On the extremality, uniqueness and optimality of transference plans

L. Caravenna, S.B.

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Transference plans

Let $\mu, \nu \in \mathcal{P}([0,1])$.

Definition (Transference plans)

The *set of transference plans between* μ *and* ν is

$$\Pi(\mu,\nu) := \Big\{ \pi \in \mathcal{P}([0,1]^2) : (P_1)_\sharp \pi = \mu \ \wedge \ (P_2)_\sharp \pi = \nu \Big\}.$$

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We will denote the *measurable sets* of a Borel measure ξ as Θ_ξ, and the $\Pi(\mu,\nu)$*-universally measurable sets* as

$$\Theta(\mu,\nu) := \cap \big\{ \Theta_\pi : \pi \in \Pi(\mu,\nu) \big\}.$$

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Optimality for a fixed $\Theta(\mu, \nu)$-measurable cost $c : [0, 1]^2 \to [0, \infty]$, find sufficient conditions on a transference plan π such that

$$\int c\pi = \inf\left\{\int c\pi : \pi \in \Pi(\mu, \nu)\right\}$$

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one sees a similar structure in the above problems:

there is a set A, a measure π with $\pi(A) = 1$ and its carriage $\Gamma \subset A$.

Admissible perturbations

The constraints

$$\mu = (P_1)_\sharp \pi \quad \nu = (P_2)_\sharp \pi$$

imply that the admissible perturbations are in the set

$$\Lambda := \Big\{ \lambda \in \mathcal{M}([0,1]) : (P_1)_\sharp \lambda = (P_2)_\sharp \lambda = 0 \Big\}.$$

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Finally we have to require that

$$\pi + \lambda \text{ is concentrated on } A.$$

Cyclical perturbations

A particular family of perturbations are the *n-cyclical perturbations*: for $n \in \mathbb{N}$ and $m \in \mathcal{M}^+([0,1]^{2n})$, then

$$\lambda = \int_{[0,1]^{2n}} \frac{1}{n} \sum_{i=1}^{n} \big(\delta_{(x_{i+1 \mod n}, y_i)} - \delta_{(x_i, y_i)}\big) m(dx_1 dy_1 \dots dx_n dy_n).$$

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In order to have λ concentrated on A, one requires

$$m\Big(\Big\{(x_1, y_1, \dots, x_n, y_n) : (x_i, y_i), (x_{i+1 \mod n}, y_i) \in A\Big\}\Big) = 1.$$

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In order to have $\pi + \lambda \geq 0$, one requires

$$\frac{1}{n} \sum_{i=1}^{n} (P_{(x_i, y_i)})_\sharp m \leq \pi.$$

Acyclicity and cyclical monotonicity

The analysis if a set Γ can carry perturbations of the above types leads naturally to the following definition for sets.

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c-cyclically monotone $\forall n \in \mathbb{N}, \{(x_i, y_i)\}_{i=1}^n \subset \Gamma$

$$\sum_{i=1}^{n} \big(c(x_{i+1 \mod n}, y_i) - c(x_i, y_i)\big) \geq 0.$$

In [5] it is proved that if A is analytic, then

$$\sup \left\{ \xi(A), (P_i)_\sharp \xi = \eta_i, i = 1, \dots, n \right\} = \inf \left\{ \sum_{i=1}^{n} \int h_i \eta_i, \sum_{i=1}^{n} h_i \geq \chi_A \right\},$$

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The following holds:

- **Extremality** π *extremal* $\Rightarrow \exists \Gamma \subset [0,1]^2$ σ*-compact acyclic carriage*
- **Uniqueness** A *analytic,* π *extremal* $\Rightarrow \exists \Gamma \subset [0,1]^2$ σ*-compact* A*-acyclic carriage*
- **Optimality** $-c$ *Souslin,* π *optimal* $\Rightarrow \exists \Gamma \subset [0,1]^2$ σ*-compact* c*-cyclically monotone carriage*

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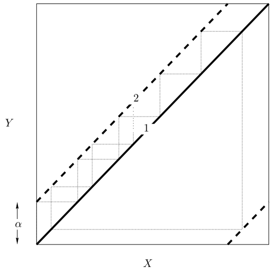

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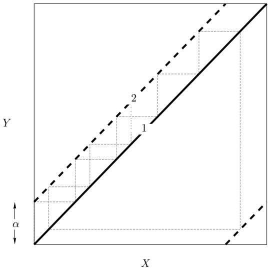

Γ is acyclic in A, but the measure

$$(x, x + \alpha \mod 1)_\sharp \mathcal{L}^1$$

is nor unique in $\Pi^f(\mathcal{L}^1, \mathcal{L}^1)$ nor optimal for

$$c(x,y) = \begin{cases} 1 & y = x \\ 2 & y = x + \alpha \mod 1 \\ +\infty & \text{otherwise} \end{cases}.$$

Disintegration Theorem [3]

Let $\{X_\alpha\}_{\alpha \in \mathtt{A}}$ be a partition of $[0,1]$ and $\mu \in \mathcal{P}([0,1])$. Let $h : X \to \mathtt{A}$ be the quotient map and

$$m = h_\sharp \mu, \text{ meaning that } h^{-1}(B) \in \Theta_\mu \ \big(m(B) = \mu(h^{-1}(B))\big).$$

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Definition (Disintegration)

The *disintegration* of μ *consistent* with $\{X_\alpha\}_{\alpha \in \mathtt{A}}$ is a map $\alpha \mapsto \mu_\alpha$

1. for all $B \in \mathcal{B}$, $\mu_\alpha(B)$ is m-measurable;
2. for all $B \in \mathcal{B}$, $A \in \Theta_m$,

$$\mu(B \cap h^{-1}(A)) = \int_A \mu_\alpha(B) m(d\alpha).$$

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We say that the disintegration is *strongly consistent* if $\mu_\alpha(X_\alpha) = 1$.

A theorem on linear preorders

Definition (Preorder)

$R \subset [0,1]^2$ is a *preorder* if

$$(x,y),(y,z) \in R \;\Rightarrow\; (x,z) \in R.$$

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Theorem

If $R \in \Theta(\mu,\mu)$ is a linear preorder $\preccurlyeq$ on $[0,1]$, then the disintegration w.r.t. E is strongly consistent, and the image set B' in the quotient space is a set of uniqueness of $\Pi(m,m)$.

A natural preorder relation

Definition (Axial preorder)

We say that $x \preccurlyeq x'$, $x, x' \in \Gamma$, if $\exists n \in \mathbb{N}, \{(x_i, y_i)\}_{i=1}^n \subset \Gamma$ s.t.

$$\{(x_{i+1 \mod n}, y_i)\}_{i=1}^n \subset A \ \wedge \ \big(x_1 = x, x_{n+1} = x'\big).$$

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The equivalence relation E is the closed cycles relation: $(x, x') \in E$ iff $\exists \{(x_{i+1 \mod n}, y_i)\}_{i=1}^n \subset \Gamma$

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The relation E and the set Γ satisfy a *crosswise relation*: if $Y_\alpha := P_2(\Gamma \cap X_\alpha \times [0,1])$ then

$$\Gamma \cap X_\alpha \times [0,1] = \Gamma \cap [0,1] \times Y_\alpha = \Gamma \cup X_\alpha \times Y_\alpha = \Gamma_\alpha. \qquad (2)$$

A sufficient condition

Theorem (Sufficient condition)

Assume that Γ is acyclic/A-acyclic/c-cyclically monotone and the axial preorder $\preccurlyeq$ can be extended into a $\Theta(\mu, \mu)$-measurable linear preorder. Then the transference plan π concentrated on Γ is extremal/unique/optimal.

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Sketch of the proof

Step 1. Theorem 6 implies that the disintegration $\mu = \int \mu_\alpha m(d\alpha)$ is strongly supported. The crosswise structure (2) of Γ yields that the same happens for ν and π:

$$\nu = \int \nu_\alpha m(d\alpha), \quad \pi = \int \pi_\alpha m(d\alpha), \quad \pi_\alpha \in \Pi(\mu_\alpha, \nu_\alpha).$$

Step 2. If h_X, h_Y are the quotient maps, then by assumption

$$A' := (h_X \otimes h_Y)(A) \in \Theta(m, m)$$

can be extended to a linear order of class $\Theta(m, m)$,

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In the original space this means that the disintegration of any $\pi' \in \Pi^f(\mu, \nu)$ w.r.t. $h_X \otimes h_y$ is given by

$$\pi' = \int \pi'_{\alpha\beta} n(d\alpha d\beta) = \int \pi'_\alpha m(d\alpha), \quad \pi'_\alpha = \pi'_{\alpha\alpha}.$$

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acyclicity has a *Borel countable limb structure* [4]:
$\exists \{C_k\}_{k\in\mathbb{N}}, \{D_k\}_{k\in\mathbb{N}_0}$ Borel and Borel functions $f_k : C_k \to D_{k-1}$, $g_k : D_k \to C_k$, such that π is concentrated on the union of the following graphs

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It follows that π_α is extremal/unique/optimal in $\Pi(\mu_\alpha, \nu_\alpha)$.

Step 4. Finally, Step 2 implies that all perturbations occurs only in the equivalence classes $\{\Gamma_\alpha\}_{\alpha \in \mathtt{A}}$, and Step 3 implies that our measure is extremal/unique/optimal in each class.

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