ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2007, Vol. 258, pp. 13–22. © Pleiades Publishing, Ltd., 2007.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2007, Vol. 258, pp. 17–27.

Rolling Balls and Octonions

A. A. Agrachev [a,b]

Received November 2006

Abstract—In this semi-expository paper we disclose hidden symmetries of a classical nonholonomic kinematic model and try to explain the geometric meaning of the basic invariants of vector distributions.

DOI: 10.1134/S0081543807030030

1. INTRODUCTION

The paper is dedicated to Vladimir Igorevich Arnold on the occasion of his 70th birthday. This is just a small mathematical souvenir, but I hope that Vladimir Igorevich will get some pleasure looking it over. The content of the paper is well described by the cryptogram below. Figure 1 represents the root system of the exceptional Lie group G_2 (the automorphism group of octonions) and two circles touching each other whose diameters are in the ratio $3:1$.

Our starting point is a classical nonholonomic kinematic system that is rather important in robotics: a rigid body rolling over a surface without slipping or twisting. The surface is supposed to be the surface of another rigid body, so that the situation is, in fact, symmetric: one body is rolling over another. We also assume that the surfaces of the bodies are smooth and cannot touch each other at more than one point.

This system has a five-dimensional configuration space: the coordinates of the points on each surface at which the surfaces touch each other give four parameters; the fifth parameter measures the mutual orientation of the bodies at the touching point.

Now assume that one of the bodies is immovable and the other rolls along it. It is rather clear that given an initial configuration, one can roll the movable body in a unique way along any curve on the immovable surface starting from the initial touching point. In other words, for a given initial configuration, admissible motions are parameterized by the curves on the two-dimensional surface with a fixed initial point. On the other hand, it is not hard to prove (see [2, Ch. 24]) that admissible motions allow one to reach any configuration from any other provided that two bodies are not congruent. We are thus in a typical nonholonomic situation since the configuration space is five-dimensional.

Now turn to mathematics. Admissible velocities (i.e., the velocities of admissible motions) form a rank 2 vector distribution on the configuration space. This distribution is involutive if and only if our bodies are balls of equal radii. More precisely, let vector fields f and g form a local basis of our distribution, $[f,g]$ be the commutator (Lie bracket) of the fields, and q be a point of the configuration space. It turns out that the vectors $f(q)$, $g(q)$, and $[f,g](q)$ are linearly independent if and only if the curvatures of the two surfaces are not equal at their touching points corresponding to the configuration q. Moreover, if these curvatures are not equal, then

$$f(q) \wedge g(q) \wedge [f,g](q) \wedge [f,[f,g]](q) \wedge [g,[g,f]](q) \neq 0. \tag{1}$$

[a] SISSA/ISAS, via Beirut 4, 34014 Trieste, Italy.
[b] Steklov Institute of Mathematics, Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.
E-mail address: agrachev@sissa.it

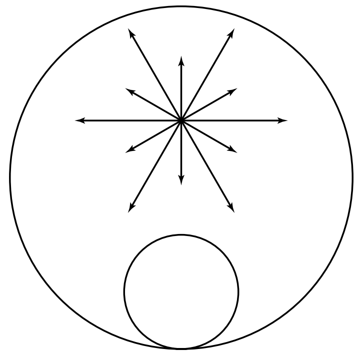

Fig. 1.

In other words, the basis fields and their first- and second-order Lie brackets are all linearly independent. All these calculations are presented in [2, Ch. 24].

The germs of rank 2 distributions in $\mathbb{R}^5$ with property (1) were first studied by Elie Cartan in his famous paper [6]. A simple count of parameters demonstrates that 5 is the lowest dimension where the classification of generic germs of distributions must have functional invariants. This happens for rank 2 and rank 3 distributions in $\mathbb{R}^5$; moreover, the classifications for these two values of the rank are essentially equivalent, and it is sufficient to study the case of rank 2.

Cartan found a fundamental tensor invariant that is a degree 4 symmetric form on the distribution: the desired functional invariant is just the cross-ratio of the roots of this form. There is exactly one equivalence class for which Cartan's form is identically zero. We call the germs from this class *flat*; a germ of the distribution is flat if and only if it admits a basis generating a five-dimensional nilpotent Lie algebra. Cartan showed that the symmetry group of a flat distribution is the real split form of the 14-dimensional exceptional simple Lie group G_2. In all other cases, the dimension of the symmetry group does not exceed 7.

How all this is related to the rolling bodies? It is rather obvious that the group of symmetries of the "no-slipping no-twisting distribution" is trivial for generic bodies. This group acts transitively on the configuration space if and only if the surfaces of both bodies have constant curvatures. Let us restrict ourselves to the case of constant nonnegative curvatures, so that the surfaces are spheres (one of them may be a plane, i.e., a sphere of infinite radius).

Natural symmetries are isometries of the spheres; they form a six-dimensional compact Lie group. It was Robert Bryant who first claimed that the no-slipping no-twisting distribution is flat in the case of spheres whose radii are in the ratio $3:1$. He insisted that he just followed Cartan's method and never published this fact in a paper or a preprint. It also remained absolutely unclear what the nature of hidden additional symmetries is (the dimension of the symmetry group jumps from 6 to 14 when the ratio of the radii becomes $3:1$!). Unfortunately, Cartan's method does not give much in this regard: the construction of the fundamental invariant is based on the involved reduction–prolongation procedures in jet spaces, and any connection with the original geometric problem is lost quite far from the end of the way.

The goal of this note is to finally untwine this puzzle. In Section 2 we give a simple twistor model for the configuration space of the rolling balls problem and for the no-slipping no-twisting distribution; the role of the group G_2 is not yet clear from this model.

The split form of G_2 is the group of automorphisms of *split-octonions*, a hyperbolic version of usual octonions, where a positive definite quadratic form (the square of the norm of octonions) is replaced by a nondegenerate sign-indefinite quadratic form. Split-octonions have nontrivial divisors of zero (the zero locus of the sign-indefinite form). A simple quadratic transformation demonstrates that the no-slipping no-twisting distribution is equivalent to the "divisors-of-zero distribution" of split-octonions if the radii of the balls are in the ratio $3:1$. This is the subject of Section 3.

In Section 4 we outline a recently developed variational way to construct differential invariants of vector distributions in order to put the rolling bodies model in a broader framework and to explain the meaning of the basic invariants.

2. TWISTOR MODEL

We study admissible motions of two balls B_r and B_R rolling one over another without slipping or twisting. Here r and R are the radii of the balls. The instantaneous configuration of the system of two balls is determined by an orientation-preserving isometry of the tangent planes to the spheres $S_r = \partial B_r$ and $S_R = \partial B_R$ at the points where the balls touch each other. In other words, the state space of our kinematic system is

$$M_{R,r} = \bigl\{\mu\colon T_{q_1}S_r \to T_{q_2}S_R \mid q_1 \in S_r,\ q_2 \in S_R, \text{ and } \mu \text{ is an isometry of oriented planes}\bigr\}.$$

It is easy to see that $M_{R,r}$ is a smooth five-dimensional manifold. The motions of the system are families of isometries $\mu(t)\colon T_{q_1(t)}S_r \to T_{q_2(t)}S_R$, $t \in \mathbb{R}$. The no-slipping condition reads

$$\mu(t)(\dot q_1(t)) = \dot q_2(t).$$

The no-twisting condition requires that $\mu(t)$ transforms parallel vector fields along $q_1(t)$ into parallel vector fields along $q_2(t)$.

These two conditions define a rank 2 vector distribution $D^{R,r}$ on $M_{R,r}$. We have $D^{R,r} = \bigcup_{\mu\in M_{R,r}} D^{R,r}_\mu$, where $D^{R,r}_\mu$ is a two-dimensional subspace of $T_\mu M_{R,r}$; the admissible motions of the two-ball system are exactly the integral curves of the distribution $D^{R,r}$. Given an initial configuration, the ball B_r can be rolled in a unique way along any smooth curve on S_R, and the same is true if we interchange r and R. In the formal geometric language this observation just means that the subspace $D^{R,r}_\mu \subset T_\mu M_{R,r}$, where $\mu\colon T_{q_1} \to T_{q_2}$, is projected one-to-one onto $T_{q_1}S_R$ and $T_{q_2}S_r$ $\forall \mu \in M_{R,r}$.

In what follows, we treat S_r and S_R as unit spheres in $\mathbb{R}^3$ with rescaled metrics:

$$M_{R,r} = \bigl\{\mu\colon q_1^\perp \to q_2^\perp \mid q_i \in \mathbb{R}^3,\ |q_i| = 1,\ i = 1,2,\ R|\mu(v)| = r|v|\ \forall v \in q_1^\perp\bigr\}.$$

Let $\rho = \frac{R}{r}$; the homothety $\iota_\rho\colon \mu \mapsto \rho\mu$, $\mu \in M_{R,r}$, transforms $M_{R,r}$ into $M_{1,1}$. We set $D^\rho = \iota_{\rho*}D^{R,r}$. The distribution D^ρ on $M_{1,1}$ is determined by the "rescaled" no-slipping condition

$$\mu(\dot q_1(t)) = \rho\dot q_2(t)$$

and the no-twisting condition; the latter remains unchanged.

From now on we will deal with the fixed space $M_{1,1}$ endowed with the family of distributions D^ρ instead of the family of pairs $(M_{R,r}, D^{R,r})$. In order to explicitly describe the distributions D^ρ, we use a classical quaternion parameterization of the spherical bundle $\mathfrak{p}\colon \mathcal{S} \to S^2$ of the unit sphere $S^2 \subset \mathbb{R}^3$, where

$$\mathcal{S} = \bigl\{(q,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |q| = |v| = 1,\ \langle q,v\rangle = 0\bigr\}, \qquad \mathfrak{p}(q,v) = q.$$

Let us recall this parameterization. We identify $\mathbb{R}^3$ with the space of imaginary quaternions:

$$\mathbb{R}^3 = \{\alpha i + \beta j + \gamma k \mid \alpha,\beta,\gamma \in \mathbb{R}\} \subset \mathbb{H}.$$

Let $S^3 = \{w \in \mathbb{H} \mid |w| = 1\}$ be the group of unitary quaternions; then $\mathfrak{h}\colon S^3 \to S^2$, $\mathfrak{h}(w) = \bar w i w$ is the classical Hopf bundle, and the mapping

$$\Psi\colon S^3 \to \mathcal{S}, \qquad \Psi(w) = (\bar w i w, \bar w j w)$$

is a double covering. Moreover, the diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{\Psi} & \mathcal{S} \\ \downarrow{\scriptstyle \mathfrak{h}} & & \downarrow{\scriptstyle \mathfrak{p}} \\ S^2 & = & S^2 \end{array}$$

is commutative; hence, Ψ is a fiberwise mapping of the bundle $\mathfrak{h}\colon S^3 \xrightarrow{S^1} S^2$ onto the bundle $\mathfrak{p}\colon \mathcal{S} \xrightarrow{S^1} S^2$ which induces a double covering of the fibers. The fibers of the bundle $\mathfrak{h}\colon S^3 \to S^2$ are the residue classes $\{e^{i\theta}w \mid \theta \in \mathbb{R} \bmod 2\pi\}$ of the one-parameter subgroup generated by i.

The vectors orthogonal to the fibers form a distribution

$$\operatorname{span}\{jw, kw\} \subset T_w S^3, \qquad w \in S^3,$$

which is a connection on the principal bundle $\mathfrak{h}\colon S^3 \to S^2$. It is easy to see that Ψ transforms this connection into the Levi-Civita connection on the bundle $\mathfrak{p}\colon \mathcal{S} \to S^2$, which defines a standard parallel translation on S^2.

We are now ready to give a quaternion model of the rolling balls configuration space $M_{1,1}$ (more precisely, of a double covering of $M_{1,1}$) and of the no-slipping no-twisting distributions D^ρ. For any $w_1, w_2 \in S^3$ there exists a unique orientation-preserving isometry of the fiber

$$\mathfrak{h}^{-1}(\mathfrak{h}(w_1)) = \{e^{i\theta}w_1 \mid \theta \in \mathbb{R} \bmod 2\pi\}$$

onto the fiber $\mathfrak{h}^{-1}(\mathfrak{h}(w_2))$ that sends w_1 to w_2. This isometry sends $e^{i\theta}w_1$ to $e^{i\theta}w_2$. Moreover, pairs (w_1, w_2) and (w_1', w_2') define the same isometry if and only if $w_1' = e^{i\theta'}w_1$ and $w_2' = e^{i\theta'}w_2$ (with the same θ'). Hence, the coset space of $S^3 \times S^3$ by the action $(w_1, w_2) \mapsto (e^{i\theta}w_1, e^{i\theta}w_2)$ of the one-parameter group $\{e^{i\theta} \mid \theta \in \mathbb{R} \bmod 2\pi\}$ is a double covering of $M_{1,1}$. We use the symbol $\mathbf{M}$ for this coset space and $\pi\colon S^3 \times S^3 \to \mathbf{M}$ for the canonical projection. By $\mathbf{D}^\rho = \bigcup_{\mathbf{x}\in\mathbf{D}} \mathbf{D}^\rho_{\mathbf{x}}$ we denote the rank 2 distribution on $\mathbf{M}$ that is the pullback of D^ρ under the double covering $\mathbf{M} \to M_{1,1}$. Then

$$\mathbf{D}^\rho_{\pi(w_1,w_2)} = \pi_* \operatorname{span}\{(jw_1, \rho j w_2), (kw_1, \rho k w_2)\} \qquad \forall w_1, w_2 \in S^3.$$

Let us now treat the quaternionic space $\mathbb{H}^2 = \{(w_1, w_2) \mid w_i \in \mathbb{H}\}$ as $\mathbb{C}^4$, where $w_1 = z_1 + z_2 j$ and $w_2 = z_3 + z_4 j$, $z_l \in \mathbb{C}$, $l = 1, \dots, 4$. We see that $\mathbf{M}$ is nothing else but a complex projective conic,

$$\mathbf{M} = \big\{z_1 : z_2 : z_3 : z_4 \ \big|\ |z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2\big\} \subset \mathbb{C}\mathrm{P}^3.$$

This conic is often called the "space of isotropic twistors." Moreover,

$$\mathbf{D}^\rho_{\pi(w_1,w_2)} = \pi_* \mathbb{C} j (w_1, \rho w_2).$$

3. SPLIT-OCTONIONS

We also treat $\mathbb{H}^2$ as the algebra $\widehat{\mathbb{O}} = \{w_1 + \ell w_2 \mid w_i \in \mathbb{H}\}$ of split-octonions, where

$$(a + \ell b)(c + \ell d) = (ac + d\bar{b}) + \ell(\bar{a}d + cb). \tag{2}$$

Let $x = w_1 + \ell w_2$, $\bar{x} = \bar{w}_1 - \ell \bar{w}_2$, and $Q(x) = \bar{x}x = |w_1|^2 - |w_2|^2$. Then $Q(xy) = Q(x)Q(y)$ and $x^{-1} = \frac{\bar{x}}{Q(x)}$ as soon as $Q(x) \neq 0$; the cone $Q^{-1}(0)$ consists of the divisors of zero.

The automorphism group of the algebra $\widehat{\mathbb{O}}$ is the split form of the exceptional Lie group G_2 [10]. These automorphisms preserve the quadratic form Q and hence its polarization $\mathbf{Q}(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y))$. In particular, these automorphisms preserve the subspace $\mathbb{R}^7 = \{x \in \widehat{\mathbb{O}} \mid \mathbf{Q}(1,x) = 0\}$ and the conic

$$K = \{x \in \mathbb{R}^7 \mid Q(x) = 0\} = \{x \in \widehat{\mathbb{O}} \mid xx = 0\}.$$

Moreover, the automorphism group of $\widehat{\mathbb{O}}$ acts transitively on the "spherization" $\mathbf{K} = \{\mathbb{R}_+ x \mid x \in K \setminus 0\} = S^2 \times S^3$ of the cone K.

With any $x \in K \setminus 0$ there is associated a three-dimensional subspace of divisors of zero,

$$\Delta_x = \{y \in \mathbb{R}^7 \mid xy = 0\};$$

the "spherization" turns Δ_x into a two-dimensional subspace $\mathbf{\Delta_x} \subset T_{\mathbf{x}}\mathbf{K}$, where $\mathbf{x} = \mathbb{R}_+ x$. Obviously, the automorphism group of $\widehat{\mathbb{O}}$ preserves the vector distribution $\mathbf{\Delta} = \{\mathbf{\Delta_x}\}_{\mathbf{x}\in\mathbf{K}}$.

Proposition 1. *The mapping* $\Phi\colon (w_1+\ell w_2) \mapsto (w_1^{-1} i w_1 + \ell(w_1^{-1} w_2))$ *induces a diffeomorphism of* $\mathbf{M}$ *onto* $\mathbf{K}$. *Moreover, the differential of this diffeomorphism transforms the no-slipping no-twisting distribution* $\mathbf{D}^3$ *into the divisors-of-zero distribution* $\mathbf{\Delta}$.

Proof. Let $\widehat{\Phi}\colon \mathbf{M} \to \mathbf{K}$ be the mapping induced by Φ. We give an explicit formula for $\widehat{\Phi}^{-1}$: take $v_1 \in S^2$, $v_1 = \mathfrak{h}(w_1)$; then

$$\widehat{\Phi}^{-1}(v_1 + \ell v_2) = \pi(w_1 + \ell(w_1 v_2)) \qquad \forall v_2 \in S^3.$$

Now we must prove that $\Phi(x)(D_x\Phi y) = 0$ for any $x = w_1 + \ell w_2$ and $y = zjw_1 + 3\ell(zjw_2)$ such that $|w_1| = |w_2|$ and $z \in \mathbb{C}$. It is sufficient to make calculation in the case of $|w_1| = |w_2| = 1$. We have

$$D_{w_1+\ell w_2}\Phi(zjw_1 + \ell(zjw_2)) = 2\bar{w}_1 zkw_1 + 2\ell(\bar{w}_1 zjw_2).$$

The desired result now follows from the multiplication rule (2).

Let us give an explicit parameterization of the distribution $\mathbf{\Delta}$ on $\mathbf{K}$. First, we parameterize $\mathbf{K}$ itself:

$$\mathbf{K} = \{v_1 + \ell v_2 \mid v_1 \in \mathbb{R}^3,\ v_2 \in \mathbb{H},\ |v_1| = |v_2| = 1\}.$$

Then $\mathbf{\Delta}_{v_1+\ell v_2} = \{v_1 u + \ell(uv_2) \mid u, (v_1 u) \in \mathbb{R}^3\}$; this is a simple corollary to the multiplication rule (2). Let $v \in \mathbb{R}^3$ with $|v| = 1$; the mapping $w \mapsto w + vwv$ maps $\mathbb{H}$ onto the subspace $\{u \mid u, (vu) \in \mathbb{R}^3\}$. Now we replace u by $w + v_1 w v_1$ in the above description of $\mathbf{\Delta}_{v_1+\ell v_2}$ and obtain the final parameterization

$$\mathbf{\Delta}_{v_1+\ell v_2} = \{[v_1, w] + \ell((w + v_1 w v_1)v_2) \mid w \in \mathbb{H}\}.$$

4. JACOBI CURVES

In this section we briefly describe the variational approach to differential invariants of vector distributions (see [1, 3, 11]) in order to put the rolling balls model in a wider perspective. This approach is based on the contemporary optimal control techniques and suggests an alternative to the classical equivalence method (see [6–8]) of Elie Cartan.

A rank k vector distribution Δ on the n-dimensional smooth manifold M is just a smooth vector subbundle of the tangent bundle TM:

$$\Delta = \bigcup_{q\in M} \Delta_q, \qquad \Delta_q \subset T_q M, \qquad \dim \Delta_q = k.$$

Distributions Δ and Δ' are called locally equivalent at $q_0 \in M$ if there exists a neighborhood $O_{q_0} \subset M$ of q_0 and a diffeomorphism $\Phi\colon O_{q_0} \to O_{q_0}$ such that $\Phi_*\Delta_q = \Delta'_{\Phi(q)}$ $\forall q \in O_{q_0}$.

A local basis of Δ is a k-tuple of smooth vector fields $f_1, \dots, f_k \in \operatorname{Vec} M$ such that

$$\Delta_q = \operatorname{span}\{f_1(q), \dots, f_k(q)\}, \qquad q \in O_{q_0}.$$

Given a local basis, one may compute the *flag of the distribution*:

$$\Delta^l_q = \operatorname{span}\bigl\{(\operatorname{ad} f_{i_j} \dots \operatorname{ad} f_{i_1} f_{i_0})(q) \ \big|\ 0 \le j < l\bigr\}, \qquad l = 1, 2, \dots,$$

where $\operatorname{ad} f g \stackrel{\text{def}}{=} [f, g]$ is the Lie bracket.

It is easy to see that the subspaces Δ^l_q do not depend on the local basis. We set $\Delta^l = \bigcup_{q\in M} \Delta^l_q$, a growing sequence of subsets in TM. This sequence stabilizes as soon as $\Delta^{l+1} = \Delta^l$. The distribution is involutive if and only if $\Delta^2 = \Delta$ and is completely nonholonomic (our subject) if $\Delta^l = TM$ for sufficiently large l. Generic distributions are, of course, completely nonholonomic.

The integral curves of a distribution are often called *horizontal paths*. It is convenient to consider all paths of class H^1 rather than only smooth ones. We thus have a Hilbert manifold Ω_Δ of horizontal paths:

$$\Omega_\Delta = \bigl\{\gamma \in H^1([0,1]; M) \ \big|\ \dot\gamma(t) \in \Delta_{\gamma(t)} \text{ for a.e. } t \in [0,1]\bigr\}.$$

Now consider *boundary mappings*

$$\partial_t\colon \Omega_\Delta \to M \times M$$

defined by the formula $\partial_t(\gamma) = (\gamma(0), \gamma(t))$. It is easy to show that ∂_t are smooth mappings.

The critical points of the mapping ∂_1 are called *singular curves* of Δ. Any singular curve is automatically a critical point of ∂_t $\forall t \in [0,1]$. Moreover, any singular curve possesses a *singular extremal*, i.e., an H^1-curve $\lambda\colon [0,1] \to T^*M$ in the cotangent bundle to M such that

$$\lambda(t) \in T^*_{\gamma(t)}M \setminus \{0\}, \qquad (\lambda(t), -\lambda(0))D_\gamma\partial_t = 0 \qquad \forall t \in [0,1].$$

We set

$$\Delta^\perp_q = \{\nu \in T^*_qM \ |\ \langle \nu, \Delta_q\rangle = 0,\ \nu \ne 0\}, \qquad \Delta^\perp = \bigcup_{q\in M} \Delta^\perp_q.$$

Obviously, $\Delta^\perp$ is a smooth $(n+k)$-dimensional submanifold of T^*M (the annihilator of Δ).

Let σ be the canonical symplectic structure on T^*M. The Pontryagin maximum principle implies that a curve λ in T^*M is a singular extremal if and only if it is a characteristic curve of the form $\sigma|_{\Delta^\perp}$; in other words,

$$\lambda(t) \in \Delta^\perp, \qquad \dot\lambda(t) \in \ker\bigl(\sigma|_{\Delta^\perp}\bigr), \qquad 0 \le t \le 1.$$

All singular extremals are contained in the *characteristic variety*

$$C_\Delta = \bigl\{z \in \Delta^\perp \ \big|\ \ker \sigma_z|_{\Delta^\perp} \ne 0\bigr\}.$$

We have $C_\Delta = \Delta^{2\perp}$ if $k = 2$ and $C_\Delta = \Delta^\perp$ if k is odd; typically, C_Δ is a codimension 1 submanifold of $\Delta^\perp$ if k is even.

A complete description of singular extremals is a difficult task; to simplify the job, we focus only on the regular part of the characteristic variety. We set

$$C^0_\Delta = \bigl\{z \in C_\Delta \ \big|\ \dim\ker\sigma_z|_{\Delta^\perp} \le 2,\ \dim\bigl(\ker\sigma_z|_{\Delta^\perp} \cap T_zC_\Delta\bigr) = 1\bigr\}$$

(see Fig. 2). If $k = 2$, then $C^0_\Delta = \Delta^{2\perp} \setminus \Delta^{3\perp}$.

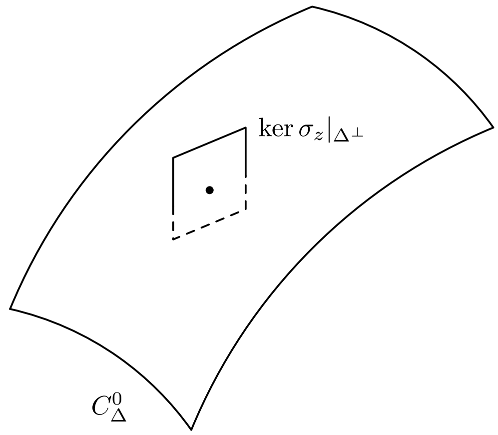

Fig. 2.

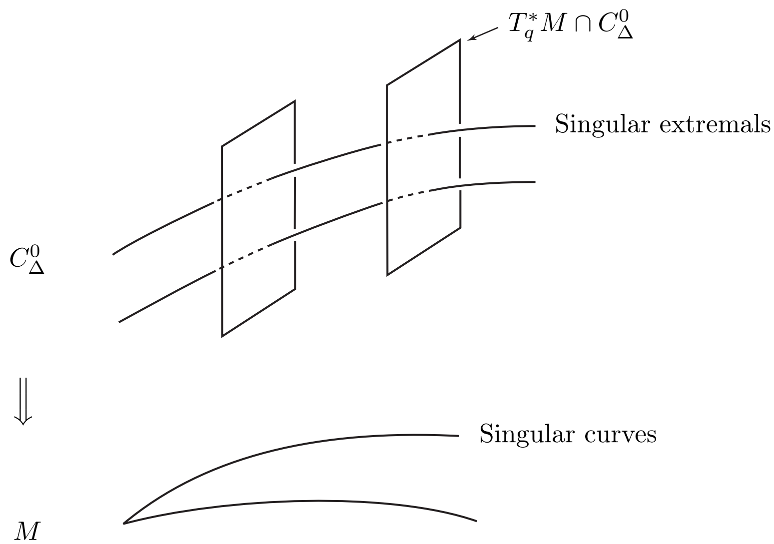

Fig. 3.

Note that C^0_Δ is a smooth submanifold of $\Delta^\perp$; it is foliated by singular extremals and by the fibers $T^*_qM \cap C^0_\Delta$.

A motion along singular extremals defines a local flow on C^0_Δ; typically, this flow is not fiberwise, i.e., it does not transform fibers into fibers (see Fig. 3).

Given $z \in C^0_\Delta$ and an appropriate small neighborhood $\mathcal{C}^0_z$ of z in C^0_Δ, we consider the canonical projection

$$F\colon \mathcal{C}^0_z \to \mathcal{C}^0_z/\{\text{Singular extremals foliation}\}$$

of $\mathcal{C}^0_z$ onto the space of singular extremals contained in $\mathcal{C}^0_z$.

Assume that λ is a singular extremal that passes through z and is associated with a singular curve γ; i.e., $\lambda(0) = z$ and $\lambda(t) \in T^*_{\gamma(t)}M$. Consider a family of subspaces

$$J^0_\lambda(t) = T_\lambda F\big(T^*_{\gamma(t)}M \cap \mathcal{C}^0_z\big)$$

of the space

$$T_\lambda\mathcal{C}^0_z/\{\text{Singular extremals foliation}\} \cong T_zC^0_\Delta/T_z\lambda.$$

Then $t \mapsto J^0_\lambda(t)$ is a smooth curve in the corresponding Grassmann manifold. The geometry of the curves $t \mapsto J^0_\lambda(\cdot)$ reflects the dynamics of the fibers along singular extremals and contains important information about the distribution Δ.

In what follows we assume that $k = 2$, $n \geq 5$, and Δ^2_q and Δ^3_q have maximum possible dimensions, i.e., $\dim \Delta^2_q = 3$ and $\dim \Delta^3_q = 5$.

I. First we consider the case $n = 5$, which was studied by Cartan (see the Introduction). Let $z \in T^*_q M$ and $\pi\colon T_z(T^*M) \to T_q M$ be the differential at z of the projection $T^*M \to M$; then $\pi(J^0_\lambda(t)) \subset z^\perp \subset T_q M$. Moreover, $T_q\gamma \subset \pi(J^0_\lambda(t))$ and $t \mapsto \pi(J^0_\lambda(t))$ is a curve in the projective plane $\mathrm{P}(z^\perp/T_q\gamma)$.

Proposition 2 (see [4])**.** *A rank 2 distribution Δ on the five-dimensional manifold is flat if and only if the curve $\pi(J^0_\lambda(\cdot))$ is a quadric for any singular extremal λ.*

In general, $\pi(J^0_\lambda(\cdot))$ is not a quadric. Let $q = \gamma(0)$ and $K_z(q) \subset z^\perp$ be the best approximating quadric for the curve $\pi(J^0_\lambda(\cdot))$ near the point corresponding to the zero value of the parameter t (the *osculating quadric* of classical projective geometry); then $K_z(q)$ is the zero locus of a signature $(2,1)$ quadratic form on $z^\perp/T_q\gamma$. We can, of course, treat $K_z(q)$ as the zero locus of a degenerate quadratic form on $z^\perp$. Finally, $\mathcal{K}(q) = \bigcup_{z \in \Delta^{2\perp}_q} K_z(q)$ is the zero locus of a $(3,2)$ quadratic form on $T_q M$ (see [4] for details).

The family of quadratic cones $\mathcal{K}(q)$, $q \in M$, is a conformal structure on M intrinsically "raised" from Δ, and $\Delta_q \subset K(q)$. This conformal structure was first found by Nurowski [9], who applied Cartan's equivalence method.

Remark. Nurowski's conformal structure has a particularly simple description for the divisors-of-zero distribution $\boldsymbol{\Delta}$ from Section 3. Namely, $\mathcal{K}(\mathbf{x}) = Q^{-1}(0) \cap T_{\mathbf{x}}\mathbf{K}$, $\mathbf{x} \in \mathbf{K}$, in this case.

II. From now on n is any integer greater than or equal to 5. Let $z \in C^0_\Delta$, λ be the singular extremal through z, and γ be the corresponding singular curve. We set

$$J_\lambda(t) = D_\lambda F\bigl(\pi^{-1}\Delta_{\gamma(t)}\bigr) \subset T_z C^0_\Delta / T_z\lambda.$$

Then $J_\lambda(t) \supset J^0_\lambda(t)$ and $J_\lambda(t)$ is a Lagrangian subspace of the symplectic space $T_z C^0_\Delta/T_z\lambda$. In other words, $J_\lambda(t)^\angle = J_\lambda(t)$, where

$$\mathcal{S}^\angle \stackrel{\text{def}}{=} \bigl\{\zeta \in T_z C^0_\Delta \colon\ \sigma(\zeta, \mathcal{S}) = 0\bigr\}, \qquad \mathcal{S} \subset T_z.$$

If $s \in \mathbb{R} \setminus \{0\}$, then $s\lambda$ is a singular extremal through $sz \in C^0_\Delta$. Hence, $T_z(\mathbb{R}z) \subset J_\lambda(t)\ \forall t$ and $J_\lambda(t) \subset T_z(\mathbb{R}z)^\angle$. This inclusion allows one to make another useful reduction. Namely, we set $\Sigma_z = T_z(\mathbb{R}z)^\angle / T_z\mathbb{R}z$, a symplectic space of dimension $2(n-3)$; then $J_\lambda(t)$ is a Lagrangian subspace of Σ_z.

Let $L(\Sigma_z)$ be the Lagrange Grassmannian, i.e., the manifold formed by all Lagrangian subspaces of Σ_z. The curve $t \mapsto J_\lambda(t)$ considered as a curve in $L(\Sigma_z)$ is called the *Jacobi curve* associated with the extremal λ.

The dimension of a Lagrangian subspace is one-half of the dimension of the ambient symplectic space. In particular, a generic pair of Lagrangian subspaces has zero intersection. Jacobi curves are not at all generic; nevertheless, under very mild regularity assumption on the distribution (see [11]), they satisfy the following important property: $J_\lambda(t) \cap J_\lambda(\tau) = 0$ for sufficiently small $|t - \tau| \neq 0$.

Let $\pi_{t\tau}$ be the linear projector of Σ_z onto $J_\lambda(\tau)$ along $J_\lambda(t)$. In other words, $\pi_{t\tau}\colon \Sigma_z \to \Sigma_z$ and

$$\pi_{t\tau}\big|_{J_\lambda(t)} = 0, \qquad \pi_{t\tau}\big|_{J_\lambda(\tau)} = \mathbf{1}.$$

Lemma 1 (see [3]). *We have*

$$\operatorname{tr}\left(\frac{\partial^2 \pi_{t\tau}}{\partial t\,\partial\tau}\bigg|_{J_\lambda(\tau)}\right) = \frac{(n-3)^2}{(t-\tau)^2} + g_\lambda(t,\tau),$$

where $g_\lambda(t,\tau)$ is a symmetric function of (t,τ) that is smooth in a neighborhood of (t,t) for all t outside a discrete subset of the domain of $J_\lambda(\cdot)$.

In what follows, we tacitly assume that the value of t is taken outside the discrete subset mentioned in the lemma. A basic invariant of the parameterized singular extremal λ is the *generalized Ricci curvature*

$$\mathfrak{r}_\lambda(\lambda(t)) \stackrel{\text{def}}{=} g_\lambda(t,t).$$

The generalized Ricci curvature depends on the parameterization of the extremal; this dependence is controlled by the following chain rule. Let $\varphi\colon \mathbb{R}\to\mathbb{R}$ be a change of the parameter; then

$$\mathfrak{r}_{\lambda\circ\varphi}(\lambda(\varphi(t))) = \mathfrak{r}_\lambda(\lambda(\varphi(t)))\dot\varphi^2(t) + (n-3)^2\mathbb{S}(\varphi),$$

where $\mathbb{S}(\varphi) = \frac{\dddot\varphi(t)}{2\dot\varphi(t)} - \frac{3}{4}\big(\frac{\ddot\varphi(t)}{\dot\varphi(t)}\big)^2$ is the Schwarzian derivative. The chain rule implies that the generalized Ricci curvature $\mathfrak{r}_\lambda$ can always be made zero by a local reparameterization of the extremal λ. We say that a local parameter t is a *projective parameter* if $\mathfrak{r}_\lambda(t)\equiv 0$; such a parameter is defined up to a Möbius transformation.

Let t be a projective parameter; then the quantity

$$A(\lambda(t)) = \frac{\partial^2 g}{\partial\tau^2}(t,\tau)\bigg|_{\tau=t}(dt)^4$$

is a well-defined degree 4 differential on λ; we call it the *fundamental form* on λ.

Under an arbitrary, not necessarily projective, parameterization, the fundamental form has the following expression:

$$A(\lambda(t)) = \left(\frac{\partial^2 g}{\partial\tau^2}\bigg|_{\tau=t} - \frac{3}{5(n-3)^2}\mathfrak{r}_\lambda(t)^2 - \frac{3}{2}\ddot{\mathfrak{r}}_\lambda(t)\right)(dt)^4.$$

Assume that $A(\lambda(t))\neq 0$; then the identity $\big|A(\lambda(s))\big(\frac{d}{ds}\big)\big| = 1$ defines a unique (up to a translation) *normal parameter* s.

Let $z\in C^0_\Delta$ and λ_s be the normally parameterized singular extremal through z. We set

$$\bar{\mathfrak{r}}(z) = \mathfrak{r}_{\lambda_s}(z).$$

The function $z\mapsto\bar{\mathfrak{r}}(z)$ defined on C^0_Δ depends only on Δ and is called the *projective generalized Ricci curvature.*

Now come back to the case of $k=2$ and $n=5$. In this case, the fundamental form A is reduced to the famous Cartan's degree-four form on the distribution constructed in [6] by the method of equivalence. The distribution is flat if and only if $A\equiv 0$.

Zelenko [11] performed detailed calculations for the rolling balls model. As before, let ρ be the ratio of the radii of the balls. We assume that $1<\rho\le+\infty$. It turns out that

$$\operatorname{sgn}(A) = \operatorname{sgn}(\rho-3).$$

Singular curves are just rolling motions along geodesics (i.e., along great circles). The symmetry group acts transitively on the space of geodesics; therefore, the function $\mathfrak{r}$ must be constant in this case. We have

$$\bar{\mathfrak{r}} = \frac{4\sqrt{35}(\rho^2+1)}{3\sqrt{(\rho^2-9)(9\rho^2-1)}}.$$

In particular, the distributions corresponding to different ρ are mutually nonequivalent, and only the distribution corresponding to $\rho = 3$ is flat.

ACKNOWLEDGMENTS

I am grateful to Robert Bryant, Aroldo Kaplan, and Igor Zelenko for very interesting stimulating discussions.

REFERENCES

1. A. A. Agrachev, "Feedback-Invariant Optimal Control Theory and Differential Geometry. II: Jacobi Curves for Singular Extremals," J. Dyn. Control Syst. **4** (4), 583–604 (1998).
2. A. A. Agrachev and Yu. L. Sachkov, *Control Theory from the Geometric Viewpoint* (Fizmatlit, Moscow, 2004; Springer, Berlin, 2004).
3. A. A. Agrachev and I. Zelenko, "Geometry of Jacobi Curves. I, II," J. Dyn. Control Syst. **8** (1), 93–140 (2002); **8** (2), 167–215 (2002).
4. A. Agrachev and I. Zelenko, "Nurowski's Conformal Structures for (2, 5)-Distributions via Dynamics of Abnormal Extremals," in *Developments of Cartan Geometry and Related Mathematical Problems: Proc. RIMS Symp., Kyoto, 2005* (Kyoto Univ., Kyoto, 2006), pp. 204–218.
5. R. Bryant and L. Hsu, "Rigidity of Integral Curves of Rank 2 Distributions," Invent. Math. **114** (2), 435–461 (1993).
6. E. Cartan, "Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre," Ann. Sci. Ec. Norm. Super., Sér. 3, **27**, 109–192 (1910).
7. R. B. Gardner, *The Method of Equivalence and Its Applications* (SIAM, Philadelphia, PA, 1989).
8. R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications* (Am. Math. Soc., Providence, RI, 2002).
9. P. Nurowski, "Differential Equations and Conformal Structures," J. Geom. Phys. **55**, 19–49 (2005).
10. T. A. Springer and F. D. Veldkamp, *Octonions, Jordan Algebras and Exceptional Groups* (Springer, Berlin, 2000).
11. I. Zelenko, "On Variational Approach to Differential Invariants of Rank Two Distributions," Diff. Geom. Appl. **24** (3), 235–259 (2006).

Translated by the author