Entanglement and Indistinguishability of Quantum States

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Entanglement is:

$$\boxed{\hat{\rho} \neq \sum_k p_k \; \hat{\rho}_k^{(A)} \otimes \hat{\rho}_k^{(B)}}$$

Why ?

Alice and Bob have two independent devices which prepare the state $\rho^{(A)}$ and $\rho^{(B)}$.
Independent measurements in terms of $\rho = \rho^{(A)} \otimes \rho^{(B)}$

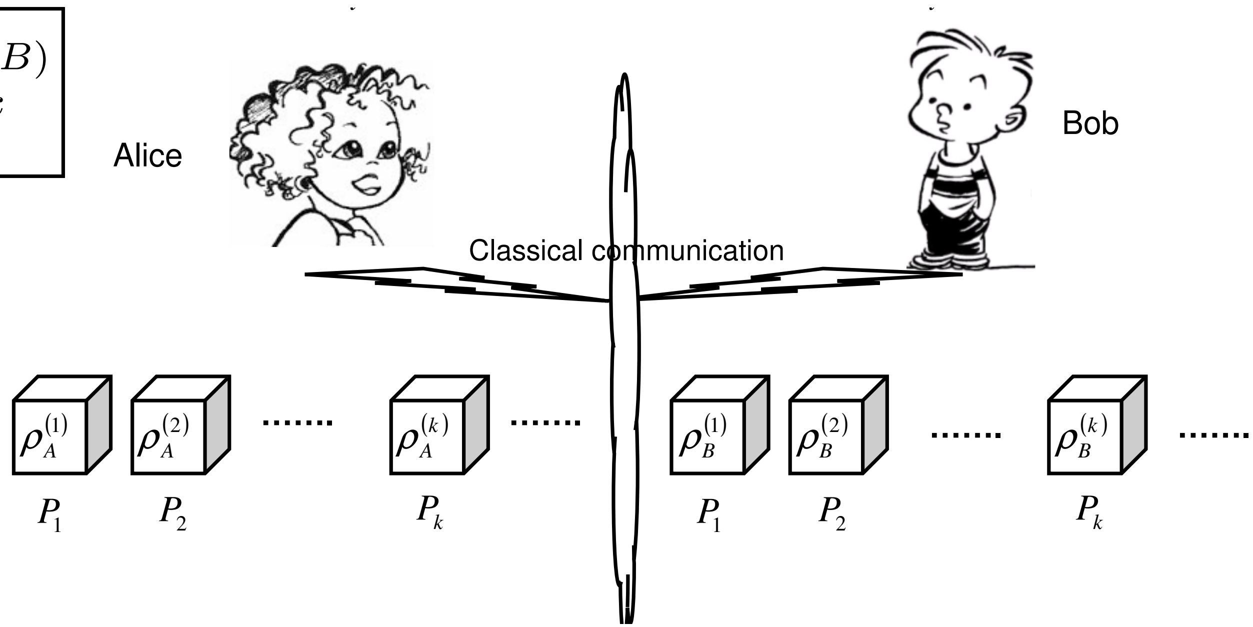
Charlie calls Alice and Bob by phone, and ask Alice and Bob to create the states $\rho_K^{(A)}$ and $\rho_K^{(B)}$, respectively, with probability p_K .

This creates correlations between the results of measurements obtained by Alice and Bob. The state of this process is classically correlated by LOCC.

How a classically correlated (separable) state looks like ?

$$\boxed{\hat{\rho} = \sum_k p_k \; \hat{\rho}_k^{(A)} \otimes \hat{\rho}_k^{(B)}}$$

Werner (1989)



LOCC: Local Operations and Classical Communication

Entanglement is:

$$\hat{\rho} \neq \sum_k p_k \; \hat{\rho}_k^{(A)} \otimes \hat{\rho}_k^{(B)}$$

J. Bell: … a correlation stronger than any classical correlation

P. Shor: … a global structure that allows for faster algorithms

C. Bennett: … a resource that enables quantum teleportation

A. Ekert: … a tool for secure communication

A. Peres: … used by quantum magicians to do tricks that cannot be imitated by classical magicians

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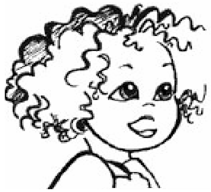
this talk: ... high distinguishability of quantum states

-> ultra-sensitive interferometry, quantum Zeno dynamics, ecc.

Can we distinguish two quantum states?

Alice



Alice and Bob play with the state $\hat{\rho}$.

Bob

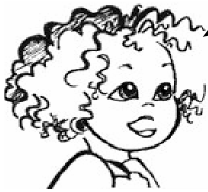
the witch steals and shifts the state by a <u>secret</u> amount θ .
Then she gives the state back to Alice and Bob.

$$\hat{\rho}(\theta) = e^{-i\hat{H}\theta} \; \hat{\rho} \; e^{i\hat{H}\theta}$$

has the state been changed ?

??

Alice



Bob

Entangled states can be more distinguishable than classically correlated states.

1) Statististical distinguishability of quantum states

How much different are $|\psi_0\rangle$ and $|\psi_f\rangle = e^{-i\hat{H}\theta}|\psi_0\rangle$?

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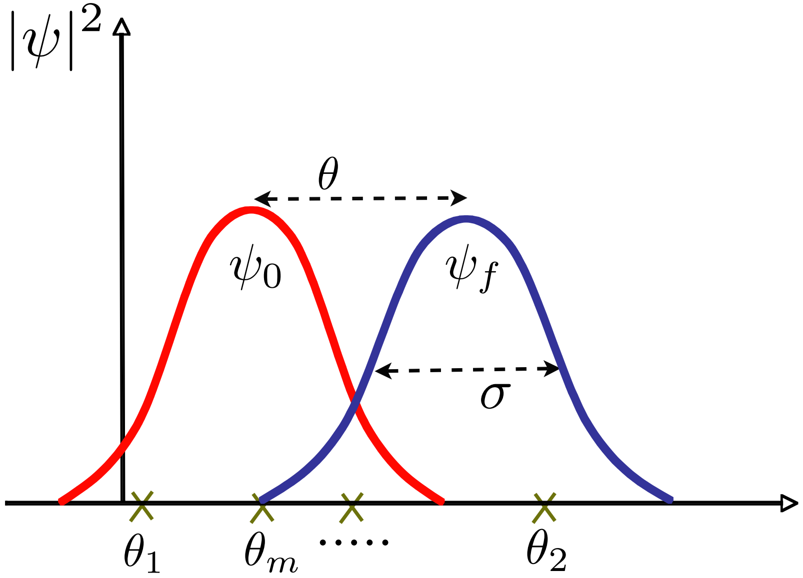
The simplest example: two Gaussian states

the states are distinguishable if their "distance" is larger than their "noise"



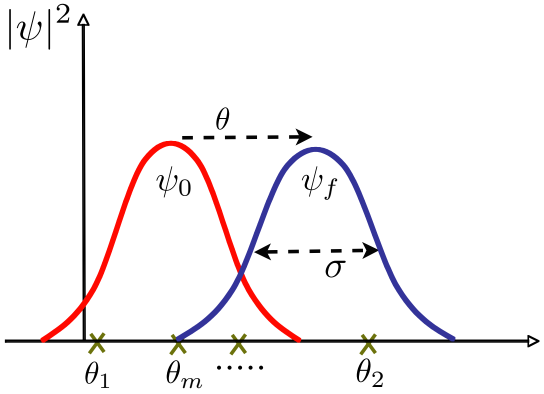
$$\theta > \Delta\theta_0 + \Delta\theta_f$$

Wooters (1981)

$$\Delta\theta \sim \frac{\sigma}{\sqrt{m}}$$

The "noise" $\Delta\theta$ decreases with the number of measurements m
and increases with quantum fluctuations σ

the states are distinguishable if their "distance" is larger than their "noise"



$$\theta > \Delta\theta_0 + \Delta\theta_f$$

$$\Delta\theta \sim \frac{\sigma}{\sqrt{m}}$$

In general, the "noise" is given by the Cramer-Rao lower bound:

$$\boxed{\Delta\theta \geq \frac{1}{\sqrt{m}}\frac{1}{\sqrt{F}}}$$

Fisher information

the larger is the Fisher the more the states are distinguishable

e.g., with pure states and optimal measurements: $F = 4\,\Delta^2 \hat{H}$

distance and scalar product: $|\langle\psi_0|\psi_{\delta\theta}\rangle|^2 = 1 - \frac{F}{4}\,\delta\theta^2$

2) Multi particle Entanglement

$$\boxed{\hat{\rho} \neq \sum_k p_k \ \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)} \otimes \ldots \hat{\rho}_k^{(N)}}$$

N particles in two modes (N qbits) are entangled
if their state cannot be written as a
convex combination of product states

Can we give
i) a simple criterion to recognize multiparticle entanglement and
ii) recognize "useful" entanglement for distinguishing states ?

Consider an Hermitian operator: $\hat{H} = \sum_{k=1}^{N} \hat{\sigma}_i$

sum of Pauli matrices along arbitrary directions rotating locally each qbit

if $\hat{\rho} = \sum_k p_k \ \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)} \otimes ... \hat{\rho}_k^{(N)}$ classically correlated $\rightarrow \ F[\hat{H}] \leq N$

The upper bound is $F \leq N^2$

$$\text{if } \ F[\hat{H}] > N \ \rightarrow \ \hat{\rho} \neq \sum_k p_k \ \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)} \otimes ... \hat{\rho}_k^{(N)}$$

If the Fisher information > the number of q-bits, the state is entangled (sufficient condition)

Luca Pezze`, AS, PRL 102, 100401 (2009)

Physical meaning ?

remember the original question:

How much different are $|\psi_0\rangle$ and $|\psi_f\rangle = e^{-i\hat{H}\theta}|\psi_0\rangle$?

Entangled states can be more distinguishable along a path in the Hilbert space than classically correlated states

Entangled states can evolve faster than separable states under unitary transformations $|\langle\psi_0|\psi_{\delta\theta}\rangle|^2 = 1 - \frac{F}{4}\,\delta\theta^2$

What this entanglement can be useful for ?

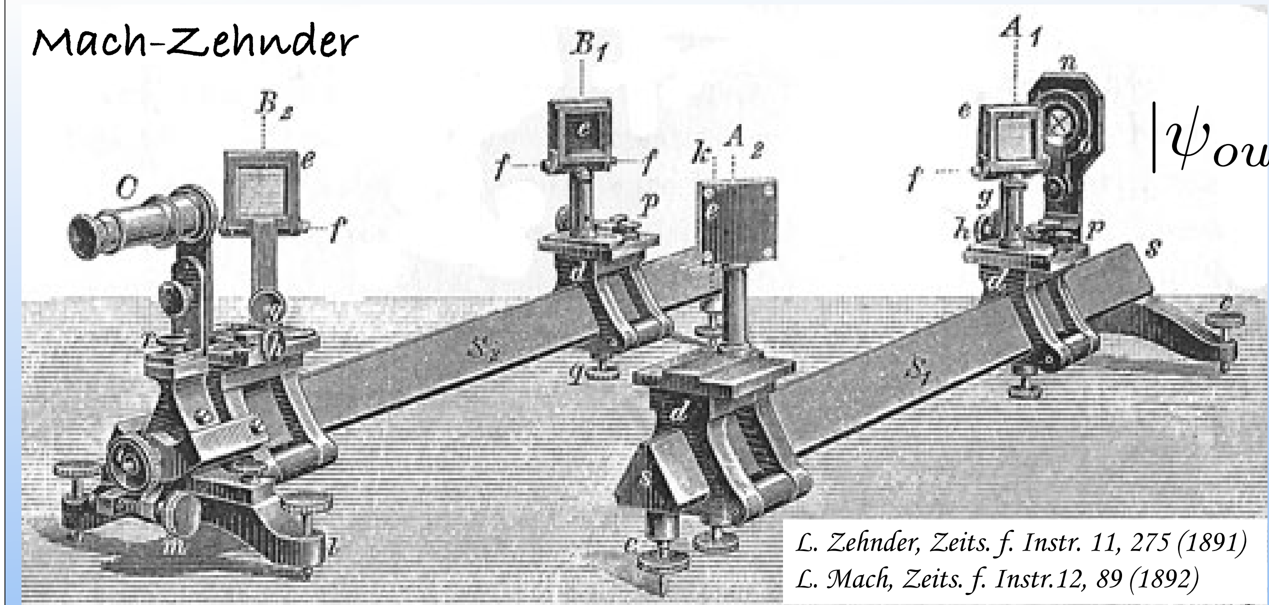
$E.g.:$

Interferometry

Zeno dynamics

What is interferometry ?

Mach-Zehnder



L. Zehnder, Zeits. f. Instr. 11, 275 (1891)
L. Mach, Zeits. f. Instr.12, 89 (1892)

$$|\psi_{out}\rangle = e^{-i\hat{S}_y \theta} |\psi_{inp}\rangle$$

$$\hat{S}_y = \frac{1}{2i}(\hat{a}^+\hat{b} - \hat{b}^+\hat{a}) = \frac{1}{2}\sum_{i=1}^{N} \hat{\sigma}_{y,i}$$

The sensitivity depends on:

i) m number of measurements

ii) F quantum fluctuations
(Fisher information)

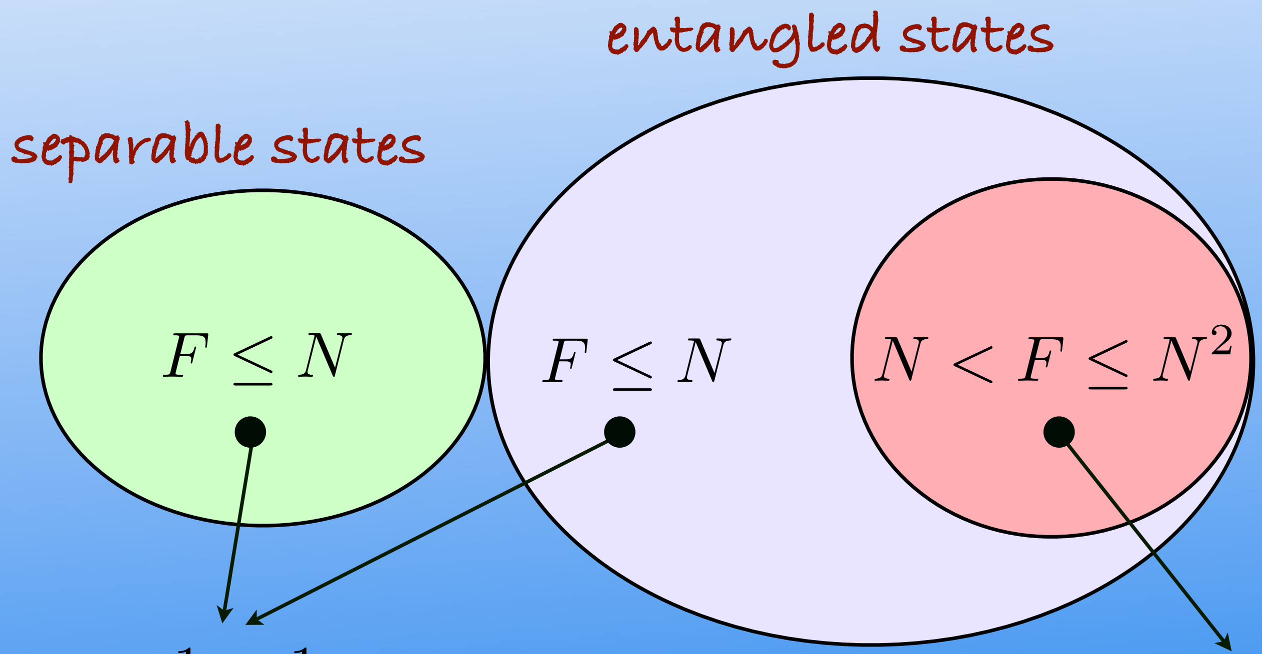
Cramer-Rao lower bound

$$\Delta\theta \geq \frac{1}{\sqrt{m}} \frac{1}{\sqrt{F}}$$

with pure states and optimal measurements : $\Delta\theta \geq \frac{1}{2}\frac{1}{\sqrt{m}} \frac{1}{\Delta\hat{S}_y}$

Highest sensitivity allowed by Quantum Mechanics

Putting together entanglement and distinguishability:



$$\Delta\theta \geq \frac{1}{\sqrt{m}} \frac{1}{\sqrt{N}}$$

shot-noise

$$\frac{1}{\sqrt{m}} \frac{1}{\sqrt{N}} > \Delta\theta \geq \frac{1}{\sqrt{m}} \frac{1}{N}$$

sub shot-noise -> Heisenberg limit

Examples: a few input states for Heisenberg limit with Mach-Zehnder

Yurke, McCall, Klauder, PRA 1987

$$|\Psi_{inp}\rangle \approx \left|\frac{N}{2}+1, \frac{N}{2}-1\right\rangle + \left|\frac{N}{2}, \frac{N}{2}\right\rangle$$

Holland & Burnett, PRL 1993

$$|\Psi_{inp}\rangle = \left|\frac{N}{2}, \frac{N}{2}\right\rangle$$

Wineland et al.
Spin-Squeezing, PRL 1994

$|\Psi_{BS}\rangle \approx |N, 0\rangle + |0, N\rangle$, PRA 1995

Pezze' & Smerzi, PRA 2006

$$|\Psi_{inp}\rangle \approx \left|\frac{N}{2}+1, \frac{N}{2}-1\right\rangle + \left|\frac{N}{2}-1, \frac{N}{2}+1\right\rangle$$

Pezze' & Smerzi, PRL 2007
Squeezed vacuum $\otimes$ coherent state
Number squeezed $\otimes$ coherent state (to be sub.)

Suggest your own state !!!

Spin-squeezed states

$$\xi = \frac{N\Delta^2 S_z}{\langle \hat{S}_x \rangle^2} < 1$$

Spin squeezing is also a sufficient condition to recognize useful multi-particle entanglement

Sorensen, Duan, Cirac, Zoller (2001)

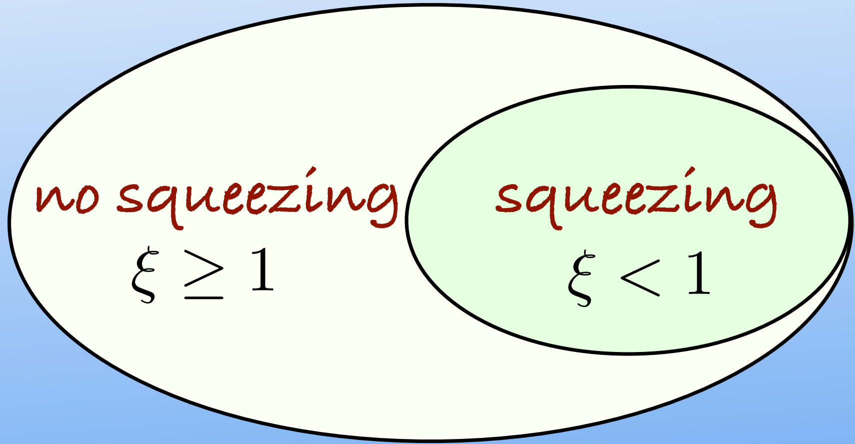
Wineland et al. 1994, Kitagawa & Ueda, 1993

Experiments in Munich, Heidelberg, Florence, (atoms), Munich (photons)

Recent related theory by Giovannetti, Maccone, Lloy, Dowling, Paris, ...

Spin squeezing vs. Fisher

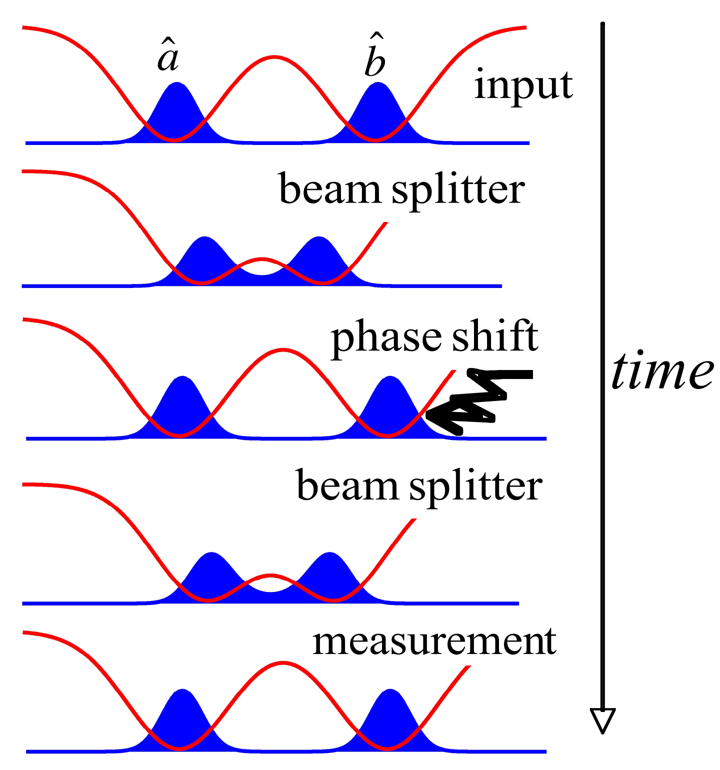
useful entanglement ($F > N$)



the Fisher information criterion
includes all spin-squeezed states

but spin-queezing is easier to measure

Mach-Zehnder interferometry with Bose-Einstein condensates trapped in a double well potential (or in two hyperfine levels):



$$\hat{H} = E_c(t)\ \hat{S}_z^2 - K(t)\ \hat{S}_x + \Delta E(t)\ \hat{S}_z$$

interatomic interaction

$$\hat{S}_z^2 = \frac{1}{4}(\hat{a}^+\hat{a} - \hat{b}^+\hat{b})^2 = \frac{1}{4}(\hat{N}_a - \hat{N}_b)^2$$

tunneling

$$\hat{S}_x = \frac{1}{2}(\hat{a}^+\hat{b} + \hat{b}^+\hat{a})$$

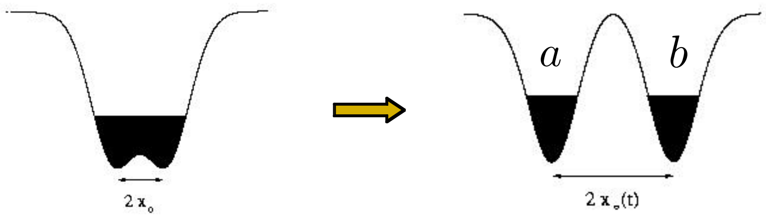
energy off-set

$$\hat{S}_z = \frac{1}{2}(\hat{N}_a - \hat{N}_b)$$

Two-modes Hamiltonian of a BEC tunneling trough the barrier of a double well potential

a protocol for creating entanglement with BEC:

1) Splitting $\hat{H} = \cancel{E_c(t)\ \hat{S}_z^2} - K(t)\ \hat{S}_x$



$$|\psi_0\rangle \sim (|1_a, 0_b\rangle + |0_a, 1_b\rangle)^N$$

Spin-coherent state
(Poisson distribution)

2) Nonlinear dynamics of the two decoupled condensates

$$\hat{H} = E_c(t)\ \hat{S}_z^2 - \cancel{K(t)\ \hat{S}_x}$$

$$|\psi_{inp}\rangle = e^{-iE_c\hat{S}_z^2 t}|\psi_0\rangle$$

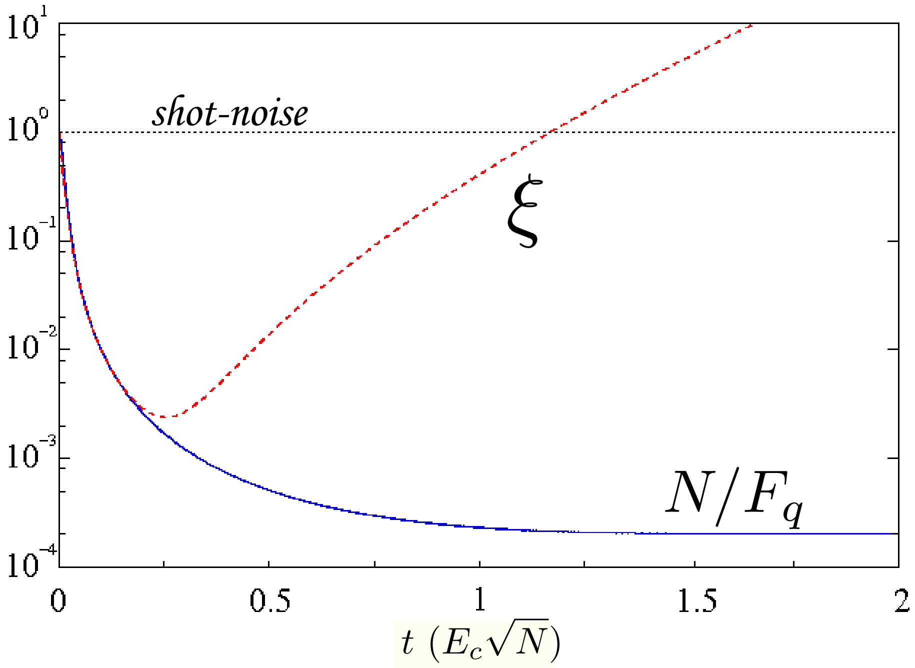
Philipp Treutlein et al., Nature 2010
Markus Oberthaler et al., Nature 2010

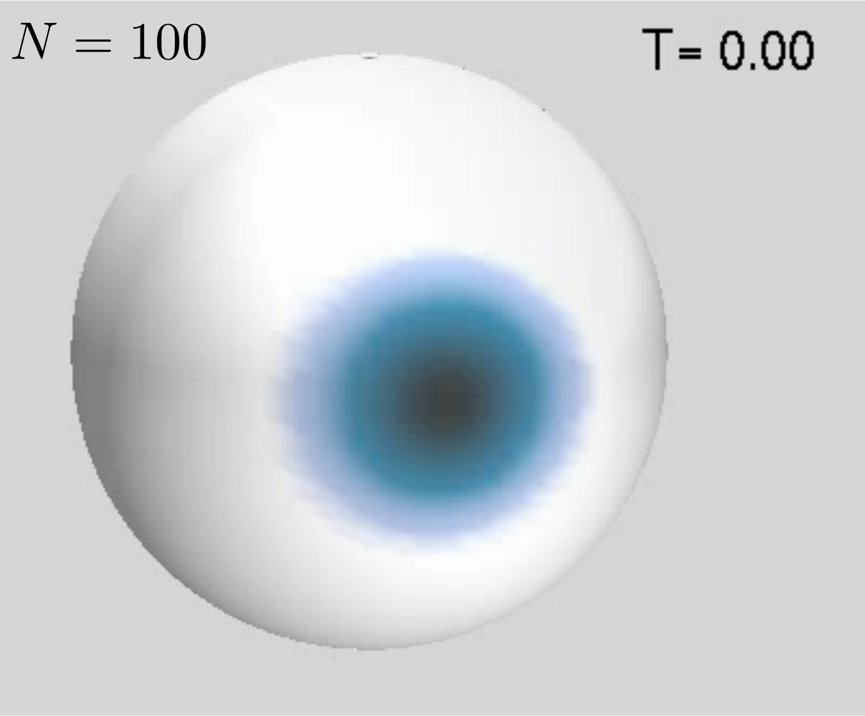
3) Use the entangled state for sub shot-noise phase estimation with the BEC Mach-Zehnder interferometer

$$\hat{H} = \cancel{E_c(t)\ \hat{S}_z^2} - K(t)\ \hat{S}_x + \Delta E(t)\ \hat{S}_z$$

Oberthaler et al., Nature 2010
sub shot-noise Ramsey

$$|\psi_{inp}\rangle = e^{-iE_c\hat{S}_z^2 t}|\psi_0\rangle$$





Particle entanglement persists longer than spin-squeezing

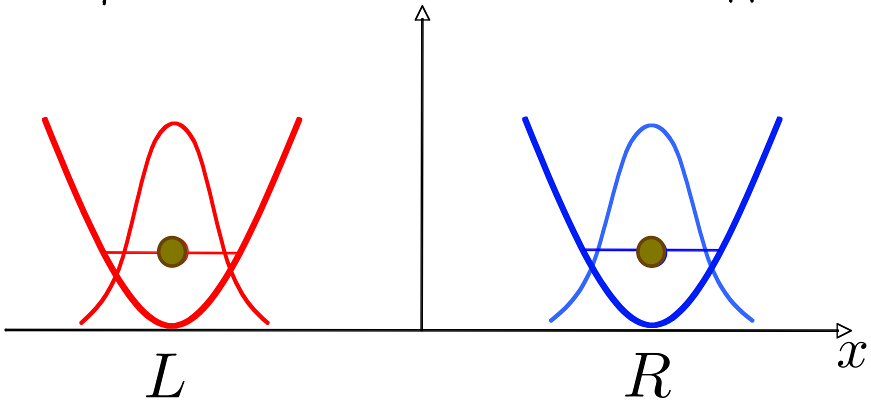
Spin squeezing $\xi = \dfrac{N\Delta^2 S_\theta}{\langle \hat{S}_x\rangle^2} < 1$

$$\Delta\theta = \frac{1}{\sqrt{m}}\sqrt{\frac{\xi}{N}}$$

Kitagawa & Ueda, 1993
Sorensen, Duan, Cirac, Zoller (2001)

Is entanglement due to symmetrization physical?

Example: 2 identical bosons in different harmonic traps: twin-Fock states



2nd quantization | 1st quantization

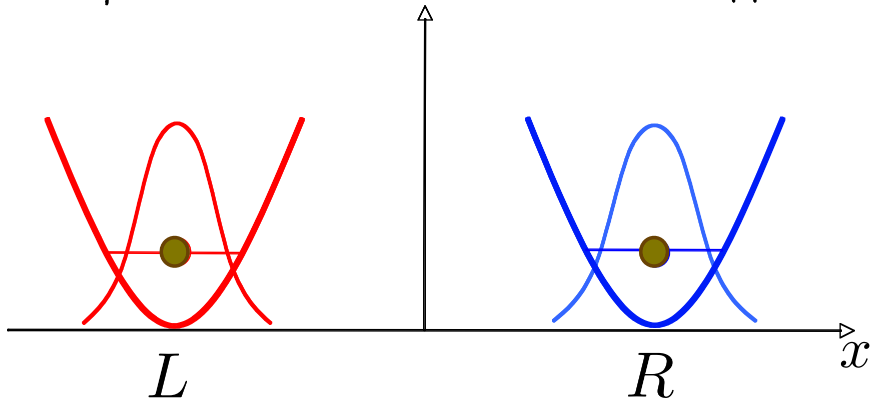
$$|1\rangle_L \; |1\rangle_R \longleftrightarrow |R_1, L_2\rangle + |L_1, R_2\rangle$$

separable | entangled

It is sometime claimed that entanglement which arises from symmetrization alone is unphysical

Is entanglement due to symmetrization physical?

Example: 2 identical bosons in different harmonic traps: twin-Fock states



2nd quantization 1st quantization

$$|1\rangle_L \; |1\rangle_R \longleftrightarrow |R_1, L_2\rangle + |L_1, R_2\rangle$$

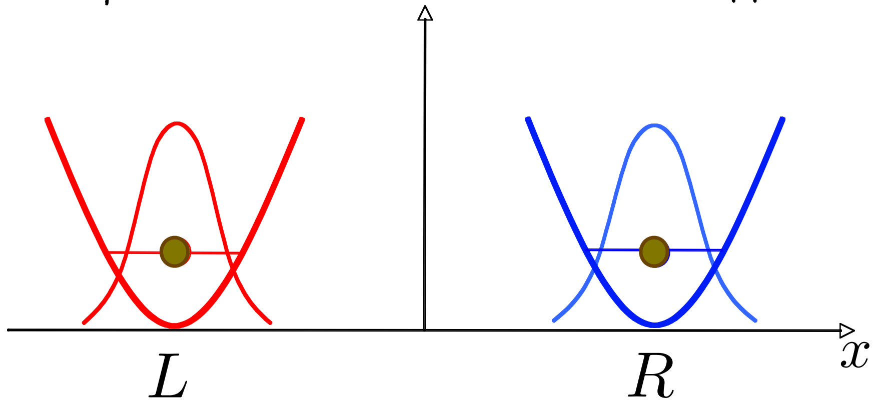
separable entangled

Why ? Because entanglement is often related to local addressability (required for quantum computation, violation of Bell inequalities, etc.

Indistinguishable particles are not locally addressable !

Is entanglement due to symmetrization physical?

Example: 2 identical bosons in different harmonic traps: twin-Fock states



2nd quantization — 1st quantization

$$|1\rangle_L\ |1\rangle_R \longleftrightarrow |R_1, L_2\rangle + |L_1, R_2\rangle$$

separable — entangled

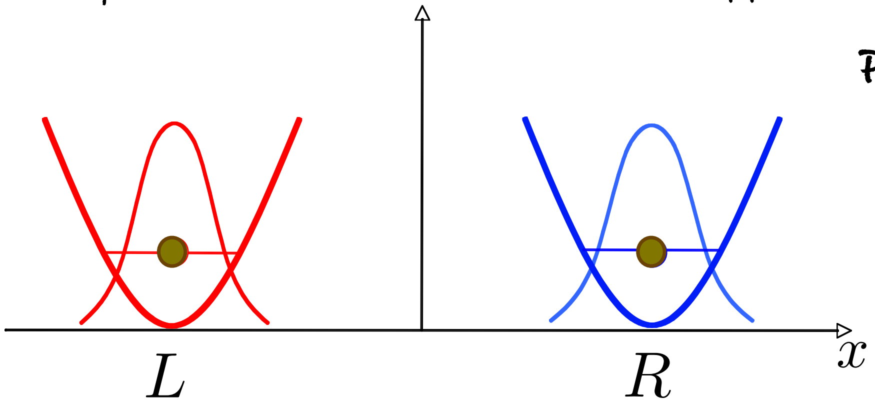
entanglement (e.g. twin-Fock states) which can be created with BEC in double wells is due to symmetrization.

Notice that the spin-squeezing & Fisher entanglement conditions require collective operations (not local operations)

Is entanglement due to symmetrization physical?

Example: 2 identical bosons in different harmonic traps: twin-Fock states

Philipp Hyllus, AS, unpublished



2nd quantization 1st quantization

$$|1\rangle_L \; |1\rangle_R \longleftrightarrow |R_1, L_2\rangle + |L_1, R_2\rangle$$

separable entangled

Particle entanglement due to symmetrization is -useful- for distinguishing quantum states
(e.g.: necessary for sub shot-noise interferometry)
where only collective operations are required

What this entanglement can be useful for ?

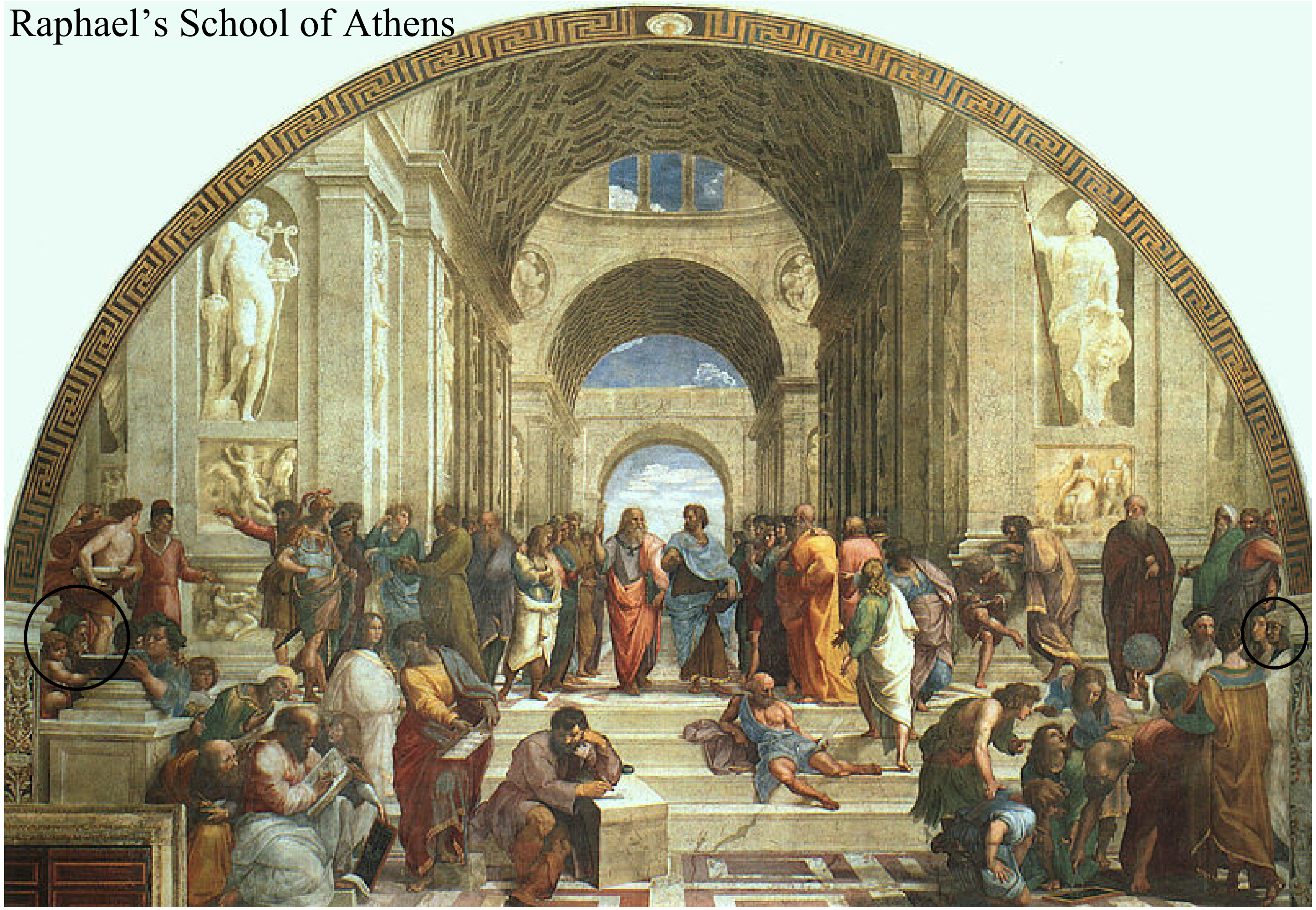
$E.g.:$

Interferometry

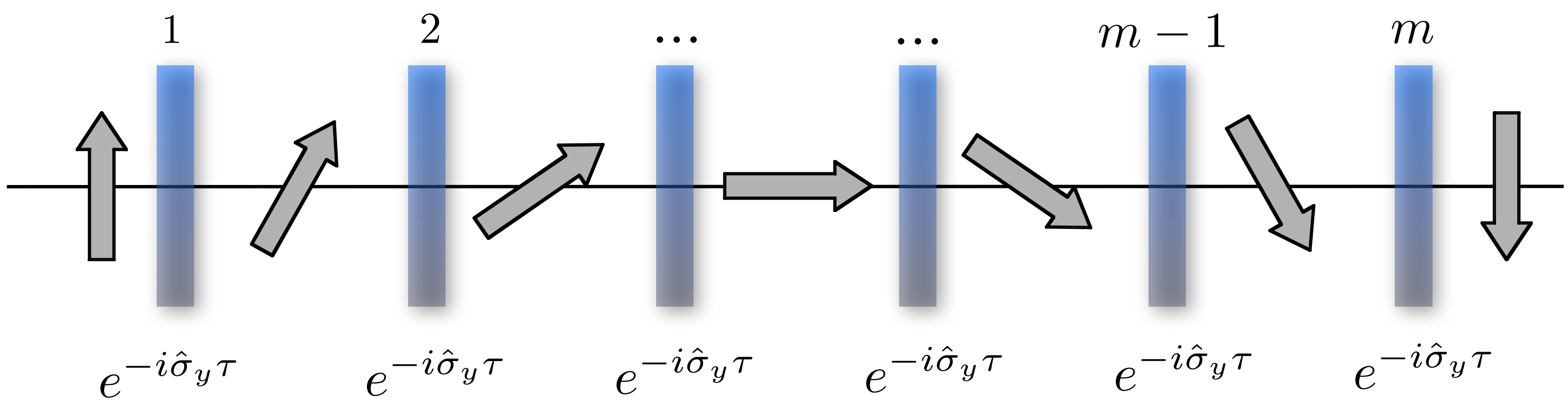
Zeno dynamics

Quantum Zeno dynamics

Raphael's School of Athens



A flying arrow is at rest. At any given moment the arrow is in a space equal to its own length, and therefore is at rest at that moment. So, it is at rest at all moments.



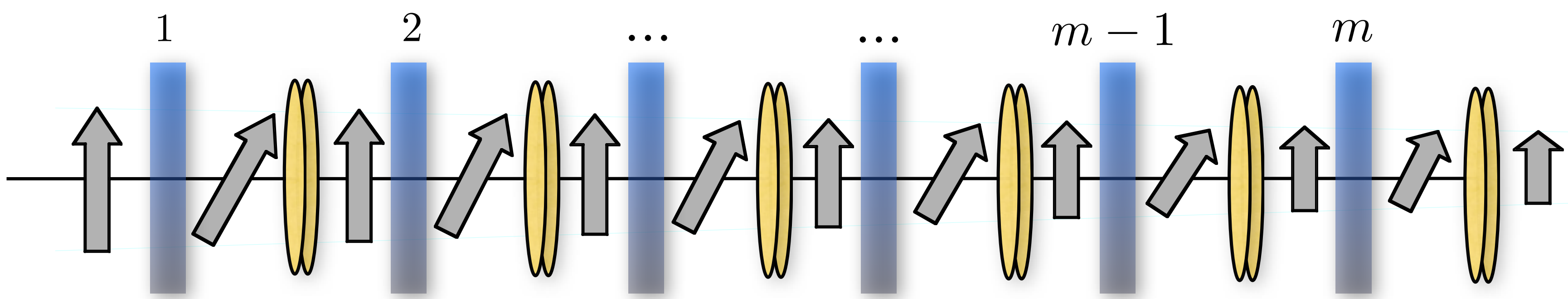
Example: spin

Consider a spin $|\psi_0\rangle = |\uparrow\rangle_z$ rotated by $\hat{U} = e^{-i\hat{\sigma}_y t} = \prod_{k=1}^{m} e^{-i\hat{\sigma}_y\tau}$

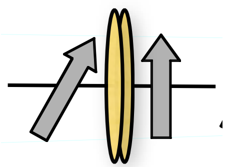
$$|\psi(t)\rangle = e^{-i\hat{\sigma}_y\tau}\ e^{-i\hat{\sigma}_y\tau} \dots\ e^{-i\hat{\sigma}_y\tau}|\psi_0\rangle = e^{-i\hat{\sigma}_y t}|\psi_0\rangle$$

(total time : $t = m\tau$)

Peres, Am. J. Phys. **48**, 931 (1980)



Consider the projective measurement:



$$\hat{\Pi} = |\psi_0\rangle\langle\psi_0|$$

The projective measurement has eigenvalue "yes", corresponding to the state projected back to $|\psi_0\rangle$ with probability $|\langle\psi_0|\psi(\tau)\rangle|^2$

$$P(yes|t) = |\langle\psi_0|\psi(\tau)\rangle|^{2m} \simeq 1 - m\ \Delta^2\hat{\sigma}_y\ \tau^2$$

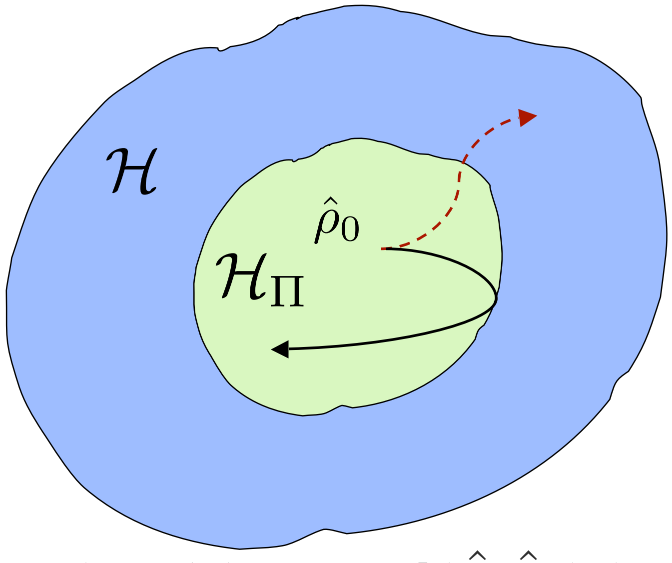
$m \to \infty,\ \tau \to 0$ *so that* $t = m\tau = const$ $\Rightarrow$ $P(yes|t) \to 1$

Zeno "paradox": the arrow does not rotate if watched !!!

Consider a system living in $\mathcal{H}$ with dynamics $\hat{U} = e^{-i\hat{\mathcal{H}}t}$

and a projector $\hat{\Pi}$ onto the subspace $\mathcal{H}_{\Pi}$

The initial state $\hat{\rho}_0 = \hat{\Pi}\hat{\rho}_0\hat{\Pi}$ is in $\mathcal{H}_{\Pi}$



Effective Zeno Hamiltonian

$$\hat{\mathrm{H}} = \hat{\mathcal{H}} - \hat{\Pi}\hat{\mathcal{H}}\hat{\Pi}$$

$$P(yes|t) = Tr[(\hat{\Pi}\hat{U}(\tau)\ \hat{\Pi})^m\ \hat{\rho}_0\ (\hat{\Pi}\hat{U}^\dagger(\tau)\hat{\Pi})^m] \simeq 1 - m\ \Delta^2\hat{\mathrm{H}}\ \tau^2$$

Quite generally,

The effective Zeno Hamiltonian is the Fisher information

$$F(\tau) = \left(\frac{\mathcal{P}(\mathrm{yes}|\tau)}{\partial\tau}\right)^2 \frac{1}{\mathcal{P}(\mathrm{yes}|\tau)[1 - \mathcal{P}(\mathrm{yes}|\tau)]} = 4\ \Delta^2\hat{\mathrm{H}} + O(\tau^4)$$

A. Smerzi, arXiv:1002.2760

Quantum Zeno dynamics

$$\boxed{P(yes|t) \simeq 1 - \frac{F}{4m}\, t^2 = 1 - \left(\frac{\tau}{\tau_{qz}}\right)^2}$$

when $\tau/\tau_{qz} << 1$ (Interval among measurements: $\tau = t/m$)

The small parameter of Zeno depends on the Cramer-Rao lower bound

$$\tau_{qz} = 2\ \Delta\tau_{crlb} = \frac{2}{\sqrt{m}\sqrt{F}}$$

The Quantum Zeno dynamics is strictly related with indistinguishability and entanglement

physical interpretation

entanglement affects Zeno

A physical interpretation of Zeno:

$$\boxed{P(yes|t) \simeq 1 - \frac{F}{4m}\, t^2 = 1 - \left(\frac{\tau}{\tau_{qz}}\right)^2}$$

when $\tau/\tau_{qz} << 1$ (Interval among measurements: $\tau = t/m$)

The projective measurements bring the state back to its initial value (the dynamics is frozen) when the two states are statistically indistinguishable with -m- measurements

Zeno for separable and entangled states:

Consider a state of N qbits

Separable states have a Fisher information bounded by $F = N$

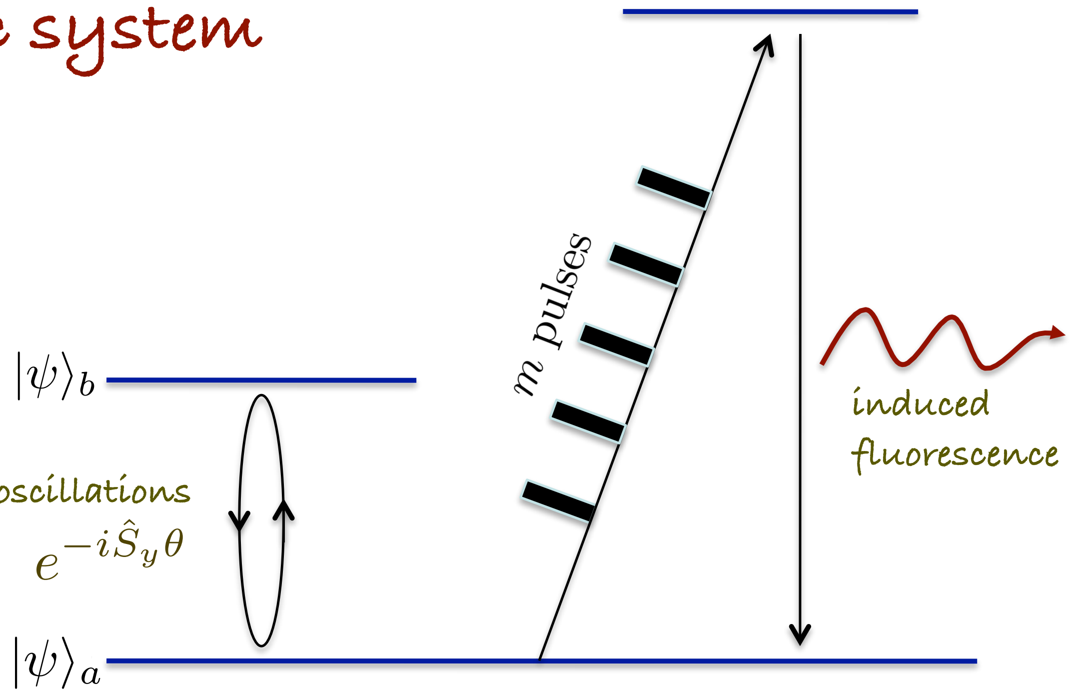
Zeno dynamics when $$\left(\frac{\tau}{\tau_{qz}}\right)^2 = \frac{t^2}{4}\frac{N}{m} << 1$$

Entangled states have a Fisher information bounded by $F = N^2$

Zeno dynamics when $$\left(\frac{\tau}{\tau_{qz}}\right)^2 = \frac{t^2}{4}\frac{N^2}{m} << 1$$

The number of measurements -m- needed to create the Zeno dynamics can be quite larger for entangled states that for separable states

This prediction can be tested with QND measurements in a three levels atomic system



Itano, Heinzen, Bollinger, and Wineland 1990

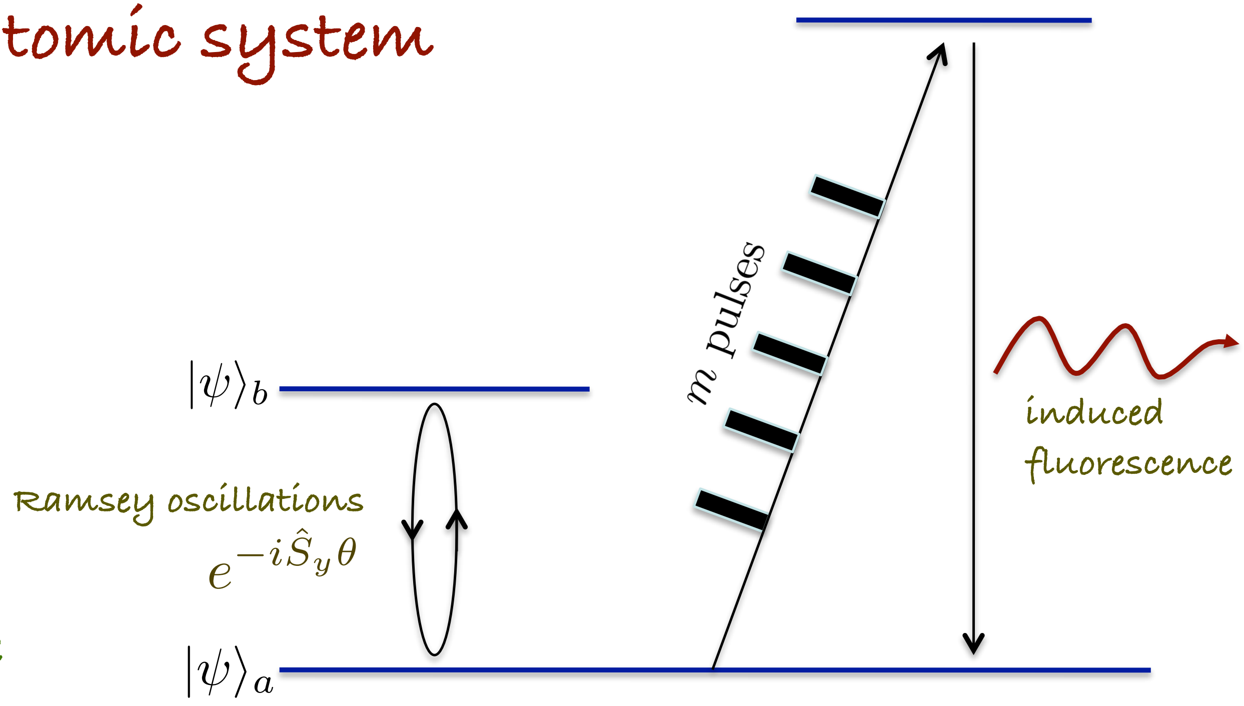
particle-separable state:

$$|\psi_0\rangle = |0\rangle_a |N\rangle_b$$

$$P(yes|t) = |\langle\psi_0|e^{-i\hat{S}_y\theta/m}|\psi_0\rangle|^{2m} \simeq 1 - \frac{N}{4m}\,\theta^2$$

Zeno dynamics with a number of measurements of the order of the number of particles

This prediction can be tested with QND measurements in a three levels atomic system



particle-entangled state (twin-Foch):

$$|\psi_0\rangle = |\frac{N}{2}\rangle_a|\frac{N}{2}\rangle_b$$

$$P(yes|t) = |\langle\psi_0|e^{-i\hat{S}_y\theta/m}|\psi_0\rangle|^{2m} \simeq 1 - \frac{N^2}{8m}\,\theta^2$$

Zeno dynamics with a number of measurements of the order of the -square- of number of particles *a few more references...*

von Neumann,1932

Beskow and Nilsson,1967

Khalfin 1968

Friedman 1972

Misra and Sudarshan, 1977

Kofman and Kurizki, 2000

Facchi and Pascazio, 2002

theory

(Cook 1988)

Itano, Heinzen, Bollinger, and Wineland 1990

Nagels, Hermans, and Chapovsky 1997

Balzer, Huesmann, Neuhauser, and Toschek, 2000

Wunderlich, Balzer, and Toschek, 2001

Wilkinson, Bharucha, Fischer, Madison, Morrow, Niu, Sundaram, and Raizen, 1997.

Fischer, B. Gutierrez-Medina, and Raizen, 2001

experiments

Summary

1) Particle entanglement <--> distinguishability of states

2) How to recognize useful entanglement: Fisher information

Applications in interferometry: shot noise versus Heisenberg limit

3) Distinguishability, entanglement and the Zeno paradox.

The Zeno dynamics is the result of projective measurements among quantum states which are indistinguishable.

The physical time scale is provided by the Cramer-Rao lower bound, which measures the distinguishability of states along a path in the Hilbert space.

Zeno dynamics with particle entangled states might require a quite smaller measurement intervals than classically correlated states.

Summary

1) Particle entanglement <--> distinguishability of states

2) How to recognize useful entanglement: Fisher information

Applications in interferometry: shot noise versus Heisenberg limit

3) Distinguishability, entanglement and the Zeno paradox.

There are different technologies which are based on efficiently distinguish quantum states. For instance:

In quantum control theories, when searching the optimal path to generate a target quantum state

Setting the conditions for adiabatic approximations

Adiabatic quantum computation

In the estimation of the speed limits of quantum computation