Geometric Control and Geometry of Vector Distributions

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Rank k vector distribution Δ on the n-dimensional smooth manifold M is a vector subbundle of the tangent bundle TM :

$$\Delta = \{\Delta_q\}_{q \in M}, \ \ \Delta_q \subset T_q M, \ \ \dim \Delta_q = k.$$

Distributions Δ and Δ' are called locally equivalent at q_0 if $\exists$ a neighborhood O_{q_0} and a local diffeomorphism $\Phi : O_{q_0} \to O_{q_0}$ such that $\Phi_* \Delta_q = \Delta'_{\Phi(q)}, \ \forall q \in O_{q_0}$.

Horizontal paths: $t \mapsto q(t), \ \dot{q}(t) \in \Delta_{q(t)}$.

Local bases: $f_1, \ldots, f_k \in \mathrm{Vec}M$,

$$\Delta_q = span\{f_1(q), \ldots, f_k(q)\}, \ \ q \in O_{q_0}.$$

Horizontal paths are admissible trajectories of the control system: $\dot{q} = \sum_{i=1}^{k} u_i f_i(q)$.

Let $\Delta'_q = span\{f'_1(q), \dots, f'_k(q)\}$. We have $\Phi_* \Delta_q = \Delta'_{\Phi(q)}$ iff $\Phi_* f_i = \sum_{j=1}^{k} a_{ij} f'_j$, where $a_{ij} \in C^\infty(O_{q_0})$,

$$\det \begin{pmatrix} a_{11}(q) & \dots & a_{1n}(q) \\ \dots & \dots & \dots \\ a_{n1}(q) & \dots & a_{n1}(q) \end{pmatrix} \neq 0.$$

In other words, the distributions are equivalent iff the control systems are equivalent by the feedback and state transformations.

Flag of the distribution:

$$\Delta^l_q = span \left\{ (\mathrm{ad} f_{i_j} \cdots \mathrm{ad} f_{i_1} f_{i_0})(q) : 0 \le j < l \right\},$$

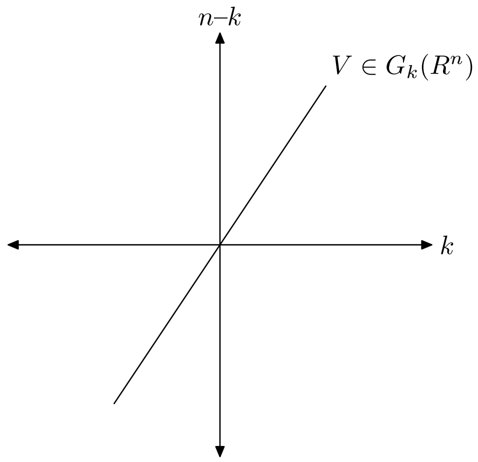
where $\mathrm{ad} f\, g \stackrel{def}{=} [f, g]$ is the Lie bracket.

Subspaces Δ^l_q do not depend on the basis of Δ since $\mathrm{ad} f(ag) = a \mathrm{ad} f\, g + (fa)g$ but the structure of the generated by $f_1, \dots, f_k$ Lie subalgebra of $\mathrm{Vec} M$ essentially depends on the basis.

Local parameterization of the space of distributions: $M \approx \mathbb{R}^n$, $TM \approx \mathbb{R}^n \times \mathbb{R}^n$,

$$\Delta : \mathbb{R}^n \to G_k(\mathbb{R}^n),$$

where $G_k(\mathbb{R}^n)$ is the Grassmann manifold of k-dim. subspaces of $\mathbb{R}^n$. Recall that $G_k(\mathbb{R}^n)$ is a smooth $k(n-k)$-dim. manifold. Indeed, all k-dim. subspaces that are transversal to a fixed $(n-k)$-dim. subspace can be identified with graphs of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$ (i. e. with $k \times (n-k)$ -matrices) and form a coordinate chart of $G_k(\mathbb{R}^n)$:



The space of rank k distributions is thus locally parameterized by $C^\infty(\mathbb{R}^n; \mathbb{R}^{k(n-k)})$.

On the other hand, local diffeomorphisms of $\mathbb{R}^n$ form an open subset of $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$. A smooth change of coordinates allows to normalize no more than n of $k(n-k)$ functions. The space of equivalence classes should be at least as "massive" as $C^\infty(\mathbb{R}^n;\mathbb{R}^{k(n-k)-n})$.

1. $k(n-k) \leq n$, i. e. $k = 1$ or $k = n-1$ or $n = 4, k = 2$. Generic distributions can be completely normalized:

   $k = 1$ –rectification of vector fields;

   $k = n-1$ –Darboux normal forms for differential 1-forms;

   $n = 4, k = 2$ –Engel structure.

2. $k(n-k) > n$. Any classification of generic distributions must contain "functional parameters".

First nontrivial case: $n = 5,\ k = 2 \ \vee\ 3.$

Theorem. Let $\mathcal{D}_k(\mathbb{R}^n)$ be the space of germs of k-distributions in $\mathbb{R}^n$. If $k(n-k) > n$, then $\exists$ a residual subset $\mathcal{U} \subset \mathcal{D}_k(\mathbb{R}^n)$ s. t. no one distribution from $\mathcal{U}$ possesses a basis generating a finite dimensional Lie algebra.

Main steps of the proof:

1. If two distributions possess bases which generate finite dimensional Lie algebras and have equal bracket relations, then the distributions are locally equivalent.

2. Take a Hall basis of Lie polynomials in k indeterminates and consider the set of all multiplication tables of Lie algebras additively generated by first m elements of this basis. The set of pairs:

   $$\langle \text{multipl. table}, \quad \text{codim. } n \text{ Lie subalgebra} \rangle$$

forms a semi-algebraic subset of the appropriate vector space. Each pair generates a germ of a k-tuple of vector fields in $\mathbb{R}^n$. Moreover, $\forall N > 0$ the set of N-jets of these germs is a semi-algebraic subset of the space of N-jets and dimension of this subset does not depend on N.

3. The group of N-jets of diffeomorphisms acts on the space of jets of distributions and codimension of the orbits of this action tends to ∞ as $N \longrightarrow \infty$.

Looking for invariants

The growth vector:

$$(\dim \Delta_q,\ \dim \Delta_q^2,\ \dim \Delta_q^3, \ldots).$$

We mainly study distributions with maximal growth vector (generic case). If $k = 2$, then maximal growth is: $(2, 3, 5, 8, \ldots)$; in general: $(k,\ k(k+1)/2,\ k(k+1)(2k+1)/6, \ldots)$.

If $k(n-k) \leq n,\ k > 1$, then any maximal growth vector distribution possesses a basis generating the nilpotent n-dimensional Lie algebra. This is not true, if $k(n-k) > n$.

Natural questions:

- Equivalence problem for the maximal growth vector distributions: Given two distributions, how to check are they locally equivalent or not?

- How to characterize the distributions which possess bases generating the n-dim. nilpotent Lie algebra?

- Is there a chance to make effective the above theorem?

Cartan equivalence method, in principle, provides the answer to first two questions for the following values of (k, n): $(2, 5)$ (E. Cartan), $(3, 6)$ (R. Bryant), and $(4, 7)$ (R. Montgomery).

"Optimal control" approach

The space of horizontal paths:

$$\Omega_\Delta = \{\gamma : [0,1] \to M : \dot{\gamma}(t) \in \Delta_{\gamma(t)},\ 0 \leq t \leq 1\},$$

$\Omega_\Delta \subset H^1([0,1]; M)$. Boundary mappings:

$$\partial_t : \gamma \mapsto (\gamma(0), \gamma(t)) \in M \times M.$$

Critical points of $\partial_1\big|_{\Omega_\Delta}$ are *singular curves* of Δ. Any singular curve is a critical point of $\partial_t\ \forall t \in [0,1]$.

Moreover, any singular curve possesses a *singular extremal*, i. e. a curve $\lambda : [0,1] \to T^*M$ in the cotangent bundle to M s. t. $\lambda(t) \in T^*_{\gamma(t)}M$,

$$(\lambda(t), -\lambda(0))D_\gamma \partial_t = 0, \ \forall t \in [0,t].$$

We set:

$$\Delta_q^\perp = \{\nu \in T_q^*M : \langle \nu, \Delta_q \rangle = 0, \nu \neq 0\},$$

$$\Delta^\perp = \bigcup_{q \in M} \Delta_q^\perp.$$

Let σ be the canonical symplectic structure on T^*M. The PMP implies: A curve λ in T^*M is a singular extremal iff it is a characteristics of the form $\sigma\big|_{\Delta^\perp}$; in other words,

$$\dot{\lambda}(t) \in \ker\left(\sigma\big|_{\Delta^\perp}\right), \quad 0 \le t \le 1.$$

Characteristic variety:

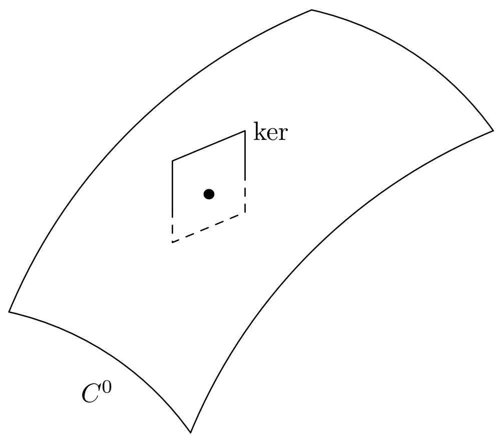
$$C_\Delta = \left\{ z \in \Delta^\perp : \ker \sigma_z\big|_{\Delta^\perp} \neq 0 \right\}.$$

We have: $C_\Delta = \Delta^{2\perp}$ if $k=2$; $C_\Delta = \Delta^{\perp}$ if k is odd; typically, C_Δ is a codim 1 submanifold of Δ if k is even.

Regular part of the characteristic variety:

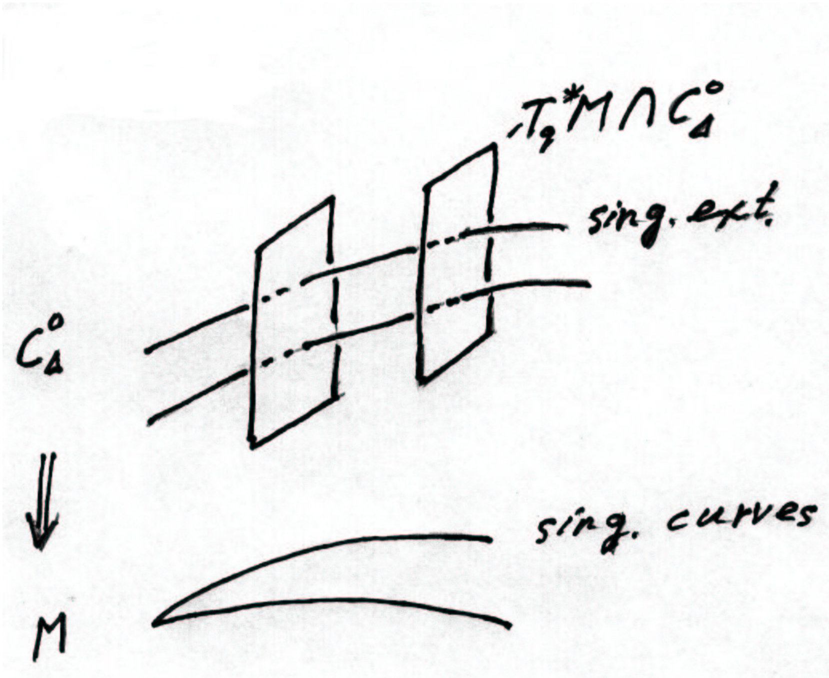
$$C^0_\Delta = \Big\{ z \in C_\Delta : \dim \ker \sigma_z\Big|_{\Delta^\perp} \le 2,$$

$$\dim \ker \sigma_z\Big|_{\Delta^\perp} \cap T_z C_\Delta = 1 \Big\}.$$



If $k=2$, then $C^0_\Delta = \Delta^{2\perp} \setminus \Delta^{3\perp}$.

Submanifold C^0_Δ is foliated by singular extremals and by the fibers $T^*_q M \cap C^0_\Delta$.



The movement along singular extremals is not fiber-wise!

Canonical projection:

$$F : C^0 \to C^0_\Delta / \{\text{sing. ext. foliation}\}.$$

Let λ be a sing. extremal associated to a sing. curve γ. Consider a family of subspaces

$$J^0_\lambda(t) = T_\lambda F(T^*_{\gamma(t)} M \cap C^0_\Delta)$$

of the space

$$T_\lambda C^0_\Delta/\{\text{sing. ext. foliation}\} \cong T_{\lambda(0)}C^0_\Delta/T_{\lambda(0)}\lambda.$$

Then $t \mapsto J^0_\lambda(t)$ is a curve in the Grassmannian. Geometry of these curves reflects the dynamics of the fibers along sing. extremals and contains the fundamental information about distribution Δ.

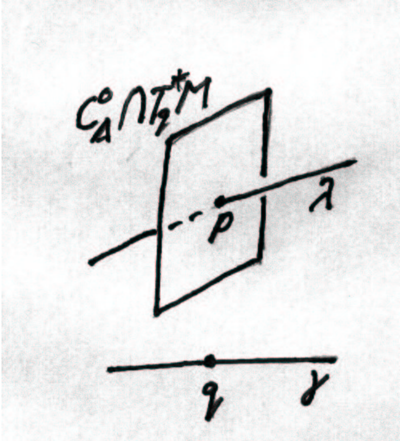
Let $n = 5$; two interesting cases $k = 2$ and $k = 3$ are essentially equivalent:

$$\dim \Delta_q = 2 \;\Rightarrow\; \dim \Delta^2_q = 3;$$

$$C^0_\Delta = C^0_{\Delta^2} = \Delta^{2\perp},\; \dim(C^0_{\Delta^2} \cap T^*_q M) = 2.$$

Reconstruction of the 2-distribution from the 3-distribution:

$$\Delta = \{\dot\gamma(t) \in TM : \gamma \text{ is a sing. curve of } \Delta^2\}.$$



Let $\pi : T_p(T^*M) \to T_qM$ be the differential of the projection $T^*M \to M$; then $\pi(J^0_\lambda(t)) \subset p^\perp \subset T_qM$ and $t \mapsto \pi\left(J^0_\lambda(t)\right)$ is a curve in the projective plane $\mathbb{P}\left(p^\perp/\dot{\gamma}\right)$.

Proposition: Distribution Δ has a basis generating the 5-dim. nilpotent Lie algebra iff this curve is a quadric $\forall p, q$.

In general, let $K_p(q) \subset p^\perp$ be the osculating quadric to this curve: $K_p(q)$ is zero locus of a signature $(2,1)$ quadratic form on $p^\perp/\dot{\gamma}$. Finally, $\mathcal{K}(q) = \bigcup_{p \in \Delta_q^{2\perp}} K_p(q)$ is zero locus of a $(3,2)$ quadratic form on T_qM.

$\mathcal{K}(q)$, $q \in M$ is and intrinsically "raised" from Δ conformal structure on M; $\Delta_q \subset K(q)$.

Assume that $k = 2$, $n \geq 5$. Let $p \in C^0_\Delta$, λ the sing. extremal through p and γ the corresponding singular curve. We set:

$$J_\lambda(t) = D_\lambda F\left(\pi^{-1}\Delta_{\gamma(t)}\right) \subset T_p C^0_\Delta / T_p\lambda.$$

Then $J_\lambda(t) \supset J^0_\lambda(t)$ and $J_\lambda(t)$ is a Lagrangian subspace of the symplectic space $T_p C^0_\Delta / T_p\lambda$. In other words, $J_\lambda(t)^\angle = J_\lambda(t)$, where

$$\mathcal{S}^\angle \stackrel{\text{def}}{=} \{\zeta \in T_p C^0_\Delta : \sigma(\zeta, \mathcal{S}) = 0\},\ \mathcal{S} \subset T_p.$$

Given $s \in \mathbb{R} \setminus \{0\}$, $s\lambda$ is the singular extremal through $sp \in C^0_\Delta$. Hence $T_p(\mathbb{R}p) \subset J_\lambda(t)$, $\forall t$ and $J_\lambda(t) \subset T_p(\mathbb{R}p)^\angle$.

Final reduction: $\Sigma_p = T_p(\mathbb{R}p)^\angle / T_p\mathbb{R}p$ is a symplectic space, $\dim \Sigma_p = 2(n-3)$. Then $J_\lambda(t)$ is a Lagrangian subspace of Σ_p.

Important property: $J_\lambda(t) \cap J_\lambda(\tau) = 0$ for sufficiently small $|t-\tau| \neq 0$.

Take projectors: $\pi_{t\tau} : \Sigma_p \to \Sigma_p$,

$$\pi_{t\tau}\Big|_{J_\lambda(t)} = 0, \quad \pi_{t\tau}\Big|_{J_\lambda(\tau)} = \mathbf{1}.$$

Lemma:

$$\operatorname{tr}\left(\frac{\partial^2 \pi_{t\tau}}{\partial t \partial \tau}\right) = \frac{(n-3)^2}{(t-\tau)^2} + g_\lambda(t,\tau),$$

where $g(t,\tau)$ is a smooth symmetric function of (t,τ).

"Ricci curvature" on λ: $\rho_\lambda(\lambda(t)) \stackrel{\text{def}}{=} g_\lambda(t,t)$.

Chain rule: let $\varphi : \mathbb{R} \to \mathbb{R}$ be a change of the parameter; then $\rho_{\lambda\circ\varphi}(\lambda(\varphi(t))) =$

$$\rho_\lambda\left((\varphi(t))\right)\dot{\varphi}^2(t) + (n-3)^2\mathbb{S}(\varphi),$$

where $\mathbb{S}(\varphi) = \frac{\dddot{\varphi}(t)}{e\dot{\varphi}(t)} - \frac{3}{4}\left(\frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)}\right)^2$.

“Ricci curvature” ρ can be killed by a local reparametrization. A parametrization which kills ρ is called *projective*; it is defined up to a Möbius transformation.

Let t be a projective parameter, then the quantity:

$$A(\lambda(t)) = \frac{\partial^2 g}{\partial \tau^2}(t,\tau)\Big|_{\tau=t}(dt)^4$$

is called the *fundamental form* on λ.

For arbitrary parameter: $A(\lambda(t)) =$

$$\left(\frac{\partial^2 g}{\partial \tau^2}\Big|_{\tau=t} - \frac{3}{5(n-3)^2}\rho_\lambda(t)^2 - \frac{3}{2}\ddot{\rho}_\lambda(t)\right)(dt)^4.$$

Assume that $A(\lambda(t)) \neq 0$, then the identity $|A(\lambda(s))\left(\frac{d}{ds}\right)| = 1$ defines a unique (up to a translation) *normal parameter* s.

Let $z \in C^0_\Delta$ and λ_s is the normally parameterized singular extremal through z. We set

$$\bar{\rho}(z) = \rho_{\lambda_s}(z),$$

the *projective Ricci curvature*. Then $z \mapsto \bar{\rho}(z)$ is a function on C^0_Δ which depends only on Δ.

Back to the $(2,5)$ distributions. Such a distribution admits a basis generating the 5-dim. nilpotent Lie algebra iff $A \equiv 0$.

Example. A radius 1 ball is rolling over the radius r ball without slipping or twisting, $1 < r \leq \infty$. Admissible velocities form a $(2,5)$ distribution. Then (I. Zelenko): $\mathrm{sgn}(A) = \mathrm{sgn}(r-3)$;

$$\bar{\rho} = \frac{4\sqrt{35}(r^2+1)}{3\sqrt{(r^2-9)(9r^2-1)}}.$$

In particular, the distributions corresponding to different r are mutually non equivalent and the distribution corresponding to $r = 3$ admits a basis generating the 5-dim. nilpotent Lie algebra.