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TOPOLOGY OF QUADRATIC MAPS AND HESSIANS OF SMOOTH MAPS

A. A. Agrachev

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In this paper we study sets of real solutions of systems of quadratic equations and inequalities. The results are used for the local study of more general systems of smooth equations and inequalities.

1. Introduction

1. A large part of the present paper is devoted to the topological study of systems of quadratic equations and inequalities. The results are also used for the local study of systems of smooth equations and inequalities. By quadratic maps here we mean vector-functions all of whose components are real quadratic forms. The number of variables can be finite or infinite, the number of forms is always finite.

We are interested in the following questions: 1) Characterize essentially surjective quadratic maps (i.e., those which are themselves surjective and any map which is sufficiently close to them is also surjective). 2) For a given quadratic map $p: E \to \mathbf{R}^k$ and a convex closed cone $K \subset \mathbf{R}^k$ one learns to calculate the homology groups of the set $p^{-1}(K) \setminus 0$. The case $K = 0$ is not excluded and in this case $p^{-1}(K)$ is the intersection of k real quadrics.

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Throughout the entire paper we adhere to the following conventions: wherever nothing is said to the contrary, the homology (cohomology) groups are singular homology (cohomology) groups of a topological space; the assertion that a typical element of a given topological space has some property means that this property holds for all elements of an open dense subset.

The results of the present paper are the natural development of my joint paper with R. V. Gamkrelidze [2] to whom I express gratitude for guidance and stimulating discussions.

2. Let B be a Banach space and $\Phi_0: B \to \mathbb{R}^d$ be a smooth map. We denote by $D_{b_0}\Phi_0$ the first, and by $D^2_{b_0}\Phi_0$ the second differentials of the map Φ_0 at the point $b_0 \in B$. Then $D_{b_0}\Phi_0: b \mapsto D_{b_0}\Phi_0 b$ is a linear map of B into $\mathbb{R}^d$ and $D^2_{b_0}\Phi_0: (b_1, b_2) \to D^2_{b_0}\Phi_0(b_1, b_2)$ is a symmetric bilinear map of B × B into $\mathbb{R}^d$.

Now let $\mathscr{B}$ be a Banach manifold modeled on the space B and $\Phi_1: \mathscr{B} \to \mathbb{R}^d$ be a smooth map, $\beta_0 \in \mathscr{B}$. Then the first differential

$$D_{\beta_0}\Phi_1: T_{\beta_0}\mathscr{B} \to \mathbb{R}^d$$

is a linear map of the tangent space to $\mathscr{B}$ at the point β_0 into $\mathbb{R}^d$. If we choose local coordinates in $\mathscr{B}$ then we can calculate also the second differential — a symmetric bilinear map of B × B into $\mathbb{R}^d$. However here one does not get a well-defined bilinear map of $T_{\beta_0}\mathscr{B} \times T_{\beta_0}\mathscr{B}$ into $\mathbb{R}^n$ since the quadratic part of a smooth map depends essentially on the choice of local coordinates in $\mathscr{B}$ (for example, if $\operatorname{im} D_{\beta_0}\Phi_1 = \mathbb{R}^d$ then according to the implicit function theorem in some local coordinates Φ_1 is a linear map). At the same time if we consider the composition of the second differential (calculated in local coordinates) with the canonical factorization

$$\mathbb{R}^d \to \mathbb{R}^d / \operatorname{im} D_{\beta_0}\Phi_1 \overset{\text{def}}{=} \operatorname{coker} D_{\beta_0}\Phi_1,$$

then we get a well-defined symmetric bilinear map

$$D^2_{\beta_0}\Phi_1: T_{\beta_0}\mathscr{B} \times T_{\beta_0}\mathscr{B} \to \operatorname{coker} D_{\beta_0}\Phi_1$$

We call the map $D^2_{\beta_0}\Phi_1$ the invariant second differential of the smooth map Φ_1 at the point β_0.

Finally, let $\Phi: \mathscr{B} \to M$ be a smooth map of a Banach manifold into a d-dimensional manifold. This means that not only in the domain of definition but also in the range of values of Φ the linear structure is missing and there is only a differential structure. In this case the first differential $D_{\beta_0}\Phi: T_{\beta_0}\mathscr{B} \to T_{\Phi(\beta_0)}M$ is a linear map of the tangent spaces. Choosing local coordinates in M we can calculate the invariant second differential; however only the restriction of the second differential to $\ker D_{\beta_0}\Phi \times \ker D_{\beta_0}\Phi$ is well-defined (i.e., independently of the choice of local coordinates not only in $\mathscr{B}$ but also in M). We get a symmetric bilinear map

$$D^2_{\beta_0}\Phi: \ker D_{\beta_0}\Phi \times \ker D_{\beta_0}\Phi \to \operatorname{coker} D_{\beta_0}\Phi,$$

which is called the Hessian of the map Φ at the point $\beta_0 \in \mathscr{B}$.

In what follows in those cases when it is clear from the context at which point we are calculating the differential and the Hessian, we shall as a rule use the abbreviated notation: $D_{\beta_0}\Phi = \Phi'$, $D^2_{\beta_0}\Phi = \Phi''$.

If $\operatorname{coker} D_{\beta_0}\Phi = 0$ (i.e., β_0 is a regular point of the map Φ), then in some local coordinates in neighborhoods of the points β_0 and $\Phi(\beta_0)$ the map Φ is a linear map of B onto $\mathbb{R}^d$. The next case in complexity is the case $\dim \operatorname{coker} D_{\beta_0}\Phi = 1$. In this case $D^2_{\beta_0}\Phi$ is a bilinear map of $\ker D_{\beta_0}\Phi \times \ker D_{\beta_0}\Phi$ into the line, i.e., essentially a real bilinear form. More precisely, in order to get a bilinear form it is necessary to take the composition of $D^2_{\beta_0}\Phi$ with a nonzero element of the one-dimensional space

$$(\operatorname{coker} D_{\beta_0}\Phi)^* = (\operatorname{im} D_{\beta_0}\Phi)^{\perp} \subset T_{\Phi(\beta_0)}M.$$

The pairing of an arbitrary vector $x \in T_\mu M$ with a covector $\xi \in T_\mu^* M$ we denote by ξx, like the multiplication of a row by a column. Throughout this section the critical point β_0 of the map Φ is considered to be fixed so that the composition of the Hessian with the covector $\omega \in (\operatorname{im} D_{\beta_0}\Phi)^{\perp}$ can be written simply as $\omega\Phi''$.

We recall that the index of inertia (or simply index) of a symmetric bilinear form q on a space B means the maximal dimension of a subspace on which the corresponding quadratic form is negative:

$$\operatorname{ind} q=\max\{\dim V \mid V\subset B, q(v,v)<0, \forall v\in V\setminus 0\}.$$

If $\operatorname{ind} q = 0$, then the form q is called nonnegative, $q\geqslant 0$, and if $\operatorname{ind}(-q) = 0$, then nonpositive, $q\leqslant 0$, and in both cases sign semidefinite; now if $\operatorname{ind} q \neq 0$ and $\operatorname{ind}(-q) \neq 0$ then the form q is said to be sign definite. The following proposition is a straightforward generalization of the fact that the Hessian of a scalar function is sign semidefinite at a local extreme point.

Proposition 1. Let us assume that $\Phi(\beta_0)\in\partial\Phi(\mathcal{O}_{\beta_0})$ for some neighborhood $\mathcal{O}_{\beta_0}$ of the point β_0 in $\mathcal{B}$. If $\dim\operatorname{coker}\Phi'=1$ then Φ'' is a sign semidefinite form.

This proposition occurs many times in the literature on extremal problems; its proof is a simple exercise which we omit.

The following modification of the assertion formulated, which distinguishes a "local minimum" from a "local maximum," is often useful.

Proposition 2. Let $\gamma:[0,+\infty)\to M$ be a smooth curve in M where $\gamma(0)=\Phi(\beta_0)$, $\left.\frac{d\gamma}{d\theta}\right|_{\theta=0}=\gamma_0'\in T_{\Phi(\beta_0)}M$. Let us assume that $\Phi(\mathcal{O}_{\beta_0})\cap\gamma((0,\varepsilon))=\varnothing$ for some neighborhood $\mathcal{O}_{\beta_0}$ of the point β_0 in $\mathcal{B}$ and some $\varepsilon > 0$. If $\dim\operatorname{coker}\Phi' = 1$ then

$$(\omega\gamma_0')\,\omega\Phi''\leqslant 0 \quad \forall\omega\in(\operatorname{im}\Phi')^{\perp}.$$

Proposition 2 is most often used in the form of the law of Lagrange multipliers for conditional extreme points, i.e., when $M=R^d, \Phi=(\varphi_0,\varphi_1,\ldots,\varphi_{d-1})^T$,

$$\gamma(\theta)=(\varphi_0(\beta_0)-\theta,\varphi_1(\beta_0),\ldots,\varphi_{d-1}(\beta_0))^T$$

(φ_0 is a minimizing function and $\varphi_1(\beta)=\text{const},\ldots,\varphi_{d-1}(\beta)=$ const are constraints].

By the kernel of the symmetric bilinear form q on the space B is meant the subspace $\ker q=\{b\in B \mid q(b, B)=0\}$. If $\ker q = 0$, then the form q is said to be nondegenerate. The form q is called sign-definite if $|q(b,b)|\geqslant\varepsilon\|b\|^2$ for some $\varepsilon > 0$ and any $b\in B$ (here $\|\cdot\|$ is the norm in the Banach space B). A nonnegative (nonpositive) sign-definite form is called positive (negative) definite.

We return to the study of the critical point β_0 of the map $\Phi:\mathcal{B}\to M$.

Proposition 3 (Generalized Morse Lemma). Let $\dim\operatorname{coker}\Phi' = 1$. If the symmetric bilinear form Φ'' is nondegenerate and its restriction to a subspace of finite codimension in $\ker\Phi'$ is a sign-definite form, then there exist local coordinates

$$S:\mathcal{O}_{\beta_0}\to R^{d-1}\oplus\ker\Phi', \quad s:O_{\Phi(\beta_0)}\to R^{d-1}\oplus\operatorname{coker}\Phi'$$

defined in neighborhoods of the points β_0 and $\Phi(\beta_0)$ such that

$$s\circ\Phi\circ S^{-1}(x, v)=(x, \Phi''(v, v))\,\forall x\in R^{d-1},\ v\in\ker\Phi'.$$

One can find a proof of the generalized Morse lemma, for example, in [7].

As the simplest consequence we get a partial converse of Proposition 1.

COROLLARY 1. Under the hypotheses of Proposition 3 the point $\Phi(\beta_0)$ lies on the boundary of the set $\Phi(\mathbf{V})$ for some neighborhood $\mathbf{V}$ of the point β_0 in $\mathcal{B}$ if and only if the form Φ'' is sign-definite.

The generalized Morse lemma implies that the level set $\mathcal{O}_{\beta_0}\cap\Phi^{-1}(\Phi(\beta_0))$ and the preimage of zero under the quadratic map $v\mapsto\Phi''(v,v)$, $v\in\ker\Phi'$ are homeomorphic.

COROLLARY 2. Under the hypotheses of Proposition 3 let $\omega\in(\operatorname{im}\Phi')^{\perp}\setminus 0$ where $\operatorname{ind}\omega\Phi''=i<+\infty$. Then if $\dim B = +\infty$ then the set $\Phi^{-1}(\Phi(\beta_0))\cap(\mathcal{O}_{\beta_0}\setminus\beta_0)$ has the homotopy type of the sphere S^{i-1} and

$$H_j(\Phi^{-1}(\Phi(\beta_0)), \Phi^{-1}(\Phi(\beta_0))\setminus\beta_0)=\begin{cases}Z, & j=i\\ 0, & j\neq i.\end{cases}$$

Now if $\dim B = N < +\infty$ then the set $\Phi^{-1}(\Phi(\beta_0))\cap(\mathcal{O}_{\beta_0}\setminus\beta_0)$ has the homotopy type of the product of spheres: $S^{i-1}\times S^{N-d-i}$.

Proof. We need to describe the homotopy type of a cone with vertex removed $C=\{v\in\ker\Phi' \mid \Phi''(v,v)=0, v\neq 0\}$. Let V be an i-dimensional subspace of $\ker\Phi'$ such that the restriction of Φ'' to V is a negative definite form. We set $W=\{w\in\ker\Phi' \mid \Phi''(w, V)=\}$. By hypothesis the restriction of Φ'' to W is a positive definite form and $\ker\Phi'=V\oplus W$. Let Σ be the unit sphere in $\ker\Phi'$; it is clear that $C \cap \Sigma$ is a homotopy retract of C. At the same time, $C \cap \Sigma$ is homeomorphic to $(\Sigma\cap V)\times(\Sigma\cap W)$. It remains to note that the restriction of the form Φ'' to W defines a Hilbert structure on W and the infinite-dimensional Hilbert sphere is contractible.

Comparison of Corollaries 1 and 2 leads to the following interesting fact.

COROLLARY 3. Under the hypotheses of Proposition 3 the following assertions are equivalent:

1) $\Phi(\beta_0)\in\partial\Phi(V)$ for any neighborhood V of the point β_0 in $\mathcal{B}$;

2) β_0 is an isolated point of the level set $\Phi^{-1}(\Phi(\beta_0))$.

For $\dim\operatorname{coker}\Phi' > 1$ the Hessian Φ'' is a vector (not scalar) quadratic map. The structure of such maps is considerably more complicated than that of quadratic forms; Sec. 2 is completely devoted to them. We return to general smooth maps in Sec. 3: despite the fact that the Morse lemma does not generalize to the case $\dim\operatorname{coker}\Phi' > 1$ in general, we are able to get a surrogate of it which is completely sufficient for many purposes.

2. Topology of Quadratic Maps

The structure of a real quadratic form in a finite number of variables is completely determined by its index of inertia and the dimension of the kernel of this form. In the infinite-dimensional case the index of inertia is also the most important invariant of a scalar quadratic form; in paticular, it uniquely determines the homotopy type of the set of points at which the form assumes negative values. Vector-valued quadratic maps however have whole families of indices corresponding to projections of the maps in different one-dimensional directions. The problem of reconstructing various properties of the original maps from these indices is far from trivial, but as we hope to show here entirely within view.

1. Let H be a Hilbert space (possibly finite-dimensional). By the symbol $\mathcal{P}(H)$ we denote the set of all continuous symmetric bilinear real forms on H; correspondingly, $\mathcal{P}^k(H)$ is the space of continuous symmetric bilinear maps of $H \times H$ into $\mathbb{R}^k$. If $p\in\mathcal{P}^k(H)$ then the quadratic map $x\mapsto p(x, x)$, $x\in H$ will be denoted by the same symbol p so that $\mathcal{P}^k(H)$ can be identified with the space of continuous quadratic forms on H, which is canonically isomorphic to it (in concrete situations the meaning of the symbol p is uniquely determined from the context). If $H=\mathbb{R}^{N+1}$ we use the abbreviated notation

$$\mathcal{P}(\mathbb{R}^{N+1})=\mathcal{P}(N),\ \mathcal{P}^k(\mathbb{R}^{N+1})=\mathcal{P}^k(N).$$

Let $p\in\mathcal{P}^k(H)$ so that by convolving p with an arbitrary row $\omega\in\mathbb{R}^{k*}$, we get a scalar quadratic form ωp. We set $p^*:\omega\mapsto\omega p$, $\omega\in\mathbb{R}^{k*}$ so the correspondence $p \to p^*$ defines a natural isomorphism of the space of quadratic maps $\mathcal{P}^k(H)$ and the space of linear systems of quadratic forms $\operatorname{Hom}(\mathbb{R}^{k*}, \mathcal{P}(H))$.

Further, to any form $q\in\mathcal{P}(H)$ corresponds a bounded self-adjoint operator $Q:H \to H$ defined by the identity $q(x, y)=(Qx, y)$; $x, y\in H$, where $(\cdot, \cdot)$ is the scalar product in H. The correspondence $q \to Q$ is obviously an isomorphism of the linear space $\mathcal{P}(H)$ onto the space of bounded self-adjoint operators on H. Here $\ker q = \ker Q$, and $\operatorname{ind} q$ is infinite if the continuous spectrum of the operator Q has nonempty intersection with the negative half-line, and is equal to the sum of the multiplicities of the negative eigenvalues of this operator otherwise. If $p\in\mathcal{P}^k(H)$, $\omega\in\mathbb{R}^{k*}$ then the self-adjoint operator corresponding to the form ωp is denoted by ωP.

Finishing with notation, we occupy ourselves seriously with the finite-dimensional case. We call a quadratic map p of the space $\mathbb{R}^{N+1}$ into $\mathbb{R}^k$ essentially surjective if for any $\tilde{p}$ sufficiently close to p in $\mathcal{P}^k(N)$ we have $\tilde{p}(\mathbb{R}^{N+1})=\mathbb{R}^k$. The property of essential surjectivity is useful for studying optimizational problems and is closely connected with the structure of the level set $p^{-1}(0)$ of the quadratic map $p:\mathbb{R}^{N+1}\to\mathbb{R}^k$.

LEMMA 1. For a typical quadratic map $p\in\mathcal{P}^k(N)$ it follows from the condition $p^{-1}(0) \neq 0$ that p is essentially surjective.

Proof. Let us assume that $p(x, x) = 0$ for some $x \neq 0$, where x is a regular point of the quadratic map p, i.e., the linear map $y\mapsto p(x, y)$, $y\in\mathbb{R}^{N+1}$ is surjective. Then according to the implicit function theorem the image of a fixed neighborhood of the point x under any map $\tilde{p}$ sufficiently close to p contains a neighborhood of zero in $\mathbb{R}^k$. Since quadratic maps are homogeneous, it follows from this that p is essentially surjective. On the other hand, it is easy to deduce from Sard's theorem that for a typical quadratic $p\in\mathcal{P}^k(N)$ all points of the set $(p^{-1}(0)\setminus 0)\subset\mathbb{R}^{N+1}$ are regular.

The assertion of Lemma 1 is true of course not only for quadratic, but also for many classes of homogeneous mappings. The converse assertion is false in general even for quadratic maps; essentially surjective maps may not have nontrivial zeros at all. The essentially surjective map $z \to z^2$, where $z \in \mathbf{C} = \mathbf{R}^2$, is the simplest example; a more interesting example is the map $(z_1, z_2) \mapsto (z_1 z_2, |z_2|^2 - |z_1|^2)$ from $\mathbf{C}^2 = \mathbf{R}^4$ to $\mathbf{C} \oplus \mathbf{R} = \mathbf{R}^3$, whose restriction to the sphere $S^3 \subset \mathbf{C}^2$ realizes the famous Hopf bundle $S^3 \xrightarrow{(S^1)} S^2$. However it turns out that for $N \gg k$ such examples do not exist.

Proposition 1. Let $k^2 \leqslant N+1$ and the quadratic map $p \in \mathcal{P}^k(N)$ be essentially surjective. Then $p^{-1}(0) \neq 0$.

Before proving this proposition we give a very useful fact about the local structure of the set of singular quadratic forms in $\mathcal{P}(N)$, which is used constantly in what follows.

LEMMA 2. Let $q_0 \in \mathcal{P}(N)$, $V = \ker q_0$. Then there exist a neighborhood $\mathcal{O}_{q_0}$ of the point q_0 in $\mathcal{P}(N)$ and an analytic map $\Phi: \mathcal{O}_{q_0} \to \mathcal{P}(V)$ such that

$$1)\ \Phi(q_0) = 0. \quad 2)\ \operatorname{ind} q = \operatorname{ind} q_0 + \operatorname{ind} \Phi(q).$$
$$3)\ \operatorname{corank} q = \operatorname{corank} \Phi(q). \quad 4)\ D_{q_0}\Phi(q) = q|V.$$

Proof. Let γ be a closed contour in the complex plane which separates the origin from the nonzero eigenvalues of the operator Q_0. We set

$$\pi_q = \frac{1}{2\pi i} \int_\gamma (Q - \xi\, \mathrm{id})^{-1} d\xi$$

for any operator Q which does not have eigenvalues on the contour γ. It is clear that π_q is an operator which depends analytically on q and commutes ~~symmetrically~~ with q; moreover, $\pi_q^2 = \pi_q$. Indeed π_q is the orthogonal projector of the space $\mathbf{R}^{N+1}$ onto the invariant subspace of the operator Q corresponding to the eigenvalues which lie inside the contour γ, in particular $\pi_{q_0}|V = \mathrm{id}$. We set

$$\Phi(q)(v_1, v_2) = q(\pi_q v_1, \pi_q v_2) \ \forall v_1, v_2 \in V.$$

For q near q_0 the map $\pi_q|V$ is nondegenerate so the form $\Phi(q)$ is equivalent to the form $q|\operatorname{im} \pi_q$. At the same time the form $q|\operatorname{im} \pi_q$ in nonsingular and has index of inertia equal to ind_{q_0} for q close to q_0. Equations 2) and 3) now follow from the fact that $\operatorname{im} \pi_q, \operatorname{im} \pi_q^\perp$ are invariant subspaces of the operator Q. Equations 1) and 4) are verified directly. ∎

Let $0 \leqslant r \leqslant N+1$ and set

$$D_r(N) = \{q \in \mathcal{P}(N) \mid \operatorname{corank} q = r\}, \quad D(N) = \bigcup_{r=1}^{N+1} D_r(N).$$

COROLLARY. The set $D_r(N)$ is a real-analytic submanifold of codimension $r(r+1)/2$ in $\mathcal{P}(N)$ for $0 \leqslant r \leqslant N+1$.

This follows directly from Lemma 2 if one notes that $\dim \mathcal{P}(\mathbf{R}^r) = \frac{r(r+1)}{2}$.

The proof of Proposition 1 is based on the following assertion.

LEMMA 3. We set $\nu_k = \frac{1}{2}(\sqrt{1 + 8(k-1)} - 1)$; if $n > 0$ is such that $nk + \max(n, \nu_k) \leqslant N$ then for a typical $p \in \mathcal{P}^k(N)$ the closed convex cone $K_n(p) = \operatorname{conv}\{\omega \in \mathbf{R}^{k*} \mid \operatorname{ind} \omega p \leqslant n\}$ is acute [we recall that a closed convex cone K is said to be acute if $K \cap (-K) = 0$).

Proof. Let $S^{k-1} = \{\omega \in \mathbf{R}^{k*} \mid |\omega| = 1\}$ be the unit sphere in $\mathbf{R}^{k*}$. It follows from Sard's theorem that for a typical $p \in \mathcal{P}^k(N)$ the map $p^*|S^{k-1}: S^{k-1} \to \mathcal{P}(N)$ is transverse to the submanifolds $D_r(N)$, $0 \leqslant r \leqslant N+1$. At the same time, if $p^*|S^{k-1}$ is transverse to the submanifolds $D_r(N)$, $0 \leqslant r \leqslant N+1$ then according to the corollary to Lemma 2, $\operatorname{corank} \omega p \leqslant \nu_k \ \forall \omega \neq 0$. Let $p \in \mathcal{P}^k(N)$ be such that $p^*|S^{k-1}$ is transverse to $D_r(N)$, $0 \leqslant r \leqslant N+1$ and here the cone K_n is not acute. Then one can find $\omega_0, \ldots, \omega_k \in K_n$, $\omega_0 \neq 0$ such that $\sum_{i=0}^k \omega_i = 0$. We consider two cases separately:

1) $K_n(p) = \mathbf{R}^{k*}$. In this case by moving the point ω_i slightly if necessary one can arrange that all the forms $\omega_i p$, $i = 0, \ldots, k$ are nonsingular. But then each of the forms $\omega_i p$ is positive definite on some subspace V_i of codimension n in $\mathbf{R}^{N+1}$, $i = 0, 1, \ldots, k$. Since $n(k+1) \leqslant N$ one has $\bigcap_{i=0}^k V_i \neq 0$. Consequently, for some $x \in \mathbf{R}^{N+1}$ one has $\omega_i p(x, x) > 0$, $i = 0$, $i = 0, 1, \ldots, k$ which contradicts the equation $\sum_{i=0}^k \omega_i = 0$.

2) $K_n(p) \neq R^{k*}$. In this case one can assume that $\omega_k = 0$. Each of the forms $\omega_i p$, $0 \leqslant i < k$ is nonnegative on some subspace of codimension n of R^{N+1} and the form $\omega_0 p$ is positive definite on a subspace of codimension $n + \nu_k$ in R^{N+1}. Since $nk + \nu_k \leqslant N$ for some $x \in R^{N+1}$ we get $\omega_0 p(x, x) > 0$, $\omega_i p(x, x) \geqslant 0$, $i = 1, \ldots, k-1$. The latter contradicts the equation $\sum_{i=0}^{k-1} \omega_i = 0$.

The contradictions obtained prove that if $p^* | S^{k-1}$ is transverse to $D_r(N)$, $0 \leqslant r \leqslant N+1$ then $K_n(p)$ is an acute cone.

Let $p \in \mathcal{P}^k(N)$; by $\hat{p}: S^N \to R^k$ we denote the restriction of the quadratic map p to the sphere $S^N = \{x \in R^{N+1} \,|\, |x| = 1\}$. The point $x \in S^N$ is critical for the map $\hat{p}$ if and only if $\omega Px = \lambda x$ for some $\omega \neq 0$, $\lambda \in R$; here the Hessian of the scalar function $\omega\hat{p}$ at the point x has the form: $(\omega\hat{p})''_x(y, y) = \omega p(y, y) - \lambda(y, y)$.

LEMMA 4. For a typical $p \in \mathcal{P}^k(N)$ the collection of those points $x \in S^N$ such that $\operatorname{rank} \hat{p}'_x \leqslant k - 2$ can be represented as the union of a finite number of smooth submanifolds in S^N of dimension no more than $2k - N - 4$.

Proof. It is easy to see that for any $x \in S^N$ the map $p \mapsto \hat{p}'_x$, $p \in \mathcal{P}^k(N)$ is a linear surjective map of $\mathcal{P}^k(N)$ into the space of N × k matrices. As is known, the matrices of rank r form a smooth submanifold of codimension $(N - r)(k - r)$ in the space of N × k-matrices. Ordinary application of Sard's theorem and calculation of the number of parameters gives the result required.

Now everything is ready for the proof of Proposition 1. Since $\nu_k \leqslant k - 1$ and $2k - k^2 - 3 < 0$, according to Lemmas 3 and 4, for $k^2 \leqslant N + 1$ a typical $p \in \mathcal{P}^k(N)$ satisfies the following conditions:

a) the cone $K_{n-1}(p)$ is acute;

b) $\operatorname{rank} \hat{p}'_x \geqslant k - 1$ $\forall x \in S^N$.

We show that if a quadratic map satisfies conditions a), b) and in addition $p(R^{N+1}) = R^k$ then $p^{-1}(0) \neq 0$.

The acuteness of the cone $K_{n-1}(p) \subset R^{k*}$ means that for some vector $l \in R^k$ one has the inequalities: $\omega l < 0$, $\forall \omega \in K_{k-1}(p) \setminus 0$. Let us assume that $p^{-1}(0) = 0$; then $\hat{p}^{-1}(0) = \varnothing$ and $\min\{\alpha \geqslant 0 \,|\, \alpha l \in \operatorname{im} \hat{p}\} > 0$. Let $S^N \ni x$ be a point at which $\min\{\alpha \geqslant 0 \,|\, \alpha l \in \operatorname{im} \hat{p}\}$ is achieved; then x is a critical point of the map $\hat{p}$ and by b), $\operatorname{rank} \hat{p}_x = k - 1$. According to Proposition 1.2 one can find an $\omega \neq 0$ such that $\omega\hat{p}'_x = 0$, $\omega l \geqslant 0$, $\omega(\hat{p})''_x \geqslant 0$. Condition $\omega\hat{p}'_x = 0$ denotes that $\omega Px = \lambda x$ for some $\lambda \in R$; since $p(x, x) = \alpha l$, $\alpha > 0$, then $\lambda = \alpha\omega l \geqslant 0$. Further, $\omega(\hat{p}''_x) = (\omega\hat{p})''_x | \ker \hat{p}'_x$, and at the same time $(\omega\hat{p})''_x(y, y) = \omega p(y, y) - \lambda(y, y) \, \forall y \perp x$. Since the subspace $\ker \hat{p}'_x$ has codimension $k - 1$ at $x^\perp$ we get that the form ωp is nonnegative on a subspace of codimension $k - 1$ in R^{N+1}, i.e., $\operatorname{ind} \omega p \leqslant k - 1$. Consequently, $\omega \in K_{n-1}(p)$ which contradicts the condition $\omega l \geq 0$. ∎

2. Definition. A map $p \in \mathcal{P}^k(N)$ is called degenerate if for some $\omega \in R^{k*} \setminus 0$ $x \in R^{N+1} \setminus 0$ one has the equations $\omega Px = 0$, $p(x, x) = 0$; otherwise the map p is called nondegenerate.

It is easy to see that $p \in \mathcal{P}^k(N)$ is degenerate if and only if in R^k there is a critical value of the map $\hat{p}: S^N \to R^k$. Consequently, if q is nondegenerate, then $\hat{p}^{-1}(0)$ is a smooth manifold of dimension $N - k$ or the empty set. For $k = 1$ the definition given is equivalent to the ordinary definition of degeneracy of a quadratic form.

It is easy to see that the degenerate maps form a proper algebraic subset of $\mathcal{P}^k(N)$; in particular, a typical $p \in \mathcal{P}^k(N)$ is nondegenerate.

Now we consider a more general situation: let K be a convex polyhedral cone in R^k and $K^\circ = \{\omega \in R^{k*} \,|\, \omega l \leqslant 0 \; \forall l \in K\}$ be the dual cone.

Definition. The map $p \in \mathcal{P}^k(N)$ is said to be degenerate with respect to K, if for some $\omega \in K^\circ \setminus 0$, $x \in R^{N+1} \setminus 0$ one has $\omega Px = 0$, $p(x, x) \in K$; otherwise the map p is said to be nondegenerate with respect to K. Clearly degeneracy of p with respect to K is equivalent to simply degeneracy of p if K = 0. The concept of degeneracy with respect to a cone also has an intuitive interpretation in terms of the map $\hat{p}$: the map $p \in \mathcal{P}^k(N)$ is nondegenerate with respect to K if and only if $\hat{p}: S^N \to R^k$ is transverse to K [a smooth map $f: M \to R^k$ of a smooth manifold M into R^k is said to be transverse to a convex closed subset $S \subset R^k$ if for any $x \in M$ it follows from the condition $f(x) \in S$ that $\operatorname{im} f_x'$ is not contained in any support hyperplane of S at the point f(x)].

It follows from Proposition A1 of the Appendix to the present section that for a map $p\in\mathcal{P}^k(N)$ which is nondegenerate with respect to the cone K, the set

$$p^{-1}(K)=\{x\in S^N \mid \omega p(x, x)\leqslant 0 \ \forall\omega\in K^0\}$$

is a topological manifold with boundary. The basic problem of the present section is to learn how to calculate the homology groups of these sets (especially the homology groups with coefficients in $\mathbf{Z}_2$).

The solution of this problem uses some structures on the space of quadratic forms $\mathcal{P}(N)$, to whose description we proceed.

Let $Q:\mathbf{R}^{N+1}\to\mathbf{R}^{N+1}$ be a self-adjoint operator and $\lambda_1(Q)\leqslant\ldots\leqslant\lambda_{N+1}(Q)$ be the eigenvalues of this operator in increasing order. We recall that $\lambda_i(Q)$ depends continuously (but not smoothly!) on Q. We introduce the notation

$$\mathcal{P}_n(N)=\{q\in\mathcal{P}(N)\mid \text{ind}\, q\leqslant n\}=\{q\in\mathcal{P}(N)\mid\lambda_{n+1}(Q)\geqslant 0\}$$

for $0\leqslant n\leqslant N$, $\mathcal{P}(N)=\varnothing$ for $j < 0$.

The sets $\mathcal{P}_n(N)$, $0\leqslant n\leqslant N$, define a filtration of the space $\mathcal{P}(N)$ by closed subsets. Here $\mathcal{P}_0(N)$ is the collection of all nonnegative quadratic forms on $\mathbf{R}^{N+1}$, which is a closed cone in $\mathcal{P}(N)$. By $\text{id}\in\mathcal{P}(N)$ we denote the quadratic form $x\to(x, x)$ (this is the form which corresponds to the identity operator in $\mathbf{R}^{N+1}$). For $n = 0, 1,\ldots,N$ the map

$$q\mapsto q+(\lambda_{n+1}(Q)-\lambda_0(Q))\,\text{id},\ q\in\mathcal{P}(N),$$

defines a homeomorphism of the space $\mathcal{P}(N)$ onto itself which carries $\mathcal{P}_n(N)$ onto $\mathcal{P}_0(N)$. It follows in particular from this that $\mathcal{P}_n(N)$ is homeomorphic to a closed half-space in $\mathbf{R}^{\frac{(N+1)(N+2)}{2}}$.

Further, we set

$$\Lambda_n(N)=\{q\in\mathcal{P}(N)\mid\lambda_n(Q)\neq\lambda_{n+1}(Q)\},\ n=1,\ldots, N.$$

It is easy to see that

$$\Lambda_n(N)=\mathcal{P}_n(N)\setminus\mathcal{P}_{n-1}(N)+\mathbf{R}\cdot\text{id}. \tag{1}$$

It follows from Lemma 2 that the closed set $\mathcal{P}(N)\setminus\Lambda_n(N)$ is a pseudomanifold of codimension 2 in $\mathcal{P}(N)$. We denote by $\gamma_n(N)$ the cohomology class from $H^1(\Lambda_n(N);\mathbf{Z}_2)$, dual to this pseudomanifold: the value of $\gamma_n(N)$ on an arbitrary one-dimensional $\mathbf{Z}_2$-cycle in $\Lambda_n(N)$ is equal to the linking coefficient of this cycle with $\mathcal{P}(N)\setminus\Lambda_n(N)$.

LEMMA 5. For any $\alpha_1 < \alpha_2$ the set

$$\{q\in\mathcal{P}(N)\mid\lambda_1(Q)=\lambda_n(Q)=\alpha_1,\ \lambda_{n+1}(Q)=\lambda_{N+1}(Q)=\alpha_2\}, \tag{2}$$

which is homeomorphic to the Grassmanian $G_n(\mathbf{R}^{N+1})$, is a homotopy retract of the space $\Lambda_n(N)$.

Proof. A homeomorphism of the space (2) onto $G_n(\mathbf{R}^{N+1})$ is given by associating to each form q belonging to (2), the invariant space of the operator Q corresponding to the eigenvalue α_1. Let $q\in\mathcal{P}(N)$; we recall that the self-adjoint operator Q is determined uniquely by its eigenvalues and the invariant subspaces corresponding to these eigenvalues. We get a homotopy retraction of $\Lambda_n(N)$ onto the space (2) if we associate with each form $q\in\Lambda_n(N)$ a family of forms q_t, $t\in[0, 1]$ such that

$$\lambda_i(Q_t)=\begin{cases}t\alpha_1+(1-t)\lambda_i(q), & i=1,\ldots,n\\ t\alpha_2+(1-t)\lambda_j(q), & j=n+1,\ldots,N+1,\end{cases}$$

and the invariant subspace corresponding to the eigenvalue $\lambda_i(Q_t)$ coincides with the invariant subspace of the operator Q corresponding to the eigenvalue $\lambda_i(Q)$, $i=1,\ldots,N+1$ $t\in[0, 1)$. ■

Lemma 5 lets us give a different description of the cohomology classes $\gamma_n(N)$ introduced above, which is more convenient for calculations.

COROLLARY. Let $\mathcal{L}_n(N)$ be a vector bundle whose base is $\Lambda_n(N)$ and whose fiber over the point $q\in\Lambda_n(N)$ is the n-dimensional invariant subspace $L_n(Q)$ of the operator Q corresponding to the eigenvalues $\lambda_1(Q),\ldots,\lambda_n(Q)$. Then $\gamma_n(N)$ coincides with the one-dimensional Stiefel–Whitney class of the bundle $\mathcal{L}_n(N)$, i.e., $\gamma_n(N)=w_1(\mathcal{L}_n(N))$.

Indeed it is easy to see that the restriction of the bundle $\mathcal{L}_n(N)$ over $\Lambda_n(N)$ to the subspace (2) reduces to the tautological bundle of the Grassmanian $G_n(\mathbf{R}^{N+1})$. Consequently,

$w_1(\mathcal{L}_n(N))\neq 0$, while $w_1(\mathcal{L}_n(N))$ is the unique nonzero class in $H^1(\Lambda_n(N);\ Z_2)\approx H^1(G_n(R^{N+1});\ Z_2)$. Since according to Alexander–Pontryagin duality the class $\gamma_n(N)$ is also nonzero, one has $\gamma_n(N)=w_1(\mathcal{L}_n(N))$ ■

Let $\theta\mapsto q_\theta$, $\theta\in S^1$, be a continuous closed curve in $\Lambda_n(N)$ and $L_n(Q_\theta)$, $\theta\in S^1$, be the corresponding family of n-dimensional subspaces of R^{N+1}. Let $\theta_0\in S^1$; any continuous transport of the subspaces $L_n(Q_\theta)$, $\theta\in S^1$ along S^1 defines the monodromy transformation $I_{\theta_0}:L_n(Q_{\theta_0})\to L_n(Q_{\theta_0})$. It follows from the corollary to Lemma 5 that the value of the class $\gamma_n(N)$ on the curve q_θ coincides with the sign of the determinant of the transformation I_{θ_0}. Further, since the sets $\mathcal{P}(N)\setminus\Lambda_i(N)$, $i=1,\ldots,N$ have codimension 2 in $\mathcal{P}(N)$, a typical curve q_θ, $\theta\in S^1$, satisfies the condition

$$\lambda_i(Q_\theta)\neq\lambda_j(Q_\theta) \text{ for } i\neq j,\ 1\leqslant i,\ j\leqslant n.$$

If the curve q_θ in $\Lambda_n(N)$ satisfies this condition, then the calculation of the value of the class $\gamma_n(N)$ on this curve simplifies. Indeed let $e_i(\theta)$ be an eigenvector of the operator Q_θ corresponding to the eigenvalue $\lambda_i(Q_\theta)$ while $|e_i(\theta)| = 1$ and $e_i(\theta)$ depends continuously on θ for $\theta\in S^1\setminus\theta_0$, $i = 1,\ldots,n$. Then

$$e_i(\theta_0+0)=\pm e_i(\theta_0-0)$$

and

$$\langle\gamma_n(N),q_\cdot\rangle=\prod_{i=1}^{n}(e_i(\theta_0-0),\ e_i(\theta_0+0)). \tag{3}$$

3. Let $R^k\supset K$ be a convex polyhedral cone and $\mathcal{P}^k(N)\ni p$ be a quadratic map which is nondegenerate with respect to K. Throughout the present point the cone K and the map p are considered fixed. This frees us from the necessity of explicitly indicating the dependence on K and p in the notation for the objects introduced below; the argument N in the notation $\mathcal{P}^k(N)$, $\mathcal{P}_n(N)$, $\Lambda_n(N)$, $\gamma_n(N)$, etc. is also omitted as a rule.

Let $S^{k-1}=\{\omega\in R^{k*}\,|\,|\omega|=1\}$ be the unit sphere in R^{k*} and $\Omega=K^\circ\cap S^{k-1}$. We set

$$B=\{(\omega,\ x)\in\Omega\times S^N\,|\,\omega p(x,\ x)>0\}$$

and we define maps

$$\beta_l:B\to\Omega,\quad \beta_r:B\to S^N$$

as follows: $\beta_l(\omega,\ x)=\omega$, $\beta_r(\omega,\ x)=x$, $\forall(\omega,\ x)\in B$ (the indices ℓ and r are from "left" and "right").

LEMMA 6. $\beta_r(B)=S^N\setminus\hat{p}^{-1}(K)$, the map β_r defines a homotopy equivalence of the spaces B and $S^N\setminus\hat{p}^{-1}(K)$.

Proof. The equation $\beta_r(B)=S^N\setminus\hat{p}^{-1}(K)$ follows from the definitions and the relation $K^{\circ\circ}=K$. Let $x\in S^N\setminus\hat{p}^{-1}(K)$ so $\beta_r^{-1}(x)$ is the intersection of the set $(\Omega,\ x)$ and an open half-space in $(R^{k*},\ x)$. Let $(\omega_x,\ x)$ be the center of gravity of the set $\beta_r^{-1}(x)$. It is easy to see that ω_x depends continuously on $x\in\beta_r(B)$. Further, it follows from convexity considerations that $\left(\frac{\omega_x}{|\omega_x|},\ x\right)\in B$ and for any $(\omega,\ x)\in B$ the arc $\left(\frac{t\omega_x+(1-t)\omega}{|t\omega_x+(1-t)\omega|},\ x\right)$, $0\leqslant t\leqslant 1$ lies entirely in B. Now it is obvious that the map $x\mapsto\left(\frac{\omega_x}{|\omega_x|},\ x\right)$, $x\in\beta_r(B)$ is a homotopy inverse to β_r. ■

We set

$$\Omega_n=\{\omega\in\Omega\,|\,\lambda_{n+1}(\omega P)\geqslant 0\},\quad n=0,\ 1,\ \ldots,\ N;\quad \Omega_j=\varnothing \text{ for } j<0.$$

We recall that the symbol $p^*:S^N\to\mathcal{P}(N)$ denotes the map $\omega\to\omega p$; therefore, $\Omega_n=\Omega\cap p^{*-1}(\mathcal{P}_n)$. It is easy to show that the subsets Ω_n are homeomorphic to finite simplicial complexes and are deformation neighborhood retracts in the sphere S^{k-1}. In particular, there is a natural isomorphism of cohomology groups:

$$H^i(\Omega_n,\ \Omega_{n-1})\approx H_c^i(\Omega_n\setminus\Omega_{n-1}),\quad i,\ n=0,\ 1,\ \ldots,$$

where H_c^* denotes cohomology with compact supports.

We note that $p^*(\Omega_n\setminus\Omega_{n-1})\subset\mathcal{P}_n\setminus\mathcal{P}_{n-1}\subset\Lambda_n$; let $\pi_n\in H^1(\Omega_n\setminus\Omega_{n-1};\ Z_2)$ be the image of the class $\gamma_n\in H^1(\Lambda_n;\ Z_2)$ under the homomorphism induced by the map p^*; in other words,

$$\pi_n=(p^*\,|\,\Omega_n\setminus\Omega_{n-1})^*\gamma_n.$$

The cohomology multiplication $\xi_c \to \pi_n \cup \xi_c$, $\xi_c \in H_c^i(\Omega_n \setminus \Omega_{n-1}; Z_2)$ defines a homomorphism of the group $H_c^i(\Omega_n \setminus \Omega_{n-1}; Z_2)$ into the group $H_c^{i+1}(\Omega_n \setminus \Omega_{n-1})$. The natural identification $H^*(\Omega_n, \Omega_{n-1}; Z_2) \approx H_c^*(\Omega_n \setminus \Omega_{n-1}; Z_2)$ leads to the fact that the multiplication $\xi \mapsto \pi_n \cup \xi$ $\xi \in H^i(\Omega_n, \Omega_{n-1}; Z_2)$ is also well-defined and gives a homomorphism of the group $H^i(\Omega_n, \Omega_{n-1}; Z_2)$ into $H^{i+1}(\Omega_n, \Omega_{n-1}; Z_2)$.

By the symbol

$$\delta_n : H^i(\Omega_n, \Omega_{n-1}; Z_2) \to H^{i+1}(\Omega_{n+1}, \Omega_n; Z_2), \quad i = 0, 1, \ldots$$

we denote the connecting homomorphism in the exact sequence of the triple $(\Omega_{n+1}, \Omega_n, \Omega_{n-1})$.

We proceed finally to the calculation of the homology groups of the space $\hat{p}^{-1}(K)$. According to Alexander duality, instead of this one can calculate the cohomology groups of the space $S^N \setminus \hat{p}^{-1}(K)$. By Lemma 6, the cohomology groups of the space $S^N \setminus \hat{p}^{-1}(K)$ (with any coefficients) coincide with the cohomology groups of the space B. We consider the map $\beta_\ell : B \to \Omega$. The map β_ℓ in contrast with β_r is not by any means a homotopy equivalence, but with it, as with any continuous map, there is associated a Leray spectral sequence which converges to the cohomology groups of the space B.

Let $\mathfrak{A}$ be an Abelian group and $(E_r(\mathfrak{A}), d_r)$, $r \geqslant 1$ be the Leray cohomology spectral sequence for the map β_ℓ with coefficients in the group $\mathfrak{A}$. Then $(E_r(\mathfrak{A}), d_r)$ converges to $H^*(B; \mathfrak{A}) \approx H^*(S^N \setminus \hat{p}^{-1}(K); \mathfrak{A})$ as $r \to \infty$.

THEOREM 1. i) $E_2^{ij}(\mathfrak{A}) = H^i(\Omega_{N-j}, \Omega_{N-j-1}; \mathfrak{A})$ for $i \geqslant 0$, $j > 0$ for any Abelian group $\mathfrak{A}$.

ii) If $\mathfrak{A} = Z_2$ then the differential $d_2 : H^i(\Omega_n, \Omega_{n-1}; Z_2) \to H^{i+2}(\Omega_{n+1}, \Omega_n; Z_2)$, where $0 \leqslant n \leqslant N-2$, $i \geqslant 0$, is defined by the formula $d_2(\xi) = \delta_n(\pi_n \cup \xi) + \pi_{n+1} \cup \delta_n \xi$, $\forall \xi \in H^i(\Omega_n, \Omega_{n-1}; Z_2)$.

This theorem is the basic result of the present section. To prove it it is necessary to develop some technique for handling families of quadratic forms as a preliminary.

4. Let V be a compact convex subset of R^{k*} and $f : V \to \mathcal{P}(N)$ be a smooth map.

Definition. The map f is said to be nondegenerate at the point $v_0 \in V$, if there exists an $x \in \ker f(v_0) \setminus 0$, such that $(f'_{v_0} v)(x, x) \leqslant 0$ for any $v \in V - v_0$; otherwise f is said to be nondegenerate at the point v_0. The map f is said to be nondegenerate on V (or simply nondegenerate), if it is nondegenerate at each point of the set V (cf. the definition of nondegeneracy of a quadratic map with respect to a cone).

It is easy to see that the nondegenerate maps form an open subset of the space of all smooth maps of V into $\mathcal{P}(N)$.

Proposition 2. Let $f_t : V \to \mathcal{P}(N)$, $t \in [0, 1]$ be a smooth homotopy where all maps f_t, $t \in [0, 1]$, are nondegenerate. We set

$$B_t = \{(v, x) \in V \times S^N \mid f_t(v)(x, x) > 0\}.$$

Then in $R^{k*} \times S^N \supset V \times S^N$ there exists a flow F_t^*, $t \in [0, 1]$, $F_0 = \mathrm{id}$ such that $F_t(B_0) \subset B_t$, $t \in [0, 1]$.

Proof. We shall seek F_t in the form $F_t^* = \overrightarrow{\exp} \int_0^t Z_\tau d\tau$, where Z_τ is a nonstationary vector field on $R^{k*} \times S^N$. We set $Z_\tau = X_\tau + Y_\tau$, where for any $(v, x) \in R^{k*} \times S^N$ the vector $X_\tau(v, x)$ is tangent to S^N and the vector $Y_\tau(v, x)$ is tangent to R^{k*}. Proposition 2 will be proved if we construct a field Z_τ such that for any t, v, and x satisfying the conditions $f_t(v)(x, x) = 0$, $v \in V$ one has

$$(f'_{tv} Y_t(v, x))(x, x) + 2 f_t(v)(x, X_t(v, x)) > 0, \; Y_t(v, x) \in (V - v).$$

Moreover, it suffices to construct such a field locally in a small neighborhood of fixed t, v, and x, and afterwards glue the fields defined in small neighborhoods together with the help of a partition of unity.

If $x \notin \ker f_t(v)$, then we set $Y_t = 0$ the inequality $f_t(v)(x, X_t) > 0$ being obviously solvable. Now if $x \in \ker f_t(v)$ then it is necessary to satisfy the relations

$$(f'_{tv} Y_t(v, x))(x, x) > 0, \quad Y_t(v, x) \in V - v.$$

One can do this by virtue of the nondegeneracy of the map f_t. It is necessary to extend the field Y smoothly to a small neighborhood of the point (t, v, x). This is simple: one can assume that the vector $Y_t(v, x)$ lies in the relative interior of the set V − v, so the field $Y \equiv Y_t(v, x)$ satisfies all the conditions.

COROLLARY. Under the hypotheses of Proposition 2 the spaces B_0 and B_t are homotopy equivalent. Indeed let G_t, $t\in[0,1]$ be a flow on $R^{k*}\times S^N$ such that $G_t(B_1)\subset B_{1-t}$, $G_0=\mathrm{id}$. The maps $G_1\circ F_1: B_0\to B_0$ and $F_1\circ G_1: B_1\to B_1$ are obviously homotopic to the identities.

LEMMA 7. Let us assume that the smooth map $f: V\to\mathcal{P}(N)$ is nondegenerate at the point $v_0\in V$ and $O_{v_0}(\varepsilon)$ is a closed neighborhood of radius ε of the point v_0 in V. Then for any sufficiently small $\varepsilon > 0$ and any sufficiently small nonnegative form $q\in\mathcal{P}(N)$ the map

$$v\mapsto f(v)+q,\quad v\in O_{v_0}(\varepsilon),$$

is nondegenerate on $O_{v_0}(\varepsilon)$.

Proof. One can assume that $v_0 = 0$. Moreover, applying Lemma 2, one can reduce everything to the case $f(v_0) = 0$. In this case the nondegeneracy of f at the point $v_0 = 0$ means that for any $x\in S^N$ one can find a $v_x\in V$ such that $(f_0'v_x)(x,x)>0$. Since S^N is compact, one can assume that this inequality holds uniformly in x, i.e., $(f_0'v_x)(x,x)\geqslant\delta>0$ $\forall x\in S^N$ for some $\delta > 0$; moreover, one can assume that $\varepsilon v_x\in O_0(\varepsilon)$ $\forall x\in S^N$ for all sufficiently small $\varepsilon > 0$.

Let $q\in\mathcal{P}(N)$, $q\geqslant 0$, $v\in O_0(\varepsilon)$, and $x\in S^N$ be such that $f(v)(x,x) + q(x,x) = 0$. Then $0 = f(0)\ (x,x) = -q(x,x) - (f_v'v)(x,x) + o(\varepsilon)$. Consequently, $f_v'(\varepsilon v_x - v)(x,x)\geqslant q(x,x)+\varepsilon\delta+o(\varepsilon)>0$ if ε is sufficiently small. ∎

Let $f: V\to\mathcal{P}(N)$ be a smooth map and $V\supset W$ be a closed convex set. We set

$$B_f(W)=\{(v,x)\in W\times S^N \mid f(v)\,(x,x)>0\}.$$

COROLLARY. If the smooth map $f: V\to\mathcal{P}(N)$ is nondegenerate at the point $v_0\in V$ then for any sufficiently small convex closed neighborhoods $O_{v_0}^1\subset O_{v_0}^2$ of the point v_0 in V the inclusion $B_f(O_{v_0}^1)\subset B_f(O_{v_0}^2)$ is a homotopy equivalence.

Indeed, let $O_{v_0}(\varepsilon_1)\subset O_{v_0}^1\subset O_{v_0}^2\subset O_{v_0}(\varepsilon_2)$. According to Lemma 7 the maps $f\,|\,O_{v_0}(\varepsilon)$, $\varepsilon_1\leqslant\varepsilon\leqslant\varepsilon_2$, are nondegenerate. After a simple change of variables this family of maps becomes a homotopy consisting of nondegenerate maps with a fixed domain of definition. It remains to use Proposition 2.

Proposition 3. If the smooth map $f: V\to\mathcal{P}(N)$ is nondegenerate at the point $v_0\in V$ then for any sufficiently small convex closed neighborhood O_{v_0} of the point v_0 in V the set $B_f(O_{v_0})$ has the homotopy type of the sphere $S^{N-\mathrm{ind}\,f(v_0)}$.

Proof. Let $\varepsilon_0 > 0$ and $q\in\mathcal{P}(N)$, $q>0$ be such that for any $0<\varepsilon\leqslant\varepsilon_0$, $0\leqslant\tau\leqslant 1$ the maps $v\mapsto f(v)+\tau q$ are nondegenerate on $O_{v_0}(\varepsilon)$ and $\mathrm{ind}\,f(v_0)=\mathrm{ind}(f(v_0)+q)$. If ε is sufficiently small, then $\ker(f(v)+q)=0$ $\forall v\in O_{v_0}(\varepsilon)$, and the map $v\mapsto f(v)+q$ is homotopic in the class of maps which are nondegenerate on $O_{v_0}(\varepsilon)$ to the constant map $v\mapsto f(v_0)+q$, $v\in O_{v_0}(\varepsilon)$. On the other hand the homotopy $f+\tau q$, $\tau\in[0,1]$, also consists of maps which are nondegenerate on $O_{v_0}(\varepsilon)$. According to the corollary to Proposition 2 the space $B_f(O_{v_0}(\varepsilon))$ is homotopy equivalent to the space $\{x\in S^N \mid f(v_0)(x,x)+q(x,x)>0\}$. Since the form $f(v_0) + q$ is nondegenerate and has the same index of inertia as the form $f(v_0)$, the latter space has the homotopy type of the sphere $S^{N-\mathrm{ind}\,f(v_0)}$.

Remark. The definition of nondegeneracy of the map $f: V\to\mathcal{P}(N)$ has a local character, so the definition of nondegeneracy and also all assertions of the present point obviously generalize to a larger class of compacta V than the convex compacta considered above: it suffices that each point $v_0\in V\subset R^{k*}$ have a closed neighborhood U_{v_0} in R^{k*}. such that for some diffeomorphism $\Phi: U_{v_0}\to U_{v_0}$ the set $\Phi(U_{v_0}\cap V)$ is a convex compactum.

Assertion i) of Theorem 1 follows almost directly from the results of this point. Indeed, the map $p^*|\Omega:\Omega\to\mathcal{P}(N)$ is nondegenerate. Applying the corollary to Lemma 7 and Proposition 3 to this map (and an arbitrary point $\omega_0\in\Omega$) according to the definition of the Leray spectral sequence we get:

$$E_1^{i,j}(\mathfrak{A})=C^i(\Omega_{N-j},\ \Omega_{N-j-1};\ \mathfrak{A}),\quad i\geqslant 0,\ j>0,$$

where $C^i(\Omega_{N-j}, \Omega_{N-j-1}; \mathfrak{A})$ is the complex of i-dimensional Čech cochains of the pair $(\Omega_{N-j}, \Omega_{N-j-1})$ with values in $\mathfrak{A}$. Consequently,

$$E_2^{i,j}(\mathfrak{A})=H^i(\Omega_{N-j},\ \Omega_{N-j-1};\ \mathfrak{A}),\quad i\geqslant 0,\ j>0$$

5. We concern ourselves with the differential d_2 of the spectral sequence $E_r(Z_2)$. We shall use the description of the Leray spectral sequence with the help of bicomplexes given in the Appendix to this section. In the present point all cohomology, homology, cochains,

and chains are considered only over the field $\mathbf{Z}_2$, so in the interest of simplifying the formulas we shall omit the argument $\mathbf{Z}_2$.

Let $\mathcal{O}=\{O_\iota, \iota\in I\}$ be a covering of the space Ω by open sets. For any finite subset $\alpha = \{\alpha_0, \dots, \alpha_m\}\subset I$ we set $O_\alpha=\bigcap_{i=0}^{m} O_{\alpha_i}$, $B_\alpha=O_\alpha\times S^N\cap B$; the symbol $C^n_s(B_\alpha)$ denotes the group of n-dimensional singular cochains of the space B_α. With a covering $\mathcal{O}$ there is connected a bicomplex $E^{i,j}(\mathcal{O})$, $i, j\geqslant 0$, where the group $E^{i,j}(\mathcal{O})$ consists of all those sequences ξ_α, $\alpha=\{\alpha_0, \dots, \alpha_i\}\subset I$, for which $\xi_\alpha\in C^j_s(B_\alpha)$. The group $C^i(\mathcal{O})$ of i-dimensional Čech cochains of the covering $\mathcal{O}$ is the collection of all sequences ε_α, $\alpha=\{\alpha_0, \dots, \alpha_i\}\subset I$ such that $\varepsilon_\alpha\in\{0, 1\}$. Consequently, the group $E^{i,j}(\mathcal{O})$ is a module over $C^i(\mathcal{O})$ for any $i,j\geqslant 0$. The "horizontal" boundary operator is denoted by the sumbol $\delta: E^{i,j}(\mathcal{O})\to E^{i+1,j}(\mathcal{O})$ and the "vertical" one by the symbol $d: E^{i,j}(\mathcal{O})\to E^{i,j+1}(\mathcal{O})$; cf. the appendix to the present section for their definition.

In what follows it is assumed that the elements O_ι of the covering $\mathcal{O}$ are convex sets in S^{k-1} (we recall that $\Omega\subset S^{k-1}$). Moreover, the following arguments are only true for sufficiently fine coverings; specific conditions on how fine the covering $\mathcal{O}$ must be will arise in the course of the calculations.

We set $\Lambda=\bigcap_{n=1}^{N}\Lambda_n=\{q\in\mathcal{P}(N)\,|\,\lambda_i(Q)\neq\lambda_j(Q)$ for $i\neq j\}$ so $\mathcal{P}(N)\setminus\Lambda$ can be represented as a union of a finite number of smooth submanifolds of codimension not less than two in $\mathcal{P}(N)$. We recall also that $D=\{q\in\mathcal{P}(N)\,|\ker q\neq 0\}$ is the union of N + 1 proper smooth submanifolds in $\mathcal{P}(N)$. An arbitrary smooth map of a smooth manifold into $\mathcal{P}(N)$ is said to be transverse to the set $\mathcal{P}(N)\setminus\Lambda$ (respectively, to the set D), if it is transverse to each of these submanifolds. The relative interior ri(Ω) of the set Ω is obviously a smooth manifold. From the nondegeneracy of the map $p^*|\Omega:\Omega\to\mathcal{P}(N)$ it does not generally follow that the map $P^*|\mathrm{ri}(\Omega)$ is transverse to the set $\mathcal{P}(N)\setminus\Lambda$ and D. However one can always choose an arbitrarily small positive quadratic form q_0 such that the map $(p^*|\Omega+q_0):\omega\to p^*(\omega)+q_0$, $\omega\in\mathrm{ri}\Omega$ is transverse to the set $\mathcal{P}(N)\setminus\Lambda$ and D, and the filtration of $(p^*|\Omega+q_0)^{-1}(\mathcal{P}_n)$, $0\leqslant n\leqslant N$ has the same homotopy type as the filtration $\Omega_n=p^*|\Omega^{-1}(\mathcal{P}_n)$, $0\leqslant n\leqslant N$ of the set Ω. Replacing $p^*|\Omega$ by $p^*|\Omega+q_0$, if necessary, we can (and shall) assume that $p^*|\mathrm{ri}(\Omega)$ is transverse to $\mathcal{P}(N)\setminus\Lambda$ and D. In this case the set $p^{*-1}(\mathcal{P}(N)\setminus\Lambda)\cap\mathrm{ri}(\Omega)$ [the set $p^{*-1}(D)\cap\mathrm{ri}(\Omega)$] can be represented as the union of a finite number of submanifolds of codimension not less than two (one) in ri(Ω).

For each $\alpha=\{\alpha_0, \dots, \alpha_m\}\subset I$ such that $O_\alpha\neq\varnothing$ we set $n_\alpha=\min_{\omega\in O_\alpha}\mathrm{ind}\,\omega p$ and we choose a point $\omega_\alpha\in O_\alpha$, satisfying the condition $\mathrm{ind}\,\omega_\alpha p=n_\alpha$.

Let

$$\omega_\alpha^{\alpha_i}(\tau)=\frac{\tau\omega_{\alpha\setminus\alpha_i}+(1-\tau)\,\omega_\alpha}{|\tau\omega_{\alpha\setminus\alpha_i}+(1-\tau)\,\omega_\alpha|},\quad 0\leqslant\tau\leqslant 1$$

be an arc of a great circle in S^{k-1} joining ω_α and $\omega_{\alpha\setminus\alpha_i}$, i = 0, 1,...,m. The transversality of the map $p^*|_{\mathrm{ri}(\Omega)}$ to the sets D and $\mathcal{P}\setminus\Lambda$ permits us by slightly changing the point ω_α, if neccessary to arrange that $\omega_\alpha\in\mathcal{P}\setminus D$, $\omega_\alpha^{\alpha_i}(\tau)\in\Lambda$ for all α, α_i, $0\leqslant\tau\leqslant 1$.

Thus, $\lambda_{n_\alpha}(\omega P)<0$ $\forall\omega\in O_\alpha$, $\lambda_{n_\alpha+1}(\omega_\alpha P)>0$,

$$\lambda_{j_1}(\omega_\alpha^{\alpha_i}(\tau)P)\neq\lambda_{j_2}(\omega_\alpha^{\alpha_i}(\tau)P)\ \text{for}\ j_1\neq j_2,\ \tau\in[0, 1].$$

Sufficiently fine coverings also satisfy the condition

$$\lambda_{n_\alpha}(\omega P)\neq\lambda_{n_\alpha+1}(\omega P)\quad\forall\omega\in O_\alpha.$$

In what follows all the conditions listed on the covering $\mathcal{O}$ and the point ω_α are assumed to hold. We denote by $R_n(\omega)$ [respectively, $S_n(\omega)$] the intersection of the sphere S^N with the invariant subspace of the operator ωP corresponding to the eigenvalues $\lambda_1(\omega P), \dots, \lambda_n(\omega P)$ (respectively, $\lambda_{n+1}(\omega P), \dots, \lambda_{N+1}(\omega P)$). In addition, let

$$R_n(O_\alpha)=\{(\omega, x)\,|\,\omega\in O_\alpha,\ x\in R_n(\omega)\},$$

$$S_n(O_\alpha)=\{(\omega, x)\,|\,\omega\in O_\alpha,\ x\in S_n(\omega)\},\quad R_\alpha=R_{n_\alpha}(\omega_\alpha),\quad S_\alpha=S_{n_\alpha}(\omega_\alpha).$$

LEMMA 8. For sufficiently fine coverings $\mathcal{O}$ the inclusions $(\omega_\alpha, S_\alpha)\subset B_\alpha\cap S_{n_\alpha}(O_\alpha)\subset B_\alpha$, where $\alpha\subset I$, $O_\alpha\neq\varnothing$, are homotopy equivalence.

Proof. Since $\omega p|R_{n_\alpha}(\omega)<0$ $\forall\omega\in O_\alpha$ one has $R_{n_\alpha}(O_\alpha)\cap B_\alpha=\varnothing$. At the same time, $S_{n_\alpha}(\omega)$ is the intersection of S^N with the orthogonal complement to $R_{n_\alpha}(\omega)$, and the obvious homotopy retraction of the space $(O_\alpha\times S^N)\setminus R_{n_\alpha}(O_\alpha)$ onto $S_{n_\alpha}(O_\alpha)$ carries B_α into $B_\alpha\cap S_{n_\alpha}(O_\alpha)$. Further, the map $\omega\mapsto\omega p|\operatorname{span}S_{n_\alpha}(\omega)$, $\omega\in O_\alpha$ can be considered as a map of O_α into $\mathcal{P}(N-n_\alpha)$ which in addition is nondegenerate (this follows instantly from the nondegeneracy of the map $p^*|\Omega$).. It remains to use the corollary to Lemma 7 and to recall that $\lambda_{n_\alpha+1}(\omega_\alpha P)>0$. ■

The space S_α is a sphere of dimension $N-n_\alpha$ and according to Lemma 8, B_α has the homotopy type of a sphere. Now we choose in each group $C_s^{N-n_\alpha}(B_\alpha)$, $\alpha\subset I$, $O_\alpha\neq\varnothing$ an element which represents the fundamental cohomology class of the homotopy sphere B_α.

An eigenvector of unit length of the operator $\omega_\alpha P$ corresponding to the eigenvalue $\lambda_{n_\alpha+1}(\omega_\alpha P)$ is defined up to sign. For each α we fix once and for all one of the two possible vectors and we denote it by e_α, $\omega_\alpha Pe_\alpha=\lambda_{n_\alpha+1}(\omega_\alpha P)e_\alpha$; by Σ_α we denote the hemisphere in $R_{n_\alpha+1}(\omega_\alpha)$ containing R_α and the vector e_α and by $\hat\Sigma_\alpha$ the hemisphere containing R_α and the vector $(-e_\alpha)$.

Let $\alpha_l\in\alpha$ and $A_\alpha^{\alpha_l}(\tau):\mathrm{R}^{N+1}\to\mathrm{R}^{N+1}$, $0\leqslant\tau\leqslant1$ be a family of orthogonal operators depending continuously on τ, satisfying the following conditions: $A_\alpha^{\alpha_l}(0)=\mathrm{id}$, the operator $A_\alpha^{\alpha_l}(\tau)$ carries eigenvectors of the operator $\omega_\alpha P$ into eigenvectors of the operator $\omega_\alpha^{\alpha_l}(\tau)P$, $0\leqslant\tau\leqslant1$. These conditions define $A_\alpha^{\alpha_l}(\tau)$ uniquely, since the symmetric operators $\omega_\alpha^{\alpha_l}(\tau)P$ do not have multiple eigenvalues. We set $A_\alpha^{\alpha_l}=A_\alpha^{\alpha_l}(1)$; it is easy to see that

$$A_\alpha^{\alpha_l}S_\alpha=S_{\alpha\setminus\alpha_l},\quad A_\alpha^{\alpha_l}e_\alpha=\pm e_{\alpha\setminus\alpha_l}\text{ for } n_{\alpha\setminus\alpha_l}=n_\alpha;$$
$$A_\alpha^{\alpha_l}S_\alpha\subset S_{\alpha\setminus\alpha_l}\setminus e_\alpha\text{ for } n_{\alpha\setminus\alpha_l}<n_\alpha.$$

The hemisphere Σ_α is an n_α-dimensional cycle in $(S^N, S^N\setminus\beta_r(B_\alpha))$, and we denote the $(N-n_\alpha)$-dimensional cocycle dual to it in $\beta_r(B_\alpha)$ by Σ_α^+: the value of the cocycle Σ_α^+ on a singular simplex in $\beta_r(B_\alpha)$ is equal to the intersection index of this simplex with Σ_α. Finally, we set $\sigma_\alpha=\beta_r^*(\Sigma_\alpha^+)$. The index of intersection of Σ_α with the sphere S_α is obviously equal to one. Consequently, the restriction of σ_α to the sphere $(\omega_\alpha, S_\alpha)$ represents the fundamental cohomology class on this sphere. According to Lemma 8, the cocycle σ_α represents the fundamental cohomology class in $H^{N-n_\alpha}(B_\alpha)$. The cocycle $\hat\sigma_\alpha=\beta_r^*(\hat\Sigma_\alpha^+)$ is obviously cohomologous to the cocycle σ.

We introduce some more auxiliary cochains in B_α. By Σ_α^+ we denote one of the hemispheres in $R_{n_\alpha+2}(\omega_\alpha)$ with boundary $R_{n_\alpha+1}(\omega_\alpha)$. Let Σ_α^{++} be the cochain dual to Σ_α^+ in $\beta_r(B_\alpha)$ and $\sigma_\alpha^+=\beta_r^*(\Sigma_\alpha^{++})$. Since $\partial\Sigma_\alpha^+=R_{n_\alpha+1}(\omega_\alpha)=\Sigma_\alpha+\hat\Sigma_\alpha$ one has

$$\sigma_\alpha^+=\sigma_\alpha+\hat\sigma_\alpha \tag{4}$$

Further, for $\alpha_l\in\alpha$ we set $\Sigma_\alpha^{\alpha_l}=\bigcup_{0<\tau<1}\left(A_\alpha^{\alpha_l}(\tau)\left(A_\alpha^{\alpha_l}\right)^{-1}\Sigma_{\alpha\setminus\alpha_l}\right)$. Again $\Sigma_\alpha^{\alpha_l+}$ is the dual cochain to $\Sigma_\alpha^{\alpha_l}$ in $\beta_r(B_\alpha)$ and $\sigma_\alpha^{\alpha_l}=\beta_r^*(\Sigma_\alpha^{\alpha_l+})$. Then

$$\sigma_\alpha^{\alpha_l}=\sigma_{\alpha\setminus\alpha_l}|B_\alpha+\begin{cases}\sigma_\alpha, & A_\alpha^{\alpha_l}e_\alpha=e_{\alpha\setminus\alpha_l}\\ \hat\sigma_\alpha, & A_\alpha^{\alpha_l}e_\alpha=-e_{\alpha\setminus\alpha_l}\\ 0, & n_{\alpha\setminus\alpha_l}<n_\alpha.\end{cases} \tag{5}$$

We recall that $\lambda_{n_\alpha}(\omega P)\neq\lambda_{n_\alpha+1}(\omega P)$ for $\omega\in O_\alpha$.. We call $\alpha=\{\alpha_0,\ldots,\alpha_m\}$ regular if the inequality $\lambda_{n_\alpha}(\omega P)\neq\lambda_{n_\alpha+1}(\omega P)$ holds for any $\omega\in\bigcup_{i=0}^m O_{\alpha_i}$. The following assertion is obvious.

<u>LEMMA 9.</u> If $\alpha=\{\alpha_0,\ldots,\alpha_m\}$ is regular, then the inclusion

$$R_{n_\alpha}(O_\alpha)\subset\bigcup_{i=0}^m R_{n_\alpha}(O_{\alpha_i})$$

is a homotopy equivalence. ▪

It is shown in the appendix to the present section how to calculate the differential d_2 in the spectral sequence of a bicomplex. In our situation $E_1^{m,N-n}(\mathcal{O})=C_{\mathcal{O}}^m(\Omega_n,\Omega_{n-1})$, where the group of Čech cochains $C_{\mathcal{O}}^m(\Omega_n,\Omega_{n-1})$ consists of sequences $\varepsilon_\alpha\in\{0,1\}$, $\alpha=\{\alpha_0,\ldots,\alpha_m\}\subset I$ such that $\varepsilon_\alpha=0$ for $n_\alpha\ ^2\ n$. Correspondingly, $E_2^{m,N-n}(\mathcal{O})=H_{\mathcal{O}}^m(\Omega_n,\Omega_{n-1})$ and $d_2:H_{\mathcal{O}}^m(\Omega_n,\Omega_{n-1})\to H_{\mathcal{O}}^{m+2}(\Omega_{n+1},\Omega_n)$.

Let the cocycle ε_α in $C_{\mathcal{O}}^m(\Omega_n,\Omega_{n-1})$ define the cohomology class $\bar\varepsilon\in H_{\mathcal{O}}^m(\Omega_n,\Omega_{n-1})$, then the d-cocycle $\varepsilon_\alpha\sigma_\alpha$, $\alpha=\{\alpha_0,\ldots,\alpha_m\}\subset I$ in $E^{m,N-n}(\mathcal{O})$ represents the class ε_α from $E_1^{m,N-n}(\mathcal{O})=C_{\mathcal{O}}^m(\Omega_n,\Omega_{n-1})$

[we recall that $E(\mathcal{O})$ is the d-cohomology of the bicomplex $E_1(\mathcal{O})$. Here $\delta(\varepsilon.\sigma.)$ is the d-coboundary in $E^{m+1,N-n}(\mathcal{O})$, $\delta(\varepsilon.\sigma.)=dE$. Then $\delta E.$ is a d-cocycle and the class of the corresponding d-cocycle $\delta E.$ in $E_1^{m+2,N-n-1}(\mathcal{O})=C_{\mathcal{O}}^{m+2}(\Omega_{n+1},\Omega_n)$ is a cocycle in this group of Čech cochains. Let $\overline{\delta E}.\in H_{\mathcal{O}}^{m+2}(\Omega_{n+1},\Omega_n)$ be the cohomology class of the Čech cocycle of $C_{\mathcal{O}}^{m+2}(\Omega_{n+1},\Omega_n)$ corresponding to the d-cocycle $\delta E.$. Then by definition

$$d_2\bar{\varepsilon}.=\overline{\delta E}.$$

Thus, our next problem is to define a sequence E_α, $\alpha=\{\alpha_0,\dots,\alpha_{m+1}\}$ explicitly such that $dE.=\delta(\varepsilon.\sigma.)$. We recall that

$$\delta(\varepsilon.\sigma.)_\alpha=\sum_{i=0}^{m+1}\varepsilon_{\alpha\setminus\alpha_i}\sigma_{\alpha\setminus\alpha_i}\,|\,B_\alpha.$$

Let

$$\nu_\alpha^{\alpha_i}=\begin{cases}1, & A_\alpha^{\alpha_i}e_\alpha=-e_{\alpha\setminus\alpha_i}\\ 0, & A_\alpha^{\alpha_i}e_\alpha\neq-e_{\alpha\setminus\alpha_i}.\end{cases}$$

We set $E_\alpha=\sum_{i=0}^{m+1}\varepsilon_{\alpha\setminus\alpha_i}(\sigma_\alpha^{\alpha_i}+\nu_\alpha^{\alpha_i}\sigma_\alpha^+)$. It follows from (4) and (5) that $(d\varepsilon.)_\alpha=\sum_{i=0}^{m+1}\varepsilon_{\alpha\setminus\alpha_i}\sigma_{\alpha\setminus\alpha_i}\,|\,B_\alpha=\delta(\varepsilon.\sigma.)$ [we of course learned that $\varepsilon.$ is a cocycle in $C_{\mathcal{O}}^m(\Omega_n,\Omega_{n-1})$ and hence $\sum_{i=0}^{m+1}\varepsilon_{\alpha\setminus\alpha_i}\equiv 0 \bmod 2$ for $n_\alpha = n$].

Let $\alpha=\{\alpha_0,\dots,\alpha_{m+2}\}$ be regular in $n_\alpha = n + 1$. It follows from Lemma 9 that the cocycle

$$(\delta E.)_\alpha=\sum_{\substack{i,j=0\\ i\neq j}}^{m+2}\varepsilon_{\alpha\setminus\{\alpha_i,\alpha_j\}}(\sigma_{\alpha\setminus\alpha_i}^{\alpha_j}+\nu_{\alpha\setminus\alpha_i}^{\alpha_j}\sigma_{\alpha\setminus\alpha_i}^+)\,|\,B_\alpha$$

is cohomologous to the cocycle

$$\sum_{\substack{i,j=0\\ i\neq j}}^{m+2}\varepsilon_{\alpha\setminus\{\alpha_i,\alpha_j\}}\nu_{\alpha\setminus\alpha_i}^{\alpha_j}\sigma_{\alpha\setminus\alpha_i}^+\,|\,B_\alpha.$$

The quantity $\varepsilon_{\alpha\setminus\{\alpha_i,\alpha_j\}}\nu_{\alpha\setminus\alpha_i}^{\alpha_j}$ is different from zero only for $n_{\alpha\setminus\alpha_i}=n$, but in this case the cocycle $\sigma_{\alpha\setminus\alpha_i}^+$ is obviously cohomologous to the cocycle σ_α in $H^{m+2}(B_\alpha)$. Conditionally, by $d_2\varepsilon.\in C_{\mathcal{O}}^{m+2}(\Omega_{n+1},\Omega_n)$ we denote the d-cohomology class of the cocycle $\delta\varepsilon.$ and by $\overline{d_2\varepsilon}.\in H_{\mathcal{O}}^{m+2}(\Omega_{n+1},\Omega_n)$ the corresponding Čech cohomology class. Then $d_2\bar{\varepsilon}=\overline{d_2\varepsilon}$, and for regular $\alpha=\{\alpha_0,\dots,\alpha_{m+2}\}$ we have established the equation

$$(d_2\varepsilon.)_\alpha=\sum_{\substack{i,j=0\\ i\neq j}}^{m+2}\varepsilon_{\alpha\setminus\{\alpha_i,\alpha_j\}}\nu_{\alpha\setminus\alpha_i}^{\alpha_j}. \tag{6}$$

Any point $\omega_0\in\Omega_{n+1}\setminus\Omega_n$ has a neighborhood U_{ω_0} such that $\lambda_{n+2}(\omega P)\neq\lambda_{n+1}(\omega P)$ for $\omega\in U_{\omega_0}$. Upon suitable refinement of the covering $\mathcal{O}$ it turns out that for O_α intersecting with a sufficiently small neighborhood of the point ω_0 it follows from the condition $n_\alpha = n + 1$ that α is regular. Hence we can restrict ourselves to calculating $(d_2\varepsilon.)_\alpha$ for regular collections α despite the fact that any fixed covering $\mathcal{O}$ generally has irregular collections.

Let $\alpha^j=\{\alpha_0^j,\dots,\alpha_l^j\}\subset I$, $j=0,1,\dots,r$ be a chain of collections of indices such that $\alpha^r = \alpha^0$ and any two neighboring collections differ by a unique element:

$$\alpha^i\cup\alpha^{i+1}=\alpha^i\cup\{\alpha_0^{i+1}\}=\{\alpha_0^i\}\cup\alpha^{i+1},\quad i=0,1,\dots,r.$$

We define a continuous closed curve $\omega(\tau)$, $0\leqslant\tau\leqslant 2r$ in Ω by the formulas

$$\omega(\tau)=\begin{cases}\omega_{\alpha^j\cup\alpha^{j+1}}^{\alpha_0^{j+1}}(2j+1-\tau), & 2j\leqslant\tau\leqslant 2j+1,\\ \omega_{\alpha^j\cup\alpha^{j+1}}^{\alpha_0^j}(\tau-2j-1), & 2j+1\leqslant\tau\leqslant 2j+2,\end{cases}$$

$j = 0, 1,\dots,r - 1$. Analogous formulas define a continuous family $A(\tau):\mathbf{R}^{N+1}\to\mathbf{R}^{N+1}$, $0\leqslant\tau\leqslant 2r$ of orthogonal transformations: $A(0) = \mathrm{id}$,

$$A(\tau)=\begin{cases} A^{\alpha_0^{j+1}}_{\alpha^j\cup\alpha^{j+1}}(2j+1-\tau)\left(A^{\alpha_0^{j+1}}_{\alpha^j\cup\alpha^{j+1}}\right)^{-1}A(2j),\ 2j+1\leqslant\tau\leqslant 2j+2\\ A^{\alpha_0^{j}}_{\alpha^j\cup\alpha^{j+1}}(\tau-2j-1)\left(A^{\alpha_0^{j}}_{\alpha^j\cup\alpha^{j+1}}\right)^{-1}A(2j+1),\ 2j+1\leqslant\tau\leqslant 2j+2\end{cases}$$

Let us assume that $n_{\alpha^j}=n_{\alpha^j\cup\alpha^{j+1}}=n,\ j=0,1,\ldots,r-1$ so the family of transformations $A(\tau)$ effects continuous transport of the subspaces span $R_n(\omega)$ along the curve $\omega(\tau)$, $0\leqslant\tau\leqslant 2r$ and one has

$$A(2r)e_{\alpha^\bullet}=(-1)^\mu e_{\alpha^\bullet},$$

where

$$\mu\equiv\sum_{j=0}^{n-1}\left(\nu^{\alpha_0^j}_{\alpha^j\cup\alpha^{j+1}}+\nu^{\alpha_0^{j+1}}_{\alpha^j\cup\alpha^{j+1}}\right)\bmod 2.$$

According to (3) (cf. the end of point 2), it follows from this that

$$\sum_{j=0}^{r-1}\left(\nu^{\alpha_0^j}_{\alpha^j\cup\alpha^{j+1}}+\nu^{\alpha_0^{j+1}}_{\alpha^j\cup\alpha^{j+1}}\right)\equiv\langle\pi_n+\pi_{n+1},\ \omega(\cdot)\rangle\bmod 2. \tag{7}$$

One can deduce from (6) and (7) that the cohomology class in $H^{m+2}(\Omega_{n+1},\Omega_n)$ corresponding to the cocycle $d_2\varepsilon$ coincides with the cohomology class corresponding to the cocycle $\delta_n(\pi_n\cup\varepsilon)+\pi_{n+1}\cup\delta_n\bar{\varepsilon}$. This deduction is a routine exercise in combinatorial topology which we omit. ∎

6. In this point we shall illustrate the action of Theorem 1, considering quadratic mappings $p:\mathbf{R}^{N+1}\to\mathbf{R}^3$ in more detail. Let $\omega\in\mathbf{R}^{3*}$; then $\omega p\overset{\text{def}}{=}p^*(\omega)$ is a real quadratic form on $\mathbf{R}^{N+1}$ and ωP is a symmetric $(N+1)\times(N+1)$-matrix.

To any nonzero row $\omega=(\omega_1,\omega_2,\omega_3)\in\mathbf{R}^{3*}\setminus 0$ corresponds a point $\bar{\omega}=(\omega_1:\omega_2:\omega_3)=\{\alpha\omega\mid\alpha\in\mathbf{R}\}$ of the real projective plane $\mathbf{RP}^2$. The equation $\det\omega P=0$ defines, in $\mathbf{RP}^2$, an algebraic curve

$$C_p=\{\bar{\omega}\in\mathbf{RP}^2\mid\det\omega P=0\}$$

of degree $N+1$. The curve C_p is called the curve of degeneration of the quadratic map p.

<u>LEMMA 10.</u> For a typical quadratic map $p\in\mathscr{P}^3(N)$ the curve C_p is nonsingular. If C_p is a nonsingular curve, then the map p is nondegenerate.

<u>Proof.</u> The curve C_p arises from the intersection of subspaces of $p^*(\mathbf{R}^{3*})$ with the hypersurface $D(N)=\{q\in\mathscr{P}(N)\mid\ker q\neq 0\}$ in $\mathscr{P}(N)$. The curve C_p is nonsingular if and only if $p^*(\mathbf{R}^{3*}\setminus 0)$ intersects D(N) only at nonsingular points of this hypersurface, where it intersects transversely. At the same time the set of singular points of the hypersurface D(N) is the union $\bigcup_{r=2}^{N+1}D_r(N)$ of smooth manifolds $D_r(N)$ each of which has codimension not less than three in $\mathscr{P}(N)$ (cf. Lemma 2 and its corollary). All these submanifolds are cones in $\mathscr{P}(N)$ (i.e., withstand multiplication by and $\alpha\in\mathbf{R}\setminus 0$), so that for a typical $p\in\mathscr{P}^{(3)}(N)$ a three-dimensional subspace of $p^*(\mathbf{R}^{3*})$ intersects the set of singular points of the hypersurface D(N) only at the origin. Moreover, a typical p^* is transverse to the manifold D_1 of nonsingular points of this hypersurface. The nondegeneracy of such a p is obvious.

We let $\tilde{C}_p=\{\omega\in S^2\mid\det\omega p=0\}$ which is a two-sheeted covering of the curve of degeneracy C_p. The connected components of the set $S^2\setminus\tilde{C}_p$ coincide with the connected components of the sets

$$\operatorname{int}\{\omega\in S^2\mid\operatorname{ind}\omega p=n\}=\operatorname{int}(\Omega_n(p)\setminus\Omega_{n-1}(p)),\quad n=0,1,\ldots,N+1.$$

Theorem 1 establishes a connection of the filtration $\Omega_n(p)$ of the sphere S^2 with the homology groups of the manifold $\hat{p}^{-1}(0)$ – the intersection of three real quadrics and S^N. Hence the disposition of the ovals of the curve of degeneracy turns out to be closely connected with the homology of $\hat{p}^{-1}(0)$.

We start with the calculation of the Euler characteristic of $\hat{p}^{-1}(0)$. The next corollary follows from Theorem 1.

<u>COROLLARY 1.</u> For any $k\leqslant N+1$ and nondegenerate $p\in\mathscr{P}^k(N)$ one has

$$\chi(\hat{p}^{-1}(0))=1+(-1)^N-2\sum_{n=0}^{N}(-1)^n\chi(\Omega_n(p)).$$

This equation follows directly from the description of the term $E_2(\mathbf{Z}_2)$ and the additivity of the Euler characteristic.

Let us assume that k = 3, N is odd, and the curve C_p is nonsingular. Straightforward transformations lead to the following identity:

$$\chi(\hat{p}^{-1}(0)) = 4\chi(\{\bar{\omega}\in \mathbf{RP}^2 \mid \det(\omega p) \leqslant 0\}).$$

For even N the manifold $\hat{p}^{-1}(0)$ is odd-dimensional so that $\chi(\hat{p}^{-1}(0)) = 0$.

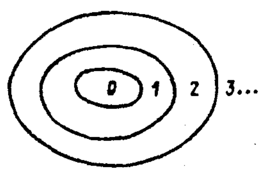
Below, until the end of the present point it is assumed everywhere that $p\in\mathcal{P}^3(N)$ while the curve C_p is nonsingular. All homology and cohomology are considered only with coefficients from $\mathbf{Z}_2$ so that the argument $\mathbf{Z}_2$ is omitted as a rule. In representing the term $E_2(\mathbf{Z}_2)$ as a table, in the (i, j) place instead of the $\mathbf{Z}_2$-space $E_2^{ij}(\mathbf{Z}_2) = H^i(\Omega_{N-j}(p), \Omega_{N-j-1}(p); \mathbf{Z}_2)$ we shall put the dimension of this space (i.e., the corresponding Betti number). We note that $E_2^{ij} = 0$ for $i > 3$ so that the table consists essentially of three columns.

COROLLARY 2. Let $N > 4$. Then

$$\hat{p}^{-1}(0) = \varnothing \Leftrightarrow H^1(\Omega_1(p) \setminus \Omega_0(p)) \neq 0.$$

Proof. The condition $\hat{p}^{-1}(0) = \varnothing$ is equivalent to the condition $H^N(S^N \setminus \hat{p}^{-1}(0)) \neq 0$, i.e., $E_3^{0,N} \oplus E_3^{1,N-1} \oplus E_3^{2,N-2} \neq 0$. We consider the following cases:

a) $\Omega_0(p) \neq \varnothing$. Since $\dim\ker\omega p \leqslant 1\ \forall\omega\in S^2$ and $\operatorname{ind}(-\omega p) = N+1-\operatorname{ind}\omega p$ if $\ker\omega p = 0$ the condition $\Omega_0(p) \neq \emptyset$ can hold only if the curve C_p contains a nest of [(N + 1)/2] ovals imbedded in one another and, if N + 1 is odd, another component which is not contractible in $\mathbf{RP}^2$. It follows from Bezout's theorem that in this case C_p is exhausted by the components indicated. All domains $\Omega_n(p)$, $0 \leqslant n \leqslant N$ turn out to be nonempty and contractible:

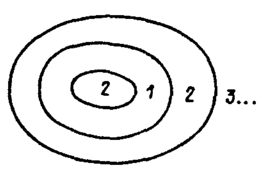


[the numbers indicate the value of $\operatorname{ind}\bar{\omega}p \overset{\text{def}}{=} \min\{\operatorname{ind}\omega p, \operatorname{ind}(-\omega p)\}$ in the corresponding domains]. The table for E_2 has the form

	0	1	2
N	1	0	0
N-1	0	0	0
N-2	0	0	0
...	.	.	.

The set $\overline{\Omega_1(p) \setminus \Omega_0(p)}$ is an annulus.

b) $\Omega_0(p) = \varnothing$, $\Omega_1(p) \neq \varnothing$. In this case C_p contains a nest of [(N + 1)/2 − 1] ovals imbedded in one another. The relations $H^1(\Omega_1, \Omega_0) = 0$, $H^2(\Omega_2, \Omega_1) \neq 0$ could only hold in the presence of a nest of [(N + 1)/2] + 1 ovals, which contradicts Bezout's theorem. There remains a unique possibility: $H^1(\Omega_1, \Omega_0) \neq 0$. In order to realize it a nest of [(N + 1)/2] ovals is necessary. The only admissible situation is:



The table for E_2 is the following:

	0	1	2
N	0	0	0
N-1	1	1	0
N-2	0	0	1
...	.	.	.

The differential $d_2: H^0(\Omega_1, \Omega_0) \to H^2(\Omega_2, \Omega_1)$ must necessarily be nonzero and $\overline{\Omega_1(p) \setminus \Omega_0(p)}$ is again an annulus.

c) $\Omega_0(p)=\Omega_1(p)=\varnothing$, $\Omega_2(p)\neq\varnothing$. The relation $H^2(\Omega_2(p))\neq 0$ is only possible for $\Omega_2(p)=S^2$. but in this case N = 3.

Remark. We have proved somewhat more than is asserted in Corollary 2: if $\hat{p}^{-1}(0)=\varnothing$ and $N\geqslant 4$ then $\Omega_1(p)\setminus\Omega_0(p)$ is an annulus and the curve C_p contains a nest of $[(N+1)/2]$ imbedded ovals. We note that all such curves are rigidly isotopic to one another (cf. [15]). For N = 3 the manifold $\hat{p}^{-1}(0)$ is zero-dimensional and its emptiness is equivalent to the equality of the Euler characteristic to zero. To the case of two ovals in one another one here adds the situation $C_p = \emptyset$, i.e., $\operatorname{ind}\omega p=2\ \forall\omega\in S^2$. It is interesting that the empty curve of degeneration corresponds to the map mentioned in point 1

$$\hat{p}:(z_1, z_2)\mapsto(z_1\bar{z}_2,\ |z_2|^2-|z_1|^2),\ (z_1, z_2)\in S^3\subset\mathbf{C}^2=\mathbf{R}^4,$$

which realizes the Hopf bundle.

Analogously to Corollary 2, by listing the various cases one proves

COROLLARY 3. Let $N\geqslant 6$. Then

$$\operatorname{rank} H_0(\hat{p}^{-1}(0))=1\Leftrightarrow H^1(\Omega_2(p)\setminus\Omega_1(p))=H^1(\Omega_1(p)\setminus\Omega_0(p))=0.$$

We cite several more simple facts which follow directly from the consideration of the term E_2 of our spectral sequence, without the participation of the differential d_2. Let $N\geqslant 4$, $b_i=\operatorname{rank} H_i(\hat{p}^{-1}(0))$. Then

1) if the curve C_p consists of $\ell > 0$ ovals situated outside one another, then

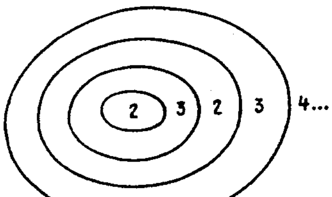
$$b_i=\begin{cases}1, & i=0,\ N-3\\ 2(2l-1) & i=(N-3)/2\\ 0, & \text{otherwise}\end{cases}$$

2) if C_p does not contain a nest of $r > 1$ ovals imbedded in one another, then $b_i = 0$ for $0 < i < [(N+1)/2] - r$;

3) if ℓ is the number of connected components of the curve C_p then

$$\sum_{i=0}^{N-3} b_i\leqslant 4l \quad \text{for } l>0 \text{ and} \sum_{i=0}^{N-3} b_i=4 \quad \text{for } l=0.$$

For the precise calculation of the Betti numbers it is necessary to use the differential d_2. Let, for example, C_p be the curve we have already encountered containing a nest of $[(N+1)/2]$ ovals imbedded in one another, and the values of $\operatorname{ind}\bar{\omega}p$, $\bar{\omega}\in\mathbf{RP}^2$ be distributed as follows:



The table for E_2 has the form

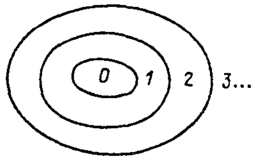
	0	1	2
N	0	0	0
N-1	0	0	0
N-2	2	1	0
N-3	0	1	1
	.	.	.

Here the differential $d_2:H^0(\Omega_2)\to H^2(\Omega_3, \Omega_2)$ can be different from zero. We renumber the ovals of the curve C_p, denoting the innermost oval by c_p^1, the one containing it by c_p^2, etc. Let $\tilde{c}=\{\omega\in S^2\,|\,\bar{\omega}\in C^2(p),\ \operatorname{ind}\omega p=2\}$ be one of the two connected components of the preimage of the oval c_p^2 under the canonical map $S^2\to\mathbf{RP}^2$. For any $\omega\in\tilde{c}$ we have $\dim\ker\omega p=1$, $\operatorname{ind}\omega p=2$, and we denote by $V^-(\omega)$ the invariant subspace of the operator ωP corresponding to the negative eigenvalues. We fix the point $\omega_0\in\tilde{c}$, and let $\mu_0:\ker\omega_0 P\to\ker\omega_0 P$ and $\bar{\mu}:V^-(\omega_0)\to V^-(\omega_0)$ be the monodromy transformations obtained upon transporting the subspaces $\ker\omega p$ and $V^-(\omega)$, respectively, along the curve τ. Using the description of the differential d_2 given in Theorem 1, by direct calculation we get:

$$d_2=0 \Leftrightarrow \det \mu_0>0 \quad \& \det \mu^->0,$$

here $\operatorname{rank} H^0(\hat{p}^{-1}(0))+\operatorname{rank} d_2=3$.

Along with the homology of the manifold $\hat{p}^{-1}(0)$ Theorem 1 lets us estimate the homology groups of the manifolds $p^{-1}(a)$, where $\mathbf{R}^k \ni a$ is a nonzero vector, i.e., get information about the sets of solutions not only of homogeneous systems of quadratic equations (or inequalities) but also inhomogeneous ones. A particularly simple situation arises in the case which is important for applications when $\Omega_0(p)\neq\varnothing$, i.e., when the linear system of quadratic forms ωp, $\omega\in\mathbf{R}^3$, contains a positive definite form. As already noted in the proof of Corollary 2, the curve C_p in this case contains a nest of $[(N+1)/2]$ ovals imbedded in one another, and the values of $\operatorname{ind}\bar{\omega}p$, $\bar{\omega}\in\mathbf{R}P^2$ are distributed as follows:



Let $a\in\mathbf{R}^3\setminus 0$ be a regular value of the quadratic map $p:\mathbf{R}^{N+1}\to\mathbf{R}^3$. We consider the quadratic map $p_a\in\mathcal{P}^3(N+1)$ defined by the formula $p_a((x_0,x),(x_0,x))=p(x,x)-x_0^2a$. It is easy to see that p_a is nondegenerate and $\hat{p}_a^{-1}(0)$ is diffeomorphic to the manifold $p^{-1}(a)\subset\mathbf{R}^{N+1}$. The vector a defines the line $A=\{\bar{\omega}\in\mathbf{R}P^2\mid\omega a=0\}$ in $\mathbf{R}P^2$; the curve of degeneration of the map p_a has the form: $C_{p_a}=C_p\cup A$. The values of $\operatorname{ind}\omega p_a$ can be calculated according to the following rule:

$$\operatorname{ind}\omega p_a=\begin{cases}\operatorname{ind}\omega p, & \omega a\leqslant 0,\\ \operatorname{ind}\omega p+1, & \omega a>0\end{cases}$$

We note that the curve $Cp_a=C_p\cup A$ is generally singular which however in no way hinders the use of Theorem 1. If A intersects (is not tangent to) the inner oval c_p^1 of the curve C_p then $\Omega_0(p_a)\neq\varnothing$ and $p^{-1}(a)=\varnothing$. Now if A does not intersect c_p^1 then either $\Omega_0(p_a)=\Omega_0(p)$ or $\Omega_0(p)$ is one of the connected components of the set $\Omega_1(p_a)$. Considering the remark made after the proof of Corollary 2, we get

COROLLARY 4. The image of the quadratic map p is a proper convex cone in $\mathbf{R}^3$ if and only if $\Omega_0(p)\neq\varnothing$.

7. Here we give infinite-dimensional versions of the results obtained in the present section. Throughout this entire point, $\mathbf{H}$ is an infinite-dimensional separable Hilbert space (but not the skew field of quaternions), S is the unit sphere in $\mathbf{H}$.

Let $p\in\mathcal{P}^k(H)$; we set $\|p\|=\sup_{\|x\|=1}|p(x,x)|$. The norm $p\to\|p\|$ defines in $\mathcal{P}^k(\mathbf{H})$ a Banach space structure. As soon as the space $\mathcal{P}^k(\mathbf{H})$ is endowed with a topology, the concept of essential surjectivity of the quadratic map $x\to p(x,x)$, whose definition is a word for word repetition of the corresponding finite-dimensional definition, acquires meaning. Just as before, to denote the quadratic map $x\to p(x,x)$ one uses the same symbol p as for the bilinear map.

LEMMA 11. Let $p\in\mathcal{P}^k(\mathbf{H})$. If the quadratic map p is essentially surjective, then the restriction of p to a finite-dimensional subspace of $\mathbf{H}$ is also essentially surjective.

Proof. Let $p=(p_1,\ldots,p_k)^T$, $p_i(x,x)=(P_ix,x)$. According to the spectral theorem, $P_i=\int_{-l}^{l}\lambda dE_\lambda^i$ for a suitable decomposition of the identity E_λ^i, $E_{-l}^i=0$, $E_l^i=\mathrm{id}$. Now let $-l=\theta_0<\theta_1<\ldots<\theta_{N+1}=l$ be a partition of the segment $[-l,l]$ which is sufficiently fine that the quadratic maps

$$x\mapsto\sum_{j=0}^{N}\left(\lambda_j^1\int_{\theta_j}^{\theta_{j+1}}d(E_\lambda^1x,x),\ldots,\lambda_j^k\int_{\theta_j}^{\theta_{j+1}}d(E_\lambda^kx,x)\right)^T,\quad x\in\mathbf{H},$$

are essentially surjective $\forall\lambda_j^i\in[\theta_j,\theta_{j+1}]$, $i=1,\ldots,k$, $j=0,\ldots,N$. We set $V_j^i=\int_{\theta_j}^{\theta_{j+1}}dE_j^i\mathbf{H}$ and we divide the set of pairs of indices (i,j), $1\leqslant i\leqslant k$, $0\leqslant j\leqslant N$ into two subsets I_f, I_∞ such that $(i,j)\in I_f$, if $\dim V_j^i<\infty$ and $(i,j)\in I_\infty$, otherwise. In each of the subspaces V_j^i for $(i,j)\in I_\infty$ we choose a nonzero vector x_j^i such that $(x_{j_1}^{i_1},x_{j_2}^{i_2})=0$ for $|(i_1,j_1)\neq(i_2,j_2)$ and we set

$$V = \sum_{(i,j)\in I_f} V_j^i + \operatorname{span}\{x_j^i \mid (i,j)\in I_\infty\}.$$

It is easy to see that the quadratic map $p|V$ is essentially surjective.

As a corollary to Lemma 11 and Proposition 1 we get

Proposition 1_∞. Let $p\in\mathscr{P}^k(\dot{H})$. If the quadratic map p is essentially surjective, then $p^{-1}(0) \neq 0$.

Let $\mathbf{R}^k\supset K$ be a convex polyhedral cone and $p\in\mathscr{P}^k(\mathbf{H})$. The definition of nondegeneracy of a quadratic map p with respect to K repeats the corresponding finite-dimensional definition from point 2 word for word. To the end of the present point the cone K and map p which is nondegenerate with respect to K are assumed fixed. We set

$$\Omega=\{\omega\in K^\circ\cap S^{k-1} \mid \operatorname{ind}\omega p<+\infty\}.$$

It is easy to see that Ω is a convex subset of the sphere S^{k-1} ; however it is not generally either open or closed. The subsets

$$\Omega_n=\{\omega\in\Omega \mid \operatorname{ind}\omega p\leqslant n\}\ n=0,\ 1,\ldots;\quad \Omega_j=\varnothing \text{ for } j<0$$

form an increasing filtration of the set Ω by closed subsets.

In contrast with the finite-dimensional case the subsets Ω_n are not generally neighborhood retracts so that the Čech cohomology may not coincide with the singular cohomogly. To denote the i-dimensional Čech cohomology group we shall use the symbol $\check{H}_i$ (recalling the Czech language).

We set $\mathscr{P}_n(\mathbf{H})=\{q\in\mathscr{P}(\mathbf{H}) \mid \operatorname{ind} q\leqslant n\}$ and let $\mathscr{L}_n(\mathbf{H})$ be the vector bundle whose base is $\mathscr{P}_n(\mathbf{H})\setminus \mathscr{P}_{n-1}(\mathbf{H})$ and whose fiber over the point $q\in\mathscr{P}_n(\mathbf{H})\setminus\mathscr{P}_{n-1}(\mathbf{H})$ is the n-dimensional invariant subspace $L_n(Q)$ of the operator Q corresponding to the negative part of the spectrum of this operator.

We recall that $p^*\omega \overset{\text{def}}{=} \omega p\ \forall\omega\in S^{k-1}$. Restricting the map p^* to $\Omega_n\setminus\Omega_{n-1}$, we get an induced bundle $(p^*|\Omega_n\setminus\Omega_{n-1})^*\mathscr{L}_n(\mathbf{H})$ over Ω_n/Ω_{n-1}. We denote by π_n the one-dimensional Stiefel–Whitney class of this bundle, so

$$\pi_n = w_1((p^*|\Omega_n\setminus\Omega_{n-1})^*\mathscr{L}_n(\mathbf{H})),\quad \pi_n\in\check{H}^1(\Omega_n\setminus\Omega_{n-1};\ \mathbf{Z}_2).$$

As in point 3, $\delta_n:\check{H}^i(\Omega_n,\ \Omega_{n-1},\ \mathbf{Z}_2)\to\check{H}^{i+1}(\Omega_{n+1},\ \Omega_n,\ \mathbf{Z}_2)$ is the connecting homomorphism in the exact sequence of the triple $(\Omega_{n+1},\Omega_n,\Omega_{n-1})$.

We set $\overline{H}_i(\hat{p}^{-1}(K);\mathbf{Z}_2) \overset{\text{def}}{=} \tilde{H}_{-i}(\hat{p}^{-1}(K),\mathbf{Z}_2)$ for any integer $i \neq 1$ (here $\tilde{H}_*$ are the reduced singular homology groups) and

$$\overline{H}_1(\hat{p}^{-1}(K);\ \mathbf{Z}_2)=\begin{cases}\mathbf{Z}_2, & \hat{p}^{-1}(K)=\varnothing\\ 0, & \hat{p}^{-1}(K)\neq\varnothing.\end{cases}$$

THEOREM 2. There exists a cohomology spectral sequence (E_r, d_r), $r\geqslant 2$ converging to $\overline{H}_*(\hat{p}^{-1}(K);\mathbf{Z}_2)$ such that

i) $E_2^{i,j}=\check{H}^i(\Omega_{1-j},\Omega_{-j};\ \mathbf{Z}_2)\ \forall i,j\in\mathbf{Z}$;

ii) the differential $d_2:\check{H}^i(\Omega_n,\ \Omega_{n-1};\ \mathbf{Z}_2)\to\check{H}^{i+2}(\Omega_{n+1},\ \Omega_n;\ \mathbf{Z}_2)$ is defined by the formula

$$d_2(\xi)=\delta_n(\pi_n\cup\xi)+\pi_{n+1}\cup\delta_n\xi,\quad \forall\xi\in\check{H}^i(\Omega_n,\ \Omega_{n-1};\ \mathbf{Z}_2).$$

Proof. Since the map $\hat{p}:S\to\mathbf{R}^k$ is transverse to the cone K, $\hat{p}^{-1}(K)$ is a submanifold with boundary in the Hilbert sphere S. Let $V \subset W$ be arbitrary finite-dimensional subspaces $\mathbf{H}$ and $\nu_V^W:H_*(\hat{p}^{-1}(K)\cap V;\mathbf{Z}_2)\to H_*(\hat{p}^{-1}(K)\cap W;\mathbf{Z}_2)$ be the homomorphism induced by the inclusion $V \subset W$. We denote by $\mathscr{V}$ the collection of all finite-dimensional subspaces of $\mathbf{H}$, partially ordered by inclusion. The groups $H_*(\hat{p}^{-1}(K)\cap V;\mathbf{Z}_2)$, $V\in\mathscr{V}$ and homomorphisms ν_V^W, $V\subset W\in\mathscr{V}$ form a direct $\mathscr{V}$-system $\mathscr{N}$. According to Theorem A2 of the appendix to the present section, the inclusions $\hat{p}^{-1}(K)\cap V\subset\hat{p}^{-1}(K)$ induce an isomorphism

$$\varinjlim \mathscr{N}\approx H_*(\hat{p}^{-1}(K)).$$

Let $\mathscr{V}\supset\mathscr{V}_0$ be the collection of all finite-dimensional subspaces of $\mathbf{H}$ such that the quadratic map $p|V$, $V\in\mathscr{V}_0$ is nondegenerate with respect to K. Since the map $\hat{p}$ is bounded, it follows from Proposition A2 (cf. the appendix to the present section) that for any $V\in\mathscr{V}$ one can find a $V_0\supset V$, $V_0\in\mathscr{V}_0$. Thus, the directed set $\mathscr{V}_0$ is cofinal in $\mathscr{V}$.

Let $V\in\mathcal{V}_0$, $\Omega_n(V)\overset{\text{def}}{=}\{\omega\in K^0\cap S^{k-1}\,|\,\operatorname{ind}(\omega p|V)\leqslant n\}$. Considering the Alexander duality, Theorem 1 guarantees the existence of a cohomology spectral sequence $(E_r(V),\ d_r(V))$, $r\geqslant 2$, with the following properties:

1) $E_2^{ij}(V)=H^i(\Omega_{1-j}(V),\ \Omega_{-j}(V);\ \mathbf{Z}_2)\ \forall i,\ j\in\mathbf{Z}$.

2) The differential $d_2(V):H^i(\Omega_n(V),\ \Omega_{n-1}(V);\ \mathbf{Z}_2)\to H^{i+2}(\Omega_{n+1}(V),\ \Omega_n(V);\ \mathbf{Z}_2)$ $n,\ i\geqslant 0$ is defined by the formula

$$d_2(V)(\xi)=\delta_n(\pi_n(V)\smile\xi)+\pi_{n+1}(V)\smile\delta_n\xi,\quad \forall\xi\in H^i(\Omega_n(V),\ \Omega_{n-1}(V);\ \mathbf{Z}_2),$$

where $\delta_n:H^j(\Omega_n(V),\ \Omega_{n-1}(V))\to H^{j+1}(\Omega_{n+1}(V),\ \Omega_n(V))$ is the connecting homomorphism in the exact sequence of the triple, $\pi_n(V)\in H^1(\Omega_n(V)\setminus\Omega_{n-1}(V))$.

3) The spectral sequence $(E_r(V),\ d_r(V))$ converges to $\bar{H}_*(\hat{p}^{-1}(K)\cap V;\ \mathbf{Z}_2)$.

Moreover, for $V\subset W\in\mathcal{V}_0$ there are defined the homomorphisms $e_V^W(r):E_r(V)\to E_r(W)$ of the spectral sequence $E_r(V)$ into the sequence $E_r(W)$ such that $e_V^W(r)$ converges to the homomorphism $\bar{v}:\bar{H}_*(\hat{p}^{-1}(K)\cap V;\ \mathbf{Z}_2)\to\bar{H}_*(\hat{p}^{-1}(K)\cap W;\ \mathbf{Z}_2)$, induced by the inclusion $V\subset W$. Here the sequences $(E_r(V),\ d_r(V))$ and homomorphisms $e_V^W(r)$ form a direct $\mathcal{V}$-system $(\mathfrak{E}_r,\ d_r)$.

We should explain where the homomorphisms $e_V^W(r)$ are taken from. Let

$$B=\{(\omega,\ x)\,|\,\omega\in K^\circ\cap S^{k-1},\ x\in S,\ \omega p(x,\ x)>0\}\subset S^{k-1}\times S$$

and

$$B(V)=B\cap(S^{k-1}\times V)\ \forall V\in\mathcal{V}_0.$$

It follows from the nondegeneracy of the quadratic map $p|V$ with respect to K that the closure $\overline{B(V)}$ of the set B in $S^{k-1}\times(S\cap V)$ is a topological submanifold with boundary in $S^{k-1}\times(S\cap V)$, in particular, the inclusion $B(V)\to\overline{B(V)}$ is a homotopy equivalence.

If $V\subset W$, then $\overline{B(V)}=B(W)\cap(S^{k-1}\cap V)$. The inclusion $\iota_V^W:S^{k+1}\times(S\cap V)\to S^{k-1}\times(S\cap W)$ defines the cohomology transfer

$$(\iota_V^W)^!:H^j(\overline{B(V)})\to H^{j+\dim(W/V)}(\overline{B(W)}).$$

We recall that the spectral sequence $(E_r(V),\ d_r(V))$ is (up to reindexing) the Leray sequence of the map

$$(\omega,\ x)\mapsto\omega,\quad(\omega,\ x)\in B(V).$$

The homomorphisms $e_V^W(r):E_r(V)\to E_r(W)$ are induced by the homomorphism $(\iota_V^W)^!$.

It is clear that $\Omega_n(W)\subset\Omega_n(V)$ for $V\subset W$ $\forall n\geqslant 0$. The homomorphism $e_V^W(2):H^i(\Omega_n(V),\ \Omega_{n-1}(V);\ \mathbf{Z}_2)\to H^i(\Omega_n(W),\ \Omega_{n-1}(W);\ \mathbf{Z}_2)$ is induced by the inclusion $(\Omega_n(W),\ \Omega_{n-1}(W))\supset(\Omega_n(V),\ \Omega_{n-1}(V))$. Since the cohomology functor commutes with direct limits, the spectral sequence $\varinjlim(\mathfrak{E}_r,\ d_r)$ converges to $\bar{H}_*(\hat{p}^{-1}(K);\ \mathbf{Z}_2)$.

Further, $E_2^{i,j}(V)=H^i(\Omega_{1-j}(V),\ \Omega_j(V);\ \mathbf{Z}_2)$ and obviously $\bigcap_{V\in\mathcal{V}}\Omega_n(V)=\Omega_n$. Consequently,

$$(\varinjlim\mathfrak{E}_2)^{ij}=\check{H}^i(\Omega_{1-j},\ \Omega_{-j};\ \mathbf{Z}_2).$$

Assertion i) of Theorem 2 is established.

LEMMA 12. Let $n\geqslant 0$, U be a topological subspace of $\mathcal{P}_n(\mathbf{H})\setminus\mathcal{P}_{n-1}(\mathbf{H})$ and $\mathcal{F}=\{F_q,\ q\in U\}$ be an n-dimensional vector bundle with base U and fibers $F_q\subset\mathbf{H}$, $q\in U$ such that $q|F_q<0$ $\forall q\in U$. Then the bundle $\mathcal{F}$ is isomorphic to the bundle $\mathcal{L}_n(\mathbf{H})|U$.

Proof. Let $L_n(Q)^\perp\subset\mathbf{H}$ be the orthogonal complement to the subspace $L_n(Q)$ in $\mathbf{H}$; then $q|L_n(Q)^\perp\geqslant 0$. Consequently, $F_q\cap L_n(Q)^\perp=0$ $\forall q\in U$, and the orthogonal projectors of the Hilbert space $\mathbf{H}$ onto $L_n(Q)$, $q\in U$, effect the isomorphism required.

It follows from the lemma proved that for any $V\in\mathcal{V}_0$ the restrictions of the classes $\pi_n(V)$ and π_n to the set $\Omega_n\setminus\Omega_{n-1}(V)$ coincide. Now assertion ii) of Theorem 2 follows directly from the definition of direct limit.

COROLLARY 1. Let us assume that p is nondegenerate with respect to the cone Ω° dual to Ω. Then the inclusion $\hat{p}^{-1}(K)\subset\hat{p}^{-1}(\Omega^\circ)$ induces a homomorphism of homology groups

$$H_i(\hat{p}^{-1}(K);\ \mathbf{Z}_2)\approx H_i(\hat{p}^{-1}(\Omega^\circ);\ \mathbf{Z}_2),\quad i>0.$$

COROLLARY 2. Let $m=\min_{\omega\in\Omega}\operatorname{ind}\omega p$. If $m>k-1$ then $\dot{p}^{-1}(K)\neq\emptyset$ and $\tilde{H}_i(\dot{p}^{-1}(K);Z_2)=0$ for $i\leqslant m-k$.

3. Application to Germs of Smooth Vector-Functions

Let O be a neighborhood of the origin of the separable Hilbert space H, $f:O\to R^k$ be a smooth map, $f(0)=0$, $f_0':H\to R^k$ be the differential, and $f_0'':\ker f_0'\times\ker f_0'\to\operatorname{coker} f_0'$ be the Hessian of the map f at zero. It is shown in this point how the properties of the quadratic map f_0'' are related to the corresponding local properties of the smooth map f. As one should expect, the condition that the Hessian f_0'' be nondegenerate plays a decisive role here.

We begin with the finite-dimensional case.

Proposition 4. Let $\dim H<+\infty$ and the quadratic map f_0'' be nondegenerate. Then

1) If $0\in\operatorname{int} f(O_0)$ for any neighborhood of the origin $O_0\subset O$ then the quadratic map f_0'' is surjective.

2) If the quadratic map f_0'' is essentially surjective and $(\operatorname{corank} f_0')^2\leqslant\dim\ker f_0'$ then $0\in\operatorname{int} f(O_0)$ for any neighborhood of the origin $O_0\subset O$.

Proof. Let $x\in O$ and set $x = u + v$, where $u\perp\ker f_0'$, $v\in\ker f_0'$ and $f(x)=g_1(u,v)+g_2(u,v)$, where $g_1(u,v)\in\operatorname{im} f_0'$, $g_2(u,v)\perp\operatorname{im} f_0'$. The smooth maps g_1, g_2 defined in this way have the following properties:

$$g_1(u,v)=\frac{\partial g_1}{\partial u}(0,0)u+O(|u|^2+|v|^2),$$

while the linear map $\frac{\partial g_1}{\partial u}(0,0):(\ker f_0')^\perp\to\operatorname{im} f_0'$ is invertible:

$$g_2(u,v)=\frac{1}{2}\frac{\partial^2 g_2}{\partial v^2}(0,0)(v,v)+O(|u|^2+|u||v|+|v|^3),$$

while the quadratic map $\frac{\partial^2 g_2}{\partial v^2}(0,0)$ turns into f_0'' under the identification of $(\operatorname{im} f_0')^\perp$ with $\operatorname{coker} f_0'$.

1) Let us assume that f_0'' is not surjective. Since the quadratic map f_0'' is nondegenerate, $f_0''(v,v)\neq 0$ for $v\neq 0$. Consequently, the image of the quadratic map f_0'' is a closed cone in $\operatorname{coker} f_0'$. So one can find a nonzero vector $l\perp\operatorname{im} f_0'$ and an $\varepsilon>0$ such that $\left|\frac{\partial^2 g}{\partial v^2}(0,0)(v,v)-\alpha l\right|\geqslant\varepsilon|v|^2$, $\forall v\in\ker f_0'$, $\alpha>0$. It is easy to show that in this case $f(x)\neq\alpha l$ for all x sufficiently close to zero and $\alpha>0$. In fact, the condition $f(x)=\alpha l$ is equivalent to the equations

$$g_1(u,v)=0,\quad g_2(u,v)=\alpha l.$$

For (u, v) sufficiently close to zero, it follows from the equation $g_1(u,v)=0$ that $|u|\leqslant c_1|v|^2$, where c_1 is a constant. In this case

$$\left|g_2(u,v)-\frac{\partial^2 g_2}{\partial v^2}(0,0)(v,v)\right|\leqslant c_2|v|^3$$

and

$$|g_2(u,v)-\alpha l|\geqslant\varepsilon|v|^2-c_2|v|^3.$$

Thus, for small (u, v), the equations $g_1(u,v)=0$ and $g_2(u,v)=\alpha\ell$ (where $\alpha>0$) are incompatible.

2) It follows from Proposition 1 that $f_0''(v_0,v_0)=0$ for some $v_0\in\ker f_0'\setminus 0$. Since f_0'' is nondegenerate, v_0 is a regular point of the quadratic map f_0''. Let

$$u_0=-\frac{1}{2}\left(\frac{\partial g_1}{\partial u}(0,0)\right)^{-1}\frac{\partial^2 g_1}{\partial v^2}(0,0)(v_0,v_0).$$

Then $f_0'u_0+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\Big|_0(v^0,v_0)=0$, while (u_0, v_0) is a regular point of the map $(u,v)\mapsto f_0'u+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\Big|_0(v,v)$.

Assertion 2) now follows from the implicit function theorem and the obvious equation

$$f(\varepsilon^2 u+\varepsilon v)=\varepsilon^2\left(f_0'u+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\Big|_0(v,v)\right)+O(\varepsilon^3).\quad\blacksquare$$

Let K be a convex polyhedral cone in $\mathbf{R}^k$ satisfying the condition $\operatorname{span} K \cap \operatorname{im} f_0' = 0$. In this case the canonical map $y \mapsto y + \operatorname{im} f_0'$ of the space $\mathbf{R}^k$ onto $\operatorname{coker} f_0'$ is one-to-one on K.

Proposition 5. Let $\dim H < +\infty$ and the quadratic map f_0'' be nondegenerate with respect to the cone $\hat{K} = K + \operatorname{im} f_0'$ in $\operatorname{coker} f_0'$. Then there exists a neighborhood $O_0 \subset O$ of the origin and a family of homeomorphisms $\Phi_t : O_0 \to O_0$, depending continuously on $t \in [0, 1]$, such that $\Phi_0 = \mathrm{id}$, $\Phi_t(0) = 0$ for $0 \leqslant t \leqslant 1$ and $\Phi_1(f_0''^{-1}(\hat{K}) \cap O_0) = f^{-1}(K) \cap O_0$.

Proof. Let $x \in H$. as in the proof of Proposition 4 let x = u + v, where $v \in \ker f_0'$, $u \perp \ker f_0'$. Let S be a sphere of unit radius in H and $\varphi : S \to \mathbf{R}^k$ be the map defined by the formula $\varphi(u, v) = f_0' u + \frac{1}{2} \frac{\partial^2 f}{\partial v^2}\Big|_0 (v, v)$, $|u|^2 + |v|^2 = 1$.

In the appendix to the present section the precise definition of transversality of a smooth map to a convex set is given. We show that the map φ is transverse to the cone K. Indeed, otherwise for some $(u_0 + v_0) \in S$, $\omega \in K^\circ \setminus 0$, $\alpha \in \mathbf{R}$. one would have

$$\omega f_0' u = \alpha (u_0, u), \quad \frac{\partial^2 f}{\partial v^2}\Big|_0 (v_0, v) = \alpha (v_0, v) \quad \forall (u + v) \in H,$$

$$\varphi(u_0 + v_0) \in K, \quad \omega \varphi(u_0 + v_0) = 0$$

(the last equation follows from the relation $\operatorname{span}\{y\} \subset T_y K \ \forall y \in K$). Consequently,

$$0 = \omega\varphi(u_0 + v_0) = \omega f_0' u_0 + \frac{1}{2} \omega \frac{\partial^2 f}{\partial v^2}\Big|_0 (v_0, v_0) = \alpha\Big((u_0, u_0) + \frac{1}{2}(v_0, v_0)\Big),$$

and $\alpha = 0$. But then $\omega \perp \operatorname{im} f_0'$, i.e., $\omega \in (\operatorname{coker} f_0')^*$, and $\omega f_0'' = \omega \frac{\partial^2 f}{\partial v^2}\Big|_0 = 0$, which contradicts the nondegeneracy of the quadratic map f_0'' with respect to the cone K.

The Taylor decomposition of the map f at zero leads to the equation

$$f(\varepsilon^2 u + \varepsilon v) = \varepsilon^2 (\varphi(u + v) + \varepsilon R_\varepsilon(u, v)),$$

where $R_\varepsilon(u, v)$ depends smoothly on ε, u, v. If $\varepsilon > 0$ is sufficiently small, then for any $t \in [0, 1]$ the map $(u, v) \to \varphi(u + v) + t\varepsilon R_\varepsilon(u, v)$, $u + v \in S$ is transverse to the cone K. According to Lemma A1 there exists a family of homeomorphisms $F_{t,\varepsilon} : S \to S$, depending continuously on $(t, \varepsilon) \in [0, 1] \times (0, \varepsilon_0]$, satisfying the conditions: $F_{0,\varepsilon} = \mathrm{id}$,

$$F_{1,\varepsilon}(\varphi^{-1}(K)) = \Delta_{\frac{1}{\varepsilon}} (f^{-1}(K) \cap \Delta_\varepsilon S),$$

where $\Delta_\varepsilon(u + v) = \varepsilon^2 u + \varepsilon v$, $u + v \in H$, $u \perp \ker f_0'$, $v \in \ker f_0'$.

We recall that $\operatorname{im} f_0' \cap \operatorname{span} K = 0$. Consequently, $\varphi^{-1}(\operatorname{span} K) = \{(u + v) \in S \mid u = p_0(v, v)\}$, where p_0 is a quadratic map from $\ker f_0$ into $(\ker f_0)^\perp$ which can obviously be calculated in terms of f_0' and f_0''. From this, the existence of a family of diffeomorphisms $G_t : S \to S$, $G_0 = \mathrm{id}$ depending smoothly on $t \in [0, 1]$ such that

$$\varphi^{-1}(\operatorname{span} K) = G_1(\ker f_0' \cap S) = G_1(f_0''^{-1}(\operatorname{coker} f_0') \cap S) \quad \text{and}$$

$$\varphi^{-1}(K) = G_1(f''^{-1}(\hat{K}) \cap S)$$

follows. Consequently,

$$f^{-1}(K) \cap \Delta_\varepsilon S = \Delta_\varepsilon \circ F_{1,\varepsilon} \circ G_1(f''^{-1}(\hat{K}) \cap S).$$

We set $O_0 = \{0\} \bigcup_{0<\varepsilon<\varepsilon_0} \Delta_\varepsilon S$ and we define homeomorphisms $\Phi_t : O_0 \to O_0$, $t \in [0, 1]$ by the rule $\Phi_t|\Delta_\varepsilon S = \Delta_\varepsilon \circ F_{t,\varepsilon} \circ G_t \circ \Delta_{\frac{1}{\varepsilon}}$, $\Phi_t(0) = 0$. Since $\Delta_\varepsilon(f''^{-1}(\hat{K}) \cap S) = f''^{-1}(\hat{K}) \cap \Delta_\varepsilon S$, one has

$$f^{-1}(K) \cap O_0 = \Phi_1(f''^{-1}(\hat{K}) \cap O_0). \ \blacksquare$$

COROLLARY. Under the hypotheses of Proposition 5, for any i > 0 there is an isomorphism of homology groups

$$H_i(f^{-1}(K), f^{-1}(K) \setminus 0) = H_i(f_0''^{-1}(\hat{K}), f''^{-1}(\hat{K}) \setminus 0) = \tilde{H}_{i-1}(f''^{-1}(\hat{K}) \cap S).$$

In the infinite-dimensional case one can assert the following

Proposition 6. Let $\dim H = +\infty$ and the quadratic map f_0'' be nondegenerate and essentially surjective. Then $0 \in \operatorname{int} f(O_0)$ for any neighborhood of the origin $O_0 \subset O$.

Proof. It follows from Proposition 1_∞ that $f_0''(v_0, v_0)=0$ for some $v_0 \in \ker f_0' \setminus 0$ and one can repeat the proof of assertion 2 of Proposition 4 word for word.

Let the symbol $\mathcal{V}$ denote the collection of all finite-dimensional subspaces of H, partially ordered by inclusion, S be a sphere of unit radius in H.

Proposition 7. Let us assume that the quadratic map f_0'' is nondegenerate with respect to the cone $\hat{K}=K+\operatorname{im} f_0' \subset \operatorname{coker} f_0'$.

Then for any i > 0 the homology group

$$H_i(f''^{-1}(\hat{K}),\ f''^{-1}(\hat{K})\setminus 0)\approx \tilde{H}_{i-1}(f''^{-1}(\hat{K})\cap S)$$

is isomorphic to the direct limit of the $\mathcal{V}$-system formed by the groups $H_i(f^{-1}(\hat{K})\cap V,\ f^{-1}(\hat{K})\cap V\setminus 0)$, $V\in\mathcal{V}$ and homomorphims $H_i(f^{-1}(\hat{K})\cap V,\ f^{-1}(\hat{K})\cap V\setminus 0)\to H_i(f^{-1}(\hat{K})\cap W,\ f^{-1}(\hat{K})\cap W\setminus 0)$ induced by the inclusions $V\subset W$ for any $V\subset W\in\mathcal{V}$.

Proof. Let $\mathcal{V}_0$ be the subset of $\mathcal{V}$ consisting of all $V_0\in\mathcal{V}$ such that the quadratic map $f_0''|V_0$ is nondegenerate with respect to K. It follows from Proposition A2 that $\mathcal{V}_0$ is cofinal in $\mathcal{V}$. The result required follows from Proposition 5 applied to the maps $f''|O\cap V_0$, $V_0\in\mathcal{V}$ and Theorem A2.

Remark. In studying a smooth directed system one has to consider maps defined on a Banach but not Hilbert space. Let H ⊃ B be a Banach space which is everywhere dense in H (the topology in B is generally stronger than that in H), $g:B\to\mathbb{R}^k$ be a smooth map, g_0' the differential and g_0'' the Hessian of the map g at zero. Let us assume that the linear map g_0' and bilinear map g_0'' are continuous in the topology of the Hilbert space H. We denote by $\overline{\ker g_0'}$ the closure of the subspace $\ker g_0'\subset B$ in H, and by $\bar{g}_0'':\overline{\ker g_0'}\times\overline{\ker g_0'}\to\operatorname{coker} g_0'$ the extension of the map g_0'' to $\overline{\ker g_0'}\times\overline{\ker g_0'}$ by continuity.

Propositions 6 and 7 remain valid if in their formulations we replace H by B, f by g, and f_0'' by $\bar{g}_0''$. The proof is an almost word for word repetition of that given above.

APPENDIX. SOME INFORMATION FROM TOPOLOGY

Most of the topological concepts and results used in the basic text are standard. In the appendix we include only those which in our view need additional clarification.

I) Spectral Sequence of a Map and Bicomplexes. Let B and A be topological spaces which are Euclidean neighborhood retracts,† and f:B → A be a continuous map. To any locally finite covering $\mathcal{O}=\{O_\iota,\ \iota\in I\}$ of the space A by open sets corresponds a covering $f^{-1}(\mathcal{O})=\{f^{-1}(O_\iota),\ \iota\in I\}$ of the space B.

Let $\alpha=\{\alpha_0,\ldots,\alpha_m\}\subset I$, $O_\alpha=\bigcap_{i=0}^{m} O_{\alpha_i}$, $C_s^n(f^{-1}(O_\alpha))$ be the group of n-dimensional singular cochains in $f^{-1}(O_\alpha)$ with values in $\mathbb{Z}_2$, $d_\alpha:C_s^n(f^{-1}(O_\alpha))\to C_s^{n+1}(f^{-1}(O_\alpha))$ be the coboundary operator in the singular complex $C_s^\bullet(f^{-1}(O_\alpha))$.

We set

$$E^{n,m}(f;\mathcal{O})=\prod_{\substack{\alpha\subset I\\ \#\alpha=m+1}} C_s^n(f^{-1}(O_\alpha)),$$

$$d=\prod_{\substack{\alpha\subset I\\ \#\alpha=m+1}} d_\alpha:E^{n,m}(f;\mathcal{O})\to E^{n+1,m}(f;\mathcal{O}).$$

We define another coboundary operator $\delta:E^{n,m}(f;\mathcal{O})\to E^{n,m+1}(f;\mathcal{O})$ by setting

$$(\delta\xi)_\beta=\sum_{i=0}^{m+1}\xi_{\beta\setminus\{\beta_i\}}\quad\text{for any}\quad \xi=\prod_{\substack{\alpha\subset I\\ \#\alpha=m+1}}\xi_\alpha,\quad \beta=\{\beta_0,\ldots,\beta_{m+1}\}.$$

The "vertical" coboundary operator d and the "horizontal" operator δ obviously commute and turn the bigraded $\mathbb{Z}_2$-module $E(f;\mathcal{O})=\bigoplus_{n,m\geq 0}E^{n,m}(f;\mathcal{O})$ into a bicomplex. We denote by $H^\bullet_{d+\delta}(f;\mathcal{O})$ the total cohomology of this bicomplex, and by $(E_r(f;\mathcal{O});d_r)$, $r\geq 1$ the spectral sequence generated by the filtration $\bigoplus_{\substack{n\geq 0\\ m\geq k}}E^{n,m}(f;\mathcal{O})$, $k=0,1,2,\ldots$ of the bicomplex $E(f;\mathcal{O})$. In this

†The facts listed below generalize to larger classes of spaces but for our purposes this is completely sufficient.

case, $E_1(f;\mathcal{O})=H_d^\bullet(f;\mathcal{O})$, $E_2(f;\mathcal{O})=H_\delta^\bullet H_d^\bullet(f;\mathcal{O})$ and the sequence (E_r, d_r) converges to $H_{d+\delta}^\bullet(f;\mathcal{O})$.

The relation of one covering being inscribed in another defines a partial order on the collection $\mathfrak{D}$ of all locally finite coverings of the space A, turning $\mathfrak{D}$ into a directed set, and the families $H_{d+\delta}^\bullet(f;\mathcal{O})$, $\mathcal{O}\in\mathfrak{D}$ and $(E_r(f;\mathcal{O}), d_r)$, $\mathcal{O}\in\mathfrak{D}$ into directed $\mathfrak{D}$-systems.

THEOREM A. The limit of the direct $\mathfrak{D}$-system of graded modules $H_{d+\delta}^\bullet(f;\mathcal{O})$, $\mathcal{O}\in\mathfrak{D}$ coincides with $H^*(B;\mathbb{Z}_2)$, and the limit of the direct $\mathfrak{D}$-system of spectral sequences $(E_r(f;\mathcal{O}), d_r)$, $\mathcal{O}\in\mathfrak{D}$ with the Leray cohomology spectral sequence of the map $f:B\to A$.

Cf., e.g., [12] for details and proofs. Analogous assertions are of course true for the cohomology with integral coefficients; it is only necessary to arrange signs properly in the construction of the bicomplexes.

We now recall the definition of the differential $d_2\colon H_\delta^n H_d^m \to H_\delta^{n-1} H_d^{m+?}$ in the spectral sequence of an arbitrary bicomplex $(E^{n,m}, d, \delta)$, $n, m\geqslant 0$ filtered by submodules $\bigoplus_{\substack{n\geqslant 0\\ m\geqslant k}} E^{n,m}$, $k=0, 1, 2, \ldots$.

Let ξ be a d-cocycle in $E^{n,m}$ such that the corresponding d-cohomology class $\bar\xi\in H_d^{n,m}=E_1^{n,m}$ is a δ-cocycle; then $\bar\xi$ represents a class $[\bar\xi]\in H_\delta^m H_d^n=E_2^{n,m}$. Since $\bar\xi$ is a δ-cocycle, $\delta\xi = d\eta$ for some $\eta\in E^{n-1,m+1}$. We have $d_2[\bar\xi]=[\overline{\delta\eta}]$. One can find the details in [12].

II) Maps Transverse to a Convex Set. Let Γ be a smooth Hilbert submanifold of the Hilbert space $\mathbf{H}$, S be a closed convex subset of $\mathbf{R}^k$ and $f:\Gamma\to\mathbf{R}^k$ be a smooth map. By T_yS we denote the cone spanned by the set $S-y$ $(y\in S)$ and by $f_x'\colon T_x\Gamma\to\mathbf{R}^k$ the differential of the map f at the point $x\in\Gamma$.

The map f is said to be transverse to the set S if for any $x\in\Gamma$ it follows from the condition $f(x)\in S$ that $\operatorname{Im} f_x'+T_{f(x)}S=\mathbf{R}^k$.

Proposition A1. If $f:\Gamma\to\mathbf{R}^k$ is transverse to the set S, then $f^{-1}(S)$ is a topological submanifold with boundary of Γ and the set $f^{-1}(\operatorname{ri} S)$ (complement of the boundary) is a smooth submanifold of Γ.

Outline of the proof. The map f is obviously transverse to a neighborhood U of the set S in the subspace span S. Consequently, $f^{-1}(U)$ is a smooth submanifold of Γ and replacing Γ by $f^{-1}(U)$ if necessary, one can assume from the beginning that S has nonempty interior in $\mathbf{R}^k$. Further, the transversality condition guarantees the existence in a neighborhood of an arbitrary point $x\in f^{-1}(S)$ of a smooth vector field X such that $f_x'X(x)\in\operatorname{int} T_{f(x)}S$, and the existence of smooth partitions of unity on Hilbert manifolds (cf. [5]) lets us construct such a field globally. The integral curves of this field define a tubular neighborhood of the set $f^{-1}(\partial S)$ and let us represent a neighborhood O_x of an arbitrary point $x\in f^{-1}(\partial S)$ as the direct product of $f^{-1}(\partial S)\cap O_x$ by an interval.

If Γ is compact (and consequently finite-dimensional) the following generalization of the standard Thom lemma on isotopies with parameters holds.

LEMMA A1. Let Γ be a smooth compact manifold, $A\subset\mathbf{R}^n$ be an arbitrary subset, and $f_{t,a}\colon \Gamma\to\mathbf{R}^k$ be a family of maps, depending smoothly on $t\in[0,1]$ and continuously on $(t,a)\in[0,1]\times A$, transverse to the closed convex set $S\subset\mathbf{R}^k$. Then there exists a family of homeomorphisms $F_{t,a}\colon\Gamma\to\Gamma$ depending continuously on $(t,a)\in[0,1]\times A$, satisfying the conditions:

$$F_{0,a}=\mathrm{id};\ F_{1,a}(f_0^{-1}(S))=f_1^{-1}(S)\ \forall a\in A.$$

If S has smooth relative boundary ∂S the assertion of Lemma A1 follows from the standard Thom lemma on isotopies. To prove the general case it suffices to approximate S by a convex set with smooth relative boundary and to use the tubular neighborhoods of the sets $f_{0,a}^{-1}(\partial S)$, $f_{1,a}^{-1}(\partial S)$ whose construction is described above in the discussion of Proposition A1.

III) Restrictions to Finite-Dimensional Subspaces. Let $f:\mathbf{H}\to\mathbf{R}^k$ be a smooth map of a separable Hilbert space $\mathbf{H}$ into $\mathbf{R}^k$, transverse to the convex closed subset $S\subset\mathbf{R}^k$, and $\mathbf{H}\supset B$ be a linear subspace which is everywhere dense in $\mathbf{H}$; the symbol $\mathcal{V}$ denotes the collection of all finite-dimensional subspaces of B, partially ordered by inclusion.

THEOREM A2. The homology group $H_i(f^{-1}(S))$, $i\geqslant 0$, is isomorphic to the direct limit of the $\mathcal{V}$-system formed by the groups $H_i(f^{-1}(S)\cap V)$, $V\in\mathcal{V}$ and homomorphisms $H_i(f^{-1}(S)\cap V)\to H_i(f^{-1}(S)\cap W)$ induced by the inclusions $V\subset W$ for any $V\subset W\in\mathcal{V}$.

Outline of the proof. The Hilbert submanifold with boundary $f^{-1}(S)$ has finite codimension in $\mathbf{H}$ and has a tubular neighborhood U. Let $\varphi\colon U\to f^{-1}(S)$ be the retraction defined by the tubular neighborhood. The usual method of construction of a tubular neighborhood using a partition of unity (cf. [5] and also II)) lets us arrange that for any compactum $K \subset U$ satisfying the conditions $K \subset B$, $\operatorname{span} K < +\infty$ one has: $\varphi(K) \subset B$, $\operatorname{span} \varphi(K) < +\infty$. At the same time by a small perturbation not leaving the limits of U one can turn any singular chain in $f^{-1}(S)$ into a chain lying in a finite-dimensional subspace of B.

Let us assume in addition that S is a bounded convex polyhedron Then one has

Proposition A2. For any finite-dimensional subspace $V \subset B$ one can find a finite-dimensional subspace W, $V \subset W \subset B$ such that $f|W$ is transverse to S.

Outline of the proof. First we find a finite-dimensional subspace $W_0 \supset V$ such that $f|W_0$ is transverse to S at all points of the subspace V. Let $\Delta_1, \ldots, \Delta_\ell$ be the affine hulls of all closed faces (of all dimensions) of the polyhedron S. Using the standard theorem on density of transverse maps (cf. [8]), we find near W_0 a finite-dimensional subspace $W \supset V$ which has the following property: for any $x \in W$, $i \leqslant 1 \leqslant l$ such that f is transverse to Δ_i at the point x, the map $f|W$ is also transverse to Δ_i at the point x.

It is easy to show that in this case $f|W$ is transverse to S. By slightly perturbing again if necessary, W can be arranged so that this subspace lies in B.

Remark. The assertions of Theorem A2 and Proposition A2 obviously extend to the case when the map f is defined not on all of **H**, but only on an open subset of **H**.

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