QUASI-EXTREMALITY FOR CONTROL SYSTEMS

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UDC 517.971:514.7

A full exposition of the authors' previously announced results about the extremality index of controls in smooth control systems and a generalization of these results to systems with constraints on the controls.

INTRODUCTION

This paper will present proofs of some results announced in [1], as well as generalizations of these results to systems in which constraints are imposed on the control parameters, as promised in [1].

Throughout the paper (and the following paper by S. A. Vakhrameev) the functional notation introduced in the first paper of this volume will be employed without special mention.

The main object of study will be the Hessian of the "input—output" map of a control system at a certain critical point (extremal of the system). Let us recall, therefore, the definition of the Hessian of a smooth map. Let $\Phi: \mathscr{B} \to M$ be a smooth map of some smooth Banach manifold into a finite-dimensional manifold and let $\beta_0 \in \mathscr{B}$. The differential of Φ at β_0 is the linear map $D_{\beta_0}\Phi: T_{\beta_0}\mathscr{B} \to T_{\Phi(\beta_0)}M$ of the tangent spaces. If we fix local coordinates in the neighborhoods of β_0 and $\Phi(\beta_0)$, we can also define the second differential (a symmetric bilinear map of a Banach space into a finite-dimensional space). However, this procedure does not yield a well-defined bilinear map of $T_{\beta_0}\mathscr{B} \times T_{\beta_0}\mathscr{B}$ into $T_{\Phi(\beta_0)}M$, since the quadratic part of a smooth map depends essentially on the choice of local coordinates (for example, if $D_{\beta_0}\Phi$ is a surjective linear map, then by the Implicit Function Theorem Φ will be represented by a linear map in certain local coordinates). But if we restrict the second differential to the kernel of the first differential and factorize its values modulo the image of the first differential, the result is a well-defined symmetric bilinear map

$$\operatorname{ges}_{\beta_0}\Phi: \ker D_{\beta_0}\Phi \times \ker D_{\beta_0}\Phi \to \operatorname{coker} D_{\beta_0}\Phi,$$

where by definition $D_{\beta_0}\Phi = T_{\Phi(\beta_0)}M/\operatorname{im} D_{\beta_0}\Phi$. The map $\operatorname{ges}_{\beta_0}\Phi$ is known as the Hessian of Φ at the point β_0. Whenever the point at which point the Hessian and the differential are being considered is clear from the context, we will use the abbreviated notation

$$\operatorname{ges}_{\beta_0}\Phi = \Phi'', \quad D_{\beta_0}\Phi = \Phi'.$$

Remark. The definition of the Hessian is invariant to smooth changes in variables both in $\mathscr{B}$, and in M. If M is a linear space, $M = \mathbf{R}^n$, and nonlinear changes in variables are allowed only in $\mathscr{B}$, the second differential turns out to be a well-defined bilinear map from $T_{\beta_0}\mathscr{B} \times T_{\beta_0}\mathscr{B}$ into $\operatorname{coker} D_{\beta_0}\Phi$ (restriction to $\ker D_{\beta_0}\Phi$ is not necessary).

The pairing of an arbitrary vector $x \in T_\mu M$ with a covector $\xi \in T_\mu^*M$ will be denoted by ξx — the product of a row and a column. If a covector ψ is orthogonal to $\operatorname{im} D_{\beta_0}\Phi$, $\psi \in (\operatorname{im} D_{\beta_0}\Phi)^\perp \subset T_{\Phi(\beta_0)}^*M$, then

$$\psi \operatorname{ges}_{\beta_0}\Phi: \ker D_{\beta_0}\Phi \times \ker D_{\beta_0}\Phi \to \mathbf{R}$$

is a real symmetric bilinear form. We will need the concept of the index of such forms. Recall that the Morse index (or simply index) of a real symmetric bilinear form $q: B \times B \to \mathbf{R}$, where B is a linear space, is the maximum dimension of a subspace of B on which the quadratic form $b \to q(b, b)$ is negative. Standard notation: $\operatorname{ind} q$. We have $0 \le \operatorname{ind} q \le \dim B$. If B is infinite-dimensional, then possibly $\operatorname{ind} q = +\infty$.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Noveishie Dostizheniya, Vol. 35, pp. 109-134, 1989.

1. Smooth Control Systems

Let M^n be a smooth n-dimensional manifold and U a smooth r-dimensional manifold. Consider the control system

$$\dot{x} = x \circ f_t(u), \quad x \in M^n, \quad u \in U, \quad t \in [0, T]. \tag{1}$$

Here $f_t(u)$ is an infinitely differentiable family, dependent on $u \in U$, of complete nonstationary vector fields on M.

An arbitrary map $u(\cdot):[0, T] \to U$ is said to be bounded on a subset $E \subset [0, T]$ if the closure of its image $\overline{u(E)}$ is compact and measurable if the preimage of every open subset of U is measurable. We will say that a map $u(\cdot)$ is in $L([0, T]; U)$ if it is measurable and bounded on some subset of full measure in $[0, T]$ (essentially bounded). The elements of $L_\infty([0, T]; U)$ will be called admissible controls. The set of admissible controls is endowed with a topology as follows. If U is embedded as a closed submanifold in $\mathbf{R}^d$, then obviously

$$L_\infty([0, T]; U) \subset L_\infty([0, T]; \mathbf{R}^d) = L_\infty^d.$$

The topology will be that induced by this embedding. In addition, $L_\infty([0, T]; U)$ is a smooth Banach submanifold of L_∞^d.

Together with this standard topology, we will sometimes find it useful to consider the space of admissible controls $L_\infty([0, T]; U)$ with a stronger, finite-dimensional-open topology. A given subset $\mathcal{O} \subset L_\infty([0, T]; U)$ is open in the finite-dimensional-open topology if its intersection with any finite-dimensional submanifold of $L_\infty([0, T]; U)$ is open in that topology, we will always mention this explicitly.

Note that the collection of all control systems of type (1) with fixed manifolds M, U and fixed time interval $[0, T]$ forms a linear space; denote this space by $CS(M, U, [0, T])$. This space $CS(M, U, [0, T])$ has a natural family of seminorms that make it into a Fréchet space. Indeed, the family of nonstationary vector fields $f_t(u)$ may be considered as a nonstationary field on $M^n \times U$ if we define, for any function $a \in C^\infty(M^n \times U)$

$$(f_t a)(x, u) = (f_t(u) a \,|\, u = \text{const})(x), \quad (x, u) \in M^n \times U.$$

Thus, there is a natural embedding of $CS(M^n, U, [0, T])$ into the Fréchet space of all nonstationary fields on $M^n \times U$ as a closed subspace. In particular, to each compact subspace $K \subset M^n \times U$ and nonnegative integer α there corresponds a seminorm $\|f\|_{K,\alpha}$.

Fix once and for all a point $x_0 \in M^n$ and consider the map $F: L_\infty([0, T]; U) \to M$ associating to each admissible control $u(\cdot)$ a point x_T, where

$$\frac{d}{dt} x_t = x_t \circ f_t(u(t)), \quad 0 \leq t \leq T.$$

Thus, $F(u(\cdot)) = x_0 \circ \overrightarrow{\exp} \int_0^T f_t(u(t))dt$.

Before moving on, let us describe the local invariants of smooth maps that will interest us.

Let $\mathcal{A}$ be a Banach manifold of class C^∞ and $a \in \mathcal{A}$. Let $C_a^\infty(\mathcal{A}, M^n)$ denote the set of germs at the point a of smooth maps from $\mathcal{A}$ into M^n. Endow $C_a^\infty(\mathcal{A}, M^n)$ with the topology of strong convergence of all derivatives at a. Suppose that $\mathcal{A}$ is modeled on a Banach space $\mathbf{A}$. Recall that the k-th derivative of a germ at a is defined only after choosing local coordinates in M^n and $\mathcal{A}$, and is a multilinear map from $\mathbf{A}^k$ to $\mathbf{R}^n$. However, the property of convergence of all derivatives at a, for a given directed family of germs, is independent of the prior choice of local coordinates. The topology is not Hausdorff, but that need not trouble us here.

In the subsequent definitions the expression "for almost every germ" means: "for any germ in some open, dense subset of the space of germs."

Definition 1. Let $\mathscr{A}$ be a Banach manifold of class C^∞ and $C_a^\infty(\mathscr{A}, M^n) \ni \mathscr{H}$ a smooth germ at a point $a \in \mathscr{A}$. The germ is said to be extremal if there exist a neighborhood $\mathcal{O}$ of a in and a representative H: $\mathcal{O} \to M^n$ of $\mathscr{H}$, such that $H(a) \in \partial H(\mathcal{O})$, i.e., the point $H(a)$ is on the boundary of the set $H(\mathcal{O})$.

Definition 2. Again, let $\mathscr{H} \in C_a^\infty(\mathscr{A}, M^n)$.

i) Let $\mathscr{H}$ be an extremal germ. We will say that $\mathscr{H}$ has extremality index $k > 0$ if k is the least number such that, for almost every germ $\Phi \in C^\infty_{\mathscr{H}(a)}(M^n, \mathbf{R}^{n-k})$ the germ $\Phi \circ \mathscr{H} \in C_a^\infty(\mathscr{A}, \mathbf{R}^{n-k})$ is not extremal.

ii) Let $\mathscr{H}$ be a germ that is not extremal. We will say that $\mathscr{H}$ has extremality index $\ell \leq 0$ if ℓ is the least number such that, for almost every germ $\Psi \in C_a^\infty(\mathscr{A}, \mathbf{R}^{-l})$, the germ $(\mathscr{H} \times \Psi) \in C_a^\infty(\mathscr{A}, M \times \mathbf{R}^{-l})$ is not extremal. If no such least ℓ exists, the extremality index is defined to be $-\infty$.

Thus, the extremality index of an arbitrary germ $\mathscr{H} \in C_a^\infty(\mathscr{A}, M^n)$ lies in the interval $[-\infty, n]$. A germ is extremal if its extremality index is positive.

We now return to the control system (1).

Definition 3. Let $\tilde{u}(\cdot) \in L_\infty([0, T]; U)$ be an admissible control. The local extremality index of the control $\tilde{u}(\cdot)$ with respect to system (1) with initial condition x_0 is the extremality index of the germ of the map F at the "point" $\tilde{u}(\cdot)$. A control with positive local extremality index is said to be locally extremal with respect to system (1) with initial condition x_0.

In the next definition we will have to consider, along with the control system (1), systems close to it in the space $CS(M^n, U, [0, T])$. The initial conditions, however, will remain fixed.

Definition 4. The quasi-extremality index of an admissible control $\tilde{u}(\cdot)$ with respect to system (1) is the maximum number $k \in [-\infty, n]$ such that, arbitrarily close to $f_t(u)$ in $CS(M^n, U, [0, T])$, there exists a control system $g_t(u)$ with respect to which $u(\cdot)$ is of local extremality index k. Controls which have positive quasi-extremality index will be called quasi-extremal with respect to system (1).

Thus, the quasi-extremality index of a control with respect to a given system $f_t(u)$ is the upper limit of the local extremality indices of $u(\cdot)$ with respect to systems $g \in CS(M^n, U, [0, T]$ for g tending to f.* In particular, the quasi-extremality index of a given control is an upper semicontinuous function of the system.

We now fix an admissible control $\tilde{u}(\cdot)$ once and for all and let $\tilde{p}_{t,\tau}$ be the corresponding family of flows in M.

$$\frac{d}{dt}\tilde{p}_{t,\tau} = \tilde{p}_{t,\tau} \circ f_t(\tilde{u}(t)), \quad \tilde{p}_{\tau,\tau} = \mathrm{id}.$$

This may be written differently as

$$\tilde{p}_{t,\tau} = \overrightarrow{\exp} \int_\tau^t f_\theta(\tilde{u}(\theta))\, d\theta.$$

Denote $\tilde{p}_t \overset{\text{def}}{=} \tilde{p}_{t,0}$, $\tilde{x}_t = x_0 \circ \tilde{p}_t$. It is easy to see that $\tilde{p}_{t,\tau} = \tilde{p}_\tau^{-1} \circ \tilde{p}_t$.

Let $u(\cdot)$ be another admissible control and denote $\delta f_t(u(t)) = f_t(u(t)) - f_t(\tilde{u}(t))$. We have the following representation:

$$\overrightarrow{\exp} \int_0^t f_\tau(u(\tau))\, d\tau = \tilde{p}_t \circ \overrightarrow{\exp} \int_0^t \mathrm{Ad}\, \tilde{p}_{t,\tau}^{-1}\, \delta f(u(\tau))\, d\tau, \tag{2}$$

where $\overrightarrow{\exp} \int_0^t \mathrm{Ad}\tilde{p}_{t,\tau}^{-1} \delta f(u(\tau)) d\tau$ is the right perturbing flow corresponding to the perturbation $\delta f_t(u(t))$ of the field $f_t(\tilde{u}(t))$.

*It is easy to see that the lower limit is always equal to $-\infty$.

Let $\tilde{f}_t'$ and $\tilde{f}_t^{(2)}$, respectively, denote the first and second differentials of the map $u \mapsto f_t(u)$ at the point $\tilde{u}(t) \in U$. Then $\tilde{f}_t': T_{\tilde{u}(t)}U \to \operatorname{Der} M^n$ is a linear map of the tangent space to U at $\tilde{u}(t)$ into the space of vector fields on M; $\tilde{f}_t^{(2)}: T_{\tilde{u}(t)}U \times T_{\tilde{u}(t)}U \to \operatorname{coker} \tilde{f}_t'$ is a symmetric bilinear map of $T_{\tilde{u}(t)}U$ into the quotient space of $\operatorname{Der}(M^n)$ by the image of $\tilde{f}_t'$.

We recall that the tangent space to the Banach manifold of admissible controls $L_\infty \times ([0, T], U)$ at the "point" $\tilde{u}(\cdot)$ is the set of all measurable and essentially bounded maps $t \mapsto v(t)$, $0 \le t \le T$ such that $v(t) \in T_{\tilde{u}(t)}U$, $\forall\, t \in [0, T]$. We denote this space by $\mathcal{L}^\infty_{\tilde{u}}$. Let $\tilde{F}': \mathcal{L}^\infty_{\tilde{u}} \to T_{\tilde{x}_T}M^n$ be the differential of the map F at the "point" $\tilde{u}(\cdot)$ and $\tilde{F}'': \ker \tilde{F}' \times \ker \tilde{F}' \to \operatorname{coker} \tilde{F}'$ the Hessian of F. It readily follows from (2) and the identity

$$\overrightarrow{\exp}\int_0^t \operatorname{Ad}\tilde{p}_{t,\tau}^{-1}\,\delta f_\tau(u(\tau))\,d\tau = \mathrm{id} + \int_0^t \left(\overrightarrow{\exp}\int_0^\tau \operatorname{Ad}\tilde{p}_{t,\theta}^{-1}\,\delta f_\theta(u(\theta))\,d\theta\right)\circ \operatorname{Ad}\tilde{p}_{t,\tau}^{-1}\,\delta f_\tau(u(\tau))\,d\tau$$

that

$$\delta f_t(\tilde{u}(t)) \equiv 0$$

$$\tilde{F}'v(\cdot) = \tilde{x}_T \circ \int_0^T \operatorname{Ad}\tilde{p}_{T,t}^{-1}\tilde{f}_t' v(t)\,dt.$$

In order to avoid needlessly cumbersome formulas, we introduce the notation $D_t^1 v(t) \overset{\text{def}}{=} \operatorname{Ad} \times \tilde{p}_{T,t}^{-1}\tilde{f}_t' v(t)$ and $D_t^2(v_1(t), v_2(t)) = \operatorname{Ad}\tilde{p}_{T,t}^{-1}\tilde{f}^{(2)}(v_1(t), v_2(t))$, $v_1(\cdot) \in \mathcal{L}^\infty_{\tilde{u}}$. Thus, $\tilde{F}'v \times (\cdot) = \tilde{x}_T \circ \int_0^T D_t^1 v(t)dt$. Clearly, $\operatorname{im}\tilde{F}' = \operatorname{span}\{\tilde{x}_T \circ D_t^1 v \mid v \in T_{\tilde{u}(t)}U$, t is a Lebesgue point of the map $\tau \to \tilde{x}_T \circ D_\tau^1\}$.

<u>LEMMA 1.</u> The Hessian $\tilde{F}''$ of the map F at the "point" $\tilde{u}(\cdot)$ has the form

$$\tilde{F}''(v_1(\cdot), v_2(\cdot)) = \tilde{x}_T \circ \int_0^T D_t^2(v_1(t), v_2(t))\,dt + \tilde{x}_T \circ \int_0^T \left[\int_0^t D_\tau^1 v_1(\tau), D_t^1 v_2(t)\right] dt + \operatorname{im}\tilde{F}',$$

$$\forall v_i(\cdot) \in \ker \tilde{F}^1 = \left\{v_i(\cdot) \in \mathcal{L}^\infty_{\tilde{u}} \,\middle|\, \tilde{x}_T \circ \int_0^T D_t^1 v(t)\,dt = 0\right\}.$$

<u>Proof.</u> Let the family of admissible controls $u_\varepsilon(\cdot)$ be such that $u_0(\cdot) = \tilde{u}(\cdot)$, $\partial/\partial\varepsilon \times u_\varepsilon(\cdot)|_{\varepsilon=0} = v(\cdot)$. Using the identity (3), we easily see that

$$\frac{\partial^2}{\partial\varepsilon^2}\Big|_{\varepsilon=0} \overrightarrow{\exp}\int_0^T \operatorname{Ad}\tilde{p}_{T,\tau}^{-1}\delta f_\tau(u_\varepsilon(\tau))\,d\tau = \int_0^T D_t^2(v(t), v(t))\,dt + 2\int_0^T \left(\int_0^t D_\tau^1 v(\tau)\,d\tau \circ D_t^1 v(t)\right) dt = \int_0^T D_t^2(v(t), v(t))\,dt +$$

$$+ \int_0^T \left[\int_0^t D_\tau^1 v(\tau)\,d\tau, D_t^1 v(t)\right] dt + \int_0^T D_t^1 v(t)\,dt \circ \int_0^T D_t^1 v(t)\,dt.$$

Thus, if $\tilde{x}_T \circ \int_0^T D_t^1 v(t)dt = 0$, then

$$\frac{\partial^2}{\partial\varepsilon^2}\Big|_{\varepsilon=0} F(u_\varepsilon(\cdot)) = \tilde{x}_T \circ \int_0^T D_t^2(v(t), v(t))\,dt + \tilde{x}_T \circ \int_0^T \left[\int_0^t D_\tau^1 v(\tau)\,d\tau, D_t^1 v(t)\right] dt.$$

For any covector $\psi \in (\operatorname{coker}\tilde{F}')^* = (\operatorname{im}\tilde{F}')^\perp \subset T_{\tilde{x}_T}^* M^n$, the product $\psi\tilde{F}''$ is a scalar quadratic form; in particular, the Morse index $\operatorname{ind}(\psi\tilde{F}'')$ is defined.

<u>THEOREM 1.</u> If $\operatorname{coker}\tilde{F}' = 0$, then the quasi-extremality index of the admissible control $\tilde{u}(\cdot)$ with respect to system (1) is $-\infty$. If $\tilde{F}' \neq 0$, then the quasi-extremality index of the control with respect to system (1) is

$$\dim \operatorname{coker}\tilde{F}' - \min\{\operatorname{ind}(\psi\tilde{F}'') \mid \psi \in (\operatorname{im}\tilde{F}'')^\perp \setminus 0\}.$$

<u>Proof.</u> The condition $\operatorname{coker}\tilde{F}' = \{0\}$ means that $\tilde{u}(\cdot)$ is a regular point of the smooth map $F: L_\infty([0, T]; U) \to M$. Let $\mathcal{U}$ be a submanifold of finite $L_\infty([0, T]; U)$ such that $\tilde{u}(\cdot) \in \mathcal{U}$ and the subspace $T_{\tilde{u}(\cdot)}\mathcal{U}$ of $\mathcal{L}^\infty_{\tilde{u}(\cdot)}$ is transversal to $\ker\tilde{F}'$ (i.e., $T_{\tilde{u}(\cdot)}\mathcal{U} + \ker\tilde{F}' = \mathcal{L}^\infty_{\tilde{u}(\cdot)}$. Then

obviously $\tilde{u}(\cdot)$ is a regular point of the map $F|\mathcal{U}$. At the same time, it follows from the Implicit Function Theorem that the germ of an arbitrary smooth map at a regular point is not extremal. Using the definition of local order, we see that the local extremality index of $\tilde{u}(\cdot)$ is $-\infty$. Moreover, for any control system $g_t(u)$ sufficiently near $f_t(u)$ the control $\tilde{u}(\cdot)$ is, as before, a regular point of the map $u(\cdot) \mapsto x_0 \circ \overrightarrow{\exp}\int_0^T g_t(u(t))dt$. Consequently, the quasi-extremality index of $\tilde{u}(\cdot)$ with respect to system (1) is also equal to $-\infty$.

The proof of theorem 1 in the case $\dim \operatorname{coker} \tilde{F}' = k > 0$ is based on the following assertion.

Proposition 1. An admissible control $\tilde{u}(\cdot)$ is quasi-extremal if and only if there exists a covector $\psi \in (\operatorname{im} \tilde{F}')^{\perp} \subset T^*_{\tilde{x}_T}M$, $\psi \neq 0$ such that the scalar quadratic form $\psi\tilde{F}''$ has index at most $k - 1$.

Proof. I) Sufficieny. Suppose that for some nonvanishing covector $\psi \in (\operatorname{im} \tilde{F}')^{\perp}$ the form $\psi\tilde{F}''$ has index $\ell \le k - 1$.

LEMMA 2. The quadratic form $\psi D_t^2(v, v)$, $v \in T_{\tilde{u}(t)}U$ is nonnegative for almost all $t \in [0, T]$. Indeed, this follows from the fact that the index of the form $\psi\tilde{F}''$ is finite and from the following easily verified fact:

LEMMA 3. Let $\bar{t} \in [0, T]$ be a Lebesgue point of the map $t \mapsto D_t^2$. Then any function $v(\cdot) \in \ker \tilde{F}'$ which also satisfies the conditions $v(t) = 0$ for $|t - \bar{t}| > \varepsilon$, $|v(t)| \le 1$ for $|t - \bar{t}| \le \varepsilon$ admits a representation

$$\tilde{F}''(v(\cdot), v(\cdot)) = \tilde{x}_T \circ \int_{\bar{t}-\varepsilon}^{\bar{t}+\varepsilon} D_{\bar{t}}^2(v(t), v(t))\, dt + o(\varepsilon).$$

Using Lemma 2, it is easy to construct a control system arbitrarily close to $f_t(u)$, say $\hat{f}_t(u)$, such that the value and first differential of the map $u \mapsto \hat{f}_t(u)$ at $\tilde{u}(t)$ coincide, respectively, with $\tilde{f}_t$ and $\tilde{f}_t'$, $t \in [0, T]$, but the second differential $\tilde{\hat{f}}_t''$ is such that the quadratic form

$$\psi\hat{D}_t^2(v, v) = \psi \operatorname{Ad} \tilde{p}_{T,t}^{-1} \tilde{\hat{f}}_t''(v, v), \quad v \in T_{\tilde{u}(t)}U,$$

is positive definite uniformly in $t \in [0, T]$.

Consequently, we may assume that the form $\psi D_t^2(v, v)$, $v \in T_{\tilde{u}(t)}U$ is positive definite uniformly in $t \in [0, T]$. We will indeed adopt this assumption from now on.

The symmetric bilinear form $\int_0^T \psi D_t^2(v_1(t), v_2(t))dt \stackrel{\text{def}}{=} (v_1(\cdot)|v_2(\cdot))$ determines a scalar product in the space $\mathcal{L}^\infty_{\tilde{u}}$; hence, also in the subspace $\ker \tilde{F}' \subset \mathcal{L}^\infty_{\tilde{u}}$. Using this scalar product, we can express the bilinear form $\psi\tilde{F}''(v_1(\cdot), v_2(\cdot))$ as

$$\psi\tilde{F}''(v_1(\cdot), v_2(\cdot)) = (v_1(\cdot)\,|\,v_2(\cdot)) + (v_1(\cdot)\,|\,Kv_2(\cdot)),$$

where

$$(v_1(\cdot)\,|\,Kv_2(\cdot)) = \int_0^T \psi\left[\int_0^t D_\tau^1 v_2(\tau)\, d\tau,\ D_t^1 v_1(t)\right] dt,$$

$K: \ker \tilde{F}' \to \ker \tilde{F}'$ is a compact symmetric operator.

Let $[\ker \tilde{F}']$ denote the completion of the space $\ker \tilde{F}'$ in the norm $\sqrt{(v(\cdot)|v(\cdot))}$. The operator K can be extended in a unique manner to a compact selfadjoint operator [K] defined in the Hilbert space $[\ker \tilde{F}']$; it is readily seen that $\operatorname{im}[K] \subset \ker \tilde{F}'$. By the Hilbert–Schmidt Theorem, every nonzero point of the spectrum of this operator is an isolated eigenvalue of finite multiplicity. Clearly, any eigenvector belonging to a nonzero eigenvalue is an element of $\operatorname{Ker} \tilde{F}'$.

Now we may assume that -1 is not an eigenvalue of K. Indeed, otherwise, we need only apply an arbitrarily small perturbation of the original system, leaving $\tilde{f}$ and $\tilde{f}'$ unchanged, in order to "correct" $\tilde{f}''$ so that the form ψD_t^2 becomes $(1 + \varepsilon)\psi D_t^2$, where $\varepsilon > 0$ is small. As the form ψD_t^2 determines our scalar product, it follows that the perturbation will transform K into the operator $(1/1 + \varepsilon)\cdot K$, i.e., all eigenvalues of K are multiplied by $1/1 + \varepsilon$.

We may thus assume that the form $\psi\tilde{F}''(v(\cdot), v(\cdot)) = (v(\cdot)|v(\cdot)) + (v(\cdot)|Kv(\cdot))$ is nonsingular and has index $\ell \leq k - 1$, where $k = \operatorname{codim}(\operatorname{im}\tilde{F}')$. Let $w_1(\cdot), \ldots, w_l(\cdot) \in \ker\tilde{F}'$ be a complete system of eigenfunctions belonging to the eigenvalues of K less than -1. Denote $W = \operatorname{span}\{w_1, \ldots, w_\ell\}$ and let $W_+ \subset \mathscr{L}^\infty_{\tilde{u}}$ and positive definite on W_- and positive definite on W_+.

Now let $X_1, \ldots, X_\ell$ be bounded vector fields on M^n such that

i) $\tilde{x}_T \circ (\psi X_i) = 0$, $i = 1, \ldots, \ell$;

ii) the tangent vectors $\tilde{x}_T \circ X_1, \ldots, \tilde{x}_T \circ X_\ell$ are linearly independent modulo $\operatorname{im}\tilde{F}'$, i.e., the subspace spanned by $\operatorname{im}\tilde{F}'$ and the vectors $\tilde{x}_T \circ X_i$, $i = 1, \ldots, \ell$, is of codimension $k - \ell$ in $T_{\tilde{x}_T}M^n$.

It is not difficult to construct a control system arbitrarily close to $f_t(u)$, say $g_t(u)$, such that $g_t(\tilde{u}(t)) = \tilde{f}_t$ and the first differential $\tilde{g}_t'$ of the map $u \mapsto g_t(u)$ at $\tilde{u}(t)$ has the fform

$$\tilde{g}_t'v = \tilde{f}_t'v + \varepsilon \sum_{i=1}^{l} \psi D_t^2(w_i(t), v) \operatorname{Ad}\tilde{p}_{T,t}X_i, \quad \forall v \in T_{\tilde{u}(t)}U,$$

where $\varepsilon \in \mathbf{R}$

Let $G: u(\cdot) \to x_0 \circ \overrightarrow{\exp}\int_0^T g_t(u(t))dt$ be the map carrying every admissible control $u(\cdot) \in L_\infty([0, T]; U)$ to the end of the corresponding trajectory. Then $G(\tilde{u}(\cdot)) = \tilde{x}_T$ and the differential $\tilde{G}'$ of G at the "point" $\tilde{u}(\cdot)$ has the form

$$\tilde{G}'v(\cdot) = \tilde{x}_T \circ \int_0^T \operatorname{Ad}\tilde{p}_{T,t}^{-1}\tilde{g}_t'v(t)\,dt = \tilde{F}'v(\cdot) + \varepsilon \sum_{i=1}^{l} (w_i(\cdot)\,|\,v(\cdot))\tilde{x}_T \circ X_i,$$

$$\forall v(\cdot) \in \mathscr{L}^\infty_{\tilde{u}}.$$

Consequently,

$$\operatorname{im}\tilde{G}' = \operatorname{im}\tilde{F} \oplus \operatorname{span}\{\tilde{x}_T \circ X_1, \ldots, \tilde{x}_T \circ X_l\},$$

$$\ker\tilde{G}' = W_+ \subset \ker\tilde{F}'.$$

In particular, the Hessian $\tilde{G}''$ of G at the "point" $\tilde{u}(\cdot)$ is a symmetric bilinear map defined on W_+,

$$\tilde{G}'': W_+ \times W_+ \to \operatorname{coker}\tilde{G}'.$$

In addition, the covector ψ is orthogonal to $\operatorname{im}\tilde{G}''$. Since $g_t(u)$ is close to $f_t(u)$, it follows that the scalar quadratic form $\psi\tilde{G}''$ is close to the form $\psi\tilde{F}''|W_+$. Since $\psi\tilde{F}''$ is positive definite on W_+, the same is true of $\psi\tilde{G}''$. Thus, to prove that $\tilde{u}(\cdot)$ is a quasi-extremal control we need only prove the following assertion.

<u>Proposition 2.</u> Let $g \in CS(M^n, U, [0, T])$ be a control system and $\tilde{G}': \mathscr{L}^\infty_{\tilde{u}(\cdot)} \to T_{G(\tilde{u}(\cdot))} \times M^n$ and $\tilde{G}'': \ker\tilde{G}' \times \ker\tilde{G}' \to \operatorname{coker}\tilde{G}'$ be the differential and Hessian, respectively, of the map $G: u(\cdot) \mapsto x_0 \circ \overrightarrow{\exp}\int_0^T g_t(u(t))dt$ at the "point" $\tilde{U}(\cdot) \in L_\infty([0, T]; U)$. If there exists a covector $\psi \in (\operatorname{im}\tilde{G}')^\perp$ such that the scalar quadratic form $\psi\tilde{G}''$ is positive definite [i.e., $\psi G''(v(\cdot), v(\cdot)) \geq \alpha \|v(\cdot)\|_2{}^2$ for some constant $\alpha > 0$], then $\tilde{u}(\cdot)$ is a locally extremal control for the system $g_t(u)$ with initial condition x_0.

<u>Proof.</u> Our assertion is purely local. Hence, by introducing suitable local coordinates we can identify the set of admissible controls with the space $\mathscr{L}^\infty_{\tilde{u}(\cdot)} = T_{\tilde{u}(\cdot)}L_\infty([0, T]; U)$ and assume, moreover, that $M^n = \mathbf{R}^n$, $G(\tilde{u}(\cdot)) = 0$. We will prove slightly more than neces-

sary, namely, we will show that there exists a (possibly nonpositive) constant c such that for all $v(\cdot) \in \mathscr{L}^{\infty}_{\tilde{u}(\cdot)}$, sufficiently close to zero we have the inequality $\psi G(\tilde{u} + v) \geq c\|v\|_{\infty}|G(\tilde{u}+ v)|$. To avoid notational complications we will write u, v, ... throughout instead of $\tilde{u}(\cdot)$, $v(\cdot)$, ...

The Taylor expansion of G at the point $\tilde{u}$ is

$$G(\tilde{u}+v)=\tilde{G}'v+\frac{1}{2}\frac{\partial^2 G}{\partial v^2}(\tilde{u})(v, v)+\int_0^1 \frac{(1-\theta)^2}{2}\frac{\partial^3 G}{\partial \theta^3}(u+\theta v)\,d\theta, \tag{4}$$

$$\forall v \in \mathscr{L}^{\infty}_{\tilde{u}}(U).$$

We also have $\tilde{G}'' = (\partial^2 G/\partial v^2)\cdot(\tilde{u})|\ker \tilde{G}' + \operatorname{im} \tilde{G}'$. It is readily deduced from the identity

$$G(u)=x_0+x_0\circ\int_0^T \overrightarrow{\exp}\int_0^t g_{\tau}(u(\tau))\,d\tau\circ g(u(t))\,dt, \quad \forall u(\cdot),$$

that

$$\left|\frac{\partial^3}{\partial u^3} G(u)(v, v, v)\right| \leqslant c_1\int_0^T |v(t)|^3 dt=c_1\|v\|_3^3$$

for some constant c_1 and all u sufficiently close to $\tilde{u}$.

The space $\mathscr{L}^{\infty}_{\tilde{u}}$ can be expressed as a direct sum $\mathscr{L}^{\infty}_{\tilde{u}} = \ker \tilde{G}' \oplus V_1$, where the finite-dimensional space V_1 is an arbitrary direct complement of $\ker \tilde{G}'$ in $\mathscr{L}^{\infty}_{\tilde{u}}$, $\dim V_1 = \dim(\operatorname{im} \times \tilde{G}')$. Let $v = v_1 + v_2$, where $v_1 \in V$, $v_2 \in \ker \tilde{G}'$. As the linear map $\tilde{G}'$ is nondegenerate on V_1 and the quadratic form $\psi(\partial^2 G/\partial v^2)\cdot(\tilde{u})$ is positive definite on $\ker \tilde{G}'$, it follows from (4) that

$$|G(\tilde{u}+v)| \geqslant 2\alpha(\|v_1\|_1+\|v_2\|_2^2) \geqslant \alpha(\|v_1\|_1+\|v\|_2^2)$$

for all v sufficiently close to zero, where $\alpha > 0$ is some constant. Multiplying (4) by ψ, we get

$$\psi G(\tilde{u}+v)=\frac{1}{2}\psi\frac{\partial^2 G}{\partial v^2}(\tilde{u})(v, v)+\int_0^1\frac{(1-\theta)^2}{2}\psi\frac{\partial^3 G}{\partial v^3}(\tilde{u}+\theta v)(v, v, v)\,d\theta \geqslant$$

$$\geqslant \psi\frac{\partial^2 G}{\partial v^2}(\tilde{u})(v_1, v_2)+\frac{1}{2}\psi\frac{\partial^2 G}{\partial v^2}(\tilde{u})(v_1, v_1)-\frac{c_1}{6}\|v\|_3^3 \geqslant$$

$$\geqslant -c_2(\|v_1\|_1\|v\|_{\infty}+\|v\|_3^3) \geqslant -c_2(\|v_1\|_1+\|v\|_2^2)\|v\|_{\infty} \geqslant$$

$$\geqslant -\frac{c_1}{\alpha}\|v\|_{\infty}|G(\tilde{u}+v)|$$

and so on.

II) <u>Necessity.</u> Let $\operatorname{codim}(\operatorname{im}\tilde{F}') = k > 0$ and suppose that for some nonvanishing covector $\psi \in (\operatorname{im}\tilde{F}')^{\perp}$ the form $\psi\tilde{F}''$ has index at least k.

<u>LEMMA 4.</u> There exists a finite-dimensional subspace $W \subset \ker \tilde{F}'$ such that for any nonvanishing covector $\psi \in (\operatorname{im}\tilde{F}')^{\perp}$ the scalar form $\psi\tilde{F}''|W$ is of index at least k.

<u>Proof.</u> Indeed, for any $\psi \in (\operatorname{im}\tilde{F}')^{\perp}$, $|\psi| = 1$, there is a k-dimensional subspace $W_{\psi} \subset \ker \tilde{F}'$ such that the form $\psi\tilde{F}''|W_{\psi}$ is negative definite. Clearly, for all $\hat{\psi}$ sufficiently near ψ the form $\hat{\psi}\tilde{F}''|W_{\psi}$ is negative definite. Choosing a finite cover of the sphere $\{|\psi| = 1\}$ by neighborhoods $O_{\psi_1}, \ldots, O_{\psi_m}$, we can write $W = W_{\psi_1} + \ldots + W_{\psi_m}$.

Let $V \subset \mathscr{L}^{\infty}_{\tilde{u}}(U)$ be a direct complement to $\ker \tilde{F}'$ in the space $\mathscr{L}^{\infty}_{\tilde{u}}$, i.e., $\mathscr{L}^{\infty}_{\tilde{u}} = V \oplus \ker \tilde{F}'$, $\dim V = k$; in addition, let $W \subset \ker \tilde{F}'$ be the subspace whose existence is guaranteed by Lemma 4.

Choose a (finite-dimensional) submanifold $\mathscr{U} \subset L_{\infty}([0, T]; U)$ such that $\tilde{u}(\cdot) \in \mathscr{U}$ and $T_{\tilde{u}(\cdot)}\mathscr{U}=V\oplus W\subset\mathscr{L}^{\infty}_{\tilde{u}}$. In the remaining part of the proof the Banach manifold of all admissible controls $L_{\infty}([0, T]; U)$ will be replaced by the submanifold $\mathscr{U}$. Therefore, from this point on until the end of the proof, only controls $u(\cdot)$ in $\mathscr{U}$ will be considered admissible. Accordingly, we get

$$F=F|_{\mathscr{U}}:\mathscr{U}\to M^n, \quad \tilde{F}':V\oplus W\to T_{\tilde{x}_T}M^n,$$

$$\ker\tilde{F}'=W, \quad \tilde{F}'':W\times W\to \operatorname{coker}\tilde{F}'.$$

LEMMA 5. Let N be a smooth manifold, $\Phi:N \to \mathbf{R}^n$ be a smooth map, $q \in N$. Let Φ_q', Φ_q'' denote the differential and Hessian, respectively, of Φ at q. Then, for all $\hat{\Phi}:N \to \mathbf{R}^n$, such that $\|\hat{\Phi} - \Phi\|_{\{q\},2}$ is sufficiently small,

i) $\operatorname{codim} \operatorname{im} \hat{\Phi}_q' \le \operatorname{codim} \operatorname{im} \Phi_q'$;

ii) if $\operatorname{ind} \psi\hat{\Phi}_q'' \ge \operatorname{codim} \operatorname{im} \Phi_q'$, $\forall \psi \in (\operatorname{im} \Phi_q')^{\perp} \backslash 0$, then also $\operatorname{ind} \psi\hat{\Phi}_q'' \ge \operatorname{codim} \operatorname{im} \hat{\Phi}_q'$, $\forall \times$ $\psi \in (\operatorname{im} \hat{\Phi}_q')^{\perp} \backslash 0$.

Part (i) of the lemma is obvious. The truth of part (ii) follows from the fact that the quadratic form $\hat{\psi}\hat{\Phi}_q''$ is close to some form $\psi\Phi_q''$, restricted to a subspace of codimension $(\operatorname{codim} \operatorname{im} \Phi_q' - \operatorname{codim} \operatorname{im} \hat{\Phi}_q')$ in $\ker \Phi_q'$.

Definition 5. Let N be a smooth manifold and $\Phi:N \to \mathbf{R}^n$ be a smooth map. We will say that Φ is essential at a point $q \in N$ if, for any neighborhood $\mathcal{O}_q$ of the point, there exist $\varepsilon > 0$, $m > 0$ such that the image of any smooth map $\hat{\Phi}: \mathcal{O}_q \to \mathbf{R}^n$, satisfying the condition $\|\hat{\Phi} - \Phi\|_{\mathcal{O}_q, m} < \varepsilon$ contains the point $\Phi(q)$, i.e., $\Phi(q) \in \hat{\Phi}(\mathcal{O}_q)$.

LEMMA 6. If the differential of a smooth map $\Phi:N \to \mathbf{R}^n$ at $q \in N$ is of rank n, then Φ is essential at q. This lemma is a simple corollary of the Implicit Function Theorem.

Since all our arguments are local, we will henceforth identify the manifold of admissible controls $\mathcal{U}$ with the vector space $T_{\tilde{u}(\cdot)} \mathcal{U} = W \oplus V$, assuming moreover that $M^n = \mathbf{R}^n$.

LEMMA 7 ("Fundamental Lemma"). The map $F: \mathcal{U} \to M^n$ is essential at the "point" $\tilde{u}(\cdot)$.

The as yet unproven part of Proposition 1 follows almost immediately from this lemma. In fact, as F is essential at $\tilde{u}(\cdot)$, the control $\tilde{u}(\cdot)$ cannot possibly be locally extremal. On the other hand, if $g_t(u)$ is a control system sufficiently close to (1) and $G:u(\cdot) \mapsto x_0 \circ \overrightarrow{\exp}\int_0^{\hat{T}} g_t(u(\cdot))dt$; then, as follows from Lemma 5, the scalar projections $\psi\tilde{G}''$ of the Hessian $\tilde{G}''$ of G at the "point" $\tilde{u}(\cdot)$ have index at least $\operatorname{codim} \operatorname{im} \tilde{G}')$ — the codimension of the image of the differential $\tilde{G}'$ of G at $\tilde{u}(\cdot)$. Thus, the map $G: \mathcal{U} \to M^n$ also falls under the sway of the Fundamental Lemma and it is therefore essential at $\tilde{u}(\cdot)$.

Proof of the Fundamental Lemma. We reason by induction on $k = \operatorname{codim}(\operatorname{im} \tilde{F}')$.

Induction Base, k = 1. In this case ψ is uniquely determined to within a scalar factor, the form $\psi\tilde{F}''$ does not have a fixed sign. Choose $w_1, w_2 \in W$ so that the numbers $\psi\tilde{F}'' \times (w_1, w_1)$ and $\psi\tilde{F}''(w_2, w_2)$ have opposite signs, $\psi\tilde{F}''(w_1, w_2) = 0$. Consider the map

$$\Phi:(v, \alpha) \mapsto \tilde{F}'v + \alpha^2\tilde{F}''(w_1, w_1) + (1-\alpha)^2\tilde{F}''(w_2, w_2),$$

where $v \in V$, $\alpha \in \mathbf{R}$.

The image of this map contains a neighborhood of the origin. Moreover, if $\Phi(v_0, \alpha_0) = 0$, then the differential of Φ at (v_0, α_0) is of rank n and, so, Φ is essential at (v_0, α_0).

Let $\varepsilon \in \mathbf{R}$, and consider the map

$$(v, \alpha) \mapsto F\left(\tilde{u}(\cdot) + \frac{\varepsilon^2}{2} v + \varepsilon\alpha w_1 + \varepsilon(1-\alpha) w_2\right).$$

Expansion in Taylor series gives

$$F\left(\tilde{u}(\cdot) + \frac{\varepsilon^2}{2} v + \varepsilon\alpha w_1 + \varepsilon(1-\alpha) w_2\right) = F(\tilde{u}(\cdot)) + \frac{\varepsilon^2}{2}\Phi(v, \alpha) + O(\varepsilon^3) \quad (\varepsilon \to 0).$$

It follows from this equality that F is essential at $\tilde{u}(\cdot)$.

Induction Step, k > 1 Arbitrary. Denote $\ker \tilde{F}'' = \{w_0 \in W | \tilde{F}''(w_0, w) = 0, \quad w \in W\}$.

i) The quadratic map

$$w \mapsto \tilde{F}''(w, w), \quad w \in W, \tag{5}$$

is essential at any point $\hat{w} \in W \backslash \ker \tilde{F}''$.

Indeed, the differential of this map at $\hat{w}$ has the form $w \mapsto 2\tilde{F}''(\hat{w}, w)$. Consequently, its image is $\tilde{F}''(\tilde{w}, W)$. Since $\hat{w} \notin \ker \tilde{F}''$, it follows that $\dim \tilde{F}''(\hat{w}, W) > 0$ and, so, codim $\times$ $\tilde{F}''(\hat{w}, W) = \hat{k} < k$.

Let $\psi \in \tilde{F}''(\hat{w}, W)^{\perp}$ and suppose that the projection of the Hessian of (5) at $\hat{w}$ in the direction ψ coincides with the quadratic form $2\psi\tilde{F}''$, restricted to the subspace $\hat{W}_0 = \{w \in W | \tilde{F}''(\hat{w}, w) = 0\}$. Since the index of $\psi\tilde{F}''$ on W is at least k and the codimension of $\hat{W}_0$ in W equals $k - \hat{k}$, it follows that the index of the form $2\psi\tilde{F}''|\hat{W}_0$ is at least $\hat{k}$. Thus, by the inductive hypothesis, the map (5) is essential at $\hat{w}$.

ii) The quadratic map (5) from W to $\operatorname{coker} \tilde{F}'$ is surjective. To prove this, we consider two cases: a) $\tilde{F}''(w, w) \neq 0$, $\forall\, w \notin \ker \tilde{F}''$. In this case the image of (5) is closed in coker $\times$ $\tilde{F}'$. If (5) is not surjective, this image contains boundary points other than the origin in $\operatorname{coker} \tilde{F}'$. This contradicts the fact that (5) is an essential map at any point $w \in \ker \tilde{F}'$. b) $\exists\, \hat{w} \notin \ker \tilde{F}''$ such that $\tilde{F}''(\hat{w}, \hat{w}) = 0$. Since (5) is an essential map at $\hat{w}$, its image must contain a neighborhood of the origin. Since (5) is homogeneous of positive degree, it is surjective.

iii) The map (5) is essential at $w = 0$. Let $\mathcal{O}$ be some neighborhood of the origin in W. It follows from (ii) that the image of this neighborhood under the map (5) contains a neighborhood of the point G(0). In addition, it can be shown that for all G close to (5) the sets $G(\mathcal{O})$ contain the balls centered at G(0) of the same radius $\rho > 0$.

iv) The map

$$\Phi\colon (v,\ w) \mapsto \tilde{F}'v + \tilde{F}''(w,\ w)$$

from $V \oplus W$ to $\mathbb{R}^n$ is essential at (0, 0). This follows directly from (iii).

That the map $F\colon \mathcal{U} \to \mathbb{R}^n$ is essential at $\tilde{u}(\cdot)$ follows from (iv) and the Taylor expansion:

$$F(\tilde{u}(\cdot) + \frac{\varepsilon^2}{2} v + \varepsilon w) = F(\tilde{u}(\cdot)) + \frac{\varepsilon^2}{2} \Phi(v,\ w) + O(\varepsilon^3),$$
$$v \in V,\ w \in W,\ (\varepsilon \to 0).$$

This completes the proof of the Fundamental Lemma and, hence, also of Proposition 1.

We return to the proof of the theorem. Recall that $\dim \operatorname{coker} \tilde{F}' = k > 0$. Denote $\underline{\operatorname{ind}}$ $\times$ $\tilde{F}'' = \min\{\operatorname{ind}(\psi\tilde{F}'') | \psi \in (\operatorname{im} \tilde{F}'')^{\perp} \backslash 0\}$.

i) Let $\mathcal{U}$ be a submanifold of finite codimension in $L_\infty([0, T]; U)$, $\tilde{u}(\cdot) \in \mathcal{U}$ and suppose that the subspace $T_{\tilde{u}(\cdot)}\mathcal{U}$ is transversal to $\ker \tilde{F}'$ in $\mathcal{L}^\infty_{\tilde{u}(\cdot)}(U)$. Let $\tilde{F}'_{\mathcal{U}}$ and $\tilde{F}''_{\mathcal{U}}$, respectively, denote the differential and Hessian of the map $F|\mathcal{U}$ at the "point" $\tilde{u}(\cdot)$. Then, as is readily seen,

$$\tilde{F}'_{\mathcal{U}} = \tilde{F}' | T_{\tilde{u}(\cdot)}\mathcal{U}, \quad \operatorname{im} \tilde{F}'_{\mathcal{U}} = \operatorname{im} \tilde{F}';$$
$$\tilde{F}''_{\mathcal{U}} = \tilde{F}'' | T_{\tilde{u}(\cdot)}\mathcal{U} \cap \ker \tilde{F}', \quad \underline{\operatorname{ind}}\, \tilde{F}''_{\mathcal{U}} \geqslant \underline{\operatorname{ind}}\, \tilde{F}'' - \operatorname{codim} \mathcal{U}.$$

Suppose now that $\ell = k - \underline{\operatorname{ind}}\tilde{F}'' \leq 0$. If $\operatorname{codim} \mathcal{U} \leqslant -\ell$, then $\dim \operatorname{coker} \tilde{F}'_{\mathcal{U}} - \underline{\operatorname{ind}}\, \tilde{F}''_{\mathcal{U}} \leq 0$. It follows from the arguments in Part II of the proof of Proposition 1 that the germ of the map $F|\mathcal{U}$ at $\tilde{u}(\cdot)$ is not extremal. Thus the local extremality index of the control $\tilde{u}(\cdot)$ with respect to system (1) is at most ℓ. Moreover, since $\dim \operatorname{coker} \tilde{F}' - \underline{\operatorname{ind}}\, \tilde{F}''$ is an upper semicontinuous function of the system, it follows that the quasi-extremality index of $u(\cdot)$ with respect to system (1) is at most ℓ.

The above inequality relating $\underline{\operatorname{ind}}\tilde{F}''_{\mathcal{U}}$ and $\operatorname{ind} \tilde{F}''$ may be written more accurately as follows:

$$\max_{\operatorname{codim} \mathcal{U} = \alpha} \underline{\operatorname{ind}}\, \tilde{F}''_{\mathcal{U}} = \underline{\operatorname{ind}}\, \tilde{F}'' - \alpha, \quad \forall \alpha \geqslant 0.$$

(This follows from the standard Courant–Fischer Theorem on the minimax representation of eigenvalues.) In particular, there exists a manifold $\mathcal{U}_l$ of codimension $1 - \ell$ such that

$$\underline{\operatorname{ind}}\, \tilde{F}''_{\mathcal{U}_l} = \operatorname{ind} \tilde{F}'' + l - 1,$$
$$\dim \operatorname{coker} \tilde{F}'_{\mathcal{U}_l} - \operatorname{ind} \tilde{F}''_{\mathcal{U}_l} = 1.$$

It follows from the arguments in Part I of the proof of Proposition 1 that there is a system $g_t(u)$, arbitrarily close to $f_t(u)$, such that the germ at $u(\cdot)$ of the map $G|\mathcal{U}_l: u(\cdot) \mapsto x_0 \circ \overrightarrow{\exp} \int_0^T g_t(u(t))dt$ is extremal. Moreover, if $\mathcal{U} \subset L_\infty([0, T]; U)$ is a submanifold sufficiently close to $\mathcal{U}_l$, then the germ at $\tilde{u}(\cdot)$ of the map $G|\mathcal{U}$ is also extremal. Thus the quasi-extremality index of the control $\tilde{u}(\cdot)$ is ℓ.

ii) Let $0 \le d \le k$ and let $\Phi: M^n \to \mathbf{R}^{n-d}$ be a smooth map which is regular at the point $F \times (\tilde{u}(\cdot))$ and such that the differential $\Phi_{F(\tilde{u})}': F_{F(\tilde{u})}M^n \to \mathbf{R}^{n-d}$ of Φ at $F(\tilde{u}(\cdot))$ satisfies the condition: $\ker \Phi_{F(\tilde{u})}' \cap \operatorname{im} \tilde{F}' = 0$. Let $\widetilde{\Phi \circ F'}$ and $\widetilde{\Phi \circ F''}$ denote the differential and Hessian, respectively, of the map $\Phi \circ F$ at the "point" $\tilde{u}(\cdot)$,

$$\underline{\operatorname{ind}}\, \widetilde{\Phi \circ F''} \overset{\text{def}}{=} \min\{\operatorname{ind} \chi \widetilde{\Phi \circ F''} \mid \chi \in (\operatorname{im} \widetilde{\Phi \circ F'})^\perp \setminus 0\}.$$

Then:

$$\widetilde{\Phi \circ F'} = \Phi'_{F(\tilde{u})} \circ \tilde{F}', \quad \operatorname{codim} \operatorname{im} \widetilde{\Phi \circ F'} = k - d;$$

$$\widetilde{\Phi \circ F''} = \Phi'_{F(\tilde{u})} \circ \tilde{F}'', \quad \underline{\operatorname{ind}}\, \widetilde{\Phi \circ F''} \geqslant \operatorname{ind} \tilde{F}''.$$

Suppose that $\ell = k - \underline{\operatorname{ind}}\, \tilde{F}'' > 0$. If $d \ge \ell$, then $\dim \operatorname{coker} \widetilde{\Phi \circ F'} - \underline{\operatorname{ind}}\, \widetilde{\Phi \circ F''} \le 0$. It follows from the arguments in Part II of the proof of Proposition 1 that the germ of $\Phi \circ F$ at $\tilde{u}(\cdot)$ is not extremal. Thus the local extremality index of the control $\tilde{u}(\cdot)$ with respect to system (1) is at most ℓ. Since $\dim \operatorname{coker} \tilde{F}' - \operatorname{ind} \tilde{F}''$ is upper semicontinuous, the quasi-extremality index of $\tilde{u}(\cdot)$ is also at most ℓ.

If $d \le k - 1$, the inequality $\underline{\operatorname{ind}}\, \widetilde{\Phi \circ F''} \ge \underline{\operatorname{ind}}\, \tilde{F}''$ can be strengthened:

$$\min\{\operatorname{ind} \widetilde{\Phi \circ F''} \mid \Phi: M^n \to \mathbf{R}^{n-d}\} = \underline{\operatorname{ind}}\, \tilde{F}'', \quad 0 \leqslant d \leqslant k-1.$$

In particular, there exists a map $\Phi_\ell: M^n \to \mathbf{R}^{n-l+1}$, such that

$$\dim \operatorname{coker} \widetilde{\Phi_l \circ F'} - \underline{\operatorname{ind}}\, \widetilde{\Phi_l \circ F''} = 1.$$

By arguments in Part I of the proof of Proposition 1, arbitrarily close to $f_t(u)$ there is a system $g_t(u)$ such that the germ at $\tilde{u}(\cdot)$ of the map

$$\Phi_l \circ G: u(\cdot) \mapsto \Phi_l\left(x_0 \circ \overrightarrow{\exp} \int_0^T g_t(u(t))\, dt\right), \quad u(\cdot) \in L_\infty([0, T]; U)$$

is extremal. Moreover, if the map $\Phi \in C^\infty(M^n, \mathbf{R}^{n-l+1})$ is sufficiently close to Φ_ℓ, the germ at $\tilde{u}(\cdot)$ of $\Phi \circ G$ is also extremal. Thus the quasi-extremality index of the control $\tilde{u}(\cdot)$ is equal to ℓ.

The proof of Theorem 1 is complete.

Remark. It is evident from the proof that the statement of the theorem remains in force if the standard topology of the space of admissible controls is replaced by the finite-dimensional-open topology.

2. Control Systems with Constraints on the Controls

We have been studying control problems in which the set of admissible control parameters is a smooth manifold. We now proceed to consider problems with sets of control parameters of a more general nature, including manifolds with boundary and all possible "angles."

1°. We will be dealing with a comparatively narrow class of manifolds with angles, whose properties we now proceed briefly to describe. For more details, see subsection 3.2 of the previous paper in this volume.

Definition 1. Let U be a smooth manifold. A closed subset $R \subset U$ is called a manifold with angles if every point $u \in R \subset U$ has a neighborhood $\mathcal{O}$ in M and local coordinates $\varphi: \mathcal{O} \to \mathbf{R}^r$, $\varphi(u) = 0$, such that $\varphi(R \cap \mathcal{O})$ is a convex polyhedral cone in $\mathbf{R}^r$ with vertex at zero.

A vector $\xi \in T_uU$ is said to be tangent to the subset R if there exists a smooth curve $\gamma: [0, \varepsilon] \to R$ such that $\gamma(0) = u$, $(d\gamma/d\xi)|_{\xi=0} = \xi$. The set of all vectors tangent to R at a point u forms a cone in T_uU, which we denote by T_uR. It is clear that if R is a manifold with angles then T_uR is a convex polyhedral cone. In addition, it follows from Definition 1 that there exists a diffeomorphism $\Phi: \mathcal{O}_u \to T_uU$ of a neighborhood of u onto T_uU such that

$$\Phi(R\cap\mathcal{O}_u)=T_uR. \tag{1}$$

Any convex polyhedral cone is given by a finite system of linear inequalities. In particular, $T_uR = \{\xi \in T_uU | \langle\omega_i, \xi\rangle \leq 0, i = 1, \ldots, N\}$ for some $\omega_1, \ldots, \omega_N \in T_u^*U$. Consequently, any manifold with angles is given locally by a finite system of smooth inequalities. Indeed, if the diffeomorphism $\Phi: \mathcal{O}_u \to T_uU$ satisfies condition (1), then

$$R\cap\mathcal{O}_u=\{v\in\mathcal{O}_u | \langle\omega_i, \Phi(v)\rangle\leqslant 0, i=1,\ldots,N\}.$$

Definition 2. We will say that two manifolds with angles R_1, $R_2 \subset U$ are transversal at a point $u \in R_1 \cap R_2$ if

$$(-T_uR_1)+T_uR_2=T_uU.$$

In particular, a smooth submanifold $N \subset U$ is transversal to a given manifold with angles R at a point $u \in N \cap R$ if and only if the plane T_uN is not contained in a supporting hyperplane of the cone T_uR.

Definition 3. An open face of a manifold with angles R is any maximal smooth connected submanifold of R (a submanifold is said to be maximal if it is not contained in any larger proper submanifold of R). A closed face of a submanifold with angles R is the closure in R of an open face.

It is not hard to show that the intersection of any two distinct open faces of a manifold with angles R is empty. Consequently, every point $u \in R$ is contained in exactly one open face, which we denote by Γ_u. In addition, any compact subset of R intersects only finitely many faces. Furthermore, if Γ is an open face in R and $\overline{\Gamma}$ its closure in R, then the set $\overline{\Gamma}\setminus\Gamma$ is a union of whole faces of R; and, moreover, both the closed face $\overline{\Gamma}$ and the set $\overline{\Gamma}\setminus\Gamma$ are themselves manifolds with angles. The set $\overline{\Gamma}\setminus\Gamma$ is called the polyhedral boundary of Γ and $\overline{\Gamma}$ (notation: $\partial_\pi\Gamma$ or $\overline{\partial}_\pi\Gamma$).

The inclusion relation "$\subset$" defines a partial ordering of the set of all closed faces of a manifold with angles. It also generates a partial ordering of the set of all open faces: a face Γ_1 is subordinate to a face Γ_2 if $\Gamma_1 \subset \overline{\Gamma}_2$. As is readily seen, any closed face is a topological manifold with boundary (the boundary need not be smooth) and the maximal closed faces are precisely the (connected) components of R.

Let $u \in R$ and let $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_k$ be all the closed faces containing u. The map taking each face $\overline{\Gamma}_1$ onto the complex cone $T_u\overline{\Gamma}_1$ determines a one-to-one correspondence between the closed faces of R containing u and the closed faces of T_uR. This correspondence preserves the inclusion relation. Moreover, $T_u\overline{\Gamma}_u = T_u\Gamma_u$ is a maximal subspace in T_uR.

Let $h: U \to M$ be a smooth map of the manifold U into a smooth manifold M, R be a manifold with angles in U, and $u \in R$, Γ_u be the open face containing u.

Definition 4. The differential of the map $h|R$ at u is the restriction to the cone T_uR of the differential $h': T_uU - T_{h(u)}M$ of h at u. The image of the differential of $h|R$ at u is a complex polyhedral cone $h'(T_uR)$.

Definition 5. The null-Hessian of the map $h|R$ is the Hessian

$$(h\,|\,\Gamma_u)'':\ker(h'\,|\,T_u\Gamma_u)\times\ker(h'\,|\,T_u\Gamma_u)\to\operatorname{coker}(h'\,|\,T_u\Gamma_u)$$

of the map $h|\Gamma_u$ at the point u.

Remark 1. Unlike the differential, the null-Hessian of $h|R$ need not coincide with the restriction of the Hessian $h'': \ker h' \times \operatorname{her} h' \to \operatorname{coker} h'$ to the appropriate subspace. In fact, since it may well occur that $h'(T_uU) \neq h'(T_u\Gamma_u)$, it follows that then also $\operatorname{coker} h' \neq \operatorname{coker} h'|T_u\Gamma_u$, so that the quadratic maps $h''|(T_u\Gamma_u \cap \ker h')$ and $(h|\Gamma_u)''$ assume values in different spaces.

Remark 2. We are using the term "null-Hessian" (rather than simply "Hessian") because the corresponding quadratic map need not contain all the invariant information about the second derivatives of $h|R$. We will not present the definition of the "true" Hessian here, since the null-Hessian is quite sufficient for dealing with quasi-extremality in problems with constraints. This situation is typical: the "true" Hessian almost always reduces to the null-Hessian, except in a few exceptional cases.

2°. Let $T > 0$ be a number and R a manifold with angles in U. Let $L_\infty([0, T]; R)$ denote the set of all maps $u(\cdot) \in L_\infty([0, T]; R)$ such that $\overline{u(E_{u(\cdot)})} \subset R$ for some subset of full measure $E_{u(\cdot)}$ in $[0, T]$. The subset $L_\infty([0, T]; R)$ of the Banach manifold $L_\infty([0, T]; U)$ possesses properties analogous to those of submanifolds with angles in finite-dimensional manifolds.

Let $\mathcal{G}$ denote the set of all open faces in R. Clearly, $\mathcal{G}$ is an at most countable set. Let $t \to \Gamma_t$ be an arbitrary measurable map of $[0, T]$ into $\mathcal{G}$, Then the set

$$\Gamma_\cdot = \{u(\cdot) \in L_\infty([0, T]; R) \mid u(t) \in \Gamma_t \text{ for almost every } t \in [0, T]\}$$

is called an open face of $L_\infty([0, T]; R)$. Similarly,

$$\overline{\Gamma}_\cdot = \{u(\cdot) \in L_\infty([0, T]; R) \mid u(t) \in \overline{\Gamma}_t \text{ for a.e. } t \in [0, T]\}$$

is a closed face of $L_\infty([0, T]; R)$. It is easy to see that any open face Γ is a Banach submanifold of $L_\infty([0, T]; U)$.

Any map $u(\cdot) \in L_\infty([0, T]; R)$ is contained in a unique open face $t \to \Gamma_{u(t)}$, which will be denoted by $\Gamma_{u(\cdot)}$.

Recall that the tangent space to $L_\infty([0, T]; U)$ at a "point" $u(\cdot)$ is the set of all measurable essentially bounded maps $t \mapsto v(t)$ such that $v(t) \in T_{u(t)}U$, $\forall t \in [0, T]$. We will denote this space by $\mathcal{L}^\infty_{u(\cdot)}$. Accordingly, the tangent cone to the set $L_\infty([0, T]; R)$ at a "point" $u(\cdot)$ is defined as the set of all measurable essentially bounded maps $t \to v(t)$ such that $v(t) \in T_{u(t)}U$, $\forall t \in [0, T]$. This cone will be denoted by $\mathcal{L}^\infty_{u(\cdot)}(R)$. A maximal subspace of $\mathcal{L}^\infty_{u(\cdot)}(R)$ is

$$T_{u(\cdot)}\Gamma_{u(\cdot)} = \{v(\cdot) \in \mathcal{L}^\infty_{u(\cdot)}(R) \mid v(t) \in T_{u(t)}\Gamma_{u(t)},\ \forall t \in [0, T]\}.$$

The following definition of transversality is essentially the same as in the finite-dimensional case.

Definition 6. Let $\mathcal{N}$ be a smooth Banach submanifold of finite codimension in $L_\infty([0, T]; U)$. We will say that $\mathcal{N}$ is transversal to the set $L_\infty([0, T]; R)$ at a "point" $u(\cdot) \in \mathcal{N} \cap L_\infty([0, T]; R)$ if $\overline{(T_{u(\cdot)}\mathcal{N} + \mathcal{L}^\infty_{u(\cdot)}(R))} = \mathcal{L}^\infty_{u(\cdot)}$ (in other words, if the subspace $T_{u(\cdot)}\mathcal{N}$ is not contained in a supporting hyperplane of the cone $\mathcal{L}^\infty_{u(\cdot)}(R)$.

Let $H: L_\infty([0, T]; U) \to M^n$ be a map, twice continuously differentiable at a "point" $u(\cdot) \in L_\infty([0, T]; U)$ of the Banach manifold $L_\infty([0, T]; U)$ into a finite-dimensional manifold M^n. Let H': $\mathcal{L}^\infty_{u(\cdot)} \to T_{H(u(\cdot))}M^n$ be the differential of H at $u(\cdot)$. Then the restriction $H'|\mathcal{L}^\infty_{u(\cdot)}(R)$ is called the differential of the map $H|L_\infty([0, T]; R)$.

The null-Hessian of the map $H|L_\infty([0, T]; R)$ at the "point" $u(\cdot)$ is defined as the Hessian

$$(H \mid \Gamma_u)'' : \ker H' \mid T_u\Gamma_u \times \ker H' \mid T_u\Gamma_u \to \operatorname{coker} H' \mid T_u\Gamma_u$$

of the map $H|\Gamma_u$ at $u(\cdot)$.

Let $C_{u(\cdot)}^\infty(L_\infty([0, T]; U); M^n) \in \mathcal{H}$ be a smooth germ at $u(\cdot) \in L_\infty([0, T]; R)$. We let $\mathcal{H}_R$ denote the germ $\mathcal{H}|L_\infty([0, T]; R)$. In the preceding section we defined the order of extremality of a germ $\mathcal{H}$; an analogous notion can be defined for $\mathcal{H}_R$.

Definition 7. The germ $\mathcal{H}_R$ is said to be extremal if there exist a neighborhood $\mathcal{O}$ of $u(\cdot)$ in $L_\infty([0, T]; U)$ and a representative H: $\mathcal{O} \to M^n$ of $\mathcal{H}$, such that $H(u(\cdot)) \in \partial H(\mathcal{O} \cap L_\infty \times ([0, T]; R))$, i.e., the point $H(u(\cdot))$ is on the boundary of the set $H|\mathcal{O} \cap L_\infty([0, T]; R)$.

Definition 8. i) Let $\mathcal{H}_R$ be an extremal germ. We will say that $\mathcal{H}_R$ has extremality index $k > 0$ if k is the least number such that, for almost every germ $\Phi \in C^\infty_{\mathcal{H}(u)}(M^n, \mathbf{R}^{n-k})$ the germ $(\Phi \circ \mathcal{H})_R$ is not extremal. ii) Assume that $\mathcal{H}_R$ is not extremal. We will say that $\mathcal{H}_R$ has extremality index $\ell \leq 0$ if ρ is the least number such that, for almost every germ $\mathcal{U}$ of a submanifold of codimension $-\ell$ at a point $u(\cdot)$, transversal to $L_\infty([0, T]; R)$, the germ $\mathcal{H}_R|\mathcal{U} \cap L_\infty([0, T]; R)$ is not extremal. If no such least ℓ exists, the extremality index is defined to be $-\infty$.

Remark 3. The transversality condition is necessary because in general it is not true that almost every germ at a point $u(\cdot)$ of a submanifold of finite codimension in $L_\infty([0, T]; U)$ is transversal to $L_\infty([0, T]; R)$.

3°. Finally, let us consider a control system with the controls subjected to geometric constraints:

$$\dot{x} = x \circ f_t(u), \ x \in M^n, \ u \in R \subset U, \ t \in [0, T]. \tag{2}$$

All the symbols here are used in the same sense as for system (1.1) and R is a given submanifold with angles in U. The elements of the set $L_\infty([0, T]; R)$ will be called admissible controls for system (2). The set of admissible controls is endowed with the topology induced by the topology of the space $L_\infty([0, T]; U)$. The linear space of all control systems of type (2) with fixed manifolds M^n, U, time interval [0, T] and manifold with angles R, will be denoted by $CS_R(M^n, U, [0, T])$. If we "forget about" the set R, the control system (2) becomes a system of type (1.1). In other words, the spaces $CS_R(M^n, U, [0, T])$ and CS × $(M^n, U, [0, T])$ consist of the same families $f_t(u)$, $t \in [0, T]$, $u \in U$ of nonstationary vector fields on M^n. In particular, the seminorms $\|f\|_{K,\alpha}$, $W \Subset M^n \times U$, $\alpha \leq 0$ (see p. 4) determine a Fréchet space structure in $CS_R(M^n, U, [0, T])$.

Fixing a point $x_0 \in M^n$, let us consider the map $F_R: u(\cdot) \to x_0 \circ \overrightarrow{\exp} \int_0^T f_t(u(t))dt$ of the set of admissible controls $L_\infty([0, T]; R)$ into M^n. Thus, $F_R = F|L_\infty([0, T]; R)$ (see p. 110).

Definition. Let $k \in [-\infty, n]$. An admissible control $\tilde{u}(\cdot)$ is said to have local extremality index k with respect to a control system (2) with initial condition x_0 if the germ of the map $F_R: L_\infty([0, T]; R) \to M^n$ at the "point" $\tilde{u}(\cdot)$ has extremality index k.

The definitions of quasi-extremality index and quasi-extremality carry over in a similar way to problems with constraints. As before, the quasi-extremality index of a control $\tilde{u}(\cdot)$ with respect to a given system $f_t(u)$, $u \in R$, is the upper limit of the local extremality indices of $\tilde{u}(\cdot)$ with respect to arbitrary systems $g_t(u)$, $u \in R$, where g tends to t. In particular, the quasi-extremality index of a given control is an upper semicontinuous functions of the system $f \in CS_R(M, U, [0, T])$.

Fixing an admissible control $\tilde{u}(\cdot)$ once and for all, we introduce the following notation:

$\tilde{F}_R': \mathcal{L}^\infty_{\tilde{u}}(R) \to T_{\tilde{x}_T}M_n$ is the differential of F_R at the "point" $\tilde{u}(\cdot)$;

$\tilde{F}_{R0}' = F_R|T_{\tilde{u}}\Gamma_{\tilde{v}}$ is the restriction of $\tilde{F}_R'$ to a maximal subspace of $\mathcal{L}^\infty_{\tilde{u}}(R)$;

$\tilde{F}_{R0}'': \ker \tilde{F}_{R0}' \times \ker \tilde{F}_{R0}' \to \operatorname{coker} \tilde{F}_{R0}'$ is the null-Hessian of F_R at $\tilde{u}(\cdot)$.

Describing expressions for $\tilde{F}_R'$ and $\tilde{F}_{R0}''$, we will use the notation introduced when we derived the formulas for the differential and Hessian in the problem without constraints (pp. 112-113), without further ado.

By the general definitions:

$$\tilde{F}_R' = \tilde{F}' \mid \mathcal{L}^\infty_{\tilde{u}}(R), \quad \tilde{F}_R' v(\cdot) = \int_0^T x_T \circ D_t^1 v(t)\, dt, \quad v(\cdot) \in \mathcal{L}^\infty_{\tilde{u}}(R),$$

$$\operatorname{im} \tilde{F}_R' = \operatorname{conv} \{x_T \circ D_t^1 v \mid v \in T_{\tilde{u}(t)} R,$$

t is a Lebesgue point of the map $t \mapsto \tilde{x}_T \circ D_\tau^1\}$.

$$\operatorname{im} \tilde{F}_{R0}' = \operatorname{span} \{x_T \circ D_t^1 v \mid v \in T_{\tilde{u}(t)} \Gamma_{\tilde{u}(t)},$$

t is a Lebesgue point of the map $\tau \mapsto \tilde{x}_T \circ D_\tau^{1}\}$.

Let

$$\tilde{f}^{(2)}_{\Gamma_{\tilde{u}(t)}} : T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)} \times T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)} \to \mathrm{Der}\, M^n / \tilde{f}'_t (T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)})$$

be the second differential of a map $u \mapsto f_t(u)$ of the smooth manifold $\Gamma_{\tilde{u}(t)}$ into $\mathrm{Der}\, M^n$ at the point $\tilde{u}(t)$ and

$$D^2_{\Gamma_{\tilde{u}(t)}}(v_1, v_2) = \mathrm{Ad}\, \tilde{p}^{-1}_{T,t} \tilde{f}^{(2)}_{\Gamma_{\tilde{u}(t)}}(v_1, v_2), \quad \forall v_1, v_2 \in T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)}.$$

Using Lemma 1.1 and the definition of the null-Hessian, we obtain

$$\tilde{F}''_{R0}(v_1(\cdot), v_2(\cdot)) = \tilde{x}_T \circ \int_0^T D^2_{\Gamma_{\tilde{u}(t)}}(v_1(t), v_2(t))\, dt + \tilde{x}_T \circ \int_0^T \left[\int_0^t D^1_\tau v_1(\tau)\, d\tau,\ D^1_t v_2(t) \right] dt + \mathrm{im}\, \tilde{F}'_{R0},$$

$$\forall v_i \in \ker \tilde{F}'_{R0}, \quad i = 1, 2.$$

Before stating our main result, we recall that the polar cone $\mathrm{im}\, \tilde{F}_r{}' \subset T_{\tilde{x}_T} M^n$ is defined by

$$(\mathrm{im}\, \tilde{F}'_R)^\circ = \{\psi \in T^*_{\tilde{x}_T} \mid \psi\xi \le 0,\ \ \forall \xi \in \mathrm{im}\, \tilde{F}'_R\} = \left\{\psi \in T^*_{\tilde{x}_T} \,\middle|\, \psi D^1_t v \le 0,\ \ \forall v \in T_{\tilde{u}(t)} R \text{ for a.e. } t \in [0, T]\right\}.$$

Clearly, $(\mathrm{im}\, \tilde{F}_R{}')^\circ \subset (\mathrm{im}\, \tilde{F}_{R0}{}')^\perp = (\mathrm{coker}\, \tilde{F}_{R0}{}')^*$.

At the same time, for any $\psi \in (\mathrm{coker}\, \tilde{F}_{R0}{}')^*$ we have a well-defined scalar quadratic form $\psi \tilde{F}_{R0}{}''$.

<u>THEOREM 1.</u> Let $\tilde{u}(\cdot)$ be an admissible control. If $(\mathrm{im}\, \tilde{F}_R)^\circ = \{0\}$, then the quasi-extremality index of the control with respect to system (2) is

$$\dim \mathrm{coker}\, \tilde{F}'_{R0} - \min \{\mathrm{ind}\,(\psi \tilde{F}''_{R0}) \mid \psi \in (-\mathrm{im}\, \tilde{F}'_R)^\circ \setminus 0\}.$$

<u>Proof.</u> If $\mathrm{coker}\, \tilde{F}_R{}' = 0$, then $\tilde{F}'(\mathscr{L}^\infty_{\tilde{u}}(R)) = T_{F(\tilde{u})} M^n$. The proof in this case differs only slightly from that of the corresponding part of Theorem 1.1. We will therefore not swell on it here, but go on at once to consider the case $\dim \mathrm{coker}\, \tilde{F}_{R0}{}' = k > 0$.

<u>Proposition 1.</u> An admissible control $\tilde{u}(\cdot)$ has positive quasi-extremality index if and only if there exists a covector $\psi \in (-\mathrm{im}\, \tilde{F}_R{}')^\circ$, $\psi \ne 0$, such that the scalar quadratic form $\psi \tilde{F}_{R0}{}''$ has index at most $k - 1$.

This proposition is the key to the proof of our theorem, which may be derived from it in almost exactly the same way as in the proof of Theorem 1.1.

<u>Proof of Proposition 1.</u>

I) <u>Sufficiency.</u> Suppose that some nonvanishing covector $\psi \in T_{F(\tilde{u})}{}^* M^n$ satisfies the conditions: $\psi \tilde{F}' v(\cdot) \ge 0$, $\forall v(\cdot) \in \mathscr{L}^\infty_{\tilde{u}}$, the form $\psi \tilde{F}_{R0}{}''$ is of index $\ell \le k - 1$.

If we have some Riemannian metric in U, we can identify the spaces $T_u U$ and $T_u{}^* U$, $u \in U$. For any $t \in [0, T]$, the cone $T_{\tilde{u}(t)} R \cap (T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)})^\perp$ is acute-angled and any vector $v \in T_{\tilde{u}(t)} R$ may be expressed uniquely as $v = v_0 + v_1$, where $v_1 \in T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)}$, $v_0 \in T_{\tilde{u}(t)} R \cap (T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)})^\perp$. Clearly, $\tilde{x}_T \circ (\psi D_t{}^1 v_0) = 0$, $\tilde{x}_T \circ (\psi D_t{}^1 v_1) \ge 0$. Using the fact that the cones $T_{\tilde{u}(t)} R \cap (T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)})^\perp$ are acute-angled, we easily construct a control system arbitrarily close to $f_t(u)$, say $\hat{f}_t(u)$, such that $\hat{f}_t(u) = f_t(u)$ $\forall\, u \in \Gamma_{\tilde{u}(t)}$, $t \in [0, T]$ and at the same time, for some $\alpha > 0$,

$$\tilde{x}_T \circ (\psi \hat{D}^1_t v) \overset{\mathrm{def}}{=} \tilde{x}_T \circ (\psi\, \mathrm{Ad}\, \tilde{p}^{-1}_{T,t} \tilde{\hat{f}}'_t v) \ge \alpha |v|,$$

$$\forall t \in [0, T], \quad v \in T_{\tilde{u}(t)} R \cap (T_{\tilde{u}(t)}\Gamma_{\tilde{u}(t)})^\perp.$$

Consequently,

$$\psi\tilde{F}'v(\cdot \geqslant \alpha\int_0^T |v(t)|\,dt = \alpha\|v(\cdot)\|_1, \tag{3}$$

$$\forall v(\cdot)\in\mathscr{L}^{\infty}_{\tilde{u}}(R)\cap(T_{\tilde{u}}\Gamma_{\tilde{u}})^{\perp}$$

(where $\tilde{F}'$ is the differential of the map

$$u(\cdot)\mapsto x_0\circ\overrightarrow{\exp}\int_0^T \hat{f}_t(u(t))\,dt = \hat{F}(u(\cdot))$$

at the "point" $\tilde{u}(\cdot)$). It follows from this inequality and the identity $\psi\,\tilde{F}'|T_{\tilde{u}}\Gamma_{\tilde{u}} = \psi\tilde{F}'|T_{\tilde{u}}\times\Gamma_{\tilde{u}} = 0$, in particular, that $\ker\tilde{F}_R' = \ker\tilde{F}_R' \cap T_{\tilde{u}}\,\Gamma_{\tilde{u}}$.

It remains to "perturb" the map $F|\Gamma_{\tilde{u}} = \hat{F}|\Gamma_{\tilde{u}}$ in a suitable way. Since $\Gamma_{\tilde{u}}$, unlike $L_\infty([0, T]; R)$, is a Banach manifold, we can reason almost exactly as at the corresponding point in the proof of Theorem 1.1, constructing a control system $g_t(u)$, arbitrarily close to $\hat{f}_t(u)$, such that the differential $\tilde{G}'$ and Hessian $\tilde{G}_{R_0}''$ of the map $G:u(\cdot)\mapsto x_0 \circ\overrightarrow{\exp}\int_0^T g_t(u(t))dt$ at $\tilde{u}(\cdot)$ satisfy the following relations:

a) $\tilde{G}'|(T_{\tilde{u}}\Gamma_{\tilde{u}})^{\perp} \equiv \tilde{F}'\,|(T_{\tilde{u}}\Gamma_{\tilde{u}})^{\perp}$;

b) $|\psi\tilde{G}_{R_0}''(v(\cdot), v(\cdot))| \geq \beta\|v(\cdot)\|_2{}^2$ for some $\beta > 0$ and $\forall\, v(\cdot) \in \ker\tilde{G}'$. These two relations, together with inequality (3), imply that $\tilde{u}(\cdot)$ is locally extremal with respect to the system $g_t(u)$, $u \in R$. The proof that this is indeed so is the same as the proof of Proposition 1.2.

II) <u>Necessity.</u> Let $\operatorname{codim}(\operatorname{im}\tilde{F}_{R_0}') = k > 0$ and suppose that for every nonvanishing covector $\psi \in (\operatorname{im}\tilde{F}_R')^{\circ}$ the form $\psi\tilde{F}_{R_0}''$ has index at least k. We must prove that the quasi-extremality index of $\tilde{u}(\cdot)$ is not positive.

The proof follows the same lines as that of the analogous statement in the problem without constraints. We need only "adjust" the statements of the appropriate lemmas so as to incorporate the constraints.

<u>LEMMA 1.4'.</u> There exists a finite-dimensional convex polyhedral cone $K \subset \mathscr{L}^{\infty}_{\tilde{u}}(R)$ such that $W = K \cap \ker\tilde{F}_{R_0}'$ is a linear space and for any nonvanishing $\psi \in \tilde{F}'(K)^{\circ}$ the quadratic form $\psi\tilde{F}''|W$ has index at least k.

Let $V \subset T_{\tilde{u}}\Gamma_{\tilde{u}}$ be a subspace such that $T_{\tilde{u}}\Gamma_{\tilde{u}} = V \oplus \ker\tilde{F}_{R_0}'$ and K be the cone whose existence is quaranteed by Lemma 4. Choose a (finite-dimensional) submanifold with angles $\subset L_\infty([0, T]; R)$ such that $T_{\tilde{u}}\mathscr{R} = V \oplus K$. In the remainder of the proof, we will use only the subset $\mathscr{R}$ instead of the set of all admissible controls.

<u>Definition 1.5'.</u> Let N be a smooth (finite-dimensional) manifold, $N \supset S$ be a manifold with angles and $\Phi:N \to \mathbf{R}^n$ be a smooth map. We will say that the map $\Phi|S$ is essential at a point $q \in S$ if, for any neighborhood $\mathcal{O}_q$ of this point in N, there exist $\varepsilon > 0$, $m > 0$ such that for any smooth map $\hat{\Phi}: \mathcal{O}_q \to \mathbf{R}^n$, satisfying the condition $\|\Phi - \Phi\|_{\mathcal{O}_q,m} < \varepsilon$ the set $\hat{\Phi}(\mathcal{O}_q \cap S)$ contains the point $\Phi(q)$.

<u>LEMMA 1.6'.</u> Let $\Phi_q':T_qN \to \mathbf{R}^n$ be the differential of the smooth map $\Phi:N \to \mathbf{R}^n$ at the point $q \in S \subset N$. If $\Phi_q'(T_qS) = \mathbf{R}^n$, then $\Phi|S$ is essential at q.

<u>LEMMA 1.7'.</u> The map $F|\,\mathscr{R}:\mathscr{R} \to M$ is essential at the "point" $\tilde{u}(\cdot)$.

The as yet unproven part of Proposition 1 follows without difficulty from Lemma 1.7'. The proof of the latter, like that of Lemma 1.7, is by induction on k. We mention only that in the induction step the quadratic map (1.5) should be replaced by the map

$$(v,\ w)\mapsto v+\tilde{F}_{R0}''(w,\ w),$$

where $w \in \ker\tilde{F}_{R_0}' \cap T_{\tilde{u}}\mathscr{R}$,

$$v\in(\tilde{F}'(T_{\tilde{u}}\mathscr{R})+\operatorname{im}\tilde{F}'_{R0})\subset\operatorname{coker}\tilde{F}'_{R0}.$$

It is essential here that $(\tilde{F}'(T_{\tilde{u}}\ \mathscr{R})+\operatorname{im}\tilde{F}_{R_0}')$ is a polyhedral and, therefore, closed cone, since it is a linear image of a polyhedral cone.

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SMOOTH CONTROL SYSTEMS OF CONSTANT RANK AND LINEARIZABLE SYSTEMS

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UDC 517.977.1;514.7

An exposition of the results of the author and A. A. Agrachev concerning systems of constant rank and bang–bang theorems for these systems. The author also presents a survey of results on the linearization of smooth systems and points out the relationship between systems which are linearizable by smooth feedback and systems of constant rank.

INTRODUCTION

This paper is devoted to a complete exposition of results partly announced in [1], [2], [5]; we will be concerned mainly with the class of smooth control systems

$$\dot{x}=f(x,u),\ x\in M,\ u\in U \tag{1}$$

(or, in the notation of the first paper in this volume, $\dot{\xi}=\xi\circ g_i=b_i\in\mathbf{R}^n$, $i=1,\ldots,m$, $\xi\circ f=A\xi$), which in the complexity of their structure most closely resemble linear control systems in $\mathbf{R}^n$ and in fact inherit some of the latter's important properties.

A linear control system in $\mathbf{R}^n$ is a system

$$\dot{\xi}=\xi\circ\left(f+\sum_{i=1}^{m}u_ig_i\right),\quad \xi\in M=\mathbf{R}^n,\quad u\in\mathbf{R}^m, \tag{2}$$

in which the fields g_i, $i=1,\ldots,m$, are constant and the field f linear; $\xi\circ g_i=b_i\in\mathbf{R}^n$, $i=1,\ldots,m$, $\xi\circ f=A\xi$. The most important property of linear systems is that the map $F_{\xi_g,T}$ which associates to an admissible control u(t), $0\le t\le T$, the right endpoint $\xi(T)=\xi(T;u,\xi_0)$ of the trajectory of the system corresponding to that control and the initial state ξ_0, is an affine map; thus an affine change of variables in the image of this map makes it linear.

Those smooth control systems of type (1) which are most naturally considered as "nearly" linear are the systems whose map $F_{\xi_0,T}$ (this map is frequently called the input–output

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Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Noveishie Dostizheniya, Vol. 35, pp. 135-178, 1989.

0090-4104/91/5504-1864$12.50