On continuous solutions to scalar balance laws

G. Alberti, L. Caravenna, S.B.

December 4, 2012

Table of Contents

Introduction

We consider the balance law

$$u_t + f(u)_x = g(t,x) \in L^\infty(\mathbb{R}^2), \quad u \in C(R^2, R), \ f : \mathbb{R} \to \mathbb{R}. \quad (1)$$

If u is smooth and g continuous, then the PDE is equivalent to

$$u_t + \lambda(u)u_x = g, \quad \lambda := \frac{df}{du}$$

$$\frac{d\gamma}{dt} = \lambda(u), \quad \frac{d}{dt}u(t,\gamma(t)) = g(t,\gamma(t)). \quad (2)$$

The converse is also true: a smooth solution $u = u(t,x)$ of the above ODE yields a solution to the PDE.

Introduction

We consider the balance law

$$u_t + f(u)_x = g(t,x) \in L^\infty(\mathbb{R}^2), \quad u \in C(R^2, R),\ f : \mathbb{R} \to \mathbb{R}. \quad (1)$$

If u is smooth and g continuous, then the PDE is equivalent to

$$u_t + \lambda(u) u_x = g, \quad \lambda := \frac{df}{du}$$

$$\frac{d\gamma}{dt} = \lambda(u), \quad \frac{d}{dt} u(t, \gamma(t)) = g(t, \gamma(t)). \quad (2)$$

The converse is also true: a smooth solution $u = u(t,x)$ of the above ODE yields a solution to the PDE.

We are interested what of the above equivalence is valid under the assumptions u continuous and g bounded Borel function.

Remark 1

By the finite speed of propagation, the results can be restated locally.

Connection to geometry

This problem arises when one considers intrinsic Lipschitz graphs in the Heisenberg group (w_1, w_2, z), with

$$W_1 = \partial_{w_1} - \frac{1}{2} w_2 \partial_z, \quad W_2 = \partial_{w_2} + \frac{1}{2} w_1 \partial_z.$$

In this setting, if $w_1 = w_1(w_2, z)$ is a (local) parameterization

- the distributional derivative is

$$\partial_{w_2} w_1 + \partial_z \left(\frac{w_1^2}{2} \right) \in L^\infty(\mathbb{R}^2),$$

- the derivative along geodesics γ in the intrinsic distance is

$$\frac{d}{dy} u(y, \gamma(y)) \in L^\infty(\mathbb{R}), \quad \frac{d\gamma}{dy} = u(y, \gamma(y)).$$

Connection to geometry

This problem arises when one considers intrinsic Lipschitz graphs in the Heisenberg group (w_1, w_2, z), with

$$W_1 = \partial_{w_1} - \frac{1}{2} w_2 \partial_z, \quad W_2 = \partial_{w_2} + \frac{1}{2} w_1 \partial_z.$$

In this setting, if $w_1 = w_1(w_2, z)$ is a (local) parameterization

- the distributional derivative is

$$\partial_{w_2} w_1 + \partial_z \left(\frac{w_1^2}{2} \right) \in L^\infty(\mathbb{R}^2),$$

- the derivative along geodesics γ in the intrinsic distance is

$$\frac{d}{dy} u(y, \gamma(y)) \in L^\infty(\mathbb{R}), \quad \frac{d\gamma}{dy} = u(y, \gamma(y)).$$

The equivalence of the two definitions reduces to prove

$$u \text{ solves the balance law} \quad \Leftrightarrow \quad u \text{ Lipschitz along characteristics.}$$

Problems we study

We will consider the relations among the following statements: for general smooth flux f

1. *u distributional solution*

$$u_t + f(u)_x = g(t,x) \in L^\infty(\mathbb{R}^2),$$

2. *u broad solution*

$$\text{if } \gamma \ \big(\dot\gamma = \lambda(u(t,\gamma))\big) \quad \Rightarrow \quad \frac{d}{dt} u \circ \gamma = \tilde g_\gamma(t) \in L^\infty(\mathbb{R}^+),$$

3. *u Lagrangian solution*: for all point $(\bar t, \bar x)$ there exists at least one characteristic γ, $\gamma(\bar t) = \bar x$, such that

$$\frac{d}{dt} u(t, \gamma(t)) \in L^\infty(\mathbb{R}^+),$$

4. there exists a *universal Borel source* $\hat g : \mathbb{R}^2 \to \mathbb{R}$

$$\int_{\mathbb{R}^2} |g - \hat g| \mathcal{L}^2 = 0 \quad \text{and} \quad \int_{\mathbb{R}} \big|\tilde g_\gamma(t) - \hat g(t, \gamma(t))\big| dt = 0.$$

Table of Contents

The case g continuous and f convex

If γ is a characteristic, the balance of $\mathrm{div}_{t,x}(u, f(u))$ in the region

$$\Gamma^\epsilon := \left\{t \in [t_1, t_2], \gamma(t) \leq x \leq \gamma(t) + \epsilon\right\}$$

yields

$$\begin{aligned}\int_{\Gamma^\epsilon} g(t,x)dtdx = & \int_0^\epsilon \big(u(t_2, \gamma(t_2) + x) - u(t_1, \gamma(t_1) + x)\big) dx \\ & + \int_{t_1}^{t_2} \Big[f(u(t, \gamma(t) + \epsilon)) - f(u(t, \gamma(t)) \\ & \quad - \lambda(u(t, \gamma(t))(u(t, \gamma(t) + \epsilon) - u(t, \gamma(t)))\Big] dt \\ \geq & \int_0^\epsilon \big(u(t_2, \gamma(t_2) + x) - u(t_1, \gamma(t_1) + x)\big) dx,\end{aligned}$$

because $f(u') \geq f(u) + \lambda(u)(u' - u)$ by convexity.

The balance on the region

$$\Gamma^{-\epsilon} := \{t \in [t_1, t_2], \gamma(t) - \epsilon \le x \le \gamma(t)\}$$

yields the opposite inequality

$$\int_{\Gamma^{-\epsilon}} g(t,x)dtdx \le \int_{-\epsilon}^{0} \big(u(t_2, \gamma(t_2) + x) - u(t_1, \gamma(t_1) + x)\big)dx.$$

Dividing by ϵ and letting $\epsilon \to 0$ one recovers

$$u(t_2, \gamma(t_2)) - u(t_1, \gamma(t_1)) = \int_{t_1}^{t_2} g(t, \gamma(t))dt,$$

which implies

$$\frac{d}{dt} u \circ \gamma = g(t, \gamma(t)).$$

Proposition 1 (Dafermos)

If f convex, g continuous then $\hat{g} = g$.

A counterexample

Let f be strictly increasing, and such that the set

$$N := \{u : f'(u) = f''(u) = 0\} \quad \text{satisfies} \quad \mathcal{L}^1(N) > 0.$$

Define

$$\tilde{f}(u) = f\big(u + \mathcal{L}^1(N \cap [0,u])\big), \quad \tilde{f}'(u) = f'(f^{-1}(\tilde{f}(u))).$$

The the function $u(x) := f^{-1}(x)$ is a solution to $u_t + f(u)_x = 1$, and the curve $\gamma(t) := \tilde{f}(t)$ is a characteristic:

$$\dot{\gamma} = \tilde{f}'(t) = f'(f^{-1}(\tilde{f}(t))) = f'(u(\gamma(t))).$$

However

$$\frac{d}{dt} f^{-1}(\tilde{f}(t)) = \mathcal{L}^1 + f_\sharp \mathcal{L}^1 \llcorner_N, \quad f_\sharp \mathcal{L}^1 \llcorner_N \perp \mathcal{L}^1.$$

Given f, partition $\mathbb{R}$ into

1. a countable family of disjoint open sets $\{I_i = (u_i^-, u_i^+)\}_{i\in\mathbb{N}}$ where $f\llcorner_{I_i}$ is either convex or concave,
2. a residual set of inflection points $\mathfrak{I}$.

Theorem 1

If $\mathcal{L}^1(\mathfrak{I}) = 0$, then u is Lipschitz along each characteristic.

Given f, partition $\mathbb{R}$ into

1. a countable family of disjoint open sets $\{I_i = (u_i^-, u_i^+)\}_{i\in\mathbb{N}}$ where $f\llcorner_{I_i}$ is either convex or concave,
2. a residual set of inflection points $\mathfrak{I}$.

Theorem 1

If $\mathcal{L}^1(\mathfrak{I}) = 0$, then u is Lipschitz along each characteristic.

Thus

$$u \text{ distributional solution} \quad \overset{\mathcal{L}^1(\mathfrak{I})=0}{\Longrightarrow} \quad u \text{ broad solution}$$

otherwise counterexamples.

Proof.

Proposition 1 implies that

$$u \circ \gamma(t_1), u \circ \gamma(t_2) \in \overline{I}_i \ \Big(\big| u \circ \gamma(t_2) - u \circ \gamma(t_1) \big| \leq |t_2 - t_1| \Big).$$

Since $\mathcal{L}^1(\mathfrak{I}) = 0$, for $v^t := u \circ \gamma(t)$, $t_1 < t_2$, $I_{i_2} \ni v^{t_2} \geq v^{t_1} \in I_{i_1}$

$$\begin{aligned} v^{t_2} - v^{t_1} &= \mathcal{L}^1\big([v^{t_1}, v^{t_2}]\big) = \bigcup_i \mathcal{L}^1\big([v^{t_1}, v^{t_2}] \cap I_i\big) \\ &= v^{t_2} - u_{i_2}^- + \sum_{I_i \subset [v^{t_1}, v^{t_2}]} \big(u_i^+ - u_i^-\big) + u_{i_1}^+ - v^{t_1} \\ &= v^{t_2} - v^{t_{i_2}^-} + \sum_{I_i \subset [v^{t_1}, v^{t_2}]} \big(v^{t_i^+} - v^{t_i^-}\big) + v^{t_{i_1}^+} - v^{t_1} \\ &\leq t_2 - t_{i_2}^- + \sum_{I_i \subset [v^{t_1}, v^{t_2}]} \big(t_i^+ - t_i^-\big) + t_{i_1}^+ - t_1 \leq t_2 - t_1. \end{aligned}$$

□

Table of Contents

Monotone flow

Consider the continuous ODE in $\mathbb{R}$

$$\dot{x} = \lambda(t, x). \tag{3}$$

Proposition 2

There exists a continuous flow $\chi(t, y)$ such that

1. $t \mapsto \chi(t, y)$ *is a solution to* (3)*,*
2. $y \mapsto \chi(t, y)$ *is increasing.*

Proof.

For every point point $(\bar{t}, \bar{x})$ consider the curve

$$\gamma_{\bar{t},\bar{x}}(t) := \begin{cases} \max\{\gamma(t) : \gamma(\bar{t}) = \bar{x}\} & t \leq \bar{t}, \\ \min\{\gamma(t) : \gamma(\bar{t}) = \bar{x}\} & t \geq \bar{t}, \end{cases}$$

and choose suitable parameterization y. □

The proof can be repeated if we restrict to a family Γ of solutions of (3) such that

$$\gamma_n \in \Gamma \;\Rightarrow\; \min\{\gamma_n\}, \max\{\gamma_n\} \in \Gamma,$$

In particular, this holds if

$$\Gamma = \left\{ \gamma \text{ characteristic}, \left| \frac{d}{dt} u \circ \gamma(t) \right| \leq 1 \right\},$$

so that the property of being a Lagrangian solution can be rewritten as:

u is a Lagrangian solution if there exists a continuous flow $\chi(t, y)$ of solutions to $\dot{x} = \lambda(t, x)$ such that

$$\forall y \in \mathbb{R} \Big(t \mapsto u \circ \chi(t, y) \; 1 - \mathit{Lipschitz} \Big).$$

Monotone approximations

Fix now two characteristics $\chi(t, y_1) \leq \chi(t, y_2)$, solutions to $\dot{x} = \lambda(u(t,x))$, and define for $u(t, \chi(t, y_1)) \leq u(t, \chi(t, y_2))$

$$u'(t,x) = u(t, \chi(t, y_1)) \vee \big(u(t,x) \wedge u(t, \chi(t, y_2))\big)$$

where $\chi(t, y_1) \leq x \leq \chi(t, \bar{y}_2)$. Let now χ' be the monotone flow for u' in this interval.

Monotone approximations

Fix now two characteristics $\chi(t, y_1) \leq \chi(t, y_2)$, solutions to $\dot{x} = \lambda(u(t,x))$, and define for $u(t, \chi(t, y_1)) \leq u(t, \chi(t, y_2))$

$$u'(t,x) = u(t, \chi(t, y_1)) \vee \big(u(t,x) \wedge u(t, \chi(t, y_2))\big)$$

where $\chi(t, y_1) \leq x \leq \chi(t, \bar{y}_2)$. Let now χ' be the monotone flow for u' in this interval.

Fixing a characteristic curve $\chi'(t, y')$ in between, define

$$u''(t,x) = \begin{cases} u'(t,x) \wedge u'(t, \chi'(t, y')) & \chi(t, y_1) \leq x \leq \chi'(t, y'), \\ u'(t,x) \vee u'(t, \chi'(t, y')) & \chi'(t, y') < x \leq \chi(t, y_2), \end{cases}$$

and let χ'' be the new monotone flow with $\chi''(t, y') = \chi'(t, y')$.

Monotone approximations

Fix now two characteristics $\chi(t, y_1) \leq \chi(t, y_2)$, solutions to $\dot{x} = \lambda(u(t,x))$, and define for $u(t, \chi(t, y_1)) \leq u(t, \chi(t, y_2))$

$$u'(t,x) = u(t, \chi(t, y_1)) \vee \big(u(t,x) \wedge u(t, \chi(t, y_2))\big)$$

where $\chi(t, y_1) \leq x \leq \chi(t, \bar{y}_2)$. Let now χ' be the monotone flow for u' in this interval.

Fixing a characteristic curve $\chi'(t, y')$ in between, define

$$u''(t,x) = \begin{cases} u'(t,x) \wedge u'(t, \chi'(t, y')) & \chi(t, y_1) \leq x \leq \chi'(t, y'), \\ u'(t,x) \vee u'(t, \chi'(t, y')) & \chi'(t, y') < x \leq \chi(t, y_2), \end{cases}$$

and let χ'' be the new monotone flow with $\chi''(t, y') = \chi'(t, y')$. By repeating countably many times, we obtain a function u^{mon} such that $x \mapsto u^{\mathrm{mon}}(t,x)$ increasing in the interval $\chi(t, y_1) \leq x \leq \chi(t, y_2)$, and

$$u \circ \gamma \text{ 1-Lipschitz} \quad \Rightarrow \quad u^{\mathrm{mon}} \circ \chi^{\mathrm{mon}} \text{ 1-Lipschitz.}$$

If $\chi^{\rm mon}$, $u^{\rm mon}$ are monotone, with $\dot{\chi}^{\rm mon} = \lambda(u^{\rm mon})$, then by writing

$$\int d_y u^{\rm mon}(t) dt = \int \upsilon_y(dt) m(dy),$$

one obtains $d_y \chi_t^{\rm mon} = \lambda'(u^{\rm mon}) d_y u^{\rm mon}(t) \in \mathcal{M}(\mathbb{R})$ and

$$\begin{aligned} \int d_y \chi^{\rm mon}(t) dt = & \int \bigg(\int_0^t \lambda'(u^{\rm mon}(s)) d_y u^{\rm mon}(s) ds \bigg) dt \\ = & \int \bigg(\int_0^t \lambda'(u^{\rm mon}(s)) \upsilon_y(ds) \bigg) m(dy) dt. \end{aligned}$$

Thus the disintegration of $\int d_y \chi^{\rm mon}(t) dt$ along characteristics is a.c. w.r.t. time (with bounded density $\int_0^t \lambda'(u^{\rm mon}(s)) \upsilon_y(ds)$).
Being the parameterization y arbitrary, we can take $m \leq \mathcal{L}^1$, and if

$$\chi^{{\rm mon},a}(t,y) = \chi^{\rm mon}(t,y) + ay \quad \big(\text{i.e. enlarging } [\chi(t,y_1), \chi(t,y_2)]\big)$$

we then have $a \leq \chi_y^{{\rm mon},a} \leq (1+a)$.

The balance for $\phi(t, \chi^{-1}(t, x))$ is estimated by

$$\int \left(\left(\phi_t - \lambda(u^{\mathrm{mon}})\phi_x\right) u^{\mathrm{mon}} + \phi_x f(u^{\mathrm{mon}}) \right) dxdt$$

$$= \int \phi_t u^{\mathrm{mon}} \chi_y dydt + \int \phi_y \left(f(u^{\mathrm{mon}}) - \lambda(u^{\mathrm{mon}}) u^{\mathrm{mon}} \right) dydt$$

$$= -\int \phi \frac{d}{dt} (u^{\mathrm{mon}} \circ \chi^{\mathrm{mon}}) \chi_y dydt$$

because if $u_y \in \mathcal{M}(\mathbb{R})$ continuous then

$$d_y(f(u) - \lambda(u)u) = -u\lambda'(u) d_y u = -u d_y \chi_t.$$

Proposition 3

If u is a 1-Lipschitz Lagrangian solution such that $x \mapsto u(t, x)$ is monotone, then is it also a distributional solution with source term $g \in [-1, 1]$.

By repeating this procedure on locally finitely many sheets

$$\mathbb{R}^2 = \cup_{j\in\mathbb{N}}\big[\chi(t, y_j), \chi(t, y_{j+1})\big]$$

we obtain a family of continuous locally BV solutions $u^{\{y_j\}}$ converging to u in C^0. Hence

Theorem 2

The function u is a distributional solution with source term g bounded by 1 in L^∞.

By repeating this procedure on locally finitely many sheets

$$\mathbb{R}^2 = \cup_{j\in\mathbb{N}}\big[\chi(t, y_j), \chi(t, y_{j+1})\big]$$

we obtain a family of continuous locally BV solutions $u^{\{y_j\}}$ converging to u in C^0. Hence

Theorem 2

The function u is a distributional solution with source term g bounded by 1 in L^∞.

Thus

$$u \text{ distributional} \quad \Longleftarrow \quad u \text{ Lagrangian} \quad \Longleftarrow \quad u \text{ broad.}$$

Remark 2

Since $u^{\{y_j\}} \in \mathrm{BV} \cap C^0$, then in the sense of measures

$$u_t^{\{y_j\}} + \lambda(u^{\{y_j\}})u_x^{\{y_j\}} = g^{\{y_j\}}\mathcal{L}^2.$$

Entropy equation

For continuous BV solution we have for $q' = \eta' \lambda$

$$\eta(u)_t + q(u)_x = \eta'(u)\big(u_t + \lambda(u)u_x\big) = \eta'(u)g(t,x), \qquad (4)$$

and since entropy solutions are stable w.r.t. strong convergence, we conclude that

Corollary 1

The solution u is entropic if $\mathcal{L}^1(\mathfrak{I}) = 0$.

Entropy equation

For continuous BV solution we have for $q' = \eta' \lambda$

$$\eta(u)_t + q(u)_x = \eta'(u)\big(u_t + \lambda(u)u_x\big) = \eta'(u)g(t,x), \qquad (4)$$

and since entropy solutions are stable w.r.t. strong convergence, we conclude that

Corollary 1

The solution u is entropic if $\mathcal{L}^1(\mathfrak{I}) = 0$.

In the general case, the entropy equation (4) holds if η is linear in a neighborhood of $\mathfrak{I}$. Since $\mathrm{int}\,\mathfrak{I} = \emptyset$, we can approximate every η with a family η^n linear in a neighborhood of $\mathfrak{I}$, and thus

Proposition 4

If u is a continuous solution to a balance laws with L^∞ source term, then it is entropic.

Entropy equation

For continuous BV solution we have for $q' = \eta' \lambda$

$$\eta(u)_t + q(u)_x = \eta'(u)\big(u_t + \lambda(u)u_x\big) = \eta'(u)g(t,x), \tag{4}$$

and since entropy solutions are stable w.r.t. strong convergence, we conclude that

Corollary 1

The solution u is entropic if $\mathcal{L}^1(\mathfrak{I}) = 0$.

In the general case, the entropy equation (4) holds if η is linear in a neighborhood of $\mathfrak{I}$. Since $\mathrm{int}\,\mathfrak{I} = \emptyset$, we can approximate every η with a family η^n linear in a neighborhood of $\mathfrak{I}$, and thus

Proposition 4

If u is a continuous solution to a balance laws with L^∞ source term, then it is entropic.

Remark 3

Since the equality holds, also $t \mapsto u(T-t,-x)$ is an entropy solution.

Continuity estimate in the strictly convex case

Let u be a broad solution and f strictly convex, and consider

$$u(t,x_1) = \bar{u} + v,\ \ u(t,x_2) = \bar{u} - v, \quad x_1 < x_2, v > 0.$$

To avoid the shock formation, the best situation is

$$u \circ \gamma_1(t+s) = \bar{u} + v - \|g\|_\infty s,\ \ u \circ \gamma_2(t+s) = \bar{u} - v + \|g\|_\infty s$$

$$\gamma_1 = x_1 + f(\bar{u}+v) - f(u \circ \gamma_1(t+s)),\ \gamma_2 = x_2 + f(u \circ \gamma_2(t+s)) - f(\bar{u}-v)$$

At the meeting point $u \circ \gamma_i = \bar{u}$, i.e.

$$x_2 - x_1 \geq f(u_1) + f(u_2) - 2f\left(\frac{u_1+u_2}{2}\right). \tag{5}$$

Lemma 1

If f is strictly convex, then u satisfies (5). In particular, if $f = u^2/2$, then u is $1/2$-Hölder continuous.

Table of Contents

Maximum principle for continuous solutions

If u_1, u_2 are two continuous solutions, then by viscosity approx.

$$u := \max\{u_1, u_2\}$$

satisfies

$$u_t + f(u)_x \leq \max\{g_1, g_2\}.$$

Since these are also entropy solutions when inverting time, then

$$\min\{g_1, g_2\} \leq u_t + f(u)_x \leq \max\{g_1, g_2\}.$$

Maximum principle for continuous solutions

If u_1, u_2 are two continuous solutions, then by viscosity approx.

$$u := \max\{u_1, u_2\}$$

satisfies

$$u_t + f(u)_x \le \max\{g_1, g_2\}.$$

Since these are also entropy solutions when inverting time, then

$$\min\{g_1, g_2\} \le u_t + f(u)_x \le \max\{g_1, g_2\}.$$

Let ω be the t, x-modulus of continuity of u: $\forall \delta > 0$, $u(\bar{t}, \bar{x})$ depends only on

$$L_{(\bar{t},\bar{x}),\pm\delta} := \Big\{\bar{t} \pm \delta\Big\} \times \Big\{\bar{x} \pm \lambda(\bar{u})\delta + \|\lambda'\|_\infty \big[-\delta\omega(\delta), \delta\omega(\delta)\big]\Big\}.$$

By maximum principle for continuous solutions, it follows that

$$\mathrm{dist}\big(\bar{u}, u(t \pm \delta, L_{(\bar{t},\bar{x}),\pm\delta})\big) \le \delta.$$

Repeating this procedure finitely many times, one constructs a sequence of points (t_k, x_k) such that

$$t_{k+1} = t_k + \delta, \ \left|x_{k+1} - x_k - \lambda(u(t_k, x_k))\delta\right| \leq \|\lambda'\|_\infty \delta\omega(\delta),$$

$$\left|u(t_{k+1}, x_{k+1}) - u(t_k, x_k)\right| \leq \delta.$$

Passing to the limit, by subsequences $\{(t_k, x_k)\}_k$ converges to the graph of a characteristic $\bar{\gamma}$, $\bar{\gamma}(\bar{t}) = \bar{x}$, and

$$t \mapsto u(t, \bar{\gamma}(t)) \quad 1 - \text{Lipschitz}.$$

Repeating this procedure finitely many times, one constructs a sequence of points (t_k, x_k) such that

$$t_{k+1} = t_k + \delta, \ \left| x_{k+1} - x_k - \lambda(u(t_k, x_k))\delta \right| \leq \|\lambda'\|_\infty \delta\omega(\delta),$$

$$\left| u(t_{k+1}, x_{k+1}) - u(t_k, x_k) \right| \leq \delta.$$

Passing to the limit, by subsequences $\{(t_k, x_k)\}_k$ converges to the graph of a characteristic $\bar{\gamma}$, $\bar{\gamma}(\bar{t}) = \bar{x}$, and

$$t \mapsto u(t, \bar{\gamma}(t)) \quad 1 - \text{Lipschitz}.$$

Theorem 3

Any distributional solution is a Lagrangian solution.

Thus

$$u \text{ distributional solution} \quad \Longleftrightarrow \quad u \text{ Lagrangian solution.}$$

Table of Contents

Uniqueness of $\{\tilde{g}_\gamma(t) : \gamma(t) = x\}$

The source term $\tilde{g}$ is a priori a function of the characteristic,

$$\tilde{G}(t,x) := \left\{\tilde{g}_\gamma(t) : \gamma(t) = x\right\} \quad \text{is a multifunction.}$$

Theorem 4
If $\mathcal{L}^1(\mathfrak{I}) = 0$, then up to a residual set N negligible along each characteristic, it holds

$$\sharp\{\tilde{g}(t) : \gamma(t) = x\} \leq 1.$$

For the proof, we subdivide the each interval I_i of convexity/concavity into

- closed intervals with non empty interior where f is linear,
- open intervals where f is strictly convex.

Proof.

We have to consider 3 cases.

Proof.

We have to consider 3 cases.

Inflection points. Since $\mathcal{L}^1(\mathfrak{I}) = 0$, for all $u \circ \gamma$ Lipschitz

$$\frac{d}{dt} u \circ \gamma \llcorner_{u \circ \gamma \in \mathfrak{I}} = 0 \quad \mathfrak{L}^1 - \text{a.e.}.$$

Proof.

We have to consider 3 cases.

Inflection points. Since $\mathcal{L}^1(\mathfrak{I}) = 0$, for all $u \circ \gamma$ Lipschitz

$$\frac{d}{dt} u \circ \gamma_{\llcorner u \circ \gamma \in \mathfrak{I}} = 0 \quad \mathfrak{L}^1 - \text{a.e.}.$$

Linear intervals. Begin λ constant, the characteristic curves do not overlaps so that $\tilde{g}$ is uniquely defined.

Proof.

We have to consider 3 cases.

Inflection points. Since $\mathcal{L}^1(\mathfrak{I}) = 0$, for all $u \circ \gamma$ Lipschitz

$$\frac{d}{dt} u \circ \gamma \llcorner_{u \circ \gamma \in \mathfrak{I}} = 0 \quad \mathfrak{L}^1 - \text{a.e.}.$$

Linear intervals. Begin λ constant, the characteristic curves do not overlaps so that $\tilde{g}$ is uniquely defined.

Strictly convex intervals. If $\tilde{g}$ is a Borel selection of $\tilde{G}$, since f is strictly convex, it is enough to prove that for fixed $\epsilon, \delta > 0$, $\bar{\gamma}$ the following set is negligible:

$$\Big\{ t : \underbrace{\frac{d}{dt} \lambda\big(u \circ \bar{\gamma}(t+s)\big) \leq \lambda(u \circ \gamma(t) + (\tilde{g} \circ \gamma(t) - \epsilon)s), |s| < \delta}_{\text{the derivative of } u \circ \gamma \text{ is } \leq \tilde{g} - \epsilon \text{ in a neighborhood of size } \delta} \Big\}.$$

The points in this set must have a distance of at least 2δ, otherwise at the crossing the curves $\tilde{\gamma}$ are transversal. □

Broad solution not differentiable $\mathcal{L}^2$-a.e. (t, x)

Since $g \in L^\infty$, then $g(t, \gamma(t))$ is meaningless, so that one cannot compute directly $\tilde{g}$ from g.

Broad solution not differentiable $\mathcal{L}^2$-a.e. (t, x)

Since $g \in L^\infty$, then $g(t, \gamma(t))$ is meaningless, so that one cannot compute directly $\tilde{g}$ from g.

On the other hand, it is possible to construct a solution u of the balance law with strictly convex flux f and source $g \in L^\infty$ such that

$$\mathcal{L}^2\left(\left\{(t,x) : \nexists\gamma\left(\dot{\gamma} = \lambda(u), \gamma(t) = x, \exists\frac{du \circ \gamma}{dt}(t)\right)\right\}\right) > 0.$$

Hence in general we cannot compute g directly from $\tilde{g}$, and the function g, $\tilde{g}$ live on different sets.

Broad solution not differentiable $\mathcal{L}^2$-a.e. (t, x)

Since $g \in L^\infty$, then $g(t, \gamma(t))$ is meaningless, so that one cannot compute directly $\tilde{g}$ from g.

On the other hand, it is possible to construct a solution u of the balance law with strictly convex flux f and source $g \in L^\infty$ such that

$$\mathcal{L}^2\Big(\Big\{(t,x) : \nexists \gamma\Big(\dot{\gamma} = \lambda(u), \gamma(t) = x, \exists \frac{du \circ \gamma}{dt}(t)\Big)\Big\}\Big) > 0.$$

Hence in general we cannot compute g directly from $\tilde{g}$, and the function g, $\tilde{g}$ live on different sets.

Remark 4

If $\mathcal{L}^1(\mathfrak{I}) \neq 0$, then in general the source depends on the Lagrangian flow χ, while for a given Lagrangian flow $\tilde{g}$ is unique.

Existence of a universal source $\hat{g}$

However the two functions are compatible: define in fact

$$\hat{g}(t,x) := \begin{cases} \tilde{g}(t,x) & \exists \tilde{g}(t,x), \\ g(t,x) & \text{otherwise.} \end{cases}$$

Theorem 5

It holds $\|\hat{g} - g\|_\infty = 0$.

Existence of a universal source $\hat{g}$

However the two functions are compatible: define in fact

$$\hat{g}(t,x) := \begin{cases} \tilde{g}(t,x) & \exists \tilde{g}(t,x), \\ g(t,x) & \text{otherwise.} \end{cases}$$

Theorem 5

It holds $\|\hat{g} - g\|_\infty = 0$.

Hence

there exists a universal source $\hat{g}$.

The result holds also in the general case, once we fix the Lagrangian flow χ.

Proof.

Since y is an arbitrary parameterization, we can assume that

$$(t, \chi^{-1}(t, y))_\sharp \mathcal{L}^2 = \int \xi_y(t) m(dy), \quad m(dy) \leq \mathcal{L}^1.$$

Thus the sets, where we need to compare g and $\tilde{g}$ are the sets which are not negligible for both, which means

$$d_y \chi(t, \chi^{-1}(t, x)) \sim a \in (0, \infty),$$
$$(t, x), (t, y = \chi^{-1}(t, x)) \text{ density point of } g, \tilde{g}, \text{ respectively.}$$

Proof.

Since y is an arbitrary parameterization, we can assume that

$$(t, \chi^{-1}(t, y))_\sharp \mathcal{L}^2 = \int \xi_y(t) m(dy), \quad m(dy) \leq \mathcal{L}^1.$$

Thus the sets, where we need to compare g and $\tilde{g}$ are the sets which are not negligible for both, which means

$$d_y \chi(t, \chi^{-1}(t, x)) \sim a \in (0, \infty),$$
$$(t, x), (t, y = \chi^{-1}(t, x)) \text{ density point of } g, \tilde{g}, \text{ respectively.}$$

For $\epsilon \ll 1$, in the set $(t, x) + [-\epsilon, \epsilon]^2$ one thus has

$$\lim_{h \to 0} \frac{1}{ah} \int_{-\epsilon}^{\epsilon} \chi(t+s, y \pm h) - \chi(t+s, y) ds = \pm 2\epsilon(1 + \mathcal{O}(\sqrt{\delta})),$$

$$\lim_{h \to 0} \frac{1}{ah} \bigg| \int_{-\epsilon}^{\epsilon} \int_{\chi(t,y)}^{\chi(t,y\pm h)} \big| g(t+s, z) - g(t, x) \big| dz ds \bigg| = \epsilon \mathcal{O}(\sqrt{\delta}),$$

up to a set of y of measure $\leq \mathcal{O}(\sqrt{\delta})$, hence $\tilde{g}$ is close to g. □

The uniformly convex case

In the case $f'' > 0$, then $\tilde{g}$ determines g completely.

Theorem 6 (Rademacher)

If f uniformly convex, then the set where $\tilde{g}$ is defined is of full Lebesgue measure in (t, x).

The uniformly convex case

In the case $f'' > 0$, then $\tilde{g}$ determines g completely.

Theorem 6 (Rademacher)

If f uniformly convex, then the set where $\tilde{g}$ is defined is of full Lebesgue measure in (t, x).

The above theorem can be extended to the following situation: there exists $p \geq 1$ such that for $\epsilon \ll 1$

$$\frac{1}{\epsilon^{2p}}\big(f(u+\epsilon v) - f(u) - \epsilon f'(u)v\big) \sim_{C^2} v^{2p}$$

Remark 5

The set where $p > 1$ has Lebesgue measure 0.

The uniformly convex case

In the case $f'' > 0$, then $\tilde{g}$ determines g completely.

Theorem 6 (Rademacher)

If f uniformly convex, then the set where $\tilde{g}$ is defined is of full Lebesgue measure in (t, x).

The above theorem can be extended to the following situation: there exists $p \geq 1$ such that for $\epsilon \ll 1$

$$\frac{1}{\epsilon^{2p}}\big(f(u+\epsilon v) - f(u) - \epsilon f'(u)v\big) \sim_{C^2} v^{2p}$$

Remark 5

The set where $p > 1$ has Lebesgue measure 0.

Hence

$$f \text{ uniformly convex} \quad \Longrightarrow \quad \tilde{g} = \hat{g} \ \mathcal{L}^2 - \text{a.e.}.$$

Proof for Burgers equation.

Proof for Burgers equation.

Step 1. The covering

$$Q^{\epsilon}_{t,x} := \Big\{ t \le s \le t + \epsilon/2, \chi(s, y_{x-\epsilon}) \le x \le \chi(s, y_{x+\epsilon}) \Big\}$$

satisfies Besicovitch covering property: in particular,

$$\lim_{\epsilon \to 0} \frac{1}{\mathcal{L}^2(Q^{\epsilon}_{t,x})} \int_{Q^{\epsilon}_{t,x}} |g(s,z) - g(t,x)| dsdz = 0 \quad \mathcal{L}^2 - \text{a.e. } (t,x).$$

Proof for Burgers equation.

Step 1. The covering

$$Q^{\epsilon}_{t,x} := \Big\{ t \leq s \leq t + \epsilon/2, \chi(s, y_{x-\epsilon}) \leq x \leq \chi(s, y_{x+\epsilon}) \Big\}$$

satisfies Besicovitch covering property: in particular,

$$\lim_{\epsilon \to 0} \frac{1}{\mathcal{L}^2(Q^{\epsilon}_{t,x})} \int_{Q^{\epsilon}_{t,x}} |g(s,z) - g(t,x)| ds dz = 0 \quad \mathcal{L}^2 - \text{a.e. } (t,x).$$

Step 2. In the above points, being $u(t,x)$ Lipschitz along characteristics and $1/2$-Hölder in x, the rescaling

$$u^{\epsilon}(\tau, z) := \frac{1}{\epsilon}\big(u(t + \epsilon s, x + \epsilon^2 z) - u(t,x)\big)$$

converges strongly to a solution to

$$u_s + \left(u^2/2\right)_z = g(t,x).$$

Proof for Burgers equation.

Step 1. The covering

$$Q^\epsilon_{t,x} := \Big\{ t \leq s \leq t + \epsilon/2, \chi(s, y_{x-\epsilon}) \leq x \leq \chi(s, y_{x+\epsilon}) \Big\}$$

satisfies Besicovitch covering property: in particular,

$$\lim_{\epsilon \to 0} \frac{1}{\mathcal{L}^2(Q^\epsilon_{t,x})} \int_{Q^\epsilon_{t,x}} |g(s,z) - g(t,x)| dsdz = 0 \quad \mathcal{L}^2 - \text{a.e. } (t,x).$$

Step 2. In the above points, being $u(t,x)$ Lipschitz along characteristics and 1/2-Hölder in x, the rescaling

$$u^\epsilon(\tau, z) := \frac{1}{\epsilon}\big(u(t + \epsilon s, x + \epsilon^2 z) - u(t,x)\big)$$

converges strongly to a solution to

$$u_s + \big(u^2/2\big)_z = g(t,x).$$

Step 3. Dafermos computation applies.

Table of Contents

L. Ambrosio, F. Serra Cassano, and D. Vittone.
Intrinsic Regular Hypersurfaces in Heisenberg Groups.

F. Bigolin, L. Caravenna, and F. Serra Cassano.
Intrinsic Lipschitz graphs in Heisenberg groups and continuous solutions of a balance equation.

F. Bigolin, and F. Serra Cassano.
Intrinsic regular graphs in Heisenberg groups vs. weak solutions of non linear first-order PDEs.

F. Bigolin, and F. Serra Cassano.
Distributional solutions of Burgers equation and Intrinsic regular graphs in Heisenberg groups.

C. Dafermos.
Continuous solutions for balance laws.

B. Franchi, R. Serapioni, and F. Serra Cassano.
Differentiability of intrinsic Lipschitz Functions within Heisenberg groups.