On Regularity Properties of Extremal Controls

A. A. Agrachev*

Abstract

We prove some regularity properties of the optimal controls for the smooth bracket generating systems with scalar control parameters, and show that the Cantor sets cannot be the sets of switching points.

Thanks to papers by H.Sussmann we know some regularity properties of optimal controls for general real-analytic systems, see [2],[3]. The same author demonstrated in [2] that optimal controls for general C^∞-systems do not possess any regularity properties. In this note, we show that the situation is not so hopeless for the bracket generating systems and establish a curious property of the sets of switching points, which is new for real-analitic systems too.

Consider a control system

$$\dot{x} = f(x) + ug(x),\ x \in M, |u| \leq 1,$$

where M is a C^∞-manifold, f, g are C^∞-vector fields on M.

Let $\mathrm{Lie}\{f, g\}$ be a Lie sub-algebra of the vector fields generated by f, g, and $L^0(f, g)$ be an ideal in $\mathrm{Lie}\{f, g\}$ generated by g. Suppose that

$$\{v(x) : v \in L^0(f, g)\} = T_x M,\ \forall x \in M. \tag{1}$$

Let $u(t), t \in \mathbf{R}$, be a measurable bounded function. A point $t_0 \in \mathbf{R}$ is called a density point for u if there exists the derivative

$$\frac{d}{dt}\int_{t_0}^{t} u(\tau)d\tau$$

for $t = t_0$.

Denote by D_u the set of all density points for u .

*Steklov Mathematical Institute, 42, ul. Vavilova, Moscow 117966, Russia

Measurable functions

$$u(\cdot) : [\alpha, \beta] \to [-1, 1]$$

satisfying the condition

$$\frac{d}{dt} \int_{\alpha}^{t} u(\tau) d\tau = u(t), \ \forall t \in D_u$$

are called admissible controls on $[\alpha, \beta]$. Solutions of the corresponding differential equations $\dot{x} = f(x) + u(t)g(x)$ are called admissible trajectories. An attainable set $A_{\beta-\alpha}(x_0)$ is, by the definition, the set of all $x(\beta)$ such that $x(\cdot)$ is an admissible trajectory and $x(\alpha) = x_0$.

An admissible control $u(\cdot)$ is called an extremal control if there exists a nonzero solution $p(t) \in T^*_{x(t)}M$ of the (nonstationary) Hamiltonian system on T^*M generated by the Hamiltonian

$$h_{u(t)}(p) = \langle p, f(x) + u(t)g(x) \rangle, \ p \in T^*_x M,$$

such that

$$u(t)\langle p(t), g(x(t)) \rangle = |\langle p(t), g(x(t)) \rangle| \ \forall t \in [\alpha, \beta].$$

It follows from the Pontryagin maximum principle [1] that any admissible control which leads to the boundary of the attainable set is an extremal control. In other words, if $\dot{x}(t) = f(x(t)) + u(t)g(x(t))$, $x(\beta) \in \partial A_{\beta-\alpha}(x(\alpha))$, then $u(\cdot)$ is an extremal control.

Further considerations are based on the following simple lemma.

Lemma 1 *Let $u(\cdot)$ be an admissible control and $p(t)$ be a trajectory of the Hamiltonian system generated by $h_{u(t)}(p)$.*

Suppose that $t_k \to \hat{t}$ is a convergent sequence in the domain of $u(\cdot)$ such that there exists a

$$\lim_{k \to \infty} \frac{1}{t_k - \hat{t}} \int_{\hat{t}}^{t_k} u(\tau) d\tau = \bar{u}.$$

Then there exists

$$\lim_{k \to \infty} \frac{\phi(p(t_k)) - \phi(p(\hat{t}))}{t_k - \hat{t}} = \{h_{\bar{u}}, \phi\}(p(\hat{t})), \ \forall \phi \in C^{\infty}(T^*M),$$

where $\{\cdot, \cdot\}$ are Poisson brackets.

Proof. Indeed,

$$\phi(p(t)) - \phi(p(\hat{t})) = \int_{\hat{t}}^{t} \{h_{u(\tau)}, \phi\}(p(\tau))d\tau =$$

$$\int_{\hat{t}}^{t} \{\langle p, f\rangle, \phi\}(p(\tau))d\tau + \int_{\hat{t}}^{t} u(\tau)\{\langle p, g\rangle, \phi\}(p(\tau))d\tau =$$

$$\int_{\hat{t}}^{t} \{\langle p, f\rangle, \phi\}(p(\tau))d\tau + \int_{\hat{t}}^{t} u(\tau)d\tau\{\langle p, g\rangle, \phi\}(p(t)) + O((t-\hat{t})^2),$$

and the desired result follows from the continuity of $p(\cdot)$.

Proposition 1 *Let u be an extremal control on $[\alpha, \beta]$, then there exists an open dense subset $O \subset [\alpha, \beta]$ such that $u|_O$ is a C^∞-function.*

Sketch of proof. The control $u(t)$ is locally constant on the open set

$$\{t : \langle p(t), g(x(t))\rangle \neq 0\},$$

hence we have to investigate only the interior of the set $\{t : \langle p(t), g(x(t))\rangle = 0\}$.

So, we can suppose, without loss of generality, that

$$\langle p(t), g(x(t))\rangle = 0 \ \forall t \in [\alpha, \beta].$$

Differentiating the last identity with respect to t in virtue of the Hamiltonian system we obtain

$$0 = \langle p(t), [f + u(t)g, g](x(t))\rangle = \langle p(t), [f, g](x(t))\rangle.$$

Differentiating one more time we obtain

$$0 = \langle p(t), [f, [f, g]](x(t))\rangle + \langle p(t), [g, [f, g]](x(t))\rangle u(t).$$

Consider the open subset

$$\{t : \langle p(t), [g, [f, g]](x(t))\rangle \neq 0\} \tag{2}$$

in $[\alpha, \beta]$. We have

$$u(t) = -\frac{\langle p, [f, [f, g]]\rangle}{\langle p, [g, [f, g]]\rangle}$$

while t belongs to (2). Substituting the last expression for t in the Hamiltonian we obtain that $p(t)$ is a solution of the smooth (stationary) Hamiltonian system generated by the Hamiltonian

$$h^1(p) = \langle p, f\rangle - \frac{\langle p, [f,[f,g]]\rangle}{\langle p, [g,[f,g]]\rangle}\langle p, g\rangle$$

for t belonging to (2).

Hence $p(t)$ and $u(t)$ are C^∞ on (2). So we can suppose, without lost of generality, that

$$\langle p(t), [g,[f,g]](x(t))\rangle \equiv \langle p(t), [f,[g,f]](x(t))\rangle \equiv 0.$$

We can continue the differentiation in t. Any time, when we meet an iterated Lie bracket which is not orthogonal to $p(t)$ we can express $u(t)$ as a smooth function of p. Hence $p(t)$ satisfies a smooth Hamiltonian system, so $p(t)$ and $u(t)$ are C^∞ wih respect to t.

On the other hand, if $p(t)$ is orthogonal to all iterated Lie brackets then $p(t) = 0$. (Actually, it is enough to consider a finite number of brackets because of compactness of the set $\{x(t) : t \in [\alpha, \beta]\}$.) $\diamondsuit$

We say that an admissible control $u(\cdot)$ is essentially discontinuous at $\bar{t} \in [\alpha, \beta]$, if $u|_{D_u \cup \{\bar{t}\}}$ is discontinuous at $\bar{t}$. Denote by S_u the set of points of essential discontinuity for u.

An admissible control $u(t)$ is called a bang-bang control if $u(t) \in \{+1, -1\}$ for almost all t. If $u(\cdot)$ is a bang-bang control then S_u is a closed set.

A nowhere dense closed subset in $\mathbf{R}$ is called a perfect subset if its intersection with any interval is empty or uncountable. Example: The Cantor sets are perfect.

A function $v(\cdot)$ is called differentiable at a point t_0 along a subset E if there exists a limit $\frac{v(t)-v(t_0)}{t-t_0}$ as $t \to t_0, t \in E$.

Proposition 2 *Let $u(\cdot)$ be a bang-bang extremal control. Then S_u is not a perfect set.*

The proposition is a corollary of the above lemma 1 and the following

Lemma 2 *Let $u(\cdot)$ be a bang-bang control such that S_u is a perfect set.*

Then there exists a perfect subset $\hat{S} \subset S_u$ such that $v(t) = \int_\alpha^t u(\tau)d\tau$ is nowhere differentiable along $\hat{S}$.

The construction of $\hat{S}$. First of all, $[\alpha,\beta]\backslash S = V_+ \cup V_-$, where V_+ and V_- are open nonempty sets, and $u(t) = 1\ (-1)$ for $t \in V_+\ (V_-)$.

In the following constructions we denote by $|\Delta|$ a length of the interval Δ and by $O_\epsilon\Delta$ a closed ϵ-neighborhood of Δ.

We construct, by the induction on n a family of closed sets $S^n_{i_1\dots i_n}$, where $i_j \in \{0,1\}, n = 1,2....$

Let $\Delta \subset V_+$ be a connected component of V_+.

1) Put $S^1 = S_u \cap O_{|\Delta|}\Delta = S^1_0 \cup S^1_1$, where $t_0 < t_1,\ \forall t_0 \in S^1_0, t_1 \in S^1_1$.

2) Let $\Delta_0 \subset V_- \cap conv(S^1_0), \Delta_1 \subset V_- \cap conv(S^1_1)$ be connected components of V_-. Put

$$S^2_0 = S^1_0 \cap O_{\frac{|\Delta_0|}{2}}\Delta_0 = S^2_{00} \cup S^2_{01};\ S^2_1 = S^1_1 \cap O_{\frac{|\Delta_1|}{2}}\Delta_1 = S^2_{10} \cup S^2_{11};\ S^2 = S^2_0 \cup S^2_1,$$

where $t_{00} < t_{01}, t_{10} < t_{11}\ \forall t_{ij} \in S^2_{ij}$.

. . . .

n+1) Let $\Delta_{i_1\dots i_n} \subset V_{(-)^n} \cap conv(S^n_{i_1\dots i_n})$ be a connected component of $V_{(-)^n}$, $S^{n+1}_{i_1\dots i_n} = S^n_{i_1\dots i_n} \cap O_{\frac{|\Delta_{i_1\dots i_n}|}{2^n}} = S^{n+1}_{i_1\dots i_n 0} \cup S^{n+1}_{i_1\dots i_n 1}$; $S^{n+1} = \cup_{(i_1\dots i_n)} S^{n+1}_{i_1\dots i_n}$, where $t_{i_1\dots i_n 0} < t_{i_1\dots i_n 1}\ \forall t_{i_1\dots i_n j} \in S^{n+1}_{i_1\dots i_n j}$.

Put $\hat{S} = \cap_{k=1}^\infty S^k$.

Proof of Proposition 2. Let $u(\cdot)$ be a bang-bang extremal control and $\hat{S}$ be a set from Lemma 2. Then $\langle p(t), g(x(t))\rangle = 0\ \forall t \in \hat{S}$. Take a subsequence

$$t_k \to t,\ t_k \in \hat{S} \tag{3}$$

and apply Lemma 1 to $\phi = \langle p, g(x)\rangle$. We obtain $\langle p(t), [f,g](x(t))\rangle = 0$. Pick various subsequences of the form (3) and apply lemma 1 to $\phi = \langle p, [f,g](x)\rangle$. We obtain

$$\langle p, [f,[f,g]](x(t))\rangle + u\langle p, [g,[f,g]](x(t))\rangle = 0$$

for, at least two, different values of u. Hence

$$\langle p, [f,[f,g]](x(t))\rangle = \langle p, [g,[f,g]](x(t))\rangle = 0.$$

Apply again lemma 1 to $\phi = \langle p, [f,[f,g]](x)\rangle$ and $\phi = \langle p, [g,[f,g]](x)\rangle$, etc. We obtain that $p(t)$ is orthogonal to $L^0(f,g)$ and hence $p(t) = 0$.

Remark 1 *Proposition 2 is new even in the real-analitic situation, unlike the proposition 1. Recall that the bracket condition (1) is equivalent to the nonemptiness of interior of attainable sets in the real-analitic case.*

References

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