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ON REDUCTION OF A SMOOTH SYSTEM LINEAR IN THE CONTROL

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ABSTRACT. A method is presented for reducing a smooth system linear in the control on an n-dimensional manifold M to a nonlinear system on an $(n-1)$-dimensional manifold. This reduction is used to obtain sufficient conditions for a high order of local controllability of the system, and the problem of a time-optimal control of the angular momentum of a rotating rigid body is investigated.

Bibliography: 7 titles.

§1. Introduction

In this article a method is presented for investigating a controllable system of the form

$$\dot{x} = f(x) + g(x)u \tag{1.1}$$

on a smooth n-dimensional manifold M. Here $x \in M$, $u \in R$, $f(x)$ and $g(x)$ are complete smooth vector fields on M, and the admissible controls $u(t)$ are bounded measurable functions of t.

It is shown that (1.1) can be reduced to a system nonlinear in the control with an $(n-1)$-dimensional phase space. This reduction is used here to obtain sufficient conditions for local controllability of high order for system (1.1), and also in the problem of time-optimal control of the rotation of an asymmetric rigid body by means of a moment applied along an axis fixed in the body.

§2. Preliminary material

We introduce some notation which mainly follows [1]. Denote by $C^\infty(M)$ the algebra of infinitely differentiable functions on M. We must deal below with operators B and families of operators B_t $(t \in R)$ mapping $C^\infty(M)$ into itself. Following [1], we define the properties of continuity, differentiability, integrability, etc., of a family of operators B_t with respect to t in the weak sense: B_t has property $(*)$ with respect to t if the function $B_t\varphi$ has property $(*)$ with respect to t for all $\varphi \in C^\infty(M)$.

A vector field on M is defined to be an arbitrary derivation of the algebra $C^\infty(M)$, i.e., a linear mapping X of $C^\infty(M)$ into itself such that $X(\varphi_1\varphi_2) = (X\varphi_1)\varphi_2 + \varphi_1(X\varphi_2)$. If we introduce local coordinates on M, then the field X can be written in the form $X = \sum_1^n X_i\partial/\partial x_i$, where $X_i \in C^\infty(M)$. The value of a vector field X at a point $x \in M$ is a vector, denoted by $x \circ X$, in the tangent space T_xM.

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The Lie bracket (commutator) $[X, Y]$ of vector fields X and Y is defined by the formula $[X, Y]\varphi = X(Y\varphi) - Y(X\varphi)$. In local coordinates, $[X, Y] = \partial Y/\partial x X - \partial X/\partial x Y$. As we know, the commutator $[X, Y]$ is also a vector field; the Lie bracket introduces the structure of a Lie algebra in the space of vector fields. For a vector field X the linear operator $\operatorname{ad} X$ is defined on the space of vector fields by the formula $(\operatorname{ad} X)Y = [X, Y]$. Finally, a nonautonomous vector field X_t $(t \in R)$ is defined to be a family of vector fields integrable with respect to t.

Consider a diffeomorphism P of M onto itself. It determines an automorphism of the algebra $C^\infty(M)$ by the formula $(P\varphi)(x) = \varphi(P(x))$ for $\varphi \in C^\infty(M)$. This automorphism of $C^\infty(M)$ is also called a diffeomorphism and is denoted by the same symbol P. So that there will be no confusion, we denote the image of a point x under a diffeomorphism P by $x \circ P$, and the value of a function φ at x by $x \circ \varphi$.

Following [1], we define a flow P_t to be an absolutely continuous family of diffeomorphisms. It is easy to show that the composition $P_t^{-1} \circ (d/dt)P_t$ is a family of derivations of $C^\infty(M)$ that is integrable with respect to t, i.e., a nonautonomous vector field X_t. It follows from the equality $P_t^{-1} \circ (d/dt)P_t = X_t$ that

$$(d/dt)P_t = P_t \circ X_t. \tag{2.1}$$

Thus, any flow P_t is generated by some nonautonomous vector field X_t in view of the differential equation (2.1). A solution of (2.1) will be denoted by $\overrightarrow{\exp}\int_0^t X_\tau \, d\tau$ and called [1] a chronological exponential. If the vector field X_t is autonomous, i.e., $X_t = X$, then the flow generated by this field is denoted by e^{tX}.

According to [1], the chronological exponential can be expanded in a series

$$\overrightarrow{\exp} \int_0^t X_\tau \, d\tau = I + \int_0^t X_\tau \, d\tau + \int_0^t \int_0^\tau X_{\tau_1} \circ X_\tau \, d\tau_1 \, d\tau + \cdots. \tag{2.2}$$

We also give a variational formula for the chronological exponential [1]:

$$\overrightarrow{\exp} \int_0^t (X_\tau + Y_\tau) \, d\tau = \overrightarrow{\exp} \int_0^t \overrightarrow{\exp} \int_0^\tau \operatorname{ad} X_s \, ds Y_\tau \, d\tau \circ \overrightarrow{\exp} \int_0^t X_\tau \, d\tau. \tag{2.3}$$

In (2.3) the operator exponential $Q_\tau = \overrightarrow{\exp}\int_0^\tau \operatorname{ad} X_s \, ds$ is an absolutely continuous family of operators on the space of vector fields that satisfies the equation

$$(d/d\tau)Q_\tau Z = Q_\tau \circ (\operatorname{ad} X_\tau)Z$$

for any vector field Z. The flow $\overrightarrow{\exp}\int_0^t \overrightarrow{\exp}\int_0^\tau \operatorname{ad} X_s \, ds Y_\tau \, d\tau$ (see (2.3)) was called a perturbation flow in [1].

We consider a controllable system on M of the form

$$\dot{x} = X(x, u), \qquad u \in U. \tag{2.4}$$

The right-hand side of (2.4) can be regarded as a family $\mathscr{X} = \{X(x, u) \colon u \in U\}$ of vector fields depending on the parameter $u \in U$. It will be assumed that the vector field $X(x, u)$ is complete for any u.

The orbit $\mathscr{O}_x$ of system (2.4) at the point $x \in M$ is defined to be the set of points of the form

$$\mathscr{O}_x = \left\{ x \circ (e^{t_1 X_1} \circ e^{t_2 X_2} \circ \cdots \circ e^{t_k X_k}) \colon t_i \in R,\ X_i \in \mathscr{X} \right\}.$$

Obviously, if $x' \in \mathscr{O}_x$, then $\mathscr{O}_{x'} = \mathscr{O}_x$.

THEOREM 2.1 (SUSSMANN [2]). *For any point $x \in M$ the orbit $\mathcal{O}_x$ is a smooth submanifold of M that is invariant for system* (2.4).

The positive orbit $\mathcal{O}_x^+$ of system (2.4) at a point $x \in M$ is defined to be the set of points of the form

$$\mathcal{O}_x^+ = \left\{ x \circ \left(e^{t_1 X_1} \circ \cdots \circ e^{t_k X_k} \right) : t_i \in R,\ t_i \geqslant 0,\ X_i \in \mathcal{X} \right\}.$$

Obviously, $\mathcal{O}_x^+ \subseteq \mathcal{O}_x$.

Denote by $\mathcal{L}[\mathcal{X}]$ the smallest Lie algebra of vector fields such that $\mathcal{L}[\mathcal{X}] \supseteq \mathcal{X}$. The rank of the controllable system (2.4) at a point $x \in M$ is defined to be

$$\dim \operatorname{span}\{ x \circ X \colon X \in \mathcal{L}[\mathcal{X}] \}.$$

THEOREM 2.2 [2]. *The value of any vector field $X \in \mathcal{L}[\mathcal{X}]$ at a point $x' \in \mathcal{O}_x$ is a tangent vector to $\mathcal{O}_x$. In particular, the rank of system* (2.4) *at a point $x' \in \mathcal{O}_x$ does not exceed* $\dim \mathcal{O}_x$.

The following condition is assumed in what follows (it is true, in particular, for all real analytic systems): the rank of (2.4) at any point $x' \in \mathcal{O}_x$ coincides with the dimension $\dim \mathcal{O}_x$. What is more, for our purposes it suffices to consider the restriction of system (2.4) to the orbit $\mathcal{O}_x$, which enables us to assume without loss of generality that the orbit $\mathcal{O}_x$ of (2.4) coincides with the manifold M, and that the rank of (2.4) is equal to $\dim M$ at each point. In this case we have

THEOREM 2.3 (KRENER [2]). *If the rank of system* (2.4) *at each point $x \in M$ is equal to* $\dim M$ *and $\mathcal{O}_x = M$, then the set of interior points of the positive orbit $\mathcal{O}_x^+$ is dense in $\mathcal{O}_x^+$.*

THEOREM 2.4 (SUSSMANN AND JURDJEVIC [2]). *If the rank of system* (2.4) *at a point x is equal to* $\dim M$, *then for any $T > 0$ the set of attainability $A_{\leqslant T, x}$ of* (2.4) *from the point x in a time $\leqslant T$ has nonempty interior, and* $\operatorname{int} A_{\leqslant T,x}$ *is dense in $A_{\leqslant T,x}$.*

Along with the Lie algebra $\mathcal{L}[\mathcal{X}]$ we consider the smallest Lie subalgebra $\mathcal{L}^0[\mathcal{X}]$ of it containing all the fields of the form $X^1 - X^2$ ($X^1, X^2 \in \mathcal{X}$) and $[Y^1, Y^2]$ ($Y^1, Y^2 \in \mathcal{L}[\mathcal{X}]$). We call $\dim \operatorname{span}\{x \circ X\colon X \in \mathcal{L}^0[\mathcal{X}]\}$ the *exact rank* of the system. Obviously, the exact rank of the system does not exceed its rank.

THEOREM 2.5 [2]. *If the exact rank of system* (2.4) *at a point x is equal to* $\dim M$, *then for any $T > 0$ the set of attainability $A_{T,x}$ of* (2.4) *from x in the time T has nonempty interior, and* $\operatorname{int} A_{T,x}$ *is dense in $A_{T,x}$.*

§3. Reduction of the controllable system (1.1)

We consider a controllable system (1.1) and an admissible control $u(t)$. The flow P_t generated by the differential equation

$$\dot{x} = f(x) + g(x)u(t) \tag{3.1}$$

can be represented in the form of the chronological exponential

$$P_t = \overrightarrow{\exp} \int_0^t (f(x) + g(x)u(\tau))\, d\tau.$$

According to the variational formula (2.3),

$$\overrightarrow{\exp}\int_0^t (f + gu(\tau))\,d\tau = \overrightarrow{\exp}\int_0^t e^{(\int_0^\tau u(s)\,ds)\operatorname{ad} g} f\,d\tau \circ e^{(\int_0^t u(\tau)\,d\tau)g}, \tag{3.2}$$

or, with the notation $v(\tau) = \int_0^\tau u(s)\,ds$,

$$\overrightarrow{\exp}\int_0^t (f + gu(\tau))\,d\tau = \overrightarrow{\exp}\int_0^t e^{v(\tau)\operatorname{ad} g} f\,d\tau \circ e^{v(t)g}. \tag{3.3}$$

The right-hand side of (3.3) is the composition of the flows generated by the respective nonautonomous vector fields $e^{v(t)\operatorname{ad} g}f$ and $v(t)g$.

Consider a neighborhood V of a point $\tilde{x} \in M$ such that $g|_V \neq 0$. We define on V an equivalence relation that puts in a single class all the points lying on a single trajectory of the vector field $g|_V$, and we denote by V^g the quotient set by this equivalence relation. We can regard V^g as a set of segments of trajectories of the vector field g. Obviously, V^g can be parametrized by the points of the set $N \cap V$, where $M \supset N$ is an $(n-1)$-dimensional submanifold of M transversal to the trajectories of the field g in the neighborhood of $\tilde{x}$.

Suppose that $g \neq 0$ on the whole manifold M and, moreover, satisfies the "nonrecurrence" conditions: for each point $x \in M$ there exist a neighborhood $V_x \ni x$ and an $(n-1)$-dimensional manifold $N_x \subset M$ $(x \in N_x)$ transversal to g such that any trajectory of g intersects the set $V_x \cap N_x$ in a unique point. In particular, the "nonrecurrence" condition holds when $M = R^n$ and g is a constant vector field. Under these conditions the equivalence relation can be defined globally on the manifold M. The corresponding quotient manifold (the manifold of trajectories of g) is denoted by M^g.

We consider the family of vector fields $F_v = e^{v\operatorname{ad} g}f$ $(v \in R)$, and prove that it is well defined on M^g, i.e., under the action of the diffeomorphism $(e^{tg})_*$ a vector field in the family F_v passes into a vector field in the same family. Indeed, under the action of $(e^{tg})_*$ the field $F_v = e^{v\operatorname{ad} g}f$ passes [1] into the field $e^{t\operatorname{ad} g}F_v = e^{t\operatorname{ad} g}e^{v\operatorname{ad} g}f = e^{(t+v)\operatorname{ad} g}f = F_{t+v}$, i.e., the group of diffeomorphisms $(e^{tg})_*$ carries the family F_v into itself. We prove

PROPOSITION 1. *Let M^g be the quotient manifold described above, π the canonical projection of M onto M^g, and $D_{T,\tilde{y}}$ $(D_{\leqslant T,\tilde{y}})$ the set of attainability in the time T $(\leqslant T)$ from a point $\tilde{y}$ for the controllable system*

$$\dot{y} = y \circ F_v = y \circ \left(e^{v\operatorname{ad} g}f\right) \tag{3.4}$$

on the manifold M^g, where essentially bounded measurable scalar functions $v(t)$ are taken as the controls. The set of attainability $A_{T,\tilde{x}}$ $(A_{\leqslant T,\tilde{x}})$ of (1.1) *in the time T $(\leqslant T)$ from a point $\tilde{x}$ is contained in the inverse image $\pi^{-1}(D_{T,\pi(\tilde{x})})$ $(\pi^{-1}(D_{\leqslant T,\pi(\tilde{x})}))$, and if the exact rank (the rank) of system* (1.1) *at $\tilde{x}$ is equal to* $\dim M$, *then the interior of $A_{T,\tilde{x}}$ $(A_{\leqslant T,\tilde{x}})$ is dense in $\pi^{-1}(D_{T,\pi(\tilde{x})})$ $(\pi^{-1}(D_{\leqslant T,\pi(\tilde{x})}))$.*

REMARK. In other words, Proposition 1 means that the sets $A_{T,\tilde{x}}$ $(A_{\leqslant T,\tilde{x}})$ and $\operatorname{int} A_{T,\tilde{x}}$ $(\operatorname{int} A_{\leqslant T,\tilde{x}})$ are contained and everywhere dense in the "cylinder"; that is "swept out" in the motion of the set $D_{T,\pi(\tilde{x})}$ $(D_{\leqslant T,\pi(\tilde{x})})$ along trajectories of g.

PROOF OF PROPOSITION 1. Let $\hat{u}(t)$ be a fixed admissible control of (1.1), and T a fixed time. Setting $\hat{v}(t) = \int_0^t \hat{u}(\tau)\,d\tau$, we get by (3.3) that

$$\overrightarrow{\exp}\int_0^T (f + g\hat{u}(\tau))\,d\tau = \overrightarrow{\exp}\int_0^T F_{\hat{v}(t)}\,dt \circ e^{\hat{v}(T)g}. \tag{3.5}$$

Obviously, the point $\tilde{x}\circ(\overrightarrow{\exp}\int_0^T F_{\hat{v}(t)}\,dt\circ e^{\hat{v}(T)g})$ is contained in $\pi^{-1}(D_{T,\pi(\tilde{x})})$, since $\tilde{x}\circ\overrightarrow{\exp}\int_0^T F_{\hat{v}(t)}\,dt\in D_{T,\pi(\tilde{x})}$; and this proves the inclusion $A_{T,\tilde{x}}\subseteq\pi^{-1}(D_{T,\pi(\tilde{x})})$.

To prove the second part of Proposition 1 we use the auxiliary

LEMMA 2 [1]. *The point*

$$\tilde{y}\circ\left(\overrightarrow{\exp}\int_0^T F_{v(t)}\,dt\right)=\tilde{y}\circ\left(\overrightarrow{\exp}\int_0^T e^{v(t)\operatorname{ad}g}f\,dt\right)$$

depends continuously on $v(\cdot)$ *in the metric of* $L_1[0,T]$.

Suppose that $\hat{x}\in\pi^{-1}(D_{T,\pi(\tilde{x})})$ and $\hat{v}(\cdot)$ is a corresponding control carrying system (3.4) from the point $\pi(\tilde{x})$ to $\pi(\hat{x})$ in time T. We consider on M the differential equation $\dot{x}=x\circ(e^{\hat{v}(t)\operatorname{ad}g}f)$ and the trajectory $\hat{x}(t)$ of it satisfying $\hat{x}(0)=\tilde{x}$. Let $\hat{x}(T)=\hat{z}$. Since $\hat{v}(\cdot)$ carries system (3.4) from $\pi(\tilde{x})$ to $\pi(\hat{x})$ in time T, the points $\hat{x}$ and $\hat{z}$ lie on a single trajectory of g in view of (3.5), i.e., $\hat{x}=\hat{z}\circ e^{sg}$. Choose an absolutely continuous function $v^\delta(\cdot)$ in the δ-neighborhood of $v(\cdot)$ in the $L_1[0,T]$-metric and satisfying the conditions $v^\delta(0)=0$ and $v^\delta(T)=s$. We let $u^\delta(t)=\dot{v}^\delta(t)$, and consider the Cauchy problem $\dot{x}=f(x)+g(x)u^\delta(t)$, $x(0)=\tilde{x}$. According to (3.3), a solution $x^\delta(t)$ of this Cauchy problem is defined by

$$x^\delta(t)=\tilde{x}\circ\left(\overrightarrow{\exp}\int_0^t e^{v^\delta(\tau)\operatorname{ad}g}f\,d\tau\circ e^{sg}\right).$$

By choosing δ sufficiently small it is possible by Lemma 2 to make the point $\tilde{x}\circ(\overrightarrow{\exp}\int_0^T e^{v^\delta(t)\operatorname{ad}g}f\,dt)$ arbitrarily close to $\hat{z}=\tilde{x}\circ\overrightarrow{\exp}\int_0^T e^{v(t)\operatorname{ad}g}f\,dt$, and thereby to make $x^\delta(T)$ arbitrarily close to $\hat{x}=\hat{z}\circ e^{sg}$.

Thus, it is proved that the set of attainability $A_{T,\tilde{x}}$ is dense in $\pi^{-1}(D_{T,\pi(\tilde{x})})$. According to Theorem 2.5, int $A_{T,\tilde{x}}$ is dense in $A_{T,\tilde{x}}$. Consequently, int $A_{T,\tilde{x}}$ is dense in $\pi^{-1}(D_{T,\pi(\tilde{x})})$. Analogous arguments can be carried out for the set $A_{\leqslant T,\tilde{x}}$. Proposition 1 is proved.

It follows from Proposition 1 that the investigation of the set of attainability of the controllable system (1.1) of order n can be reduced to an investigation of a system (3.4) of order $n-1$ which, contrary to (1.1), is nonlinear (and often nondegenerate) in the control.

Proposition 1 admits a natural generalization to the case of a system linear in the vector-valued control $u=(u_1,\dots,u_l)$ and of the form

$$\dot{x}=f(x)+\sum_{i=1}^{l}g_i(x)u_i. \tag{3.6}$$

Suppose that the fields $\{g_i(x),\ i=1,\dots,l\}$ are linearly independent at each point $x\in M$ and generate an involutory l-dimensional distribution G on M. By the Frobenius theorem, there exist functions $b_{ij}(x)$, $i,j=1,\dots,l$, such that the vector fields $\hat{g}_i(x)=\sum_{j=1}^{l}b_{ij}(x)g_j(x)$ form a basis for the distribution G and have commutator $[\hat{g}_i,\hat{g}_j]=0$ for any i and j. Obviously, the determinant of the matrix $B=\|b_{ij}(x)\|$ is nonzero on M. Let $B^{-1}(x)=C(x)=\|c_{ij}(x)\|$; then

$$g_i(x)=\sum_{j=1}^{l}c_{ij}(x)\hat{g}_j(x),\qquad i=1,\dots,l. \tag{3.7}$$

Substituting (3.7) into (3.6) and introducing the new controls $v_j=\sum_{i=1}^{l}c_{ij}(x)u_i$, $j=1,\dots,l$, we get that (3.6) can be reduced to the system

$$\dot{x}=f(x)+\sum_{i=1}^{l}\hat{g}_i(x)v_i \tag{3.8}$$

with pairwise commuting fields $\hat{g}_i(x)$, $i = 1, \ldots, l$, which generate the same distribution G on M.

According to the Frobenius theorem, M is stratified into the integral manifolds of the distribution G. A literal repetition of the arguments given above with the integral curves of g replaced by the integral manifolds of G enables us to define from G an equivalence relation on M and an $(n-l)$-dimensional quotient manifold M^G by this equivalence relation.

PROPOSITION 1′. *Let M^G be the indicated quotient manifold, π the canonical projection of M onto M^G, and $D_{T,\tilde{y}}$ ($D_{\leqslant T,\tilde{y}}$) the set of attainability in the time T ($\leqslant T$) from a point $\tilde{y} \in M^G$ for the controllable system*

$$\dot{y} = y \circ \left(e^{\sum_1^l w_i \operatorname{ad} \hat{g}_i} f \right) \tag{3.9}$$

on M^G, where the essentially bounded scalar functions $w_i(t)$ are taken as controls. The set of attainability $A_{T,\tilde{x}}$ ($A_{\leqslant T,\tilde{x}}$) of system (3.6) (*or* (3.8)) *in time T ($\leqslant T$) from a point $\tilde{x} \in M$ is contained in the inverse image $\pi^{-1}(D_{T,\pi(\tilde{x})})$ ($\pi^{-1}(D_{\leqslant T,\pi(\tilde{x})})$), and if the exact rank* (*the rank*) *of* (3.6) *at $\tilde{x}$ is equal to* $\dim M$, *then the interior of $A_{T,\tilde{x}}$ ($A_{\leqslant T,\tilde{x}}$) is dense in $\pi^{-1}(D_{T,\pi(\tilde{x})})$ ($\pi^{-1}(D_{\leqslant T,\pi(\tilde{x})})$).*

Thus, by Proposition 1′, the investigation of the system (3.6) of order n with l-dimensional control reduces to the investigation of the system (3.9) of order $n - l$.

§4. Sufficient conditions for local controllability

Let us consider a controllable system (1.1) and a trajectory $\tilde{x}(t)$ of this system with the initial condition $\tilde{x}(0) = \tilde{x}$ generated by an admissible control $\tilde{u}(t)$. We introduce a special norm in the space of controls $u(\cdot)$; namely, we let

$$\| u(\cdot) \|_{[0,T]} = \sup_{t_1, t_2 \in [0,T]} \left| \int_{t_1}^{t_2} u(\tau)\, d\tau \right|.$$

This kind of norm is used in investigating sliding regimes [3]; therefore, the metric generated by it is called the sliding regime metric.

For what follows it is convenient to introduce the notation $A^{\varepsilon}_{T,\tilde{x}}$ for the set of attainability of system (1.1) in time T from the point $\tilde{x}$ by means of a control $u(\cdot)$ with $\|u(\cdot)\|_{[0,T]} < \varepsilon$.

DEFINITION. Let $\tilde{x}(\cdot)$ be the trajectory of (1.1) generated by the zero control, $\tilde{x}(0) = \tilde{x}$. Then system (1.1) is *weakly locally controllable* from the point $\tilde{x}$ in time T if $\tilde{x}(T) \in \operatorname{int} A^{\varepsilon}_{T,x}$ for all $\varepsilon > 0$.

PROPOSITION 3. *Consider on M the two-parameter family of vector fields $Z_{t,v} = e^{t \operatorname{ad} f} e^{v \operatorname{ad} g} f - f$, and let*

$$\Theta_{T,\varepsilon}(\tilde{x}) = \operatorname{con}\left\{ \tilde{x} \circ Z_{t,v} \colon 0 \leqslant t \leqslant T,\ |v| \leqslant \varepsilon \right\}, \tag{4.1}$$

$$\Xi_{T,\varepsilon}(\tilde{x}) = \operatorname{con}\left\{ \Theta_{T,\varepsilon}(\tilde{x}) \cup \{ \tilde{x} \circ g, \tilde{x} \circ (-g) \} \right\} \tag{4.2}$$

(*here* con B *denotes the convex cone generated by a set B; $\Theta_{T,\varepsilon}(\tilde{x})$ and $\Xi_{T,\varepsilon}(\tilde{x})$ are thus convex cones lying in the tangent plane $T_{\tilde{x}}M$*).

Suppose that $\tilde{x}(t)$ is the trajectory of system (1.1) *generated by the zero control, $\tilde{x}(0) = \tilde{x}$, and $\gamma(s)$ ($s \geqslant 0$) is a curve on M with $\gamma(0) = \tilde{x}$. If $\gamma'(0) \in \operatorname{int} \Xi_{T,\varepsilon}(\tilde{x})$ for all $\varepsilon > 0$, then for any $\varepsilon > 0$ the point $\gamma(s) \circ e^{Tf}$ lies in* int $A^{\varepsilon}_{T,\tilde{x}}$ *for all sufficiently small $s \geqslant 0$.*

PROOF OF PROPOSITION 3. For an arbitrary control $u(\cdot)$ we represent the trajectory of (1.1) generated by it in the form of the chronological exponential $\overrightarrow{\exp}\int_0^t (f + gu(\tau))\,d\tau$. According to (3.5),

$$\overrightarrow{\exp}\int_0^T (f + gu(\tau))\,d\tau = \overrightarrow{\exp}\int_0^T e^{v(\tau)\operatorname{ad} g} f\,d\tau \circ e^{v(T)g}, \tag{4.3}$$

where $v(t) = \int_0^t u(\tau)\,d\tau$. We represent the vector field $e^{v(t)\operatorname{ad} g} f$ in the form

$$e^{v(t)\operatorname{ad} g} f = f + \left(e^{v(t)\operatorname{ad} g} f - f\right).$$

By the variational formula (2.3),

$$\begin{aligned}\overrightarrow{\exp}\int_0^T e^{v(t)\operatorname{ad} g} f\,dt &= \overrightarrow{\exp}\int_0^T e^{t\operatorname{ad} f}\left(e^{v(t)\operatorname{ad} g} f - f\right)dt \circ e^{Tf} \\ &= \overrightarrow{\exp}\int_0^T \left(e^{t\operatorname{ad} f} e^{v(t)\operatorname{ad} g} f - f\right)dt \circ e^{Tf}.\end{aligned} \tag{4.4}$$

Combination of (4.3) and (4.4) gives us that

$$\overrightarrow{\exp}\int_0^T (f + gu(\tau))\,d\tau = \overrightarrow{\exp}\int_0^T \left(e^{t\operatorname{ad} f} e^{v(t)\operatorname{ad} g} f - f\right)dt \circ e^{Tf} \circ e^{v(T)g},$$

or [1]

$$\begin{aligned}\overrightarrow{\exp}\int_0^T (f + gu(\tau))\,d\tau &= \overrightarrow{\exp}\int_0^T \left(e^{t\operatorname{ad} f} e^{v(t)\operatorname{ad} g} f - f\right)dt \circ e^{v(T)e^{T\operatorname{ad} f} g} \circ e^{Tf} \\ &= \overrightarrow{\exp}\int_0^T Z_{t,v(t)}\,dt \circ e^{v(T)e^{T\operatorname{ad} f} g} \circ e^{Tf}.\end{aligned} \tag{4.5}$$

We prove that

$$\operatorname{con}\left(\Theta_{T,\varepsilon}(\tilde{x}) \cup \left\{\tilde{x} \circ \left(\pm e^{T\operatorname{ad} f} g\right)\right\}\right) = \Xi_{T,\varepsilon}(\tilde{x}) \tag{4.6}$$

(cf. (4.2)). To do this we first show that $(\tilde{x} \circ (\pm e^{t\operatorname{ad} f}(\operatorname{ad} f)g)) \in \Theta_{T,\varepsilon}(\tilde{x})$. Since $(\tilde{x} \circ Z_{t,\pm v}) \in \Theta_{T,\varepsilon}(\tilde{x})$ for $t \in [0, T]$ and $|v| \leqslant \varepsilon$, and $\Theta_{T,\varepsilon}(\tilde{x})$ is a convex cone, it follows that

$$\left.\frac{\partial}{\partial v}\right|_{v=0} (\tilde{x} \circ Z_{t,\pm v}) \in \Theta_{T,\varepsilon}(\tilde{x}).$$

A direct computation yields

$$\left.\frac{\partial}{\partial v}\right|_{v=0} (\tilde{x} \circ Z_{t,\pm v}) = \tilde{x} \circ \left(\pm e^{t\operatorname{ad} f}(\operatorname{ad} g) f\right) = \tilde{x} \circ \left(\mp e^{t\operatorname{ad} f}(\operatorname{ad} f) g\right) \in \Theta_{T,\varepsilon}(\tilde{x}).$$

We now prove that

$$\left(\tilde{x} \circ \left(\pm e^{t\operatorname{ad} f} g\right)\right) \in \Xi_{T,\varepsilon}(\tilde{x}). \tag{4.7}$$

Obviously,

$$\frac{d}{dt}\left(\tilde{x} \circ \left(\pm e^{t\operatorname{ad} f} g\right)\right) = \tilde{x} \circ \left(\pm e^{t\operatorname{ad} f}(\operatorname{ad} f) g\right) \in \Theta_{T,\varepsilon}(\tilde{x}) \subseteq \Xi_{T,\varepsilon}(\tilde{x}).$$

On the other hand, $(\tilde{x} \circ (\pm e^{t\operatorname{ad} f} g)) = (\tilde{x} \circ (\pm g)) \in \Xi_{T,\varepsilon}(\tilde{x})$ for $t = 0$. Since $\Xi_{T,\varepsilon}(\tilde{x})$ is a convex cone, we get (4.7) and, in particular, $(\tilde{x} \circ (\pm e^{T\operatorname{ad} f} g)) \in \Xi_{T,\varepsilon}(\tilde{x})$, which implies that

$$\operatorname{con}\left(\Theta_{T,\varepsilon}(\tilde{x}) \cup \left\{\tilde{x} \circ \left(\pm e^{T\operatorname{ad} f} g\right)\right\}\right) \subset \Xi_{T,\varepsilon}(\tilde{x}).$$

To prove the reverse inclusion we show that $(\tilde{x} \circ (\pm(g - e^{T\operatorname{ad} f} g)))$ lies in $\Theta_{T,\varepsilon}(\tilde{x})$. Indeed,

$$\frac{d}{dt}\left(\tilde{x} \circ \left(\pm\left(g - e^{t\operatorname{ad} f} g\right)\right)\right) = \left(\tilde{x} \circ \left(\mp e^{t\operatorname{ad} f}(\operatorname{ad} f) g\right)\right) \in \Theta_{T,\varepsilon}(\tilde{x}).$$

Since $g - e^{t\,\mathrm{ad} f} g = 0$ for $t = 0$, we get that $g - e^{t\,\mathrm{ad} f} g \in \Theta_{T,\varepsilon}(\tilde{x})$ for all $t \in [0, T]$. The equality (4.6) is proved.

We consider the set

$$C^{\varepsilon}_{T,\tilde{x}} = \left\{ \tilde{x} \circ \left(\overrightarrow{\exp} \int_0^T Z_{t,v(t)}\, dt \circ e^{v(T) e^{T\,\mathrm{ad} f} g} \right) \right\},$$

where the $v(\cdot)$ are absolutely continuous functions with $|v| \leqslant \varepsilon$. By (4.5), it suffices to show that for small $s \geqslant 0$ the points of the curve $\gamma(s)$ with $\gamma'(0) \in \operatorname{int} \Xi_{T,\varepsilon}(\tilde{x})$ lie in $C^{\varepsilon}_{T,\tilde{x}}$. Let $Y_{t,v(\cdot)} = v(t) e^{t\,\mathrm{ad} f} g$. Then

$$C^{\varepsilon}_{T,\tilde{x}} = \left\{ \tilde{x} \circ \left(\overrightarrow{\exp} \int_0^T Z_{t,v(t)}\, dt \circ e^{Y_{T,v(\cdot)}} \right) \right\}. \tag{4.8}$$

Note that $Z_{t,v(\cdot)} = Y_{t,v(\cdot)} \equiv 0$ for $v(\cdot) \equiv 0$. Using formula (2.2) for the exponentials on the right-hand side of (4.8), we get that

$$\overrightarrow{\exp} \int_0^T Z_{t,v(t)}\, dt \circ e^{Y_{T,v(\cdot)}} = I + \int_0^T Z_{t,v(t)}\, dt + Y_{T,v(\cdot)} + \cdots, \tag{4.9}$$

where the dots stand for terms of higher than first order of smallness in Z and Y. Let $W(v(\cdot)) = \int_0^T Z_{t,v(t)}\, dt + Y_{T,v(\cdot)}$. The range of the mapping W when $v(\cdot)$ is replaced by the set of absolutely continuous functions with $|v| \leqslant \varepsilon$ is a convex subset of $T_{\tilde{x}} M$. The interior of the cone spanned by it coincides with $\operatorname{int} \Xi_{T,\varepsilon}(\tilde{x})$ by the definition of $\Xi_{T,\varepsilon}(\tilde{x})$. Therefore the points of any curve $\gamma(s)$ ($s \geqslant 0$) lying in $\operatorname{int} \Xi_{T,\varepsilon}(\tilde{x})$ for small $s > 0$ also lie in the interior of the range of W for $|s| < \delta$ if δ is small.

Arguments analogous to those used in proving the maximum principle (see, for example, [3], Theorem VII.1) imply the existence of a δ', $0 < \delta' < \delta$, such that for $|s| < \delta'$ the points of the curve $\gamma(s)$ lie in $C^{\varepsilon}_{T,\tilde{x}}$, and this proves Proposition 3.

The next result follows directly from the proof of Proposition 3.

PROPOSITION 4. *If*

$$\begin{aligned} 0 &\in \operatorname{int} \Xi_{T,\varepsilon}(\tilde{x}) \\ &= \operatorname{int} \operatorname{con}\left(\left\{ \tilde{x} \circ \left(e^{t\,\mathrm{ad} f} e^{v\,\mathrm{ad} g} f - f \right) : 0 \leqslant t \leqslant T, |v| \leqslant \varepsilon \right\} \cup \left\{ \tilde{x} \circ (\pm g) \right\} \right), \end{aligned} \tag{4.10}$$

then system (1.1) *is weakly locally controllable from the point $\tilde{x}$ in time T.*

§5. Algebraic conditions for weak local controllability

Everywhere in this section we consider a controllable system (1.1) with the extra condition $\tilde{x} \circ f = 0$. Denote by Φ the Jacobi matrix $\Phi = \tilde{x} \circ (\partial f / \partial x)$. Then $\tilde{x} \circ (e^{t\,\mathrm{ad} f} X) = \tilde{x} \circ (e^{t\Phi} X)$ for any vector field X on M. In this case condition (4.10) for system (1.1) takes the form

$$\begin{aligned} 0 &\in \operatorname{int} \Xi_{T,\varepsilon}(\tilde{x}) \\ &= \operatorname{int} \operatorname{con}\left(\left\{ e^{t\Phi} \left(\tilde{x} \circ \left(e^{v\,\mathrm{ad} g} f \right) \right) : 0 \leqslant t \leqslant T, |v| \leqslant \varepsilon \right\} \cup \left\{ \tilde{x} \circ (\pm g) \right\} \right). \end{aligned} \tag{5.1}$$

Since the matrix $e^{t\Phi}$ determines a linear transformation of the tangent space $T_{\tilde{x}} M$, it follows from (4.6) that

$$\Xi_{T,\varepsilon}(\tilde{x}) = \left\{ e^{t\Phi} \left(\operatorname{con}\left(\left\{ \tilde{x} \circ \left(e^{v\,\mathrm{ad} g} f \right) : |v| \leqslant \varepsilon \right\} \cup \left\{ \tilde{x} \circ (\pm g) \right\} \right) \right) : 0 \leqslant t \leqslant T \right\}.$$

Let us investigate the set $\operatorname{con}(\{\tilde{x} \circ (e^{v\,\mathrm{ad} g} f) : |v| \leqslant \varepsilon\} \cup \{\tilde{x} \circ (\pm g)\})$. To do this we consider the smallest even $j \geqslant 0$ such that

$$\left(\tilde{x} \circ \left((\mathrm{ad}\, g)^j f \right) \right) \notin \operatorname{span}\left(\left\{ \tilde{x} \circ \left((\mathrm{ad}\, g)^i f \right) : 1 \leqslant i < j \right\} \cup \left\{ \tilde{x} \circ g \right\} \right). \tag{5.2}$$

If condition (5.2) does not hold for any even j, then we set $j = +\infty$. Let

$$\hat{\mathscr{L}}_{\tilde{x}} = \begin{cases} \operatorname{span}\left(\left\{ \tilde{x} \circ \left((\operatorname{ad} g)^i f\right) : 1 \leqslant i < j \right\} \cup \{\tilde{x} \circ g\}\right) & \text{if } j < +\infty, \\ \operatorname{span}\left(\left\{ \tilde{x} \circ \left((\operatorname{ad} g)^i f\right) : 1 \leqslant i < +\infty \right\} \cup \{\tilde{x} \circ g\}\right) & \text{if } j = +\infty. \end{cases}$$

PROPOSITION 5. *The linear space* $\hat{\mathscr{L}}_{\tilde{x}}$ *and the vector* $\tilde{x} \circ ((\operatorname{ad} g)^j f)$ (*if* $j < +\infty$) *are contained in the cone*

$$\mathscr{F}_\varepsilon = \operatorname{con}\left(\left\{ \tilde{x} \circ \left(e^{v \operatorname{ad} g} f\right) : |v| \leqslant \varepsilon \right\} \cup \{\tilde{x} \circ (\pm g)\}\right) \subseteq \mathrm{T}_{\tilde{x}} M.$$

The proof is by contradiction. If this assertion is false, then, since $\mathscr{F}_\varepsilon$ is convex, there exist a vector $q \in \hat{\mathscr{L}}_{\tilde{x}}$ and a covector $\psi \in \mathrm{T}^*_{\tilde{x}} M$ ($\psi \neq 0$) such that with the notation $\varphi(v) = \langle \psi, (\tilde{x} \circ (e^{v \operatorname{ad} g} f))\rangle$ we have

$$\begin{aligned} &\left(\left(\left\langle \psi, \left(\tilde{x} \circ \left((\operatorname{ad} g)^j f\right)\right)\right\rangle > 0\right) \vee (\langle \psi, q \rangle > 0)\right) \\ &\quad \wedge \left(\forall v : |v| \leqslant \varepsilon, \varphi(v) \leqslant 0\right) \wedge (\langle \psi, \tilde{x} \circ g \rangle = 0). \end{aligned} \tag{5.3}$$

Obviously, $\varphi(0) = 0$. It follows from (5.3) that the first nonzero derivative $\varphi^{(k)}(0)$ must be even, and $\varphi^{(k)}(0) < 0$. We prove that $k \geqslant j$. Indeed, if $k < j$ is even and $\varphi^{(l)}(0) = \langle \psi, (\tilde{x} \circ ((\operatorname{ad} g)^l f))\rangle = 0$ for all $l < k$, then by the definition of j

$$\left(\tilde{x} \circ \left((\operatorname{ad} g)^k f\right)\right) \in \operatorname{span}\left(\left\{ \tilde{x} \circ \left((\operatorname{ad} g)^l f\right) : l < k \right\} \cup \{\tilde{x} \circ g\}\right),$$

and, consequently, $\varphi^{(k)}(0) = \langle \psi, (\tilde{x} \circ ((\operatorname{ad} g)^k f))\rangle = 0$. Thus, $k \geqslant j$, and hence $\varphi^{(j)}(0) = \langle \psi, (\tilde{x} \circ ((\operatorname{ad} g)^j f))\rangle \leqslant 0$, which contradicts (5.3). Proposition 5 is proved.

A consequence of Propositions 4 and 5 is

PROPOSITION 6. *Let* $\mathscr{K}$ *be the cone generated by the space* $\hat{\mathscr{L}}_{\tilde{x}}$ *and the vector* $(\tilde{x} \circ ((\operatorname{ad} g)^j f))$ (*if* $j < +\infty$). *If* $\Phi = \tilde{x} \circ \partial f / \partial x$ *and* $\operatorname{con}\{e^{t\Phi} \mathscr{K} : 0 \leqslant t \leqslant T\} = \mathrm{T}_{\tilde{x}} M$, *then system* (1.1) *is weakly locally controllable from* $\tilde{x}$ *in time* T.

PROOF. By Proposition 5, $\mathscr{F}_\varepsilon \supseteq \mathscr{K}$, and hence

$$\begin{aligned} 0 \in \operatorname{int} \mathrm{T}_{\tilde{x}} M &= \operatorname{int} \operatorname{con}\{e^{t\Phi} \mathscr{K} : 0 \leqslant t \leqslant T\} \\ &\subseteq \operatorname{int} \operatorname{con}\left\{e^{t\Phi} \mathscr{F}_\varepsilon : 0 \leqslant t \leqslant T\right\} = \operatorname{int} \Xi_{T,\varepsilon}(\tilde{x}), \end{aligned}$$

i.e., condition (4.10) of Proposition 4 holds. Proposition 6 is proved.

We deduce from Proposition 6 that system (1.1) is weakly locally controllable from the point $\tilde{x}$ in some sufficiently large time T.

Let $\hat{\mathscr{L}}_{\tilde{x}}$ be the subspace of $\mathrm{T}_{\tilde{x}} M$ defined above, and let $\mathscr{L}^0_{\tilde{x}}$ be the smallest Φ-invariant subspace of $\mathrm{T}_{\tilde{x}} M$ containing $\hat{\mathscr{L}}_{\tilde{x}}$. In this case Φ is well defined on the quotient space $\mathrm{T}_{\tilde{x}} M / \mathscr{L}^0_{\tilde{x}}$. If $\mathscr{L}^0_{\tilde{x}}$ coincides with $\mathrm{T}_{\tilde{x}} M$, then by Proposition 6 the system is weakly locally controllable from $\tilde{x}$ in any time $T > 0$. In the opposite case we have

PROPOSITION 7. *If the vector* $(\tilde{x} \circ ((\operatorname{ad} g)^j f))$ *does not belong to any nontrivial* Φ-*invariant subspace of* $\mathrm{T}_{\tilde{x}} M / \mathscr{L}^0_{\tilde{x}}$ *and all the eigenvalues of* Φ *on* $\mathrm{T}_{\tilde{x}} M / \mathscr{L}^0_{\tilde{x}}$ *are nonreal, then system* (1.1) *is weakly locally controllable from* $\tilde{x}$ *in a sufficiently large time* $T > 0$.

PROOF. We consider an arbitrary covector $\psi \in (\mathrm{T}_{\tilde{x}} M / \mathscr{L}^0_{\tilde{x}})^*$, $\psi \neq 0$. If

$$\left\langle \psi, \left(\tilde{x} \circ \left(e^{t\Phi}\left((\operatorname{ad} g)^j f\right)\right)\right)\right\rangle \equiv 0,$$

then this means that the Φ-invariant subspace $\operatorname{span}\{\tilde{x}\circ(e^{t\Phi}((\operatorname{ad} g)^j f)),\ t\in R\}$ contains the vector $(\tilde{x}\circ((\operatorname{ad} g)^j f))$ and is orthogonal to ψ, i.e., does not coincide with $\mathrm{T}_{\tilde{x}}M/\mathscr{L}_{\tilde{x}}^0$, which contradicts the condition.

Suppose that $\omega(t)=\langle\psi,(\tilde{x}\circ(e^{t\Phi}((\operatorname{ad} g)^j f)))\rangle\not\equiv 0$. We prove that $\omega(t)$ changes sign on some interval $[0,T]$. Indeed, let $R(\lambda)$ be the characteristic polynomial of Φ on $\mathrm{T}_{\tilde{x}}M/\mathscr{L}_{\tilde{x}}^0$, $R(\Phi)=0$, and consider the differential operator $R(d/dt)$. Obviously, $R(d/dt)\omega=0$, i.e., $\omega(t)$ is a nonzero solution of a linear homogeneous equation with constant coefficients. Since all the eigenvalues of Φ are nonreal, $\omega(t)$ has the form

$$\omega(t)=\sum_{k=1}^{m} e^{\alpha_k t}\left(P_{r_k}(t)\cos\beta_k t+Q_{r_k}(t)\sin\beta_k t\right)=\sum_{s=1}^{N} a_s e^{\alpha_s t}t^{r_s}\begin{pmatrix}\cos\beta_s t\\ \sin\beta_s t\end{pmatrix}. \quad (5.4)$$

On the right-hand side of (5.4) we single out all the monomials corresponding to the largest of the α_s, and then we single out those of them for which the power r_s of t is maximal. Obviously, for large t the sign of $\omega(t)$ is determined by the sum of these monomials, i.e., by an expression of the form

$$e^{\alpha t}t^r\left(\sum_{l=1}^{m}(a_l\cos\beta_l t+b_l\sin\beta_l t)\right),\qquad \beta_l\neq 0.$$

As is known, any nonzero trigonometric polynomial of the form

$$P(t)=\sum_{l=1}^{m}(a_l\cos\beta_l t+b_l\sin\beta_l t)$$

is a function of variable sign on any interval of the form $(\hat{t},+\infty)$, which proves that $\omega(t)$ is of variable sign.

Since the choice of ψ was arbitrary, what has been proved implies that the cone $\mathscr{H}_T=\operatorname{con}\{e^{t\Phi}(\tilde{x}\circ((\operatorname{ad} g)^j f))\colon 0\leqslant t\leqslant T\}$ is a complement of $\mathscr{L}_{\tilde{x}}^0$ for all sufficiently large T, i.e., $\mathscr{H}_T+\mathscr{L}_{\tilde{x}}^0=\mathrm{T}_{\tilde{x}}M$, and hence, by the inclusion $\mathscr{H}_T+\mathscr{L}_{\tilde{x}}^0\subseteq\operatorname{con}\{e^{t\Phi}\mathscr{K}\colon 0\leqslant t\leqslant T\}$, we find ourselves under the conditions of Proposition 6, i.e., system (1.1) is weakly locally controllable from $\tilde{x}$ in a sufficiently large time T. Proposition 7 is proved.

We now investigate weak local controllability of system (1.1) in an arbitrarily small time $T>0$. Obviously, if there is a number m such that $\operatorname{span}\{\Phi^k\hat{\mathscr{L}}_{\tilde{x}}\colon 0\leqslant k\leqslant m\}=\mathrm{T}_{\tilde{x}}M$, then for any arbitrarily small $T>0$ the conditions of Proposition 6 hold for the system (1.1); hence we have

PROPOSITION 8. *Let j be the index defined in Proposition 5. If there exists a number m such that*

$$\operatorname{span}\left(\left\{\tilde{x}\circ\left((\operatorname{ad} f)^k(\operatorname{ad} g)^i f\right)\colon 0\leqslant k\leqslant m,\ 1\leqslant i<j\right\}\cup\{\tilde{x}\circ g\}\right)=\mathrm{T}_{\tilde{x}}M, \quad (5.5)$$

then system (1.1) *is weakly locally controllable from $\tilde{x}$ in any* (*arbitrarily small*) *time $T>0$.*

REMARK. The following condition for local controllability of system (1.1) in an arbitrarily small time $T>0$ was presented in [4].

THEOREM [4]. *Suppose that $S^k(f,g)$ is the linear hull of the values at a point $\tilde{x}$ of all possible commutators of the vector fields f and g, with g appearing at most k times. If $\tilde{x}\circ f=0$ and*

1) *$S^k(f,g)$ coincides with $\mathrm{T}_{\tilde{x}}M$ for some k,*

2) *$S_{i+1}(f,g)=S_i(f,g)$ for any odd i,*

then system (1.1) *is locally controllable from $\tilde{x}$ in an arbitrarily small time $T>0$.*

A comparison of condition (5.5) in Proposition 8 with conditions 1) and 2) in the theorem shows that these two assertions do not reduce to each other.

§6. Time-optimality in the problem of controlling the angular momentum of a rotating rigid body

The free rotation of a rigid body is described by the Euler equation (see [5]): $\dot{K} = K \times BK$, where $K \in R^3$ is the angular momentum vector in a coordinate system connected with the body, B is the symmetric 3×3 matrix inverse to the inertia tensor of the body A, and the sign "$\times$" denotes the vector product in R^3. Denote by $I_1 < I_2 < I_3$ the principal central moments of inertia of the body (the body is dynamically asymmetric), and by $J_1 > J_2 > J_3$ the quantities inverse to them (J_1, J_2, and J_3 are the eigenvalues of the matrix B).

If a controlling moment is applied to the body along an axis $\bar{L}$ passing through the center of mass, then the controlled motion of the angular momentum vector K is described by

$$\dot{K} = K \times BK + Lu, \tag{6.1}$$

where L is the unit vector on the axis $\bar{L}$.

We assume that the axis $\bar{L}$ is in general position: $\bar{L}$ does not coincide with any of the principal axes of inertia of the body and does not lie in one of the planes of the separatrices Π_1 and Π_2 given in the principal axes by the equations $\sqrt{J_1 - J_2}K_1 \pm \sqrt{J_2 - J_3}K_3 = 0$.

It follows from results in [6] that the exact rank (and thus also the rank) of system (6.1) is equal to 3 when $\bar{L}$ is in general position. The same is obviously true for the time-reversed system (6.1), denoted by (6.1 $-$). Hence, the conditions of Theorems 2.4 and 2.5 (see §2) and Proposition 1 in §3 are satisfied for systems (6.1) and (6.1 $-$).

For a controllable system (6.1) we consider the time-optimal problem

$$T \to \min \tag{6.2}$$

with boundary conditions

$$K(0) = \tilde{K}, \qquad K(T) = \hat{K}. \tag{6.3}$$

To investigate problem (6.1)–(6.3) we apply the reduction described in §3 to system (6.1), setting $f = K \times BK$ and $g = L$. As a result we get the planar system

$$\dot{K} = K \circ \left(e^{v \operatorname{ad} g} f\right) = K \circ \left(e^{vg} \circ f \circ e^{-vg}\right),$$

which, since $g = L$ is a constant field, is equivalent to the system $\dot{K} = K \circ (e^{vg} f)$ or

$$\dot{K} = (K + vL) \times B(K + vL). \tag{6.4}$$

We remark that in the case of a constant field $g = L$ the quotient manifold $(R^3)^g$ can be identified with the plane P passing through the origin O and perpendicular to the axis $\bar{L}$. Under this identification system (6.4) on $(R^3)^g$ is carried into the system

$$\begin{aligned} \dot{K} &= (K + vL) \times B(K + vL) - \langle (K + vL) \times B(K + vL), L \rangle L \\ &= (K + vL) \times B(K + vL) - \langle K \times B(K + vL), L \rangle L, \end{aligned} \tag{6.5}$$

whose right-hand side is the projection of the right-hand side of (6.4) on P. Any trajectory of system (6.5) generated by an absolutely continuous control $v(t)$ is the projection on P of some (nonunique!) trajectory of (6.4).

We introduce the Cartesian coordinate system Oy_1y_2 on P by directing the Oy_1 axis along the vector $L \times BL$ and the axis Oy_2 along the vector $L \times (L \times BL)$. In this coordinate system (6.5) takes the form

$$\dot{y}_1 = b_{13}y_2^2 + (-b_{23}y_1 + (b_{11} - b_{33})y_2)v + v^2,$$
$$\dot{y}_2 = -b_{13}y_1y_2 + ((b_{22} - b_{11})y_1 + b_{23}y_2)v, \tag{6.6}$$

where the b_{ij} are the components of the tensor B in the basis L, $L \times BL$, $L \times (L \times BL)$. Obviously, $b_{ij} = b_{ji}$, and a direct computation gives us also that $b_{13} < 0$ and $b_{22} - b_{11} \neq 0$.

For the controllable system (6.6) let us consider the time-optimal problem with the conditions

$$y(0) = \tilde{y}, \quad y(T) = \hat{y}, \quad T \to \min \quad (y = (y_1, y_2)). \tag{6.7}$$

We establish a connection between the optimal trajectories of problems (6.1)–(6.3) and those of problem (6.6)–(6.7).

DEFINITION. A control $\bar{u}(t)$ and the trajectory $\bar{K}(t)$ generated by it for system (6.1) are said to be *strongly locally optimal* if for any points $K^1 = \bar{K}(t_1)$ and $K^2 = \bar{K}(t_2)$ there exists a δ-neighborhood of $\bar{u}(t)$ in the sliding regime metric (δ is the same for all the pairs of points K^1, K^2 of the trajectory $\bar{K}(t)$) such that $T \geqslant t_2 - t_1$ for any control $u(\cdot)$ in this δ-neighborhood that carries system (6.1) from K^1 to K^2 in the time T.

DEFINITION. A control $\bar{v}(t)$ and the trajectory $\bar{y}(t)$ generated by it for system (6.6) are said to be *locally optimal* if there exists a δ-neighborhood of $\bar{v}(t)$ in the $L_\infty[0, T]$-metric such that $T \geqslant t_2 - t_1$ for any points $y^1 = \bar{y}(t_1)$ and $y^2 = \bar{y}(t_2)$ of the trajectory $\bar{y}(t)$ and any control $v(\cdot)$ in this δ-neighborhood that carries system (6.6) from y^1 to y^2 in the time T.

Let $\bar{v}(t)$ and $\bar{y}(t)$ be locally optimal for system (6.6), with $\bar{v}(\cdot)$ absolutely continuous, and let $\bar{u}(t)$ and $\bar{K}(t)$ be a control and the corresponding trajectory of (6.1) that pass under the reduction of (6.1) to (6.6) into $\bar{v}(t)$ and $\bar{y}(t)$, respectively. By the definition of the reduction (see §3), the δ-neighborhood of $\bar{u}(\cdot)$ in the sliding regime metric is mapped under the reduction inside the δ-neighborhood of $\bar{v}(\cdot)$ in the $L_\infty[0, T]$-metric. This implies immediately that the local optimality of $\bar{v}(t)$ and $\bar{y}(t)$ for system (6.6) yields the strong local optimality of the corresponding pair $\bar{u}(t)$, $\bar{K}(t)$ for (6.1).

It turns out that the time-optimal problem (6.1)–(6.3) under consideration has many strongly locally optimal trajectories, but does not have any globally optimal ones. Namely, we have the following assertion.

PROPOSITION 9. *For any point $\tilde{K} \in R^3$ there exists a one-parameter family of strongly locally time-optimal trajectories $K^\alpha(t)$ of system* (6.1) *emanating from $\tilde{K}$ and generated by the corresponding controls $u^\alpha(t)$.*

PROOF. For the reduced controllable system (6.6) we form the Hamiltonian

$$H = \psi_1(b_{13}y_2^2 + (-b_{23}y_1 + (b_{11} - b_{33})y_2)v + v^2)$$
$$+\psi_2(-b_{13}y_1y_2 + ((b_{22} - b_{11})y_1 + b_{23}y_2)v), \tag{6.8}$$

and write out the conjugate system

$$\dot{\psi}_1 = -\partial H/\partial y_1 = b_{23}v\psi_1 + (b_{13}y_2 - (b_{22} - b_{11})v)\psi_2,$$
$$\dot{\psi}_2 = -\partial H/\partial y_2 = -(2b_{13}y_2 + (b_{11} - b_{33})v)\psi_1 + (b_{13}y_1 - b_{23}v)\psi_2. \tag{6.9}$$

Obviously, if $\psi_1 < 0$, then the Hamiltonian H, which is quadratic in v, attains for

$$v = -\frac{1}{2}(-b_{23}y_1 + (b_{11} - b_{33})y_2) - \frac{\psi_2}{2\psi_1}((b_{22} - b_{11})y_1 + b_{23}y_2) \tag{6.10}$$

a maximum equal to

$$H_{\max} = b_{13}(\psi_1 y_2^2 - \psi_2 y_1 y_2) - \beta^2/4\psi_1,$$

where, for brevity, β denotes the coefficient of v in (6.8). Obviously, the strengthened Legendre condition $\partial^2 H/\partial v^2 = \psi_1 < 0$ holds for $\psi_1 < 0$, and $H_{\max} > 0$ under the additional condition $\operatorname{sgn}\psi_2 = \operatorname{sgn} y_1 y_2$ (with the inequality $b_{13} < 0$ taken into account), i.e., the corresponding transversality condition holds in problem (6.6)–(6.7).

Substituting (6.10) into (6.6) and (6.9), we get a system of fourth-order differential equations. Specifying the initial conditions $y_1(0) = \tilde{y}_1$, $y_2(0) = \tilde{y}_2$, $\psi_1(0) = -1$, $\psi_2(0) = \alpha$ (α is a parameter, and $\operatorname{sgn}\alpha = \operatorname{sgn} y_1 y_2$), we get the family of trajectories $y^\alpha(\cdot)$, $\psi^\alpha(\cdot)$ of this system, and from (6.10) the corresponding family of controls $v^\alpha(\cdot)$. The maximum principle in combination with the strengthened Legendre condition and the transversality condition ensures the local time-optimality of some part of any of the trajectories $y^\alpha(\cdot)$.

By the foregoing, any pair $u^\alpha(\cdot)$, $K^\alpha(\cdot)$ passing under reduction into the pair $v^\alpha(\cdot)$, $y^\alpha(\cdot)$ is strongly locally time-optimal for system (6.1). Proposition 9 is proved.

PROPOSITION 10. *In problem* (6.1)–(6.3) *there exists a minimizing sequence of controls* $\{u_n(\cdot)\}$ *carrying system* (6.1) *from* $\tilde{K}$ *to* $\hat{K}$ *in time* T_n, *where* $\lim_{n\to\infty} T_n = 0$. *In other words, system* (6.1) *can be carried from* $\tilde{K}$ *to* $\hat{K}$ *in an arbitrarily small time* $T > 0$.

REMARK. Generally speaking, an assertion stronger than Propositions 9 and 10 is valid. It can be shown that for any fixed compact set $C \subset R^3$ (for example, a compact ball) containing $\tilde{K}$ and $\hat{K}$ and for the set of trajectories γ of (6.1) going from $\tilde{K}$ to $\hat{K}$ in a time T_γ while remaining in C we have that $\inf_\gamma T_\gamma = T_{C,\tilde{K},\hat{K}} > 0$. If C_n is a collection of compact balls such that $C_1 \subset C_2 \subset \cdots$ and $\bigcup_i C_i = R^3$, then $\lim_{n\to\infty} T_{C_n,\tilde{K},\hat{K}} = 0$.

PROOF OF PROPOSITION 10. We first formulate and prove an auxiliary lemma.

LEMMA 11. *The statement of Proposition* 10 *is true for the reduced system* (6.6).

PROOF OF LEMMA 11. In polar coordinates (r, φ) ($y_1 = r\cos\varphi$, $y_2 = r\sin\varphi$) system (6.6) takes the form

$$\dot{r} = r \cdot F(\cos\varphi, \sin\varphi)v + \cos\varphi v^2, \tag{6.11}$$

$$\dot{\varphi} = -b_{13} r\sin\varphi - (1/r)\sin\varphi v^2 + G(\cos\varphi, \sin\varphi)v, \tag{6.12}$$

where F and G are homogeneous polynomials of degree 2, and $G(\pm 1, 0) = b_{22} - b_{11} \neq 0$.

We prove that (6.6) has trajectories γ beginning and ending on the positive semi-axis Oy_1 and encircling the origin O. We remark that the first and second terms on the right-hand side of (6.12) have (since $b_{13} < 0$) different signs. Setting

$$\hat{v}_\varepsilon(r,\varphi) = \hat{v}_\varepsilon(\varphi) = \begin{cases} 0, & \sin\varphi > \varepsilon, \\ \pm k, & \sin\varphi < -\varepsilon, \\ (b_{22} - b_{11}), & |\sin\varphi| \leqslant \varepsilon, \end{cases} \tag{6.13}$$

we get that for all $\rho_0 > 0$ there exist a sufficiently large k and a sufficiently small $\varepsilon > 0$ such that for $|r| \geqslant \rho_0$ we have (by (6.12) and (6.13))

$$\dot{\varphi} \geqslant a > 0, \tag{6.14}$$

i.e., φ is monotonically increasing along any trajectory γ of system (6.11), (6.12) generated by the control (6.13) and contained in the region $r \geqslant \rho_0$.

We prove the existence of such a trajectory. Since $\rho_0 > 0$ is arbitrary, it suffices to prove the existence of a trajectory of system (6.11) generated by the control (6.13) and not passing through O. It follows from (6.6), (6.12), and (6.13) that any trajectory (6.6) passing through O at the time $\hat{t}$ is tangent to the axis Oy_1, and $\lim_{t \to \hat{t}-0} \varphi(t) = \pi - 0$.

Let us fix ρ_0 and take the initial point $\bar{y}$ on the axis Oy_1 with polar coordinates $r = \rho_1, \varphi = 0$ $(\rho_1 > \rho_0)$. Since $\hat{v}_\varepsilon(\varphi)$ is a bounded function, the right-hand side of (6.11) admits the estimate

$$|\dot{r}| \leqslant \mu r + \nu. \tag{6.15}$$

It follows [7] from the differential inequality (6.15) that as φ varies along the trajectory from $\varphi(0) = 0$ to $\varphi(t_\varepsilon) = \arcsin \varepsilon$ we have that $r(t) \geqslant \rho_1 e^{-\mu t_\varepsilon} - \nu t_\varepsilon$, or, by (6.14),

$$r(t) \geqslant \rho_1 e^{-\mu \arcsin \varepsilon / a} - \nu \arcsin \varepsilon / a.$$

As φ varies along the trajectory from $\arcsin \varepsilon$ to $\pi - \arcsin \varepsilon$ the control $\hat{v}_\varepsilon(\varphi)$ is equal to 0 in view of (6.13), and (6.11) implies that $r(t) = \text{const}$. As φ varies along the trajectory from $\pi - \arcsin \varepsilon$ to π we get from (6.14) and (6.15) that

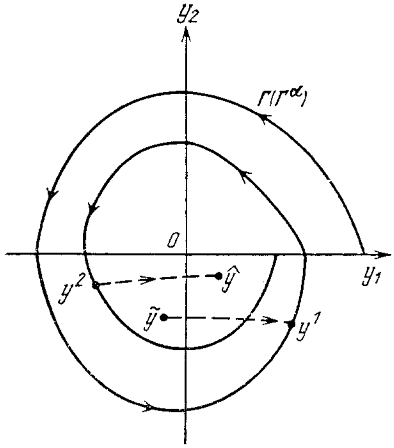
$$r(t) \geqslant r(t_\varepsilon) e^{-\mu \arcsin \varepsilon / a} - \nu \arcsin \varepsilon / a,$$

or

$$r(t) \geqslant \rho_1 \cdot e^{-2\mu \arcsin \varepsilon / a} - 2\nu \arcsin \varepsilon / a.$$

Obviously, by choosing ε sufficiently small we can get that $r(t) > \rho_0$, which is what was required.

Thus, the trajectory γ of system (6.6) generated by the control (6.13) and beginning on the positive semi-axis Oy_1 does not pass through the origin and, in view of the monotone variation of φ along γ, returns to the positive semi-axis Oy_1 in a finite amount of time t_0. Similarly, it is possible to construct a trajectory Γ of (6.6) that completes two circuits about O in a finite amount of time T_Γ (see the figure) and is generated by the control $\hat{v}(\cdot)$.



We remark that system (6.6) (as well as (6.1)) has an obvious self-similarity—it is invariant with respect to the change of variables $y_1 \to \alpha y_1$, $y_2 \to \alpha y_2$, $v \to \alpha v$, $t \to \alpha^{-1} t$ ($\alpha > 0$). Consequently, the curve $\Gamma^\alpha = \alpha\Gamma$ also is an admissible trajectory of system (6.6) generated by the control $\hat{v}^\alpha(\varphi) = \alpha\hat{v}(\varphi)$, and its circuit time is $T_{\Gamma^\alpha} = \alpha^{-1}T_\Gamma$.

We prove that if $\tilde{y}$ and $\hat{y}$ are arbitrary points of the plane P and $\varepsilon > 0$, then $\hat{y}$ can be reached from $\tilde{y}$ by means of (6.6) with the help of some control $w(\cdot)$ in a time $T \leqslant \varepsilon$.

Choose $\alpha > 0$ such that 1) the points $\tilde{y}$ and $\hat{y}$ are covered by the trajectory Γ^α, and 2) $\alpha^{-1}T_\Gamma \leqslant \varepsilon/3$. It follows from the form of the right-hand side of (6.6) that by choosing v large in absolute value we can ensure an arbitrarily rapid motion of system (6.6) in the positive direction of the axis Oy_1 along a trajectory close to the horizontal. Similarly, for the reversed-time system (6.6) a control v large in absolute value ensures an arbitrarily rapid displacement in the negative direction of the Oy_1-axis. Consequently, there exists a control $v^1(t)$ carrying system (6.6) from $\tilde{y}$ to a point y^1 on the trajectory Γ^α in a time $\tau_1 \leqslant \varepsilon/3$, as well as a control $v^2(t)$ carrying the reversed-time system (6.6) from $\hat{y}$ to a point $y^2 \in \Gamma^\alpha$ in a time $\tau_2 \leqslant \varepsilon/3$. The latter means that system (6.6) goes from y^2 to $\hat{y}$ with the help of the control $v^2(t)$ in the same time $\tau_2 \leqslant \varepsilon/3$. Passage of (6.6) from y^1 to y^2 by means of the control $\hat{v}^\alpha(t) = \alpha\hat{v}(\alpha^{-1}t)$ along the trajectory Γ^α takes place in the time $\tau_0 \leqslant T_{\Gamma^\alpha} \leqslant \varepsilon/3$ (see the figure).

The desired control $w(\cdot)$ is determined by

$$w(t) = \begin{cases} v^1(t), & 0 \leqslant t \leqslant \tau_1, \\ \hat{v}^\alpha(t), & \tau_1 < t \leqslant \tau_1 + \tau_0, \\ v^2(t), & \tau_1 + \tau_0 \leqslant t \leqslant \tau_1 + \tau_0 + \tau_2. \end{cases}$$

Obviously, $w(t)$ carries the system (6.6) from $\tilde{y}$ to $\hat{y}$ in time $\tau_1 + \tau_0 + \tau_2 \leqslant \varepsilon$. Lemma 11 is proved.

Let us consider again the time-optimal problem (6.1)–(6.3). We project the points $\tilde{K}$ and $\hat{K}$ onto the plane P into the respective points $\tilde{y} = \pi(\tilde{K})$ and $\hat{y} = \pi(\hat{K})$, and consider the δ-neighborhood $U_\delta(\hat{y})$ of $\hat{y}$. By Lemma 11, for any $\varepsilon > 0$ and any $y \in U_\delta(\hat{y})$ there exists a control $w(t)$ carrying system (6.6) from $\tilde{y}$ to y in a time $\leqslant \varepsilon/2$. Let $D_{\leqslant \varepsilon/2, \tilde{y}}$ be the set of attainability of system (6.6) from $\tilde{y}$ in a time $\leqslant \varepsilon/2$; then $U_\delta(\hat{y}) \subseteq D_{\leqslant \varepsilon/2, \tilde{y}}$. By Proposition 1, the interior of the set of attainability $A_{\leqslant \varepsilon/2, \tilde{K}}$ of system (6.1) from $\tilde{K}$ in a time $\leqslant \varepsilon/2$ is dense in $\pi^{-1}(U_\delta(\hat{y})) \subseteq \pi^{-1}(D_{\leqslant \varepsilon/2, \tilde{y}})$, i.e., in the cylinder C_δ with base $U_\delta(\hat{y}) \subset P$ and generator parallel to $\bar{L}$. Obviously,

$$\hat{K} \in \pi^{-1}(\hat{y}) \subset \operatorname{int} C_\delta, \qquad \hat{K} \in \operatorname{clos} \operatorname{int} A_{\leqslant \varepsilon/2, \tilde{K}}.$$

As mentioned above, Theorem 2.5 in §2 is applicable to system (6.1 $-$) (system (6.1) with reversed time). In particular, the set of attainability $A^-_{\leqslant \varepsilon/2, \hat{K}}$ of this system in a time $\leqslant \varepsilon/2$ from the point $\hat{K}$ has a nonempty interior that is dense in $A^-_{\leqslant \varepsilon/2, \hat{K}}$. It follows from the inclusions

$$\hat{K} \in A^-_{\leqslant \varepsilon/2, \hat{K}}, \quad \hat{K} \in \operatorname{int} C_\delta, \quad \hat{K} \in \operatorname{clos} \operatorname{int} A_{\leqslant \varepsilon/2, \tilde{K}} \supseteq C_\delta$$

that $\operatorname{int} A^-_{\leqslant \varepsilon/2, \hat{K}} \cap C_\delta \neq \varnothing$, and, consequently,

$$\operatorname{int} A^-_{\leqslant \varepsilon/2, \hat{K}} \cap \operatorname{int} A_{\leqslant \varepsilon/2, \tilde{K}} \neq \varnothing.$$

If $K^1 \in \operatorname{int} A^-_{\leqslant \varepsilon/2, \hat{K}} \cap \operatorname{int} A_{\leqslant \varepsilon/2, \tilde{K}}$, then (6.1) can be brought from $\tilde{K}$ to K^1 in a time $\leqslant \varepsilon/2$ and from K^1 to $\hat{K}$ in a time $\leqslant \varepsilon/2$, hence from $\tilde{K}$ to $\hat{K}$ in a time $\leqslant \varepsilon$. Proposition 10 is proved.

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BIBLIOGRAPHY

1. A. A. Agrachev and R. V. Gamkrelidze, *Exponential representation of flows and the chronological calculus*, Mat. Sb. **107(149)** (1978), 467–532; English transl. in Math. USSR Sb. **35** (1979), no. 6.

2. Hector J. Sussmann and Velimir Jurdjevic, *Controllability of nonlinear systems*, J. Differential Equations **12** (1972), 95–116.

3. R. V. Gamkrelidze, *Principles of optimal control theory*, 2nd ed., Izdat. Tbilis. Univ., Tbilisi, 1977; English transl., Plenum Press, 1978.

4. H. J. Sussmann, *Lie brackets, real analyticity and geometric control*, Differential Geometric Control Theory (Proc. Conf., Houghton, Mich., 1982; R. W. Brockett et al., editors), Birkhäuser, 1983, pp. 1–116.

5. V. I. Arnol'd, *Mathematical methods of classical mechanics*, "Nauka", Moscow, 1974; English transl., Springer-Verlag, 1978.

6. A. A. Agrachev and A. V. Sarychev, *The control of rotation for an asymmetric rigid body*, Problems Control Inform. Theory/ Problemy Upravlen. Teor. Inform. **12** (1983), 335–347.

7. Philip Hartman, *Ordinary differential equations*, Wiley, 1964.

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