alpha})}, \hat{\Lambda})^2$ we cannot return into $L_\tau{}^{\alpha(\theta_1)}$ without passing it all (we can return only along this circle and only from the negative side); at the same time, until we return $\tau(\theta)$ does not increase.

The given argument and (6) imply that curve $\hat{\Lambda}_\theta$, $\theta \in (0, \hat{\theta})$ contains the whole curve $\Lambda_\tau(\psi)$, $0 < \tau \le t$, and also the circle $L(\Delta_{\theta_1+0}{}^{\underline{\alpha}\angle}/\Delta_{\theta_1+0}{}^{\underline{\alpha}})$ (the orientations are uniquely determined by the fact that all the curves must be nondecreasing). It remains to note that $\operatorname{Ind} L(\Delta_{\theta_1+0}{}^{\underline{\alpha}\angle}/\Delta_{\theta_1+0}{}^{\underline{\alpha}}) = 1 > 0$; therefore, $n \ge \operatorname{ind} \hat{\Lambda}. \ge \operatorname{ind} \Lambda.(\psi) + 1 = n + 1$. Contradiction. We have not yet considered the case when $\tau(\theta_1) = t$, but this case reduces to the already considered one if from the very beginning we extent $z_\tau(\psi_i)$ and $z_\tau(\psi)$ in a suitable way to some interval on the time axis containing $(0, t]$. ■

THEOREM 1. For any $n \ge 0$ the following isomorphism of vector bundles holds: $\mathscr{L}_n \mid \Psi_n{}^s \approx J_n{}^*\mathscr{K}_n$. In particular, $\pi_n \mid \Psi_n{}^s = J_n{}^*\kappa_n$.

Proof. Suppose that $\psi \in \Psi_n{}^s$ and $\Lambda_\tau(\psi)$, $0 \le \tau \le t$ is the corresponding Jacobian curve. In order not to encumber the presentation too much we again assume that $r = 1$ and that notation $z_\tau(\psi)$, $k_\tau(\psi)$, and $\gamma_\tau(\psi)$ has the same meaning as in the proof of Proposition 3.

Let $0 = \tau_0 < \tau_1 < \ldots < \tau_{l+1} = t$ be a partition of interval $(0, t]$ compatible with curve $\Lambda.(\psi)$, $D_i = [\tau_i, \tau_{i+1}]$, $D = \{\tau_1, \ldots, \tau_l\}$. The fiber of bundle $J_n{}^*\mathscr{K}_n$ at point ψ by the definition itself of this bundle is identified with the space

$$K_D(\Lambda.(\psi)) = \bigoplus_{i=1}^{l} K_{D_i}(\Lambda.(\psi)),$$

where

$$K_{D_i}(\Lambda.(\psi)) = \sum_{\tau_i < \tau < \tau_{i+1}} (\Pi^* \cap \Lambda_\tau), \quad i = 1, \ldots, l.$$

Let $\lambda \in K_{D_i}(\Lambda.(\psi))$, and set $D_i(\lambda) = \{\tau \in D_i \mid \lambda \in \Lambda_\tau\}$. Assertion c) of Lemma 2 implies that $D_i(\lambda)$ is connected, i.e., it is a subinterval in D_i. In the case when this interval is not reducible to one point, the differentiation of identity $0 = \sigma(\lambda, \lambda_\tau(\psi))$, $\tau \in D_i(\lambda)$ with respect to τ leads to the equality $\sigma(\lambda, z_\tau{}^{(k_\tau)}(\psi)) = 0$, $\tau \in D_i(\lambda)$.

Assume that $\tau_0 \in D_i(\lambda)$, and denote by λ_τ, $0 < \tau \le \tau_0$ the solution of the differential equation

$$\frac{d\lambda_\tau}{d\tau} = \frac{\sigma\left(z_\tau^{(k_\tau)}(\psi), \lambda_\tau\right)}{\gamma_\tau(\psi)} z_\tau^{(k_\tau)},$$

satisfying the condition $\lambda_{\tau_0} = \lambda$. Finally, set

$$v_\lambda(\tau) = \begin{cases} \dfrac{\sigma\left(z_\tau^{(k_\tau)}(\psi), \lambda_\tau\right)}{\gamma_\tau(\psi)}, & 0 < \tau \le \tau_0; \\ 0, \quad \tau \notin (0, \tau_0] \end{cases}$$

to be a piecewise smooth function on $(0, t]$. From the aforesaid it follows that correspondence $\lambda \to v_\lambda(\cdot)$ unambiguously defines a linear injective mapping of space $K_{D_i}(\Lambda.(\psi))$ into a space of piecewise smooth functions on **R**. Furthermore, associating with vector $(\lambda_1 \oplus ... \oplus \lambda_l) \in K_D(\Lambda.(\psi))$ function $\sum_{i=1}^{l} v_{\lambda_i}(\cdot)$, we apparently obtain an injective linear mapping of $K_D(\Lambda.(\psi))$ into a space of piecewise smooth functions.

Now we use some notation introduced in the proof of Theorem 3.1. By $H_{-k.(\psi)}[0, t]$ (see Lemma 3.8) we denote the direct sum of some Sobolev spaces with negative weight $H_{-k.(\psi)}[0, t] = \bigoplus_{j=1}^{N+1} H_{-k_{t_j}(\psi)}[t_{j-1}, t_j]$, where $0 = t_0 < t_1 < ... < t_{N+1} = t$, and $t_1, ..., t_N$ are all the points where the smoothness of $z_\tau(\psi)$ is violated. In Lemma 3.8 we established the continuity of form $\psi G_t''$ in the norm of space $H_{-k.(\psi)}[0, t) \supset L_2[0, t]$. The closure of space $\ker G_t'$ (of the domain of form $\psi G_t''$) in $H_{-k.(\psi)}[0, t]$ consists of "distributions" $u = \bigoplus_{j=0}^{N} u_j$ such that $\langle z.u, 1\rangle \overset{\text{def}}{=} \sum_{j=0}^{N} \langle z.u_j, 1\rangle \in \Pi^*$ the extension with respect to continuity of form $\psi G_t''$ to this space of "distributions" is denoted by $Q(\psi)$.

Let $\lambda \in \Lambda_{\tau_0} \cap \Pi^*$ and let λ_τ, $0 < \tau \leq \tau_0$ be a solution of the equation

$$\frac{d\lambda_\tau}{d\tau} = v_\lambda(\tau)\, z_\tau^{(k_\tau)},$$

satisfying the condition $\lambda_{\tau_0} = \lambda$. Then $\lambda_{+0} \in \Lambda_{+0}$; consequently, vector λ_{+0} is uniquely represented in the form $\lambda_{+0} = \nu_0(\lambda) + \sum_{i=0}^{k_{+0}-1} a_i(\lambda) z_{+0}$, where $\nu_0(\lambda) \in \Pi^* \cap \Gamma_{+0}^{\perp}$, $a_i(\lambda)$ are scalars. The following formula associates with each $\lambda \in K_{D_i}(\Lambda.(\psi))$ the element u_λ of space $H_{-k.(\psi)}[0, t]$:

$$u_\lambda \overset{\text{def}}{=} \sum_{i=0}^{k_{+0}-1} a_i(\lambda)\, \partial^i \delta_{+0} + \sum_{j=1}^{N+1} \partial^{k_{\tau_j}} (v_\lambda |_{[t_{j-1}, t_j]}),$$

where $\delta_{+0} \in H_{-1}[0, t]$, a δ-function concentrated at the origin, and $\partial = -\partial/d\tau$ is the differentiation operator in the distribution space.

Associating with each $(\lambda_1 \oplus ... \oplus \lambda_l) \in K_D(\Lambda.(\psi))$ distribution $\sum_{i=1}^{l} u_{\lambda_i}$, we obtain the linear injective mapping $\mathbf{u}_\psi$: $K_D(\Lambda.(\psi)) \to H_{-k.(\psi)}[0, t]$. Observe that

$$\langle z.u_\lambda, 1\rangle = \sum_{i=0}^{k_{+0}-1} a_i(\lambda)\, z_{+0}^{(i)} + \int_0^t z_\tau^{(k_\tau)} v_\lambda(\tau)\, d\tau = (\lambda - \nu_0(\lambda)) \in \Pi^*.$$

Consequently, the n-dimensional space $\mathbf{u}_\psi(K_D(\Lambda.(\psi)))$ lies in the domain of form $Q(\psi)$.

LEMMA 4. The restriction of form $Q(\psi)$ to subspace $\mathbf{u}_\psi(K_D(\Lambda.(\psi)))$ equals zero; in this connection, $\mathbf{u}_\psi(K_D(\Lambda.(\psi))) \cap \ker Q(\psi) = 0$.

Proof. Indeed, for $\lambda' \in \Lambda_{\tau'} \cap \Pi^*$, $\lambda'' \in \Lambda_{\tau''} \cap \Pi^*$, $0 < \tau'' \leq \tau' < t$ we have (for the uniformity of formulas we assumed that $k_\tau > 0\ \forall \tau$)

$$Q(u_{\lambda'}, u_{\lambda''}) = \int_0^t \sigma\left(\int_0^\tau (-1)^{k_\theta} z_\theta v_{\lambda'}^{k_\theta}(\theta)\, d\theta + \right.$$
$$\left. + \sum_{i=0}^{k_{+0}-1} a_i(\lambda') z_{+0}^{(i)},\ (-1)^{k_\tau} z_\tau v_{\lambda''}^{(k_\tau)}(\tau)\right) d\tau = \int_0^t \left(\gamma_\tau v_{\lambda'}(\tau) + \right.$$
$$\left. + \sigma\left(\int_0^\tau z_\theta^{(k_\theta)} v_{\lambda'}(\theta)\, d\theta + \sum_{i=0}^{k_{+0}-1} a_i(\lambda') z_{+0}^{(i)},\ z_\tau^{(k_\tau)}\right)\right) v_{\lambda''}(\tau)\, d\tau =$$
$$= -\sigma(\nu_0(\lambda'),\ \int_0^t z_\tau^{(k_\tau)} v_{\lambda''}(\tau)\, d\tau =$$
$$= \sigma\left(\nu_0(\lambda'),\ \nu_0(\lambda'') + \sum_{i=0}^{k_{+0}-1} a_i z_{+0}^{(i)} - \lambda''\right) = 0.$$

The kernel of form $Q(\psi)$ was described in the proof of Theorem 3.1 (see Lemma 3.12). Curve $\Lambda.(\psi)$ belongs to $\Omega^s[0, t]$; consequently, $\Lambda_\theta(\psi) \cap \Pi^* = 0$ on some interval $\overline{\tau} < \theta < t$. In such a case the space $\ker Q$ contains no nonzero distributions equal identically to zero on the interval $(\overline{\tau}, t)$.

In Proposition 3 the continuous dependence of curve $\Lambda.(\psi)$ on $\psi \in \Psi_n^s$ is established. An analysis of the proof of this proposition shows that spaces $\{u_\lambda \mid \lambda \in K_{D_i}(\Lambda.(\psi))\}$ depend continuously on ψ as do finite-dimensional subspaces in the Sobolev space $H_{-N}[0, t]$ with sufficiently large $N > 0$.*

Consequently, n-dimensional spaces $\mathbf{u}_\psi(K_D(\Lambda.(\psi)))$ also depend continuously on ψ if they are considered as subspaces in $H_{-N}[0, t]$.

Observe that $\mathbf{u}_\psi(K_D(\Lambda.(\psi)))$ is independent of the choice of the partition of D, and set $U_\psi = \mathbf{u}_\psi(K_D(\Lambda.(\psi)))$. The family of subspaces $U_\psi \subset H_{-k.(\psi)}[0, t] \subset H_{-N}[0, t]$, $\psi \in \Psi_n^s$ determines an n-dimensional vector bundle $\mathcal{U}_n$ over Ψ_n^s and mapping $\mathbf{u}_\psi$: $K_D(\Lambda.(\psi)) \to U_\psi$ induces an isomorphism of bundles $J_n^* \mathcal{H}_n \approx \mathcal{U}_n$. It remains to construct an isomorphism of fiber U_n on $\mathcal{L}_n \mid \Psi_n^s$.

Recall that the fiber of bundles $\mathcal{L}_n$ at point $\psi \in \Psi_n^s$ is an n-dimensional invariant subspace $L_\psi \subset \overline{\ker G_t'}$ of a self-adjoint operator in $\overline{\ker G_t'}$ defined by the quadratic form ψg_t, such that $\psi g_t \mid L_\psi < 0$. In that case, $Q(\psi) \mid L_\psi = \psi g_t \mid L_\psi < 0$. Let $\operatorname{Dom} Q(\psi)$ be the closure of subspace $\ker G_t'$ in $H_{-k.(\psi)}[0, t]$, the domain of form $Q(\psi)$.

According to Lemma 3.10, form $Q(\psi)$ is positive definite on some subspace of finite codimension in $\operatorname{Dom} Q(\psi)$; subspace $(L_\psi)_{Q(\psi)}^\perp$ has codimension n in $\operatorname{Dom} Q(\psi)$, and according to Lemma 3.11,

$$Q(\psi)|(L_\psi)^\perp_{Q(\psi)} > 0, \ \ker(Q(\psi)|(L_\psi)^\perp_{Q(\psi)}) = \ker Q(\psi). \tag{8}$$

Equation (8) and Lemma 4 imply that $U_\psi \cap (L_\psi)_{Q(\psi)}^\perp = 0$.

Let $\Pr_\psi$: $\operatorname{Dom} Q(\psi) \to L_\psi$ be the projection of $\operatorname{Dom} Q(\psi)$ onto L_ψ parallel to U_ψ; in other words,

$$\ker \Pr_\psi = U_\psi, \ \operatorname{im} \Pr_\psi = L_\psi, \ \Pr_\psi \circ \Pr_\psi = \Pr_\psi.$$

Set $I_\psi = \Pr_\psi \mid U_\psi$. Family I_ψ: $U_\psi \to L_\psi$, $\psi \in \Psi_n^s$ of isomorphisms of linear spaces realizes an isomorphism of the bundles

$$\mathcal{U}_n \approx \mathcal{L}_n \mid \Psi_n^s.$$

Suppose that K is a convex polyhedral cone in $\operatorname{coker} G_t'$ and the quadratic mapping g is not degenerate relative to K. In the case when inclusions

$$\left(\bigcup_{i=0}^n \Psi_i^s\right) \cap K^\circ \subset \left(\bigcup_{i=0}^n \Psi_i\right) \cap K^\circ, \ n > 0$$

are homotopy equivalent, Theorem 1 allows us to use effectively Theorem 2 from [1] for a highly exact estimate of homotopy groups of the set $g_t^{-1}(K)\backslash 0$. Indeed, in many important problems equality $\Psi^s = \Psi$ is fulfilled and, consequently, $\Psi_n^s = \Psi_n$.

The case when $K = 0$ is especially interesting: the results from [1, Sec. 3], when applied to mapping G_t, give the following answers to questions I) and II) posed in the beginning of the present article in terms of homology groups of sets $g_t^{-1}(0)\backslash 0$.

THEOREM 2. Assume that g_t is nondegenerate; then

i) if $g_t^{-1}(0) \neq 0$, then $\tilde{\mu}_t \in \operatorname{int} F_t(\mathcal{O}_{\tilde{u}(\cdot)})$ for any neighborhood $\mathcal{O}_{\tilde{u}(\cdot)}$ of point $\tilde{u}(\cdot)$ in $L_\infty([0, t]; U)$;

ii) if $g_t^{-1}(0) = 0$, then whatever the finite-dimensional submanifold $V \subset L_\infty([0, t]; U)$, containing the point $\tilde{u}(\cdot)$, is there is a neighborhood $\mathcal{O}_{\tilde{u}(\cdot)}$ such that $\tilde{\mu}_t \in \partial F_t(V \cap \mathcal{O}_{\tilde{u}(\cdot)})$;

iii) for every $i > 0$ group $\check{H}_{i-1}(g_t^{-1}(0)\backslash 0)$ coincides with the direct limit of the family of groups

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*Allowing some liberties, we denote by $H_{-N}[0, t]$ the space $\bigoplus_{j=1}^{N+1} H_{-N}[\tau_{j-1}, \tau_j]$.

$$H_t(F_t^{-1}(\tilde{\mu}_t)\cap V, \quad F_t^{-1}(\tilde{\mu}_t)\cap V\setminus\tilde{u}(\cdot)),$$

where V runs the set of all finite-dimensional submanifolds in $L_\infty([0, t]; U)$, partially ordered with respect to inclusion, and the homomorphisms

$$H_t(F_t^{-1}(\tilde{\mu}_t)\cap V, \ F_t^{-1}(\tilde{\mu}_t)\cap V\setminus\tilde{u}(\cdot))\to$$
$$\to H_t(F_t^{-1}(\tilde{\mu}_t)\cap W, \ F_t^{-1}(\tilde{\mu}_t)\cap W\setminus\tilde{u}(\cdot))$$

for $V \subset W$ are induced by imbedding.

5. APPLICATION TO CONTROLLED SYSTEMS ON LIE GROUPS

In this section we use the methods developed above to solve some special problems. Basically we consider a neighborhood of a constant control for the stationary system

$$\dot{\mu}=\mu\circ f(u), \quad \mu\in M, \ u\in U,$$

on a semisimple Lie group, i.e., the phase manifold M is a semisimple Lie group and vector fields f(u), $u \in U$ are left-invariant. First we study the behavior of the indices of Jacobian curves as $t \to +\infty$ (the "elliptic" and "hyperbolic" situations are characterized). Then we describe a class of systems on compact groups in which all the information is extracted directly from the Stiefel diagram of group M. We do not reach for maximum generality but center our attention on situations in which calculations can be finished and the results presented in a visible form.

1. The first result refers to the derivative of a controlled system not necessarily on a Lie group. We again use notation of subsection 1 of Sec. 3; here $t > 0$ is assumed to be so large that

$$\sum_{0<\tau<+\infty}\mu_0\circ Z_\tau R^r=\sum_{0<\tau<t}\mu\circ Z_\tau R^r=\Pi.$$

Vector $\psi \in \Pi^\perp\setminus 0$ is fixed throughout the present subsection, $\Sigma = E_{\Pi,\psi}$ is a symplectic space with a skew inner product σ. The integral quadratic form $\psi G_t''$ is defined by equality (3.5) and here $\mathrm{ind}\,\psi G_t''$ does not decrease monotonically with the growth of t.

Recall the definition of integer $k_t \geq 0$ and of the quadratic form γ_t on $\mathbf{R}^r$ introduced in subsection 2 of Sec. 3: if the quadratic form ψh_θ does not equal identically zero on any interval $\tau < \theta < t$, then set $k_t = 0$, $\gamma_t = \psi h_t$; otherwise, k_t is maximal among numbers k, $1 \leq k \leq m = \dim\Pi$, such that $\sigma(z_\theta^{(i)}v_1, z_\theta v_2) \equiv 0$ for $i < 2(k-1)$, $v_1, v_2 \in \mathbf{R}^r$ on some interval $\tau < \theta < t$, $\gamma_t(v) = \sigma(z^{(k_t)}v, z^{(k_t-1)}v)$, $v \in \mathbf{R}^r$.

In the sequel we assume that $\sigma(z^{(k_t-1)}v_1, z^{(k_t-1)}v_2) = 0$ $\forall v_1, v_2 \in \mathbf{R}^r$, $t > 0$ and $\gamma_t(v) \geq \varepsilon_t |v|^2$ $\forall v \in \mathbf{R}^r$, $t > 0$, and some $\varepsilon_t > 0$, i.e., the conditions for the finiteness of ind $\psi G_t''$ formulated in Proposition 3.1 are fulfilled.

Proposition 1. If $\lim_{t\to+\infty}\mathrm{ind}\,\psi G_t''<+\infty$, then there exists a Lagrangian plane $\Lambda_\infty \subset \Sigma$, such that for every $v \in \mathbf{R}^r$ and any neighborhood $O_{\lambda_0} \subset \Sigma$ of the arbitrary point $\lambda_0 \in \Lambda_\infty$ the following relation is fulfilled:

$$\int_0^\infty \frac{1}{\gamma_t(v)}\min_{\lambda\in O_{\lambda_0}}\sigma(z_t^{(k_t)}v, \ \lambda)^2\,dt<+\infty.$$

Proof. Since γ_t is a nonsingular quadratic form on $\mathbf{R}^r$, a quadratic form γ_t^{-1} on $\mathbf{R}^{r*}$ is defined, and relation

$$x \mapsto \frac{1}{2} \gamma_t^{-1} (\sigma (z_t^{(k_t)} \cdot, x)), \quad x \in \Sigma,$$

defines a quadratic Hamiltonian on Σ [see (3.9)]. If $v_t^1, \ldots, v_t^r$ is a basis in $\mathbf{R}^r$ such that

$$\gamma_t \left(\sum_{i=1}^{r} \alpha_i v_t^i \right) = \sum \alpha_i^2 \gamma_t (v_t^i),$$

then

$$\gamma_t^{-1} (\sigma (z_t^{(k_t)} \cdot, x)) = \sum_{i=1}^{r} \frac{\sigma \left(z_t^{(k_t)} v_i, x \right)^2}{\gamma_t (v_i)}. \tag{1}$$

The restriction of quadratic form (1) to Λ is denoted, as in Sec. 3, by $\gamma_t^{-1}(\sigma(z_t^{(k_t)}\cdot, \Lambda))$. Let Λ_t, $t \geq 0$ be the Jacobian curve corresponding to Hamiltonian (1). In that case, at every point of continuity of $z_t^{(k_t)}$, γ_t equality

$$\frac{d\Lambda_t}{dt} = \frac{1}{2} \gamma_t^{-1} (\sigma (z_t^{(k_t)} \cdot, \Lambda_t)) \tag{2}$$

is fulfilled.

In subsection 2 of Sec. 4, a definition is given of the index of an arbitrary curve in the Lagrange Grassmannian $L(\Sigma)$; moreover, Theorem 3.1 implies the equality

$$\operatorname{ind} \psi G_t'' = \operatorname{ind} (\Lambda_\cdot |_{[0,t]}), \quad t > 0.$$

Suppose that $\lim\limits_{t \to +\infty} \operatorname{ind} \psi G_t'' = N < +\infty$, Proposition 4.2 implies that the length of curve Λ_t, $0 < t < +\infty$ does not exceed $\pi/2(N + m)$. Consequently, the limit of Λ_t as $t \to +\infty$ exists. Set $\Lambda_\infty = \lim\limits_{t \to +\infty} \Lambda_t$. According to Eq. (2), the length of curve Λ_t, $0 < t \leq \infty$ equals $\frac{1}{2} \int_0^{+\infty} \| \gamma_t^{-1} (\sigma (z_t^{(k_t)} \cdot, \Lambda_t)) \| \, dt < +\infty$. Consequently, for any neighborhood $\mathcal{O}_{\Lambda_\infty}$ of point Λ_∞ in $L(\Sigma)$ we have

$$\int_0^\infty \min_{\Lambda \in \mathcal{O}_{\Lambda_\infty}} \| \gamma_t^{-1} (\sigma (z_t^{(k_t)} \cdot, \Lambda_t)) \| \, dt < +\infty,$$

whence, taking into account equality (1), the proof of the assertion follows. ■

Proposition 1 gives a rather strong necessary condition for the finiteness of the index of a Jacobian curve on a half-line. However, as will be clear from what follows, this condition is far from sufficient.

Suppose now that M is a Lie group and $\mathfrak{M}$ is a Lie algebra of left-invariant vector fields on M; the controlled system has the form

$$\dot{\mu} = \mu \circ f(u), \quad u \in U, \quad \mu(0) = \mu_0, \quad f(u) \in \mathfrak{M} \quad \forall u \in U.$$

Let $\tilde{u}(t) \equiv u_0$; then $\tilde{\mu}(t) = \mu_0 e^{tf(u_0)}$, $Z_t = e^{t \,\mathrm{ad} f(u_0)} (\partial f / \partial u) \big|_{u_0}$; in particular, $Zv_t \in \mathfrak{M}$ $\forall v$. Without loss of generality we can consider that μ_0 is the unit element of group M, $T_{\mu_0} M \approx \mathfrak{M}$. In this situation the inertia index of the integral quadratic form $\psi G_t''$ can be computed using purely algebraic means. We restrict ourselves to the case when M is a semisimple Lie group and consider only the so-called bilinear systems with scalar control:

$$\dot{\mu} = \mu \circ (a + ub), \quad u \in \mathbf{R}, \quad a, b \in \mathfrak{M}, \quad u_0 = 0. \tag{3}$$

The calculations for systems of form (3) also contain basic singularities of the general nonlinear case with multidimensional control, the advantage, however, is in the fact that we use a minimal number of initial givens: everything is determined by the two elements a and b of the Lie algebra $\mathfrak{M}$.

The Killing form of semisimple Lie algebra $\mathfrak{M}$ determines a canonic identification of $\mathfrak{M}$ and $\mathfrak{M}^*$. Everywhere to the end of this section the angled brackets $\langle\cdot,\cdot\rangle$ denote the Killing form; here,

$$S^{\perp} \stackrel{\text{def}}{=} \{x \in \mathfrak{M} \mid \langle x, S\rangle = 0\}$$

for any subset $S \subset \mathfrak{M}$.

Below we assume the following general position conditions to be fulfilled:

1) a is a regular element of the semisimple Lie algebra $\mathfrak{M}$, i.e., $\dim\ker(\operatorname{ad} a) = \operatorname{rank}\mathfrak{M}$;

2) element b does not lie in any invariant subspace of operator $\operatorname{ad} a\colon \mathfrak{M} \to \mathfrak{M}$ of codimension equal to at least rank $\mathfrak{M}$.

Conditions 1) and 2) are obviously equivalent to the equality $\operatorname{codim}\operatorname{span}\{(\operatorname{ad}^i a)b \mid i \geq 0\} = \operatorname{rank}\mathfrak{M} - 1$.

We have

$$Z_t = e^{t\,\operatorname{ad} a} b,\quad G_t : u(\cdot) \mapsto \mu_0 \circ \overrightarrow{\exp} \int_0^t e^{\tau\,\operatorname{ad} a} b u(\tau)\, d\tau,$$

$$G_t' u(\cdot) = \int_0^t e^{\tau\,\operatorname{ad} a} b u(\tau)\, d\tau,$$

$$G_t''(v_1(\cdot), v_2(\cdot)) =$$

$$= \int_0^t \left[\int_0^\tau e^{\theta\,\operatorname{ad} a} b v(\theta)\, d\theta,\ e^{\tau\,\operatorname{ad} a} b v(\tau)\right] d\tau + \operatorname{im} G_t', \quad v_i(\cdot) \in \ker G_t'.$$

The notation of the initial point μ_0 in terms of expressions for G_t' and G_t can be dropped by virtue of the identification $T_{\mu_0}M \approx \mathfrak{M}$: the commutation operation as well as operators $e^{\tau\,\operatorname{ad} a}$ do not lead out of Lie algebra $\mathfrak{M}$.

Suppose that $\mathscr{H} = \ker(\operatorname{ad} a)$ is a Cartan subalgebra of Lie algebra $\mathfrak{M}$ containing a, and $b_0 \in \mathscr{H}$ is an orthogonal projection of b onto $\mathscr{H}$ [so that $(b - b_0) \perp \mathscr{H}$]. The general position conditions imply that

$$\Pi = \operatorname{im} G_t' = \operatorname{span}\{e^{\tau\,\operatorname{ad} a} b,\ 0 \leqslant \tau \leqslant t\} = \mathscr{H}^{\perp} + Rb \quad m = \dim \mathscr{H}^{\perp} + 1.$$

Thus, $\psi \in \Pi^{\perp} \subset \mathscr{H}$, $[a, \psi] = 0$. Consequently, values $\langle\psi, [z_t^{(i)}, z_t^{(j)}]\rangle = \langle\psi, [\operatorname{ad}^i ab, \operatorname{ad}^j ab]\rangle$ are independent of $t \in R$; therefore, $k_t = \text{const}$, $\gamma_t = \text{const}$. General position conditions imply that $\langle\psi, [\operatorname{ad}^k ab, b]\rangle = \langle[b, \psi], \operatorname{ad}^k ab\rangle \neq 0$ for some $k \leq m - 2$, so $k_t \leq (m-1)/2$.

Let $k_t = 1$ for certainty (the case when $k_t > 1$ is no more complicated; we simply desire to avoid superfluous parameters). Set $\gamma_i = \langle\psi, [\operatorname{ad}^{i+1} ab, \operatorname{ad}^i ab]\rangle$, $i \geq 0$; in that case, $\gamma_t = \gamma_0$.

Recall that by $\mathscr{E}_\Pi$ we denote the space of all vector fields on M, the value at point μ_0 of which lies in Π. The symplectic space Σ is the quotient space $\mathscr{E}_\Pi$ by the kernel of the skew-symmetric form $(x_1 \wedge x_2) \to \langle\psi, \mu_0 \cdot [x_1, x_2]\rangle$, $x_1, x_2 \in \mathscr{E}_\Pi$. Identifying Π with the set of left-invariant fields lying in $\mathscr{E}_\Pi$, we can assume that $\Pi \subset \mathscr{E}_\Pi$. The image of subspace Π under the canonical factorization $\mathscr{E}_\Pi \to \Sigma$ is denoted by $\hat{\Pi}$. Recall that by Π_0 we denote the image under the same factorization of the space of fields vanishing at the point μ_0. Clearly, $\Sigma = \hat{\Pi} \oplus \Pi_0$; however, in contrast to Π_0, subspace $\hat{\Pi}$ is not a Lagrange plane in a symplectic space (Σ, σ). There is an obvious isomorphism of space $\Sigma = \hat{\Pi} \oplus \Pi_0$ on $\Pi \oplus \Pi$, under which $\hat{\Pi}$ transforms into the first term, Π_0 into the second, and the symplectic structure σ into the skew-symmetric form

$$\bar{\sigma} : (x_1, \xi_1) \wedge (x_2, \xi_2) \mapsto \langle\psi, [x_1, x_2]\rangle + \langle\xi_2, x_1\rangle - \langle\xi_1, x_2\rangle,$$
$$x_i, \xi_i \in \Pi \subset \mathfrak{M}, \quad i = 1, 2.$$

The symplectic space $(\Pi \oplus \Pi, \bar{\sigma})$ is a convenient model for the space (Σ, σ). Hamiltonian h_τ has the following form in this model:

$$h_\tau = \frac{1}{2\gamma_0}(\langle \psi, [e^{\tau \operatorname{ad} a}[a,b], x]\rangle + \langle e^{\tau \operatorname{ad} a}[a,b], \xi\rangle)^2. \tag{4}$$

The linear Hamiltonian system in $\Pi \oplus \Pi$, corresponding to Hamiltonian $h_\tau(x, \xi)$, has the form

$$\begin{cases} \gamma_0 \dot{x} = (\langle \psi, [e^{\tau \operatorname{ad} a}[a,b], x]\rangle + \langle e^{\tau \operatorname{ad} a}[a,b], \xi\rangle) e^{\tau \operatorname{ad} a}[a,b], \\ \dot{\xi} = 0. \end{cases}$$

The Jacobian curve is a curve on the Lagrange Grassmannian $L(\Pi \oplus \Pi)$, smooth for $0 < \tau < +\infty$ and with initial conditions $\Lambda_0 = 0 \oplus \Pi$, $\Lambda_{+0} = \mathbf{R}b \oplus (\Pi \cap \{b\}^\perp)$, that is generated by this system.

Now we must make a small digression.

LEMMA 1. Suppose that g_τ is a nonnegative quadratic Hamiltonian on some symplectic space (Σ, σ), Γ is an isotropic subspace in Σ, such that $\Gamma \subset \ker h_\tau\ \forall \tau > 0$, and $\Delta_\tau \in L(\Sigma)$, $\tau \in [0, t]$ is a piecewise smooth curve continuous for all $\tau > 0$ (for $\tau = 0$ a discontinuity is possible). If Δ_τ satisfies the equation

$$\frac{d\Delta}{d\tau} = g_\tau(\Delta), \tag{5}$$

then curve Δ_τ^Γ also satisfies the equation. If, in addition, $\Gamma \subset \Delta_0$, then $\operatorname{ind}\Delta_\cdot = \operatorname{ind}\Delta_\cdot^\Gamma$.

Proof. Let $\xi_1, \ldots, \xi_r$ be a basis of isotropic space Γ. Consider the quadratic Hamiltonian $\frac{1}{2}\sum_{j=1}^{r} \sigma(\xi_j, x)^2$, $x \in \Sigma$. This Hamiltonian generates in Σ flux $x \mapsto x + s\sum_{j=1}^{r} \sigma(\xi_j, x)\xi_j$, $s \in \mathbf{R}$. The corresponding flux on $L(\Sigma)$ is denoted by Ξ_s, $s \in \mathbf{R}$. On the other hand, let $\mathcal{G}_\tau\colon L(\Sigma) \to L(\Sigma)$, $\tau \in [0, t]$ be a flux generated by Hamiltonian $\mathcal{G}_\tau$. Since $\xi_j \in \ker g_\tau$, $j = 1, \ldots, r$, $\tau \in (0, t]$, fluxes Ξ_s and $\mathcal{G}_\tau$ commute: $\Xi_s \cdot \mathcal{G}_\tau = \mathcal{G}_\tau \cdot \Xi_s$, $s \in \mathbf{R}$, $\tau \in (0, t]$; therefore, for every $s \in \mathbf{R}$ curve $\tau \to \Xi_s(\Delta_\tau)$ satisfies Eq. (5). At the same time, it is easy to see that $\Xi_s(\Lambda) \to \Lambda^\Gamma$ ($s \to +\infty$) $\forall \Lambda \in L(\Sigma)$. Consequently, curve Δ_τ^Γ, $\tau \in (0, t]$ indeed satisfies Eq. (5).

Note that subspace $\Delta_\tau \cap \Gamma$ is independent of τ for $\tau \in (0, t]$. Taking into account this circumstance, it is easy to show that under the condition $\Gamma \subset \Delta_0$ equality $\operatorname{ind}\Delta = \operatorname{ind}\Delta^\Gamma$ is valid. ■

There is a close connection between the indices of continuous curves on Lagrange Grassmannians generated by the same Hamiltonian system but with different initial conditions.

LEMMA 2. Suppose that g_τ is a nonstationary quadratic Hamiltonian on some symplectic space Σ and $\Delta_\tau^1, \Delta_\tau^2$, $0 \le \tau \le t$ are two continuous nondecreasing curves on $L(\Sigma)$ satisfying the equation

$$\frac{d\Delta}{d\tau} = g_\tau(\Delta).$$

Then

$$\operatorname{ind}\Delta_\cdot^1 - \operatorname{ind}\Delta_\cdot^2 = \operatorname{ind}_{\Delta_0^2}(\Delta_t^1, \Delta_0^1) - \operatorname{ind}_{\Delta_t^2}(\Delta_t^2, \Delta_0^2) + \\ + \operatorname{ind}_{\Delta_0^2}(\Delta_t^2, \Delta_t^1) - \operatorname{ind}_{\Delta_t^2}(\Delta_0^2, \Delta_0^1) + \dim\left(\bigcap_\tau \Delta_\tau^1\right) - \dim\left(\bigcap_\tau \Delta_\tau^2\right).$$

Proof. Let $\tau_{l+1} = 0 = \tau_0 < \tau_1 < \ldots < \tau_l = t$ be a partition of interval $[0, t]$ such that curves $\Delta_\tau^j|_{[\tau_i, \tau_{i+1}]}$, $i = 0, 1, \ldots, l$, $j = 1, 2$ are simple. Let $\Lambda \in L(\Sigma)$ be an arbitrary Lagrangian plane. According to Proposition 3.2, values

$$I^j = \sum_{i=0}^{l} \operatorname{ind}_\Lambda(\Delta_{\tau_i}^j, \Delta_{\tau_{i+1}}^j), \quad j = 1, 2 \tag{6}$$

are independent of Λ.

Determine piecewise smooth curves $\hat{\Lambda}_\tau^j$ on $L(\Sigma)$, $j = 1, 2$ by the following rule:

$$\hat{\Delta}_\tau^1 = \begin{cases} \Delta_0^2, & 0 \leqslant \tau \leqslant \frac{t}{2}, \\ \Delta_{2\tau - t}^1, & \frac{t}{2} < \tau \leqslant t. \end{cases} \qquad \hat{\Delta}_\tau^2 = \begin{cases} \Delta_{2\tau}^2, & 0 \leqslant \tau \leqslant \frac{t}{2}, \\ \Delta_t^1, & \frac{t}{2} < \tau \leqslant t. \end{cases}$$

Substituting Λ in (6) first with Λ_0^1 and then with Λ_0^2, we get

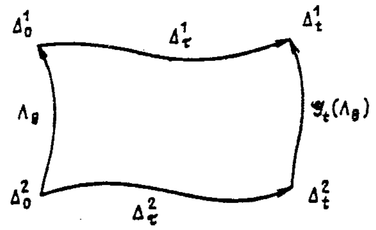
$$\operatorname{ind} \Delta_\cdot^1 + \dim \left(\Delta_0^1 / \bigcap_{0 < \tau < t} \Delta_\tau^1 \right) = I^1 = \operatorname{ind} \hat{\Delta}_\cdot^1 + \operatorname{ind}_{\Delta_0^2} (\Delta_t^1, \Delta_0^1) -$$
$$- \operatorname{ind}_{\Delta_0^2} (\Delta_0^2, \Delta_0^1) - \operatorname{ind}_{\Delta_0^2} (\Delta_t^1, \Delta_0^2) + \dim \left(\Delta_0^1 / \Delta_0^2 \bigcap_{0 < \tau < t} \Delta_\tau^1 \right).$$

At the same time,

$$\operatorname{ind} \Delta_\cdot^2 + \dim \left(\Delta_0^2 / \bigcap_{0 < \tau < t} \Delta_\tau^2 \right) = I^2 = \operatorname{ind} \hat{\Delta}_\cdot^2 + \operatorname{ind}_{\Delta_0^2} (\Delta_t^2, \Delta_0^2) -$$
$$- \operatorname{ind}_{\Delta_0^2} (\Delta_t^2, \Delta_t^1) - \operatorname{ind}_{\Delta_0^2} (\Delta_t^1, \Delta_0^2) + \dim \left(\Delta_0^2 / \Delta_t^1 \bigcap_{0 < \tau < t} \Delta_\tau^2 \right).$$

Suppose that $\mathcal{G}_\tau$: $L(\Sigma) \to L(\Sigma)$ is a flux on $L(\Sigma)$ determined by the Hamiltonian g_τ and Λ_θ is a simple nondecreasing continuous curve joining Δ_0^2 to Δ_0^1, i.e., $\Lambda_0 = \Delta_0^2$ and $\Lambda_t = \Delta_0^1$. In that case, $\forall \tau \in [0, t]$, the simple nondecreasing curve $\theta \to \mathcal{G}_t(\Lambda_\theta)$ joins Δ_τ^2 to Δ_τ^1.

Consequently, continuous nondecreasing curves $\Delta_\cdot^1 \cdot \Lambda_\cdot$ and $\mathcal{G}_\tau(\Lambda_\cdot) \cdot \Delta_\cdot^2$, having common ends, are holomorphic:



Identity (7.3) relates the index of a nondecreasing curve with the Maslov index of the appropriate closed curve. Since the Maslov index is a homotopic invariant and

$$\operatorname{ind} (\Delta_\cdot^1 \circ \Lambda_\cdot) = \operatorname{ind} \hat{\Delta}_\cdot^1, \quad \operatorname{ind} (\mathcal{G}_t (\Lambda_\cdot) \circ \Delta_\cdot^2) = \operatorname{ind} \hat{\Delta}_\cdot^2,$$

we get

$$\operatorname{ind} \hat{\Delta}_\cdot^1 + \dim \left(\Delta_0^1 / \Delta_0^2 \bigcap_{0 < \tau < t} \Delta_\tau^1 \right) = \operatorname{ind} \hat{\Delta}_\cdot^2 + \dim \left(\Delta_0^2 / \bigcap_{0 < \tau < t} \Delta_\tau^2 \right).$$

Comparing the already obtained equalities, we get an expression for $\operatorname{ind} \Delta_\cdot^1 - \operatorname{ind} \Delta_\cdot^2$. ■

COROLLARY. Suppose that the continuous curve Δ_τ, $\tau \in [0, t]$ on $L(\Sigma)$ satisfies Eq. (5) and $\bigcap_{0<\tau<t} \ker g_\tau \supset \Gamma$ is an isotropic subspace. Then

$$\operatorname{ind} \Delta_\cdot^\Gamma - \operatorname{ind} \Delta_\cdot = \operatorname{ind}_{\Delta_0} (\Delta_t^\Gamma, \Delta_0^\Gamma) - \operatorname{ind}_{\Delta_0} (\Delta_t, \Delta_0) +$$
$$+ \operatorname{ind}_{\Delta_0} (\Delta_t, \Delta_t^\Gamma) - \operatorname{ind}_{\Delta_0} (\Delta_0, \Delta_0^\Gamma) + \dim \left(\bigcap_{0 < \tau < t} \Delta_\tau^\Gamma / \bigcap_{0 < \tau < t} \Delta_\tau \right).$$

In particular,

$$| \operatorname{ind} \Delta_\cdot^\Gamma - \operatorname{ind} \Delta_\cdot - \dim (\bigcap_\tau \Delta_\tau^\Gamma / \bigcap_\tau \Delta_\tau) | \leqslant \frac{1}{2} \dim \Sigma.$$

Let us return to the Hamiltonian system under investigation. For a nonstationary Hamiltonian (4) given on $\Pi \oplus \Pi$, we get

$$\bigcap_{\tau>0} \ker h_\tau \supset (\ker \operatorname{ad} \psi) \cap \Pi) \oplus \mathbf{R} b.$$

Lemma 1 implies the equality

$$\operatorname{ind} \Lambda|_{[0,t]} = \operatorname{ind} \Lambda^{0\oplus \mathbf{R}b_0}|_{[0,t]} \quad \forall t > 0.$$

Note that curve $\Lambda_\tau^{0\oplus\mathbf{R}b_0}$ is continuous on $(0, +\infty)$ (while Λ_τ has a discontinuity at $\tau = 0$).

Set $\Gamma_h = (\ker(\operatorname{ad}\psi) \cap \Pi \oplus 0$ to be an isotropic space in $(\Pi \oplus \Pi, \hat{\overline{\sigma}})$. It is easy to see that $\Gamma_h^\perp/\Gamma_h \approx \operatorname{im}(\operatorname{ad}\psi) \oplus \operatorname{im}(\operatorname{ad}\psi) \subset \Pi \oplus \Pi$, while $\operatorname{im}(\operatorname{ad}\psi) = \ker(\operatorname{ad}\psi)^\perp$. Denote by $\hat{b}$ the orthogonal projection of vector b onto $\operatorname{im}(\operatorname{ad}\psi)$, and let $c = [a, \hat{b}]$ [subspace $\operatorname{im}(\operatorname{ad}\psi)$ is obviously invariant under both $\operatorname{ad} a$ and $\operatorname{ad}\psi$].

Suppose that Δ_τ, $\tau \geq 0$ is a curve on the Lagrange Grassmannian $L(\operatorname{im}(\operatorname{ad}\psi) \oplus \operatorname{im}(\operatorname{ad}\psi))$, generated by the Hamiltonian system

$$\begin{cases} \gamma_0 \dot{x} = \langle \xi - [\psi, x], e^{\tau \operatorname{ad} a} c \rangle e^{\tau \operatorname{ad} a} c \\ \dot{\xi} = 0, \qquad\qquad x, \xi \in \operatorname{im}(\operatorname{ad}\psi) \end{cases} \tag{7}$$

with the initial condition $\Delta_0 = \Delta_{+0} = 0 \oplus \operatorname{im}(\operatorname{ad}\psi)$.

Lemma 1 implies that Δ_τ passes into $\Lambda_\tau^{\Gamma_h}/\Gamma_h = (\Lambda_\tau^{0\oplus\mathbf{R}b})^{\Gamma_h}/\Gamma_h$ under the isomorphism of symplectic spaces $\operatorname{im}(\operatorname{ad}\psi) \oplus \operatorname{im}(\operatorname{ad}\psi) \approx \Gamma_h^\perp/\Gamma_h$.

The formula given in the corollary to Lemma 2 enables us to calculate $\operatorname{ind}\Lambda|_{[0,t]}$ in terms of $\operatorname{ind}\Delta|_{[0,t]}$; in any case,

$$|\operatorname{ind}\Delta.|_{[0,t]} - \operatorname{ind}\Lambda.|_{[0,t]} - \dim(\ker \operatorname{ad}\psi \cap \mathscr{H}^\perp)| \leqslant \dim \mathscr{H}^\perp.$$

If, however, ψ is a regular element of Lie algebra $\mathfrak{M}$, i.e., $\ker\psi = \mathscr{H}$, then $\operatorname{ind}\Delta.|_{[0,t]} = \operatorname{ind}\Lambda.|_{[0,t]}$.

Proposition 2. If the subalgebra of Cartan algebra $\mathscr{H}$ has a ~~radical~~ root $\rho \in \mathbf{C} \otimes \mathscr{H}^*$ such that $\rho(\psi) \neq 0$, and $\rho(a)$ is a purely imaginative number, then $\operatorname{ind}{}^\psi G_t'' \to \infty$ $(t \to +\infty)$

Before we prove this proposition we introduce some notation connected to the Cartan decomposition of Lie algebra $\mathfrak{M}$. Denote by $\hat{R} \subset \mathbf{C} \otimes \mathscr{H}^*$ the set of all radicals not vanishing on vector ψ.

The set $\hat{R}$, generally speaking, contains real ~~radicals~~ roots and also pairs of nonreal complex conjugate ~~radicals~~ roots. We construct subset $R \subset \hat{R}$ by including into it all the real ~~radicals~~ roots and also one of each pair $\{\rho, \overline{\rho}\}$ of nonreal complex conjugate ~~radicals~~ roots so that for every pair $\rho \in R$ the condition $\operatorname{Re}\rho(a)\operatorname{Im}\rho(a) \geq 0$ is fulfilled (we choose from a pair of purely imaginative ~~radicals~~ roots arbitrarily).

Let $R_+ = \{\rho \in R \mid \operatorname{Re}\rho(a) > 0\}$, $R_- = \{\rho \in R \mid \operatorname{Re}\rho(a) < 0\}$, and $R_0 = \{\rho \in R \mid \operatorname{Re}\rho(a) = 0\}$, so that $R = R_+ \cup R_- \cup R_0$.

Eigenvector $e_\rho \in \mathbf{C} \otimes \operatorname{im}\operatorname{ad}\psi$ corresponds to each ~~radical~~ root $\rho \in \hat{R}$ so that $\operatorname{ad}\omega e_\rho = \rho(\omega)e_\rho$, $\forall \omega \in \mathscr{H}$. We assume that vectors e_ρ, $\rho \in \hat{R}$ are normalized in a way such that

$$\langle e_\rho, e_{-\rho} \rangle \begin{cases} 2, & \rho \neq \overline{\rho} \\ 1, & \rho = \overline{\rho} \end{cases}$$

(recall that $\langle e_{\rho_1}, e_{\rho_2} \rangle = 0$ for $\rho_1 + \rho_2 \neq 0$).

For any $x \in \operatorname{im}\operatorname{ad}\psi$, $\rho \in \hat{R}$ we set

$$x_\rho = \frac{1}{2} \langle x, e_{-\rho} \rangle.$$

If ρ is a real ~~radical~~ root, then x_ρ is a real number, and if ρ is a nonreal ~~radical~~ root, then x_ρ is a complex number. Mapping $x \to \{x_\rho\}_{\rho\in R}$ defines special coordinates in space $\operatorname{im}\operatorname{ad}\psi$ (part of the coordinates are real and part are complex). Note that $x_{\bar\rho} = \overline{x}_\rho$. For any $x, y \in \operatorname{im}\operatorname{ad}\psi$ the following relations are fulfilled:

$$[\omega, x]_\rho = \rho(\omega)\, x_\rho, \quad \rho\in\hat{R}; \tag{8}$$

$$\langle x, y \rangle = \operatorname{Re}\left(\sum_{\rho\in R} x_\rho y_{-\rho}\right); \tag{9}$$

$$\langle \omega, [x, y] \rangle = \operatorname{Re}\left(\sum_{\rho\in R} \rho(\omega)\, x_\rho y_{-\rho}\right). \tag{10}$$

Finally, set

$$E_+ = \{x\in \operatorname{im}\operatorname{ad}\psi \mid x_\rho = 0\ \forall\rho\in R_-\},\ E_- = \{x\in \operatorname{im}\operatorname{ad}\psi \mid x_\rho = 0\ \forall\rho\in R_+\},\ E_0 = E_+\cap E_-$$

to be invariant subspaces corresponding to the sets of ~~radicals~~ roots $R_+ \cup R_0$, $R_- \cup R_0$, and R_0, respectively.

Proof of Proposition 2. Assume that $\lim\limits_{t\to+\infty} \operatorname{ind}\psi G_t' < +\infty$. We need to prove that $E_0 = 0$. We use Proposition 1. By virtue of this proposition there is a Lagrange plane $\Lambda \in L(\operatorname{im}\operatorname{ad}\psi \oplus \operatorname{im}\operatorname{ad}\psi)$, such that for any neighborhood O_{λ_0} of the arbitrary point $\lambda_0 \in \Lambda$ the relation

$$\int_0^\infty \min_{\lambda\in O_{\lambda_0}} \bar\sigma(\lambda, (e^{\tau \operatorname{ad} a}c, 0))^2 d\tau < +\infty$$

is fulfilled.

Equality (8) implies that $(e^{\tau\operatorname{ad}a}c)_\rho = e^{\tau\rho(a)}c_\rho$. According to the general position conditions we have $c_\rho = \rho(a)b_\rho \neq 0$, $\rho(a) \neq \rho'(a)$, $\forall\rho, \rho' \in R$. Therefore, all the integers converge only provided that $\Lambda \angle (E_+ \oplus 0)$. Since Λ is a Lagrange plane, the last relation is equivalent to the inclusion $(E_+ \oplus 0) \subset \Lambda$. Thus, space $E_+ \oplus 0$ and hence its subspace $E_0 \oplus 0$ are isotropic. At the same time, $\sigma((x, 0), (x', 0)) = \langle\psi, [x, x']\rangle$, and (10) implies that space $E_0 \oplus 0$ is isotropic if and only if it is empty.

COROLLARY 1. If $\mathfrak{M}$ is a compact Lie algebra, then

COROLLARY 2. If $\mathfrak{M} = sl_{\mathbf{R}}(n)$ and matrix $a \in sl_{\mathbf{R}}(n)$ has at most one real eigenvalue, then $\lim\limits_{t\to+\infty} \psi G_t'' = +\infty$.

We conduct further investigation under the assumption that the necessary condition for the finiteness of $\lim\limits_{t\to+\infty} \operatorname{ind}\psi G_t''$; is fulfilled; in other words, to the end of this section we assume that $R_0 = \phi$. In that case, $\operatorname{im}\operatorname{ad}\psi = E_+ \oplus E_-$,

$$\langle \omega, [x, x'] \rangle = \operatorname{Re}\left(\sum_{\rho\in R^+} \rho(\omega)(x_\rho x'_{-\rho} - x_{-\rho}x'_\rho)\right),$$
$$\forall\omega\in\mathscr{H},\quad x, x'\in\operatorname{im}\operatorname{ad}\psi$$

For every $x \in \operatorname{im}\operatorname{ad}\psi$ we denote by x^+ the orthogonal projection of x onto E_+ and by x^- the orthogonal projection of x onto E_-. Then

$$\langle x^+, x^+ \rangle = \langle x^-, x^- \rangle = 0, \quad \langle x^+, x^- \rangle = \operatorname{Re}\left(\sum_{\rho\in R^+} x_\rho x_{-\rho}\right).$$

We make a replacement of variables in Jacobi equation (7) by setting $\xi = [\eta, \psi]$, $x = y - \eta$. In variables (y, η) system (7) takes on the following form:

$$\gamma_0 \dot{y} = \langle [y, \eta], e^{\tau \operatorname{ad} a} c \rangle e^{\tau \operatorname{ad} a} c$$
$$\dot{\eta} = 0, \quad y, \eta \in \operatorname{im} \operatorname{ad} \psi. \tag{11}$$

The symplectic structure $\bar{\sigma}$ on $\operatorname{im} \operatorname{ad} \psi \oplus \operatorname{im} \operatorname{ad} \psi$ looks thus in these variables:

$$\bar{\sigma}((y, \eta), (y', \eta')) = \langle \psi, [y, y'] - [\eta, \eta'] \rangle .$$

The Hamiltonian is $h_\tau(y, \eta) = (1/2\gamma_0)\langle \psi, [y,[\psi, e^{\tau \operatorname{ad} a} c]]\rangle^2$. Curve Δ_τ, $\tau \geq 0$ on the Lagrange Grassmannian $L(\operatorname{im} \operatorname{ad} \psi \oplus \operatorname{im} \operatorname{ad} \psi)$ is generated by system (10) with initial condition $\Delta_0 = \{(y, y) \mid y \in \operatorname{im} \operatorname{ad} \psi\}$.

Note that the isotropic subspace $0 \in E_+$ lies in $\ker h_\tau$ $\forall \tau \in R$. Here $(0 \oplus E_+)^\angle = \operatorname{im} \operatorname{ad} \psi \oplus E_+$, and symplectic space $(0 \oplus E_+)^\angle/(0 \oplus E_+)$ is naturally isomorphic to space $\operatorname{im} \operatorname{ad} \psi$ with symplectic structure $\tilde{\sigma}(y, y') = \langle \psi, [y, y'] \rangle$, $y, y' \in \operatorname{im} \operatorname{ad} \psi$.

Hamiltonian system

$$\gamma_0 \dot{y} = \langle [y, \psi], e^{\tau \operatorname{ad} a} c \rangle e^{\tau \operatorname{ad} a} c \tag{12}$$

corresponds to Hamiltonian $\tilde{h}_\tau(y) = (1/2\gamma_0)\langle y, [\psi, e^{\tau \operatorname{ad} a} c]\rangle^2$ on $\operatorname{im} \operatorname{ad} \psi$.

Let $\tilde{\Delta}_\tau$ be a curve on Lagrange Grassmannian $L(\operatorname{im} \operatorname{ad} \psi)$, generated by system (12) with initial condition $\tilde{\Delta}_0 = E_+$. The corollary to Lemma 2 enables us to express $\operatorname{ind} \Delta \mid_{[0,t]}$ through $\operatorname{ind} \tilde{\Delta}_{[0,t]}$. An analysis of this expression enables us to somewhat refine the general estimate given in that corollary. We have

$$-\frac{1}{4} \operatorname{rank} \operatorname{ad} \psi \leqslant \operatorname{ind} \tilde{\Delta} |_{[0,t]} - \operatorname{ind} \Delta |_{[0,t]} \leqslant \operatorname{rank} \operatorname{ad} \psi \quad \forall t > 0.$$

Recall that point $t > 0$ is called conjugate to 0 for curve $\tilde{\Delta}_\tau$ if $\tilde{\Delta}_0 \cap \tilde{\Delta}_t \neq 0$. According to the corollary to Theorem 3.2,

$$\operatorname{ind} \tilde{\Delta} |_{[0,t]} = \sum_{0 < \tau < t} \dim (\tilde{\Delta}_\tau \cap \tilde{\Delta}_0) \quad \forall t > 0;$$

in particular, on each interval $[0, t]$ only a finite number of points conjugate to zero occur.

Let x_τ be an arbitrary solution of system (12); setting $x_\tau = e^{\tau \operatorname{ad} a} u_\tau$, we obtain

$$\gamma_0 \dot{u} = -(\gamma_0 \operatorname{ad} a + \langle \psi, [c, u] \rangle) c. \tag{13}$$

Linear transformation $u \to (1/\gamma_0)\langle \psi, [c, u]\rangle c = (1/\gamma_0)\langle [\psi, c], u\rangle c$ of the space $\operatorname{im} \operatorname{ad} \psi$ is denoted by the symbol $[\psi, c/\gamma_0] \otimes c$. Then

$$\tilde{\Delta}_t = e^{t \operatorname{ad} a} e^{t([\psi, c/\gamma_0] \otimes c - \operatorname{ad} a)} E_+ \quad \forall t > 0.$$

Since $e^{t \operatorname{ad} a} E_+ = E_+$, we have

$$\dim (\tilde{\Delta}_t \cap \tilde{\Delta}_0) = \dim (\tilde{\Delta}_t \cap E_+) = \dim (E_+ \cap e^{t([\psi, c/\gamma_0] \otimes c - \operatorname{ad} a)} E_+).$$

Consequently,

$$\operatorname{ind}\tilde{\Delta}|_{[0,t]}=\sum_{0<\tau<t}\dim(E_+\cap e^{\tau([\psi,c/\gamma_0]\otimes c-\operatorname{ad}}E_+). \quad (14)$$

LEMMA 3. Suppose that h is a quadratic stationary Hamiltonian on some symplectic space Σ and

$$\dot{x}=Hx, \quad x\in\Sigma,$$

is the corresponding Hamiltonian system.

Then there exists a Lagrangian plane $\Lambda_0 \in L(\Sigma)$, such that

1) Subspace $\Lambda_0 \cap H\Lambda_0$ is invariant under H.

2) For any $t > 0$,

$$\sum_{0<\tau<t}\dim(\Lambda_0\cap e^{\tau H}\Lambda_0/\Lambda_0\cap H\Lambda_0)=\sum_{j=1}^{l}[t\nu_j],$$

where $\pm i\pi\nu_1, \ldots, \pm i\pi\nu_l$ are all purely imaginative eigenvalues of matrix H; moreover, each is taken as many times as is its multiplicity; square brackets [ ] denote the integer part of the number appearing in them.

3) If for some $\Lambda \in L(\Sigma)$ quadratic form $h|\Lambda$ is nonnegative, $h|\Lambda \geq 0$, then also $h|\Lambda_0 \geq 0$.

We omit the proof of this lemma, plane Λ_0 is clearly calculated using the Williamson canonical forms of the linear Hamiltonian systems (the description of these forms is found, for example, in [9]).

The autonomous Hamiltonian system (13) corresponds to the Hamiltonian

$$h=\frac{1}{2\gamma_0}\langle[\psi,c],u\rangle^2+\frac{1}{2}\langle[\psi,u],[a,u]\rangle, \quad u\in\operatorname{im}\operatorname{ad}\psi.$$

Since $[\omega, E_+] \subset E_+ \ \forall\omega \in \mathcal{H}$ and $(E_+, E_+) = 0$, we have $h|_{E_+} \geq 0$.

Let $H = [\psi, c/\gamma_0] \otimes c - \operatorname{ad} a$ and $L(\operatorname{im}\operatorname{ad}\psi) \ni \Lambda_0$ be a Lagrangian plane the existence of which is guaranteed by Lemma 3. Since $h|_{E_+} \geq 0$, $h|_{\Lambda_0} \geq 0$ and Hamiltonian h is the first integral of the proper Hamiltonian system, then $e^{\tau H}\Lambda_0$, $e^{\tau H}E_+$, $\tau \geq 0$ are nondecreasing curves on $L(\operatorname{im}\operatorname{ad}\psi)$. Lemma 2 enables us to express $\operatorname{ind}(e^{\tau H}E_+|_{[0,t]})$ through $\operatorname{ind}(e^{\tau H}\Lambda_0|_{[0,t]})$.

In any case,

$$|\operatorname{ind}(e^{\tau H}E_+|_{[0,t]})-\operatorname{ind}(e^{\tau H}\Lambda_0|_{[0,t]})+\dim(\Lambda_0\cap H\Lambda_0)|\leqslant\frac{1}{2}\operatorname{rank}\operatorname{ad}\psi,$$
$$\forall t>0.$$

On the other hand, Theorems 3.1 and 3.2 imply that

$$\operatorname{ind}(e^{\tau H}E_+|_{[0,t]}),=\sum_{0<\tau>t}\dim(E_+\cap e^{\tau H}E_+),$$

$$\operatorname{ind}(e^{\tau H}\Lambda_0|_{[0,t]})=\sum_{0<\tau<t}\dim(\Lambda_0\cap e^{\tau H}\Lambda_0\Lambda_0\cap H\Lambda_0).$$

From formula (14) we derive the identity

$$\operatorname{ind}\tilde{\Delta}|_{[0,t]}=\operatorname{ind}(e^{\tau H}E_+|_{[0,t]})\ \forall t>0.$$

Assertion 2) of Lemma 3 now reduces the calculation of $\operatorname{ind}\tilde{\Delta}|_{[0,t]}$ to the calculation of purely imaginary eigenvalues of matrix $H = [\psi, c/\gamma_0] \otimes c - \operatorname{ad} a$. In any case, $\lim_{t\to+\infty}\operatorname{ind}\tilde{\Delta}|_{[0,t]}<+\infty$ if and only if matrix H has no nonzero purely imaginary eigenvalues. Collecting together the inequalities established above, we obtain

Proposition 3. If $\lim_{t\to+\infty}\operatorname{ind}\psi G_t^{\cdot}<+\infty$, then

$$\operatorname{ind} \psi G_t'' \leqslant \frac{1}{4} \min\{5 \operatorname{rank} \operatorname{ad} \psi, \operatorname{codim} \mathscr{H}\} \quad \forall t > 0.$$

Remark. Inequality $\operatorname{ind}\psi G_t'' \leq 1/4 \operatorname{codim} \mathscr{H} = 1/4(\dim \mathfrak{M} - \operatorname{rank} \mathfrak{M})$ follows directly from the estimates given above only for regular $\psi \in \mathscr{H} \cap b^{\perp}$; to nonregular ψ it extends by continuity.

It is natural to call the situation when $\operatorname{ind}\psi G_t'' \to +\infty$ ($t \to \infty$) elliptic (the Jacobian curve "oscillates") and when $\operatorname{ind}\psi G_t''$ is bounded on the half-line $(0, +\infty)$ hyperbolic (the Jacobian curve "does not oscillate"). In order to separate one situation from another we must be able to ascertain whether or not matrix $[\psi, c/\gamma_0] \otimes c - \operatorname{ad} a$ has purely imaginary eigenvalues. The knowledge of the purely imaginary eigenvalues themselves immediately gives (see Assertion 2 of Lemma 3) the magnitude of $\operatorname{ind}\psi G_t''$ to within a value of "order rank ad ψ."

Soon we shall express the characteristic polynomial of matrix $([\psi, c/\gamma_0] \otimes c - \operatorname{ad} a)$ through the numbers γ_i, $i = 0, 1, \ldots$ (see p. 2720) and the coefficients of the characteristic polynomial of matrix ad a.

According to equalities (8) and (10), the coordinate transcript of system (13) has the form

$$\gamma_0 \dot{u}_\rho = -\gamma_0 \rho(a) u_\rho + \operatorname{Re}\left(\sum_{\rho \in R} \rho(\psi) c_\rho u_{-\rho}\right) c_\rho, \quad \rho \in R.$$

For any $\rho \in R_+$ we set

$$v_\rho = \frac{\rho(\psi)}{2}(c_\rho u_{-\rho} - c_{-\rho} u_\rho)$$

$$v_\rho' = \frac{\rho(\psi)}{2}(c_\rho u_{-\rho} + c_{-\rho} u_\rho).$$

Then

$$\dot{v}_\rho = \rho(a) v_\rho'$$

$$\gamma_0 \dot{v}_\rho' = \gamma_0 \rho(a) v_\rho + \operatorname{Re}\left(\sum_{\rho \in R_+} v_\rho\right) \rho(\psi) c_\rho c_{-\rho}, \quad \rho \in R_+.$$

Consequently,

$$\gamma_0 \ddot{v}_\rho = \gamma_0 \rho(a)^2 v_\rho + \rho(a)\, \rho(\psi) c_\rho c_{-\rho} \operatorname{Re}\left(\sum_{\rho \in R^+} v_\rho\right), \quad \rho \in R_+. \tag{15}$$

Let $N = \dim E_+$; recall that

$$\gamma_k = \langle \psi, [\operatorname{ad}^{k+1} ab, \operatorname{ad}^k ab] \rangle = (-1)^{k-1} 2 \operatorname{Re}\left(\sum_{\rho \in R^+} \rho(\psi) \rho(a)^{2k-1} c_\rho c_{-\rho}\right)$$

We make a linear replacement of variables in system (15) by setting

$$w_k = \operatorname{Re} \sum_{\rho \in R_+} \rho(a)^{2k} v_\rho, \quad k = 0, 1, \ldots, N.$$

We have

$$\gamma_0 \ddot{w}_k = \gamma_0 w_{k+1} + (-1)^k \gamma_{k+1} w_0, \quad k = 0, 1, \ldots, N-1.$$

Another missing inequality is obtained from the condition that the numbers $\pm\rho(a)$, $\rho \in R_+$ are the roots of the characteristic polynomial of operator ad a. Let

$$\det(\operatorname{ad} a - sI) = s^{\operatorname{rank}\mathfrak{M}} \sum_{j=1}^{N} (-1)^j \alpha_j s^{2j},$$

then

$$\sum_{j=0}^{N} (-1)^j \alpha_j w_j = 0 \text{ and } \sum_{j=0}^{N} \alpha_j \gamma_j = 0.$$

Consequently,

$$\sum_{k=0}^{N} \sum_{j=0}^{k-1} (-1)^{k+j} \alpha_k \gamma_j w_0^{(2k-2j)} = 0.$$

Suppose that $\varphi(s)$ is the characteristic polynomial of the last equation, $\varphi(s) = \sum_{k=0}^{N} \sum_{j=0}^{k-1} (-1)^{k+j} \alpha_k \gamma_j s^{2(k-j)}$. The roots of this polynomial are the characteristic roots of system (13). After elementary transformations we get

$$\frac{1}{s} \varphi(\sqrt{-s}) = \sum_{j=1}^{N} s^{j-1} \sum_{k=j}^{N} \alpha_k \gamma_{k-j}.$$

Now we sum up the principal results obtained in this subsection. Everything is formulated in terms of the collection of numbers γ_k, $k = 0, \ldots, N-1$, where $N = 1/2(\dim \mathfrak{M} - \operatorname{rank} \mathfrak{M})$. Recall that numbers α_k depend only on $a \in \mathfrak{M}$ and numbers γ_k depend also on $b \in \mathfrak{M}$ and (linearly) on $\psi \in \mathscr{H} \cap b^{\perp}$, where $\mathscr{H}$ is a Cartan subalgebra containing a. We assume the general position condition to be fulfilled:

$$\operatorname{codim} \operatorname{span} \{\operatorname{ad}^j ab \mid 0 \leqslant j \leqslant 2N\} = \operatorname{rank} \mathfrak{M} - 1.$$

In the process of calculating we also assume that $\gamma_0 > 0$; however, it is easy to get rid of this condition.

THEOREM 1. I) Let $t > 0$. Then $\operatorname{ind} \psi G_t'' < +\infty \Leftrightarrow$ the first number different from zero in the sequence $\gamma_0, \gamma_1, \ldots, \gamma_{N-1}$ is positive.

II) Let $\operatorname{ind} \psi G_t'' < +\infty$ for some (and hence any) $t > 0$. Then $\lim_{t \to +\infty} \operatorname{ind} \psi G_t'' < +\infty \Leftrightarrow$ polynomials $\sum_{j=0}^{N} \alpha_j s^j$ and $\sum_{j=1}^{N} s^{j-1} \sum_{k=j}^{N} \alpha_k \gamma_{k-j}$ in variable $s \in \mathbf{R}$ do not have positive real roots.

III) If $\lim_{t \to +\infty} \operatorname{ind} \psi G_t'' < +\infty$ then $\operatorname{ind} \psi G_t'' \le N/2 \ \forall t > 0$.

In conclusion, we apply Theorem 1 to some Lie algebras of rank 2. In this case, $\dim(\mathscr{H} \cap b^{\perp}) = 1$, and vector ψ is uniquely determined to within a scalar factor. The general position condition guarantees that at least one of the numbers γ_k, $k = 0, 1, \ldots, N-1$ is different from zero.

1) $\mathfrak{M} = so(1, 3)$ is a Lie algebra of a Lorentz group. Here $N = 2$, $\alpha_1^2 \le 4\alpha_0$. If $2\sqrt{\alpha_0} + \alpha_1 \neq 0$, $\gamma_0 \ge 0$, $\alpha_1\gamma_0 + \gamma_1 \ge 0$, then $\operatorname{ind} \psi G_t'' \le 1 \ \forall t > 0$.

Otherwise, $\operatorname{ind} \psi G_t'' \to +\infty$ $(t \to +\infty)$.

2) $\mathfrak{M} = so(2, 2)$. Again, $N = 2$; however, $4\alpha_0 \le \alpha_1^2$. If $\alpha_0 > 0$, $\alpha_1 > 0$, $\gamma_0 \ge 0$, $\alpha_1\gamma_0 + \gamma_1 \ge 0$, then $\operatorname{ind} \psi G_t'' \le 1 \ \forall t > 0$.

Otherwise, $\operatorname{ind}\psi G_t'' \to +\infty$ $(t \to +\infty)$.

3) $\mathfrak{M} = so(4)$. This is a compact Lie algebra; therefore, $\operatorname{ind}\psi G_t'' \to +\infty$ $(t \to +\infty)$.

4) $\mathfrak{M} = sl_R(3)$. Here $N = 3$. If $\alpha_j > 0$, $j = 0, 1, 2$, $\gamma_0 \geq 0$ and either $(\alpha_2\gamma_0 + \gamma_1)^2 < 4\gamma_0(\alpha_1\gamma_0 + \alpha_2\gamma_1 + \gamma_2)$ or $(\alpha_2\gamma_0 + \gamma_1 \geq 0)$ & $(\alpha_1\gamma_0 + \alpha_2\gamma_1 + \gamma_2 \geq 0)$, then $\operatorname{ind}\psi G_t'' \leq 1$ $\forall t > 0$.

Otherwise, $\operatorname{ind}\psi G_t'' \to +\infty$ $(t \to +\infty)$.

2. If in the previous subsection we considered systems with one control parameter, now, on the contrary, we study one class of systems in which there are sufficiently many control parameters. For these systems, $\operatorname{ind}\psi G_t''$, for fixed $\psi \perp \operatorname{im} G_t'$, is computed very simply; therefore, an opportunity arises to explicitly describe the partition of $(\operatorname{im} G_t')^\perp$ into domains corresponding to various values of the index and to find homology groups of the set $\mathfrak{g}_t^{-1}(0)\backslash 0$.

Let M be a compact semisimple Lie group with a Lie algebra $\mathfrak{M}$. Angled brackets $\langle\cdot,\cdot\rangle$, as above, denote a Killing form on $\mathfrak{M}$. This form is negative definite. Suppose that a is a regular element of Lie algebra $\mathfrak{M}$, $\langle a, a\rangle = -1$, and $\mathscr{H}$ is a Cartan subalgebra in $\mathfrak{M}$ containing element a. Denote by U the intersection of sphere $\{x \in \mathfrak{M} \mid \langle x, x\rangle = -1\}$ with subspace $Ra + \mathscr{H}^\perp$,

$$U=\{\alpha a+v \mid \alpha\in R,\ v\perp\mathscr{H},\ \langle v, v\rangle=\alpha^2-1\},$$

and consider the controlled system

$$\dot{\mu}=\mu\circ u,\quad u\in U,\quad \mu(0)=\mu_0$$

on M in the neighborhood of control $\tilde{u}(\tau) \equiv a$.

We have

$$G_t: u(\cdot)\mapsto\mu_0\circ\overrightarrow{\exp}\int_0^t (e^{\tau\,\mathrm{ad}\,a}u(\tau)-a)\,d\tau,$$

$$G_t'v(\cdot)=\int_0^t e^{\tau\,\mathrm{ad}\,a}v(\tau)\,d\tau,\quad v(\tau)\in\mathscr{H}^\perp,\quad \operatorname{im} G_t'=\mathscr{H}^\perp.$$

$$\mathfrak{g}_t(v_1(\cdot), v_2(\cdot))=\int_0^t \Big(\langle v_1(\tau), v_2(\tau)\rangle a+ \left[\int_0^\tau e^{\theta\,\mathrm{ad}\,a}v_1(\theta)\,d\theta,\ e^{\tau\,\mathrm{ad}\,a}v_2(\tau)\right]\Big)d\tau+\mathscr{H}^\perp, \tag{16}$$

$$G_t''=\mathfrak{g}_t\big|_{\ker G_t'\times\ker G_t'}.$$

The notation of the initial point μ_0 in the expressions for G_t' and G_t'' can be dropped using the identification $T_{\mu_0}M \approx \mathfrak{M}$. Similarly, without special stipulations, we use below the identification $\mathfrak{M}^* \approx \mathfrak{M}$ defined by the Killing form.

Proposition 4. Let $\psi \in \mathscr{H}\backslash 0$, $t > 0$. Then

$$\operatorname{ind}\psi G_t''<+\infty \Leftrightarrow \langle \psi, a\rangle <0.$$

Proof. Since $\langle v, v\rangle < 0$ $\forall v \neq 0$, then (16) and Proposition 3.1 imply that inequality $\langle\psi, a\rangle \leq 0$ is a necessary and inequality $\langle\psi, a\rangle < 0$ is a sufficient condition for the finiteness of $\operatorname{ind}\psi G_t''$. There still remains the case when $\langle\psi, a\rangle = 0$. In that case, according to Proposition 3.1, for the finiteness of $\operatorname{ind}\psi G_t''$ the fulfillment of identity

$$\langle \psi, [e^{\tau\,\mathrm{ad}\,a}v_1, e^{\tau\,\mathrm{ad}\,a}v_2]\rangle = 0\quad \forall v_1, v_2\perp\mathscr{H},\quad \forall\tau\in[0, t].$$

is necessary. This identity, however, is not fulfilled since

$$\langle \psi, [e^{\tau\,\mathrm{ad}\,a}v_1, e^{\tau\,\mathrm{ad}\,a}v_2]\rangle = \langle \psi, e^{\tau\,\mathrm{ad}\,a}[v_1, v_2]\rangle = \langle \psi, [v_1, v_2]\rangle$$

and

$$[\mathscr{H}^{\perp}, \mathscr{H}^{\perp}] \supset \mathscr{H}. \blacksquare$$

Thus, when calculating ind $\psi G_t''$, we must consider only the case when $\langle \psi, a \rangle < 0$. Below we assume the normalization condition $\langle \psi, a \rangle = -1$ to be fulfilled.

We pass to the description of the Jacobi equation. In subsection 1 of Sec. 3, with the arbitrary subspace $\Pi \subset T_{\mu_0}M$ and covector $\psi \perp \Pi$ we associated the symplectic space $E_{\Pi,\psi}$, and in subsection 1 of the present section (see p. 130) we described a natural model of this space in the case when M is a semisimple Lie group. In our situation, $\Pi = \mathscr{H}^{\perp}$, while $E_{\Pi,\psi}$ is naturally isomorphic to space $\mathscr{H}^{\perp} \oplus \mathscr{H}^{\perp}$ with the skew inner product

$$\bar{\sigma}:(x_1, \xi_1)\wedge(x_2, \xi_2) \mapsto \langle \psi, [x_1, x_2] \rangle + \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle,$$
$$x_i, \xi_i \in \mathscr{H}^{\perp}, \quad i = 1, 2.$$

A Hamiltonian system on this space defined by the Hamiltonian

$$h(x, \xi) = -\frac{1}{2}\langle [x, \psi] + \xi, [x, \psi] + \xi \rangle \quad x, \xi \in \mathscr{H}^{\perp}$$

corresponds to the Jacobian equation.

As we see, the Hamiltonian proved to be autonomous. The Hamiltonian system has the following form:

$$\begin{cases} \dot{x} = [\psi, x] - \xi \\ \dot{\xi} = 0. \end{cases} \tag{17}$$

The Jacobian curve Λ_τ, $\tau \geq 0$ is a smooth curve on the Lagrange Grassmannian $L(\mathscr{H}^{\perp} \oplus \mathscr{H}^{\perp})$, defined by the system (17) with the initial condition

$$\Lambda_0 = 0 \oplus \mathscr{H}^{\perp} = \{(x, \xi) \mid x = 0, \ \xi \in \mathscr{H}^{\perp}\}.$$

So

$$\Lambda_t = \left\{ \left(\left(\int_0^t e^{(t-\tau)\operatorname{ad}\psi} d\tau \xi, \ \xi \right) \right) \middle| \xi \in \mathscr{H}^{\perp} \right\}.$$

According to the corollary to Theorem 3.2,

$$\operatorname{ind} \psi G_t'' = \operatorname{ind} \Lambda|_{[0,t]} = \sum_{0<\tau<t} \dim(\Lambda_0 \cap \Lambda_\tau).$$

Since $\mathfrak{M}$ is a compact Lie algebra,.all of its radicals are purely imaginary. Suppose that $\dim \mathscr{H}^{\perp} = 2N$, and $\rho_1, \ldots, \rho_N \in \mathscr{H}$ are such that the linear forms on $\mathscr{H}$ of form $\omega \to \pm 2\pi i \langle \rho_j, \omega \rangle$, $j = 1, \ldots, N$, make up a complete set of radicals of a Cartan subalgebra $\mathscr{H}$. We get

Proposition 5. If $t > 0$ is such that numbers $t\langle \rho_j, \psi \rangle$, $j = 1, \ldots, N$ are not integers, then

$$\operatorname{ind} \psi G_t'' = 2 \sum_{j=1}^{N} [t \mid \langle \rho_j, \psi \rangle \mid],$$

where [ ] is the "integer part."

Recall that ψ has the following normalization: $\langle\psi, a\rangle = -1$. Set $\Omega(t) = \{\omega \in \mathscr{H} \mid \langle\omega, a\rangle = -t\}$,

$$\Omega_k(t)=\{\omega\in\Omega(t)\mid \operatorname{ind}\omega G_t^{\cdot}\leqslant k\},\quad k=0, 1,\ldots.$$

Proposition 5 implies that

$$\Omega_{2k}(t)=\left\{\omega\in\Omega(t)\,\middle|\,\sum_{j=1}^{N}[|\langle\rho_j, \omega\rangle|]\leqslant k\right\},\quad \Omega_{2k+1}(t)=\Omega_{2k}(t),\quad k=0, 1,\ldots.$$

Without loss of generality we can (and will) assume that all the forms ρ_j are negative on vector a, i.e., $\langle\rho_j, a\rangle < 0$, $j = 1, \ldots, N$.

Let

$$\Omega^-(t)=\{\omega\in\Omega(t)\mid\langle\rho_j, \omega\rangle\leqslant 0,\ j=1,\ldots, N\}$$

and

$$\Omega_k^-(t)=\Omega_k(t)\cap\Omega^-(t),\ k=0, 1,\ldots.$$

Set $\Omega^-(t)$, the intersection of hyperplane $\Omega(t)$ and the closure of the Weyl chamber containing vector a, is a simplex of dimension $\dim\mathscr{H}-1$. It turns out that filtrations $\Omega_0(t)\subset\Omega_2(t)\subset\ldots\subset\Omega(t)$ and $\Omega_0^-(t)\subset\Omega_2^-(t)\subset\ldots\subset\Omega^-(t)$ are homotopy equivalent. Namely, the following holds:

LEMMA 3. There exists a homotopy retraction $\Omega(t)$ on $\Omega^-(t)$ preserving filtration $\Omega_k(t)$, $k = 0, 1, 2, \ldots$.

Proof. Let $\dim H = r$. Among roots $2\pi i\rho_1, \ldots, 2\pi i\rho_N$ there are exactly r simple ones; without loss of generality, we can assume that this is $2\pi i\rho_1, \ldots, 2\pi i\rho_r$. Vectors $\rho_1, \ldots, \rho_r$ form a basis of the linear space $\mathscr{H}$. Any vector ρ_j, $j = 1, \ldots, N$ is a linear combination of vectors with integer *nonnegative* coefficients. For every $\omega\in\mathscr{H}$ we denote by ω_- an element of space $\mathscr{H}$ uniquely determined by the conditions

$$\rho_j(\omega_-)=\begin{cases}\langle\rho_j, \omega\rangle, & \text{if } \langle\rho_j, \omega\rangle\leqslant 0,\\ 0, & \text{if } \langle\rho_j, \omega\rangle>0,\end{cases}\quad j=1,\ldots, r.$$

The sought homotopic retraction of hyperplane $\Omega(t)$ on $\Omega^-(t)$ has the following form:

$$\varphi_s:\omega\mapsto(1-s)\,\omega-\frac{st}{\langle a, \omega_-\rangle}\,\omega_-,\quad \omega\in\Omega(t),\quad s\in[0, 1].\ \blacksquare$$

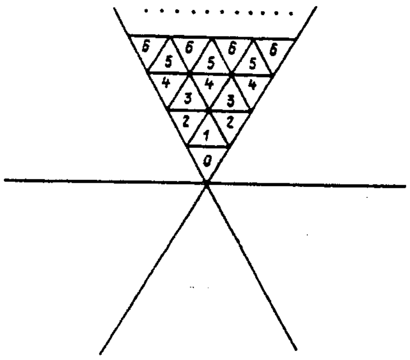
Let Z be a ring of integers. The set $\mathfrak{R} = \bigcup_{j=1}^{N}\{\omega\mid\langle\rho_j, \omega\rangle\in Z\}$ is usually called a Stiefel diagram. The integer-valued function $I(\omega)=\sum_{j=1}^{N}[|\langle\rho_j, \omega\rangle|]$ is locally constant on $\mathscr{H}/\mathfrak{R}$. The closure of each component of set $\mathscr{H}/\mathfrak{R}$ is a convex polyhedron. The values of $I(\omega)$ on a pair of neighboring components adjacent on an $(r-1)$-dimensional face lying in one Weyl chamber differ by a unit. We are only interested in the restriction of this function to a closure of the Weyl chamber $\{\omega\in\mathscr{H}\mid\langle\rho_j, \omega\rangle < 0,\ j = 1, \ldots, N\}$, on which our integer-valued function takes the following form:

$$I(\omega)=\sum_{j=1}^{N}[-\langle\rho_j, \omega\rangle].$$

For example, for Lie algebra su(r + 1) the number of radicals is 2N = r(r + 1); on the isolated Weyl chamber the function $I(\omega)$ has the following form:

$$I(\omega)=\sum_{1<\alpha<\beta<r}\left[-\sum_{j=\alpha}^{\beta}\langle\rho_j,\omega\rangle\right].$$

The case when r = 2 can be represented on the figure



Proposition 5 implies the equality ind $\omega G_t'' = 2I(\omega)$, $\forall\omega\in\Omega(t)\setminus\mathfrak{R}$.

We have already noted that the closure of any component of the connection of set $\mathcal{H}/\mathfrak{R}$ is a convex polyhedron. The vertices of these polyhedra are called nodes of diagram $\mathfrak{R}$. These are points $\omega\in\mathfrak{R}$, such that $\langle\rho_{j_1},\omega\rangle\in\mathbf{Z}$, ..., $\langle\rho_{j_r},\omega\rangle\in\mathbf{Z}$ for some linearly independent vectors ρ_{j_1}, ..., ρ_{j_r} from set $\{\rho_1, ..., \rho_N\}$.

Proposition 6. Assume that a does not lie in the linear span of any r − 1 vectors from set $\{\rho_1, ..., \rho_N\}$. If for a given t > 0 the quadratic mapping g_t is degenerate, then hyperplane $\Omega(t)$ contains a node of diagram $\mathfrak{R}$.

Proof. Making in formula (16) a substitution of parameters $\omega(\tau) = e^{\tau\,\mathrm{ad}\,a}v(\tau)$, we get the quadratic mapping

$$\begin{gathered}Q_t(w(\cdot),\ w(\cdot))\overset{\mathrm{def}}{=}g_t(v(\cdot),\ v(\cdot))=\\ =\int_0^t\left(\langle w(\tau),\ w(\tau)\rangle a+\left[\int_0^\tau w(\theta)\,d\theta,\ w(\tau)\right]\right)d\tau+\mathcal{H}^\perp,\qquad(18)\\ w(\tau)\in\mathcal{H}^\perp\ \text{for}\ 0\leqslant\tau\leqslant t,\quad\int_0^t w(\tau)\,d\tau=0.\end{gathered}$$

Vector-valued function w(·) is a critical point of mapping Q_t if and only if for some $\omega\in\mathcal{H}^\perp\setminus 0$, $b\in\mathcal{H}^\perp$ identity

$$\langle\omega,a\rangle w(\tau)+\left[\omega,\ \int_0^\tau w(\theta)\,d\theta\right]\equiv b,\quad 0\leqslant\tau\leqslant t \qquad (19)$$

is fulfilled.

The degeneracy of quadratic mapping Q_t is equivalent to the fact that $Q_t(w(\cdot), w(\cdot)) = 0$ for some nonzero critical point w(·). We consider separately two cases:

1) $\langle\omega, a\rangle = 0$. Differentiating identity $\left[\omega,\ \int_0^\tau w(\theta)\,d\theta\right]=b$ with respect to τ, we obtain $[\omega, w(\tau)] \equiv 0$.

Identity $[\omega, w(\tau)] \equiv 0$ implies that

$$[w(\tau_1),\ w(\tau_2)]\in\mathrm{span}\{\rho_j\mid\langle\rho_j,\omega\rangle=0\}+\mathcal{H}^\perp\ \ \forall\tau_1,\tau_2\in[0,t].$$

Among vectors ρ_j orthogonal to ω there are at most r — 1 linearly independent ones and the orthogonal projection of vector $\int_0^t \left[\int_0^\tau w(\theta)\,d\theta,\ w(\tau)\right] d\tau$ to $\mathscr{H}$ is contained in their linear span. At the same time, according to the hypothesis vector a is not contained in this linear span.

2) $\langle \omega, a\rangle \neq 0$. Without loss of generality, we can assume that $\langle \omega, a\rangle = -t$, i.e., $\omega \in \Omega(t)$.

Differentiating Eqs. (19) with respect to τ, we obtain $t\dot{w} = [\omega, w]$. Consequently, $w(t) = e^{(\tau/t)\mathrm{ad}\omega}b$.

Equality $\int_0^t w(\tau)\,d\tau = 0$ is equivalent to the relation $e^{\mathrm{ad}\omega}b = b$. Suppose that b_j is an orthogonal projection of vector b onto a two-dimensional invariant subspace of operator $\mathrm{ad}\,\omega$ corresponding to eigenvalues $\pm 2\pi i\langle \rho_j, \omega\rangle$, j = 1, ..., N.

We see that b_j can differ from zero only for those j for which $\langle \rho_j, \omega\rangle \in \mathbf{Z}$. Consequently,

$$[w(\tau_1),\ w(\tau_2)] \in \mathrm{span}\,\{\rho_j \mid \langle \rho_j, \omega\rangle \in \mathbf{Z}\} + \mathscr{H}^\perp, \quad \forall \tau_1, \tau_2 \in [0, t].$$

If among vectors ρ_j satisfying condition $\langle \rho_j, \omega\rangle \in \mathbf{Z}$ there are r linearly independent vectors, then ω is a node of diagram $\mathscr{R}$. Otherwise,

$$a \notin \mathrm{span}\,\{[w(\tau_1), w(\tau_2)],\ \tau_1, \tau_2 \in [0, t]\} + \mathscr{H}^\perp. \ \blacksquare$$

COROLLARY. If a does not lie in the linear span of any r — 1 vectors from set $\{\rho_1, \ldots, \rho_N\}$, then on the half-line $[0, +\infty)$ there are at most an even number of points t such that the quadratic mapping $\mathfrak{g}_t$ is degenerate.

In the case when $\mathfrak{g}_t$ is nondegenerate, Theorem 2 from [1] enables us to estimate very accurately the homology groups of set $\mathfrak{g}_t^{-1}(0)]0$ in terms of filtration $\Omega_k^-(t)$, k = 0, 1, 2, ... In particular,

$$\mathrm{rank}\,\bar{H}_n(\mathfrak{g}_t^{-1}(0)\setminus 0) \leqslant \sum_{j=0}^{r-1} \mathrm{rank}\, H^j(\Omega^-_{n+j+1}(t), \Omega^-_{n+j}(t)),\ n = 0, 1, 2, \ldots;$$

here, if $r \leq 3$ (and also in many other cases), the inequality becomes an equality.* In its turn, Theorem 4.2 connects homology groups $\bar{H}_n(\mathfrak{g}_t^{-1}(0)\setminus(0) \approx H_{n+1}(\mathfrak{g}_t^{-1}(0), \mathfrak{g}_t^{-1}(0)\setminus 0)$ with the local structure of a set of level $G_t^{-1}(\mu_0)$.

In conclusion of the present section we examine to the end the case of $\mathfrak{M} = su(3)$.

Cartan subalgebra $\mathscr{H}$ is two-dimensional, N = 3. We introduce in $\mathscr{H}$ coordinates $\omega = (\omega_1, \omega_2)$ by setting $\omega_1 = \langle \rho_1, \omega\rangle$, $\omega_2 = \langle \rho_2, \omega\rangle$; then $\langle \rho_3, \omega\rangle = \omega_1 + \omega_2$.

The Killing form has the following form:

$$\langle \omega, \omega\rangle = -2\left(\omega_1^2 + \omega_2^2 + (\omega_1 + \omega_2)^2\right);$$

$$\rho_1 = \left(-\frac{1}{3}, \frac{1}{6}\right), \quad \rho_2 = \left(\frac{1}{6}, -\frac{1}{3}\right), \quad \rho_3 = \left(-\frac{1}{6}, -\frac{1}{6}\right).$$

Let $a_1 = \langle \rho_1, a\rangle$, $a_2 = \langle \rho_2, a\rangle$; by hypothesis $a_1 < 0$, $a_2 < 0$. We assume that the general position conditions formulated in Proposition 6 are fulfilled; in this case, they reduce to the relation $a_1 \neq a_2$.

Line $\Omega(t)$ is defined by the equation

$$(2a_1 + a_2)\,\omega_1 + (a_1 + 2a_2)\,\omega_2 = \frac{t}{2}.$$

Set

$$\alpha = \frac{2a_1 + a_2}{a_1 + 2a_2}, \quad \tau = \frac{-t}{2(a_1 + 2a_2)}.$$

*Recall that $\bar{H}_n(\cdot)$ denotes an n-dimensional group of singular homologies.

By hypothesis $\alpha > 0$, $\tau > 0$, $\alpha \neq 1$. When changing the places of roots ρ_1 and ρ_2, the number α changes into $1/\alpha$; therefore, without loss of generality, we can assume that $0 < \alpha < 1$.

Let $\nu = -\omega_1$; segment $\Omega^-(t)$ has the form

$$\Omega^-(t)=\left\{(-\nu,\ \alpha\nu-\tau)\mid 0\leqslant\nu\leqslant\frac{\tau}{\alpha}\right\}.$$

Identify $\Omega^-(t)$ with segment $0 \leq \nu \leq \tau/\alpha$.

We have

$$I(\nu)=[\nu]+[\tau-\alpha\nu]+[\tau+(1-\alpha)\nu],$$
$$\Omega_{2k}^-(t)=\mathrm{cl}\left\{\nu\in\left[0,\frac{\tau}{\alpha}\right]\,\Big|\,[\nu]+[\tau-\alpha\nu]+[\tau+(1-\alpha)\nu]\leqslant k\right\},$$
$$\Omega_{2k+1}^-(t)=\Omega_{2k}^-(t),\quad k=0,1,\ldots.$$

Assume that none of the numbers of form $\tau-\alpha j$, $j = 0, \pm1, \pm2, \ldots$ is an integer. Then, according to Proposition 6, the quadratic mapping $\mathfrak{g}_t$ is nondegenerate.

Furthermore, $\mathfrak{g}_t^{-1}(0) = 0$ if and only if $\Omega_0^-(t) = \varnothing$. Consequently,

$$\mathfrak{g}_t^{-1}(0)=0\Leftrightarrow\tau<\alpha.$$

Below we assume that $\tau > \alpha$.

Recall that

$$\tilde{H}_n(\mathfrak{g}_t^{-1}(0)\setminus 0)\approx H^0(\Omega_{n+1}^-(t),\ \Omega_n^-(t))\oplus H^1(\Omega_{n+2}^-(t),\ \Omega_{n+1}^-(t)),$$
$$n=0,1,2\ldots$$

(here there is indeed an isomorphism and not only an equality of ranks, since the groups on the right-hand side have no torsion).

Where it is convenient instead of homology groups $\tilde{H}_n(\cdot)$ separately for each dimension n, we will consider graded groups $\tilde{H}(\cdot)=\bigoplus_{n=0}^{+\infty}\tilde{H}^n(\cdot)$.

We introduce the notation: $\varepsilon = [\tau - [\tau] - \alpha]$; $\iota(n) = 4n + 4[(\tau - n)/\alpha] - 2$ for $n = 0, 1, \ldots, [\tau]$.

Then

$$\tilde{H}(\mathfrak{g}_t^{-1}(0)\setminus 0)\approx\tilde{H}(S^{4[\tau]-2+\varepsilon})\bigoplus_{n=1}^{[\tau]+\varepsilon}(\tilde{H}(S^{\iota(n)-1}\vee S^{\iota(n)})).$$

In particular,

$$\mathrm{rank}\,\tilde{H}(\mathfrak{g}_t^{-1}(0)\setminus 0)=2([\tau]+\varepsilon)-1.$$

Consequently, the rank of group $\tilde{H}(\mathfrak{g}_t^{-1}(0)\setminus 0)$ tends to infinity as $t \to +\infty$ (whereas for a scalar quadratic form this rank does not exceed one!). It is interesting to consider the asymptotic properties of this graded group as $t \to +\infty$ (for a fixed a). Recall that τ is a linear function of t. Setting $d(t) = \min\{n \mid \tilde{H}_n(\mathfrak{g}_t^{-1}(0)\setminus 0) \neq 0\}$, $D(t) = \max\{n \mid \tilde{H}_n(\mathfrak{g}_t^{-1}(0)\setminus 0) \neq 0\}$, we get

$$\mathrm{rank}\,\tilde{H}(\mathfrak{g}_t^{-1}(0)\setminus 0)\approx 2\tau\quad(t\to+\infty)$$
$$d(t)\approx 4\tau\quad(t\to+\infty)$$
$$D(t)\approx\frac{4\tau}{\alpha}\quad(t\to+\infty).$$

Thus,

$$\frac{d(t)}{D(t)} \to \alpha \quad (t \to +\infty).$$

It is interesting that for large t homology groups behave as continuous: parameter α definable by the position of vector a in Cartan subalgebra can be restored from these groups. The family of quadratic mappings $\mathfrak{g}_t$, $t > 0$, proves to be completely different topologically for different values of α.

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