The Monge problem in Metric Spaces

F. Cavalletti - S. B.

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Outline

The Monge problem

Let (X, d) be a Polish space, $d_L : X \times X \to [0, +\infty]$ a Borel distance on X such that (X, d_L) is a geodesic space:

$$d_L(x, y) = \min_{\mathrm{Lip}_{d_L}([0,1],X)} \left\{ \mathrm{Lenght}(\gamma), \gamma(0) = x, \gamma(1) = y \right\}.$$

Given $\mu, \nu \in \mathcal{P}(X)$, find $T : X \to X$ Borel map such that $T_\sharp \mu = \nu$ and

$$\int d_L(x, T(x))\mu(dx) = \min \left\{ \int d_L \pi, \pi \in \Pi(\mu, \nu) \right\},$$

where

$$\Pi(\mu, \nu) := \left\{ \pi \in \mathcal{P}(X \times X), (P_1)_\sharp \pi = \mu, (P_2)_\sharp \pi = \nu \right\}.$$

The fact that d_L is degenerate along geodesics (it is equivalent to the usual norm in $\mathbb{R}$) implies that optimal transference plans are not unique.
To avoid further degeneracies, we assume that (X, d_L) is not branching, i.e.

$$\forall r > 0 \Big(d_L(x,y) = \frac{r}{2} \;\Rightarrow\; \sharp\{B_{d_L}(x,r) \cap B_{d_L}(y, r/2)\} = 1 \Big).$$

To have a strong consistent disintegration of the transport problem along the geodesics, it is further assumed that if γ is a geodesic the $\gamma \in C(\mathbb{R}, (X,d))$ and

$$\forall t \exists r \Big(\gamma(\mathbb{R}) \cap \bar{B}_d(\gamma(t), r) \in \mathcal{K}(X) \Big),$$

where $\mathcal{K}(X)$ is the family of compact sets of X.

The Wiener space

A prototype of these spaces is

$$X = \ell^2, \quad d(x,y) = \|x-y\|_{\ell^2}, \quad d_L(x,y) = \|x-y\|_{h^1}.$$

The fact that $d \leq d_L$ and that geodesics of infinite length are straight lines implies that the conditions on the geodesics are automatically satisfied.
Note that the disintegration ℓ^2/h^1 is not strongly consistent: otherwise from the fact $d_L < +\infty$ one obtains the existence of a potential ϕ.
Remarks. The space (X,d) plays a support role, in order to use the standard measure theory.
The construction works for d_L-cyclically monotone sets, not necessarily optimal.

Transport rays

Let $\Gamma \subset X \times X$ be a d_L-cyclically monotone set, and define

$$\Gamma' := \Big\{(x,y) : \exists I \in \mathbb{N}_0, (w_i, z_i) \in \Gamma \text{ for } i = 0, \dots, I,\ z_I = y$$

$$w_{I+1} = w_0 = x,\ \sum_{i=0}^{I} d_L(w_{i+1}, z_i) - d_L(w_i, z_i) = 0\Big\},$$

$$G := \Big\{(x,y) : \exists (w,z) \in \Gamma', d_L(w,x) + d_L(x,y) + d_L(y,z) = d_L(w,z)\Big\}.$$

Both sets are d_L-cyclically monotone (by the triangle inequality) and of Souslin class: Γ' concatenate points in Γ which belongs to the same geodesic and G takes all the points of each geodesic. The set G replaces the set $\{\phi(x) - \phi(y) = d_L(x,y)\}$ in the case a potential ϕ exists: clearly we cannot say that every optimal transport satisfies $\pi(G) = 1$.

For each $x \in X$, the set $G(x)$ is the set of geodesics used by the transference plan exiting from x, while $G^{-1}(x)$ are the geodesics arriving in x.
Define the Souslin sets

$$\mathcal{T}_e := P_1\big(G^{-1} \setminus \{x = y\}\big) \cup P_1\big(G \setminus \{x = y\}\big),$$
$$\mathcal{T} := P_1\big(G^{-1} \setminus \{x = y\}\big) \cap P_1\big(G \setminus \{x = y\}\big).$$

The first set is made of points $z \in X$ such that there exists $(x, y) \in G$ and z belongs to a geodesic connecting x to y.
The second set instead requires also that $z \neq x, y$.
The assumption that d_L is not branching implies
Lemma. *If $x \in \mathcal{T}$, then $R(x) := G(x) \cup G^{-1}(x)$ is a single geodesic.*
In particular R is an equivalence relation on $\mathcal{T}$, while G is a partial order relation on $\mathcal{T}_e$.

Define the multivalued *endpoint graphs* by:

$$a := \big\{(x,y) \in G^{-1} : G^{-1}(y) \setminus \{y\} = \emptyset\big\},$$
$$b := \big\{(x,y) \in G : G(y) \setminus \{y\} = \emptyset\big\}.$$

We call $P_2(a)$ the set of *initial points* and $P_2(b)$ the set of *final points*.

The following holds:

1. $a \cap b \cap \mathcal{T}_e \times X = \emptyset$;
2. $a(x)$, $b(x)$ are singleton or empty when $x \in \mathcal{T}$;
3. $a(\mathcal{T}) = a(\mathcal{T}_e)$, $b(\mathcal{T}) = b(\mathcal{T}_e)$;
4. $\mathcal{T}_e = \mathcal{T} \cup a(\mathcal{T}) \cup b(\mathcal{T})$, $\mathcal{T} \cap (a(\mathcal{T}) \cup b(\mathcal{T})) = \emptyset$.

In particular we can assume

$$\mu(b(\mathcal{T})) = \nu(a(\mathcal{T})) = 0.$$

By a countable partition, we reduce the disintegration problem to the following case: for x_i dense, $j, k \in \mathbb{N}$

$$\begin{aligned}\mathcal{T}' := \Big\{ x \in \mathcal{T} \cap \bar{B}(x_i, 2^{-j}) : \; & L(G(x)), L(G^{-1}(x)) \geq 2^{2-k}, \\ & L\big(R(x) \cap \bar{B}(x_i, 2^{1-j})\big) \leq 2^{-k} \\ & \bar{B}(x_i, 2^{-j}) \cap R(x) \text{ is compact} \Big\}\end{aligned}$$

The map

$$\mathcal{T}' \ni x \mapsto R(x) \cap B(x_i, 2^{-j})$$

is thus universally measurable and with compact sections: by Kuratowski-Ryll-Nardzewski selection principle, there exists a universally measurable selection $f : \mathcal{T}' \to B(x_i, 2^{-j})$.
In particular, the disintegration

$$\mu \llcorner_{\mathcal{T}} = \int \mu_y m(dy), \quad m := f_\sharp \mu \llcorner_{\mathcal{T}}$$

satisfies $\mu_y(f^{-1}(y)) = 1$. i.e. it is strongly consistent.

Let $\mathcal{S} := f(\mathcal{T})$ be the corresponding cross section.
Define the *ray map* g by the formula

$$
\begin{aligned}
g := & \Big\{(y,t,x) : y \in \mathcal{S}, t \in [0,+\infty), x \in G(y) \cap \{d_L(x,y) = t\}\Big\} \\
& \cup \Big\{(y,t,x) : y \in \mathcal{S}, t \in (-\infty,0), x \in G^{-1}(y) \cap \{d_L(x,y) = -t\}\Big\} \\
= & \, g^+ \cup g^-.
\end{aligned}
$$

Proposition. *The following holds.*

1. *The set g is the graph of a map with range $\mathcal{T}_e$.*
2. *$t \mapsto g(y,t)$ is d_L 1-Lipschitz G-order preserving.*
3. *$(t,y) \mapsto g(y,t)$ is bijective on $\mathcal{T}$, and its inverse is*

$$
x \mapsto g^{-1}(x) = \big(f(y), \pm d_L(x, f(y))\big).
$$

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For $A \subset \mathcal{T}_e$, $t \in \mathbb{R}$ define the *t-evolution* A_t *of* A by

$$A_t := g\big(g^{-1}(A) + (0,t)\big).$$

If A is Souslin, then A_t is Souslin, and $t \mapsto \mu(A_t)$ is Souslin.
Theorem. *Assume that for all Borel sets* A *such that* $\mu(A) > 0$ *the set* $\{t \in \mathbb{R}^+ : \mu(A_t) > 0\}$ *has cardinality* $> \aleph_0$. *Then* μ *is concentrated on* $\mathcal{T}$ *and the conditional probabilities of the disintegration*

$$\mu = \int \mu_y m(dy), \quad m := f_\sharp m$$

are continuous.
The key argument is that one can reduce the problem to a single δ along each geodesic.

Under a stronger assumption we obtain the absolute continuity of the conditional probabilities.

Theorem. *Assume that for every Borel set* $A \subset \mathcal{T}_e$

$$\mu(A) > 0 \quad \Longrightarrow \quad \int_0^{+\infty} \mu(A_t) dt > 0.$$

Then for m-a.e. $y \in \mathcal{S}$ *the conditional probabilities* μ_y *are absolutely continuous w.r.t.* $\mathcal{H}^1_{R(y)}$.

The Hausdorff measure is constructed by using the metric d_L.

Proof. The argument follows from the following contradiction: if C, $\mu(C) > 0$ and $\mathcal{L}^1(C) = 0$, then

$$0 < \int \mu(C_t) dt = \mu \times \mathcal{L}^1(\{x - t \in C\}) = \int \mathcal{L}^1(x - C)\mu(dx) = 0.$$

If $d_L = d$ and (X, d, η) satisfies $MCP(K, N)$, then we have

Proposition. *The η-measure of the end points $a(\mathcal{T}) \cup b(\mathcal{T})$ is 0 and the disintegration*

$$\eta = \int \eta_y m(dy), \quad m := f_\sharp \eta \llcorner_{\mathcal{T}}$$

satisfies

$$\eta_y = q(y)\mathcal{H}^1 \llcorner_{R(y)}, \quad q(y,t) \geq \left\{ \frac{s_K(d(g(y,t),\bar{x}))}{s_K(d(g(y,s),\bar{x}))} \right\}^{N-1} q(y,s),$$

where

$$s_K(t) := \begin{cases} (1/\sqrt{K})\sin(\sqrt{K}t) & \text{if K} > 0, \\ t & \text{if K} = 0, \\ (1/\sqrt{-K})\sinh(\sqrt{-K}t) & \text{if K} < 0. \end{cases}$$

If (X, d_L, η) has Ricci curvature $\geq R$, then we have

Theorem. *If the disintegration*

$$\eta = \int \eta_y m(dy), \quad m := f_\sharp \eta \llcorner_{\mathcal{T}}$$

satisfies $\eta_y = q(y)\mathcal{H}^1 \llcorner_{R(y)}$, *then*

$$\frac{d^2}{dt^2} \log q(y,t) \leq -R.$$

The absolute continuity of the disintegration of η are invariant under Measure Gromov-Hausdorff convergence, if the Ricci curvature is bounded from below.

Using the fact that the space is non-branching, one can prove uniqueness of the transport set for the optimal transference plan.

Proposition. *If μ, ν are concentrated on $\mathcal{T}$ and μ_y, ν_y are continuous, then G is a carriage for all optimal transference plans.* More precisely, if

$$\mu = \int \mu_y m(dy), \quad \nu = \int \nu_y m(dy), \quad m := f_\sharp \mu = f_\sharp \nu,$$

then every optimal transference plan π can be represented as

$$\pi = \int \pi_y m(dy), \quad \pi_y \in \Pi(\mu_y, \nu_y).$$

In particular it is easy to solve the Monge problem, just piecing together the 1-d monotone rearrangements along each geodesic.

If $d \leq d_L$, $\mu \leq \eta$ and η has absolutely continuous disintegration, we can solve the equation

$$\partial U = \mu - \nu$$

in the sense of currents.
Define the *flow* $\dot{g}$ as

$$\langle \dot{g}, (h, \omega) \rangle = \int_{S \times \mathbb{R}} h(g(y,t)) \partial_t \omega(g(y,t)) q(y,t) dt m(dy)$$

where h, ω are Lipschitz functions of (X, d) with h bounded, and

$$\eta = \int q \mathcal{H}^1_{R(y)} m(dy).$$

If $t \mapsto q(y,t)$ is BV with m-integrable total variation, then $\dot{g}$ is a normal current: this is the case of MCP and Ricci curvature bounds far from the end points.

Under the continuity of the disintegration of η, a solution to $\partial U = \mu - \nu$ is given by the current U defined as

$$\langle U, (h, \omega) \rangle = \int_S \left(\int_{\mathbb{R}} (F(y,t) - H(y,t)) h(g(y,t)) \partial_t \omega(g(y,t)) dt \right) m(dy),$$

where

$$H(y,t) := \mu_y \big(g(y, (-\infty, t)) \big),$$
$$F(y,t) := \nu_y \big(g(y, (-\infty, t)) \big).$$

Outline

S. Bianchini and F. Cavalletti.
The Monge problem for distance cost in geodesic spaces.

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The Monge problem in metric spaces with curvature bounds.

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A strategy for non-strictly convex distance transport cost and the obstacle problem.

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The Monge problem in Wiener space.