## An introduction to Hamilton-Jacobi equations

Stefano Bianchini

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Stefano Bianchini An introduction to Hamilton-Jacobi equations

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#### Introduction

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Existence in the Lipschitz class

Viscosity solutions

Lagrangian formulation

Regularity

Some simple computations

A regularity result

Regularity for hyperbolic conservation laws

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The Hamilton-Jacobi equation (HJ equation) is a special fully nonlinear scalar first order PDE. It arises in many different context:

- 1. Hamiltonian dynamics
- 2. Classical limits of Schrödinger equation
- 3. Calculus of variation
- 4. Control theory
- 5. Optimal mass transportation problems
- 6. Conservation laws in one space dimension
- 7. etc...

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Even if it is fully nonlinear, there is a satisfactory theory of existence and regularity of solutions.

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## Hamilton's principal function

The function S = S(q, P, t) defining a canonical transformation of coordinates  $(p, q) \mapsto (Q, P)$ 

$$p = \nabla_q S, \quad Q = \nabla_P S,$$

yields a canonical transformation with the new Hamiltonian H'=0 if

$$\partial_t S + H(t,q,\nabla_q S) = 0.$$

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The above equation is the **Hamilton-Jacobi equation**: the function *H* is called the *Hamiltonian*, and depending on the context the solution can be called *minimizer*, *value function*, *potential*, or in this case *Hamilton principal function*.

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## Schrödinger equation

The Schrödinger equation for a single particle in a potential U can be written as

$$i\hbar\partial_t\psi=-rac{\hbar^2}{2m}
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If we look for a solution of the form  $\psi = \psi_0 e^{iS/\hbar}$ , where S is the *phase* and we let  $\hbar \to 0$  (classical limit), then (formally) we obtain

$$-\partial_t S = \frac{1}{2m} |\nabla S|^2 + U,$$

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$$-\partial_t S = \frac{1}{2m} |\nabla S|^2 + U,$$

which is the Hamilton-Jacobi equation for the Hamiltonian

$$H=\frac{p^2}{2m}+U.$$

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## Calculus of variation

Consider the minimization problem in  $\Omega \subset \mathbb{R}^d$ 

$$\min\bigg\{\int \big(\mathbf{1}_{|p|\leq 1}(\nabla u(x))+u(x)\big)dx:u|_{\partial\Omega}=u_0\bigg\}.$$

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The solution satisfies the time independent Hamilton-Jacobi equation

$$1-|\nabla u|=0,$$

with the Hamiltonian |p| and boundary data  $u_0$ .

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$$\operatorname{div}(\rho d) = 1,$$

with d the direction of the optimal ray (see later).

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## Control theory

Consider the ODE

$$\dot{x} = f(x, u), \quad u \text{ control},$$

and the problem is to minimize the functional

$$A(t) = \min_{u} \left\{ \int_{t}^{T} L(x, u) dt + F(x(T)) \right\}$$

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Defining

$$H(t,x,p) := \min_{u} \big\{ p \cdot f(x,u) + L(x,u) \big\},\$$

the function A(t) satisfies the Hamilton-Jacobi-Bellman equation

$$\partial_t A + H(t, x, \nabla A) = 0, \quad A(T) = F(x).$$

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## Optimal mass transportation

Let 
$$\|\cdot\|: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$
 be a norm,  $\mu, \nu \in \mathcal{P}([0,1])$  and  
 $\Pi(\mu, \nu) := \left\{ \pi \in \mathcal{P}([0,1]^2) : (P_1)_{\sharp} \pi = \mu, (P_2)_{\sharp} \pi = \nu \right\}.$ 

The problem is to minimize

$$\int \|x-y\|\pi(dxdy), \quad \pi\in\Pi(\mu,\nu).$$

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The problem is to minimize

$$\int \|x-y\|\pi(dxdy), \quad \pi\in\Pi(\mu,
u).$$

By duality, this is equivalent to maximize

$$\int \phi(x)(\mu-
u)(dx), \quad ig|\phi(x)-\phi(y)ig|\leq \|x-y\|,$$

and one can show that  $\phi$  is the solution to the Hamilton-Jacoby equation

$$1 - \|\nabla \phi\| = 0.$$

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### Conservation laws

Consider the scalar conservation laws

$$u_t + f(u)_x = 0, \quad u: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}.$$

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## Conservation laws

Consider the scalar conservation laws

$$u_t + f(u)_x = 0, \quad u: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}.$$

By the change of variable

$$U(x)=\int^x u(y)dy,$$

we can transform the PDE into

$$U_t + f(U_x) = 0,$$

which is a Hamilton-Jacobi equation with Hamiltonian H(p) = f(p).

Existence in the Lipschitz class Viscosity solutions Lagrangian formulation

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Existence in the Lipschitz class Viscosity solutions Lagrangian formulation

#### What we can expect

The natural space of functions where the solutions lives is Lipschitz.

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Existence in the Lipschitz class Viscosity solutions Lagrangian formulation

#### What we can expect

The natural space of functions where the solutions lives is Lipschitz.

**Example.** The model Hamiltonian is  $\frac{p^2}{2}$ , and the function

$$u(t,x) = -\int_0^x \min\left\{1, -\frac{y}{t}\right\} dy$$

is a regular solution for t < 1 to

$$u_t + \frac{|u_x|^2}{2} = 0.$$

At t = 1 the solution becomes only 1-Lipschitz.

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Existence in the Lipschitz class Viscosity solutions Lagrangian formulation

A solution to Hamilton-Jacobi can be defined as a Lipschitz function  $u: \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$  such that

 $u_t + H(t, x, \nabla u) = 0$ 

is satisfied at every differentiable point of u, i.e.  $\mathcal{L}^{d+1}$ -a.e..

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is satisfied at every differentiable point of u, i.e.  $\mathcal{L}^{d+1}$ -a.e.. A solution in the above "a.e.-sense" is not unique. **Example.** The function

$$u(t,x)=\min\left\{|x|-\frac{t}{2},0\right\},\,$$

satisfies

$$u_t + \frac{|u_x|^2}{2} = 0, \quad u(0,x) = 0.$$

Clearly the expected solution is u(t, x) = 0.

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Existence in the Lipschitz class Viscosity solutions Lagrangian formulation

## Maximum principle

For scalar equation, a natural requirement is

$$u(0,x) \leq v(0,x) \quad \Rightarrow \quad u(t,x) \leq v(t,x).$$

We can restrict the possible solutions to the ones generating a semigroup satisfying the maximum principle.

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We can restrict the possible solutions to the ones generating a semigroup satisfying the maximum principle.

If  $\phi$  is a regular function such that  $u^{\epsilon} - \phi$  has a local minimum in  $(\bar{t}, \bar{x})$ , then it follows

$$\Delta(u^{\epsilon}-\phi)\geq 0,$$

Since  $\nabla u^{\epsilon} = \nabla \phi$ ,  $u^{\epsilon}_t = \phi_t$ , we recover

$$\phi_t + H(\bar{t}, \bar{x}, \nabla \phi) - \epsilon \Delta \phi \ge 0,$$

and in the limit

$$\phi_t + H(\bar{t}, \bar{x}, \nabla \phi) \geq 0.$$

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#### Definition

A function u is a viscosity solution to the HJ equation if for all  $\phi$  smooth such that

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A function u is a viscosity solution to the HJ equation if for all  $\phi$  smooth such that

1.  $u - \phi$  has a local maximum in  $(\bar{t}, \bar{x})$ , then

 $\partial_t \phi(\overline{t}, \overline{x}) + H(\overline{t}, \overline{x}, \nabla \phi(\overline{t}, \overline{x})) \leq 0,$ 

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$$\partial_t \phi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, \nabla \phi(\bar{t}, \bar{x})) \geq 0,$$

Under mild assumptions on H and  $u_0$ ,

Theorem (Crandall-Lions)

The viscosity solution exists and is unique.

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Existence in the Lipschitz class Viscosity solutions Lagrangian formulation

## Lax formula

If u is a viscosity solution and H convex in p, then it can be obtained by the formula

$$u(t,x) = \min \left\{ u(0,y) + \int_0^t L(s,\gamma(s),\dot{\gamma}(s)) ds, \\ \gamma: [0,t] \to \mathbb{R}^d, \gamma(0) = y, \gamma(t) = x \right\},$$

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where the Lagrangian L is given by the Legendre transform of H

$$L(t, x, a) = \sup_{p} \left\{ a \cdot p - H(t, x, p) \right\}.$$

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The curve  $\gamma$  for which the minimum

$$u(t,x) = u(0,y) + \int_0^t L(s,\gamma(s),\dot{\gamma}(s)) ds$$

is called *characteristic* or *optimal ray*.

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Existence in the Lipschitz class Viscosity solutions Lagrangian formulation

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is called *characteristic* or *optimal ray*.

By Euler-Lagrange equation, it is a solution to the ODE system

$$\begin{cases} \dot{x} = \nabla_{p} H(t, x, p) \\ \dot{p} = -\nabla_{x} H(t, x, p) \end{cases}$$

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$$\begin{cases} \dot{x} = \nabla_{p} H(t, x, p) \\ \dot{p} = -\nabla_{x} H(t, x, p) \end{cases}$$

In the special case where H = H(p), this curve is a straight line, and the min-formula reads as

$$u(t,x) = \inf_{y} \left\{ u(0,y) + tL\left(\frac{x-y}{t}\right) \right\}.$$

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Some simple computations A regularity result Regularity for hyperbolic conservation laws

If H is convex in p, then the solution u is not only Lipschitz, but enjoys more regularity.

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**Example.** Let  $H = p^2/2$ , so that  $L = a^2/2$  and the function

$$x \mapsto tL\left(\frac{x-y}{t}\right) = \frac{|x-y|^2}{2t}$$

is semiconcave of parameter 1/t.

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is semiconcave of parameter 1/t.

Since the minimum of semiconcave functions is semiconcave, it follows that the solution u to the HJ equation

$$\partial_t u + \frac{|\nabla_x u|^2}{2} = 0$$

is semiconcave.

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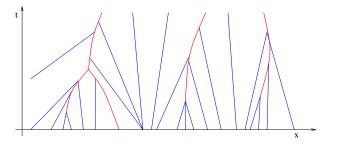
# A regularity result

The following result can be proved: if the Hamiltonian is uniformly convex in p and the initial data is sufficiently regular then there exists piecewise smooth hypersurfaces  $\{S_k\}_k$  of codimension 1 such that  $\nabla u$  is regular outside  $\bigcup_k S_k$  (Cannarsa-Sinestrari).

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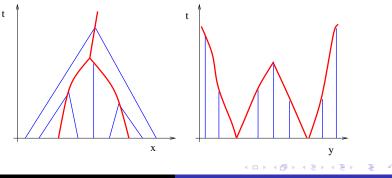
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Due to the regularity outside the jumps of  $\nabla_x u$ , it is clear that the change of variable

$$\begin{cases} t = \tau, \\ x = \gamma(t, y), \end{cases} y \text{ initial point of the characteristic } \gamma, \end{cases}$$

is regular.



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## Solutions to hyperbolic conservation laws

For strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \ u \in \mathbb{R}^m,$$

one expects a similar structure: countably many shock curves and regularity of the solution in the remaining set.

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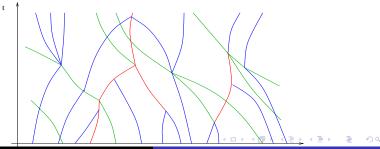
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one expects a similar structure. However the presence of the other characteritic families generates a complicated structure.



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1. the structure of solutions if the initial data is only Lipschitz and H convex

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In the second part of the talk we will be concerned with:

- 1. the structure of solutions if the initial data is only Lipschitz and H convex
- 2. the regularity of the decomposition of  $\mathbb{R}^+\times\mathbb{R}^d$  given by the characteristics

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In the second part of the talk we will be concerned with:

- 1. the structure of solutions if the initial data is only Lipschitz and *H* convex
- 2. the regularity of the decomposition of  $\mathbb{R}^+\times\mathbb{R}^d$  given by the characteristics
- 3. the applications/extension of these results to conservation laws and optimal transport on manifolds