SBV regularity of Solutions to Strictly Hyperbolic Systems of Conservation Laws

L. Caravenna, S. B.

August 19, 2011

Outline

Several kinds of regularities

For the solutions to the system

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \; u \in \mathbb{R}^n,$$

it is possible to consider different regularity properties of u:

1. u is BV
2. decay properties of positive waves
3. differentiability of $t \mapsto u(t,\gamma(t))$, where γ is a characteristic
4. rectifiability properties of the jump set
5. fractional differentiability of L^∞-solutions
6. ...

Here we are interested in the absence of Cantorian part in the derivative u_x.

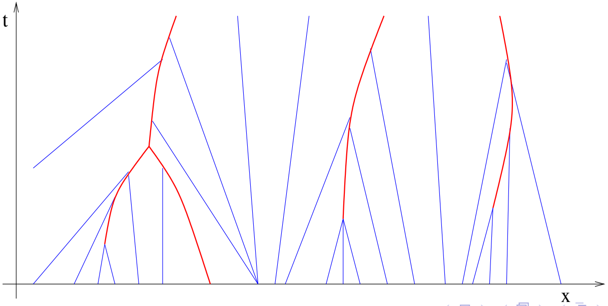
Structure of BV solutions to HCL

For solutions of strictly hyperbolic systems of conservation laws in one space dimension one expects the following structure:
countably many shock curves and regularity of the solution in the remaining set.

Structure of BV solutions to HCL

For solutions of strictly hyperbolic systems of conservation laws in one space dimension one expects the following structure:
countably many shock curves and regularity of the solution in the remaining set.

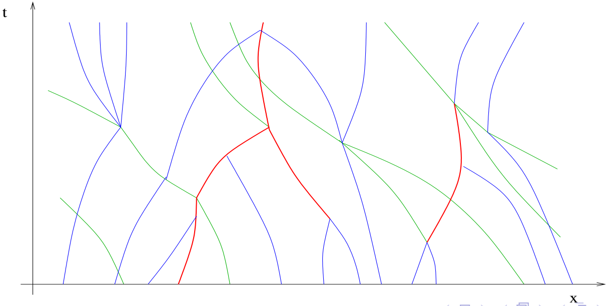
Scalar case:



Structure of BV solutions to HCL

For solutions of strictly hyperbolic systems of conservation laws in one space dimension one expects the following structure: countably many shock curves and regularity of the solution in the remaining set.

Vector case:



Most important observation

All the fundamental ideas can be understood in the scalar case:

$$u_t + f(u)_x = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \; u \in \mathbb{R}.$$

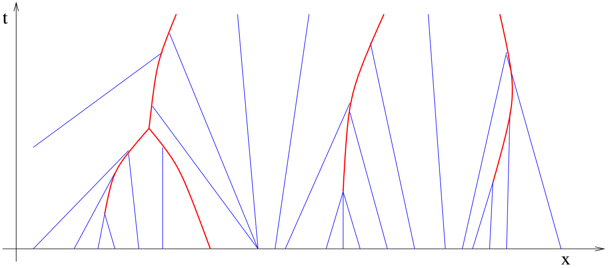
The rest are technical details.

Structure in the scalar case with convex flux

For the solution to

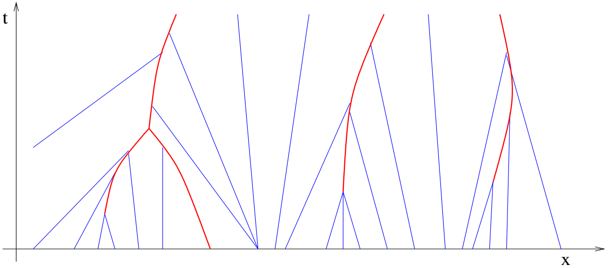
$$u_t + f(u)_x = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \ u \in \mathbb{R},$$

we expect a smooth function outside countably many regular curves.



Structure in the scalar case with convex flux

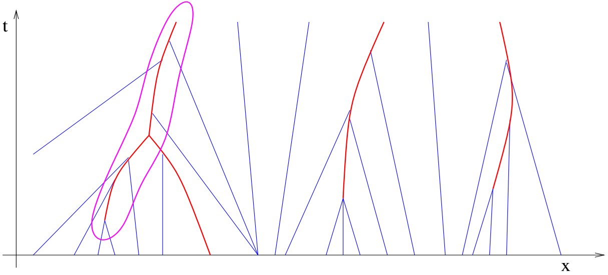
The picture below can be interpreted as:



Structure in the scalar case with convex flux

The picture below can be interpreted as:

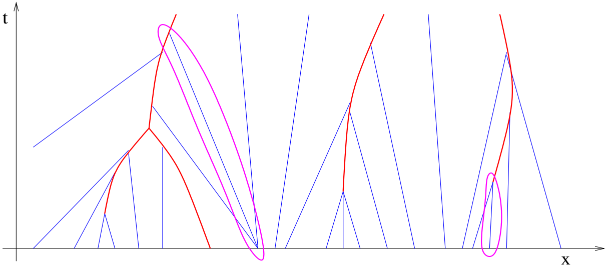
- shocks are concentrated on countably many Lipschitz curves (with first derivative in BV)



Structure in the scalar case with convex flux

The picture below can be interpreted as:

- shocks are concentrated on countably many Lipschitz curves (with first derivative in BV)
- decay of positive and negative waves as t^{-1}



Structure in the scalar case with convex flux

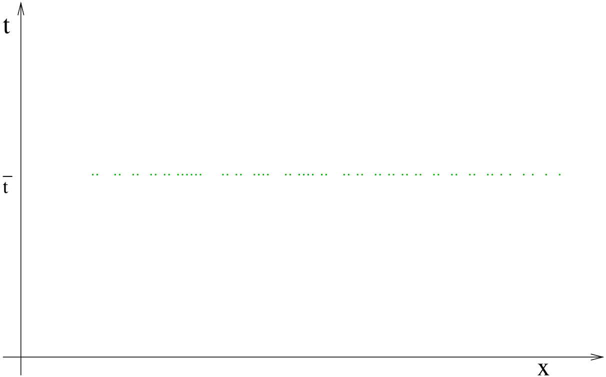
The picture below can be interpreted as:

- shocks are concentrated on countably many Lipschitz curves (with first derivative in BV)
- decay of positive and negative waves as t^{-1}
- no other terms in the derivative, i.e. no Cantorian part [Ambrosio-De Lellis]

Outline

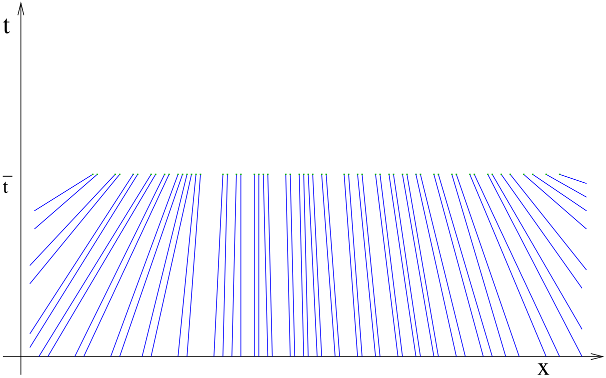
Short proof of SBV regularity for convex flux

The idea of the proof is the following [Ambrosio-De Lellis]:



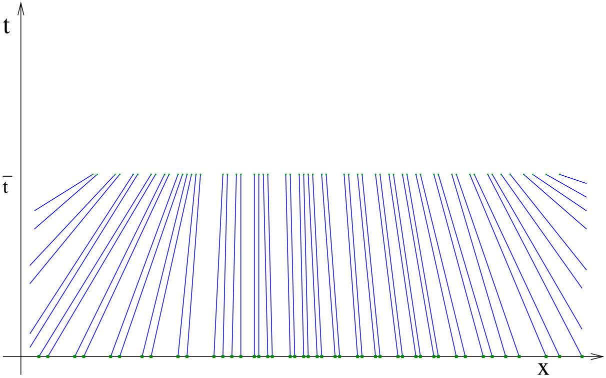
Short proof of SBV regularity for convex flux

The idea of the proof is the following [Ambrosio-De Lellis]:



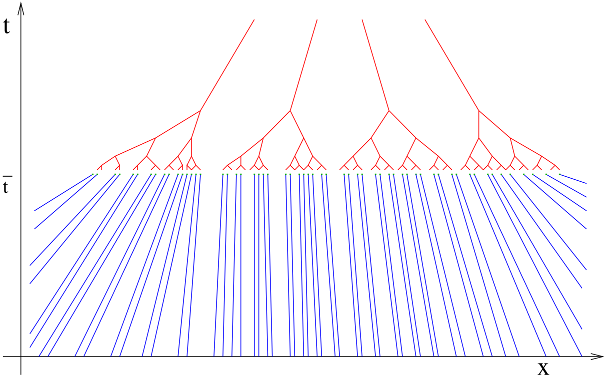
Short proof of SBV regularity for convex flux

The idea of the proof is the following [Ambrosio-De Lellis]:



Short proof of SBV regularity for convex flux

The idea of the proof is the following [Ambrosio-De Lellis]:



Short proof of SBV regularity for convex flux

The idea of the proof is the following [Ambrosio-De Lellis]:

1. if $\partial_x u(\bar{t})$ has a Cantorian part in the Cantor set $A \subset \mathbb{R}$, then the area of the initial points of the characteristics ending in A has positive $\mathcal{L}^1$-measure;
2. the characteristics ending in A cannot be prolonged $\partial_x u(\bar{t})$-a.e.;
3. thus the decreasing function

$$t \mapsto \mathcal{L}^1\big(\big\{x : x \text{ starting point of a characteristic arriving at } t\big\}\big)$$

has a jump downward at $\bar{t}$.

Hence a Cantor part in $\partial_x u(t)$ can appear only at countably many times.

A reformulation of the above proof

Since $x \mapsto -f'(u(t,x))$ is a quasi-monotone operator, it follows that the ODI

$$\dot{x} \in -f'(u(t,x))$$

generates a unique Lipschitz semigroup $X(t,x)$.
In particular we can consider the transport solution of

$$\rho_t + (f'(u(t))\rho)_x = 0, \quad \rho(0) = \mathcal{L}^1,$$

which can be represented as $X(t)_\sharp \mathcal{L}^1$, i.e. the Jacobian of $X^{-1}(t)$.

A reformulation of the above proof

If we split $\rho(t) = \rho^c(t) + \rho^a(t)$, ρ^a atomic part, then

$$\rho^c + (f'(u)\rho^c)_x = -\mu, \quad \rho^a + (f'(u)\rho^a)_x = \mu,$$

where μ is a distribution.

A reformulation of the above proof

If we split $\rho(t) = \rho^c(t) + \rho^a(t)$, ρ^a atomic part, then

$$\rho^c + (f'(u)\rho^c)_x = -\mu, \quad \rho^a + (f'(u)\rho^a)_x = \mu,$$

where μ is a distribution.
Using the fact that the atomic part of ρ can only increase (because of monotonicity), then μ is a positive Radon measure.

A reformulation of the above proof

If we split $\rho(t) = \rho^c(t) + \rho^a(t)$, ρ^a atomic part, then

$$\rho^c + (f'(u)\rho^c)_x = -\mu, \quad \rho^a + (f'(u)\rho^a)_x = \mu,$$

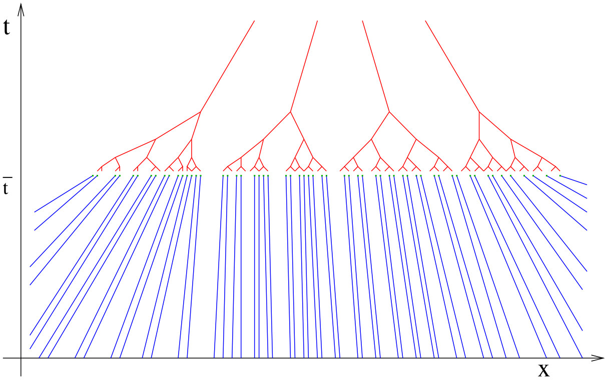
where μ is a distribution.
Using the fact that the atomic part of ρ can only increase (because of monotonicity), then μ is a positive Radon measure.
The previous proof shows that if a Cantor part appears in ρ^c, then

$$\mu(\{t\} \times A) \geq \rho^{\text{cantor}}(A),$$

and the local boundedness of μ allows to conclude as in the previous proof.

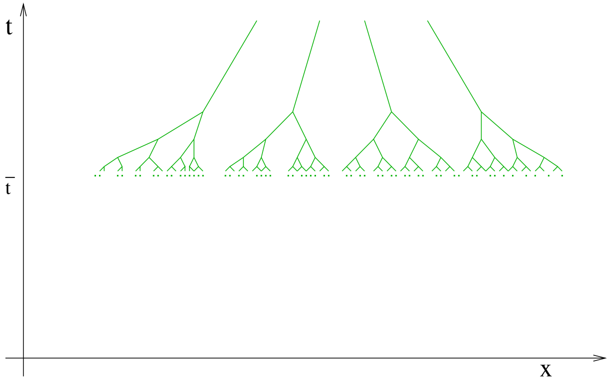
A reformulation of the above proof

In this model case the measure μ is concentrated on the Cantor set and in the jump set.



A reformulation of the above proof

In this model case the measure μ is concentrated on the Cantor set and in the jump set.



Equation for $D_x u(t)$

The measure $v := D_x u(t)$ satisfies the same transport equation in conservation form

$$v_t + (f'(u(t))v)_x = 0, \quad v(0) = D_x u(0),$$

but since it has a sign the equations for its atomic and non atomic part are a little more complicated. In fact cancellation among negative and positive waves should be considered.

Equation for $D_x u(t)$

The measure $v := D_x u(t)$ satisfies the same transport equation in conservation form

$$v_t + (f'(u(t))v)_x = 0, \quad v(0) = D_x u(0),$$

but since it has a sign the equations for its atomic and non atomic part are a little more complicated. In fact cancellation among negative and positive waves should be considered.
By using the wavefront tracking approximation, one can prove that if $v = v^c + v^a$, v^a atomic part of v, then

$$v^c_t + (f'(u(t))v^c)_x = -\mu^{CJ}, \quad v^a_t + (f'(u(t))v^a)_x = \mu^{CJ},$$

with μ^{CJ} signed locally bounded measure such that

$$\mu^{CJ} - \{\text{measure of cancellation of waves}\} \leq 0.$$

Equation for $D_x u(t)$

Summing up, we have 3 equations

$$v_t + (f'(u(t))v)_x = 0$$

$$|v|_t + (f'(u(t))|v|)_x = -\mu^C \leq 0,$$

$$v^a_t + (f'(u(t))v^a)_x = \frac{1}{2}\mu^C + \mu^J,$$

with $\mu^J \leq 0$.

Equation for $D_x u(t)$

Summing up, we have 3 equations

$$v_t + (f'(u(t))v)_x = 0$$

$$|v|_t + (f'(u(t))|v|)_x = -\mu^C \leq 0,$$

$$v^a_t + (f'(u(t))v^a)_x = \frac{1}{2}\mu^C + \mu^J,$$

with $\mu^J \leq 0$.
The proof of SBV regularity can be thus restated as

$$\mu^J(\{t\} \times A) \leq v^{\text{cantor}}(t, A).$$

Refined decay estimates

For convex conservation laws the decay of positive waves reads as

$$v(t,A) \leq \frac{1}{c_0}\frac{\mathcal{L}^1(A)}{t}, \quad f'' \geq c_0.$$

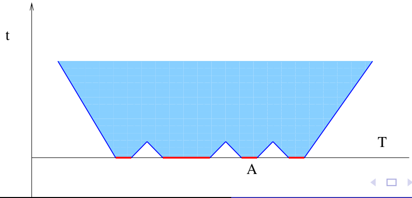
Refined decay estimates

For convex conservation laws the decay of positive waves reads as

$$v(t, A) \leq \frac{1}{c_0} \frac{\mathcal{L}^1(A)}{t}, \quad f'' \geq c_0.$$

The measure μ^J allows to obtain the corresponding decay estimate for the negative part v^c:

$$v^c(T, A) \geq -\frac{1}{c_0} \frac{\mathcal{L}^1(A)}{t - T} + \mu^J \big(\text{domain of influence of A}\big).$$



Equation along rays

Using now the fact that $u(t)$ is absolutely continuous outside the jump part, one can write the equation for v^c along each ray γ:

$$v_t^c + (f'(u(t))v^c)_x = 0, \quad \frac{d}{dt}v^c(t, \gamma(t)) = -f''(u)(v^c)^2.$$

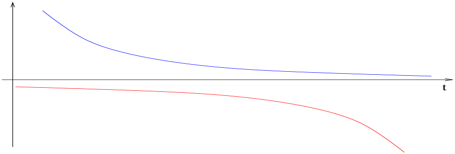
Equation along rays

Using now the fact that $u(t)$ is absolutely continuous outside the jump part, one can write the equation for v^c along each ray γ:

$$v_t^c + (f'(u(t))v^c)_x = 0, \quad \frac{d}{dt}v^c(t,\gamma(t)) = -f''(u)(v^c)^2.$$

This yields that if the ray $\gamma(t)$ has a life span of $[0, T]$, then

$$-\frac{1}{c_0}\frac{1}{T-t} \leq v^c(t,\gamma(t)) \leq \frac{1}{c_0}\frac{1}{t}.$$



Dynamical interpretation

We can thus give the following dynamic representation of the evolution of the derivative $D_x u$.

If we consider the measures

$$\omega^c(t) := v^c_\sharp(v^c \mathcal{L}^1), \quad \omega^a(t) := v^a(t, \mathbb{R}^1)$$

then it follows that

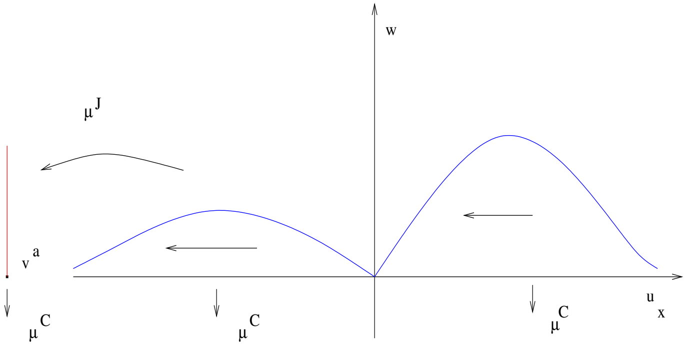
$$\omega^c_t + y^2 \omega^c = -\tilde{\mu}, \quad \omega^a_t = \tilde{\mu},$$

with (formally)

$$\tilde{\mu} = v(t)_\sharp \left(\frac{1}{2} \mu^C + \mu^J \right).$$

Dynamical interpretation

We can thus give the following dynamic representation of the evolution of the derivative $D_x u$.



Outline

Assumptions and main result

We consider the hyperbolic system

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \; u \in \mathbb{R}^n,$$

and we assume that the $\bar{i}$-eigenvalue λ_i of $Df(u)$ is g.n.l.: by choosing the direction of the unit eigenvector $r_{\bar{i}}$,

$$D\lambda_{\bar{i}}(u) r_{\bar{i}}(u) \leq c_0 < 0.$$

Assumptions and main result

We consider the hyperbolic system

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \; u \in \mathbb{R}^n,$$

and we assume that the $\bar{i}$-eigenvalue λ_i of $Df(u)$ is g.n.l.: by choosing the direction of the unit eigenvector $r_{\bar{i}}$,

$$D\lambda_{\bar{i}}(u) r_{\bar{i}}(u) \leq c_0 < 0.$$

We moreover decompose the derivative of the solution as [Bressan]

$$u_x(t) = \sum v_i(t) \tilde{r}_i,$$

with $\tilde{r}_i = r_i$ where u is continuous, otherwise is the direction of the jump of the i-th family. Each $v_i(t)$ is a bounded measure.

Assumptions and main result

We consider the hyperbolic system

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \; u \in \mathbb{R}^n,$$

and we assume that the $\bar{i}$-eigenvalue λ_i of $Df(u)$ is g.n.l.: by choosing the direction of the unit eigenvector $r_{\bar{i}}$,

$$D\lambda_{\bar{i}}(u) r_{\bar{i}}(u) \leq c_0 < 0.$$

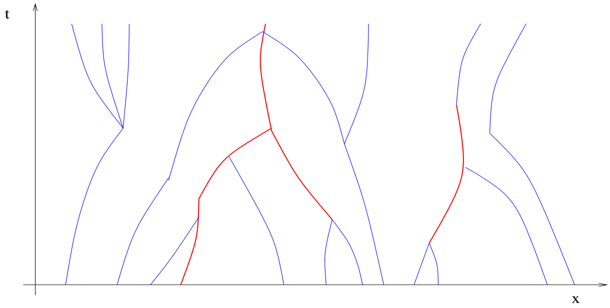
We moreover decompose the derivative of the solution as [Bressan]

$$u_x(t) = \sum v_i(t) \tilde{r}_i,$$

with $\tilde{r}_i = r_i$ where u is continuous, otherwise is the direction of the jump of the i-th family. Each $v_i(t)$ is a bounded measure.
Our aim is to prove that $v_{\bar{i}}(t)$ has a Cantor part only at countably many times.

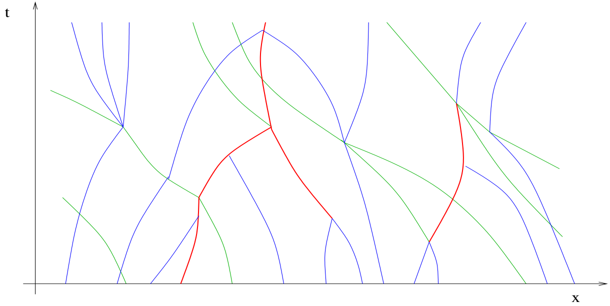
Assumptions and main result

In general the situation is more complicated than in the scalar case, due to the presence and the interaction of the waves of different families.



Assumptions and main result

In general the situation is more complicated than in the scalar case, due to the presence and the interaction of the waves of different families.



Equation for wave measures

Let $\tilde{\lambda}_i$ be the i-th eigenvector if u is continuous or the speed of the i-th shock. By the wavefront approximation, one obtain the following balance equation

Equation for wave measures

Let $\tilde{\lambda}_i$ be the i-th eigenvector if u is continuous or the speed of the i-th shock. By the wavefront approximation, one obtain the following balance equation

- conservation of v_i:

$$(v_i)_t + (\tilde{\lambda}_i v_i)_x = \mu_i^I$$

  where μ_i^I is a signed measure bounded by the decrease of the interaction potential $Q(u)$;

Equation for wave measures

Let $\tilde{\lambda}_i$ be the i-th eigenvector if u is continuous or the speed of the i-th shock. By the wavefront approximation, one obtain the following balance equation

- conservation of v_i:

$$(v_i)_t + (\tilde{\lambda}_i v_i)_x = \mu_i^I$$

  where μ_i^I is a signed measure bounded by the decrease of the interaction potential $Q(u)$;

- conservation of $|v_i|$:

$$(|v_i|)_t + (\tilde{\lambda}_i |v_i|)_x = \mu_i^{IC}$$

  where μ_i^{IC} is a signed measure bounded by the decrease of the potential $\mathrm{Tot.Var.}(u) + CQ(u)$.

Equation for the atomic part

If $\bar{i}$ is genuinely nonlinear, the equation for the atomic part $v_{\bar{i}}^{a}$ is

$$(v_{\bar{i}}^{a})_t + (\tilde{\lambda}_{\bar{i}} v_{\bar{i}}^{a})_x = \mu_{\bar{i}}^{ICJ},$$

where $\mu_{\bar{i}}^{ICJ}$ is a distribution satisfying

$$\mu^J := \mu_{\bar{i}}^{ICJ} - |\mu_{\bar{i}}^{I}| - |\mu_{\bar{i}}^{IC}| \leq 0.$$

Equation for the atomic part

If $\bar{i}$ is genuinely nonlinear, the equation for the atomic part $v^a_{\bar{i}}$ is

$$(v^a_{\bar{i}})_t + (\tilde{\lambda}_{\bar{i}} v^a_{\bar{i}})_x = \mu^{ICJ}_{\bar{i}},$$

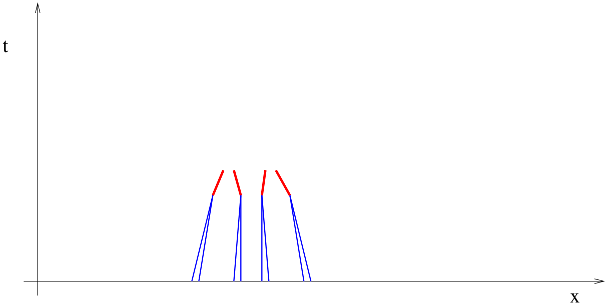
where $\mu^{ICJ}_{\bar{i}}$ is a distribution satisfying

$$\mu^J := \mu^{ICJ}_{\bar{i}} - |\mu^I_{\bar{i}}| - |\mu^{IC}_{\bar{i}}| \leq 0.$$

Hence $\mu^J_{\bar{i}}$ is a bounded measure (*jump measure*), which measures the amount of jumps created.

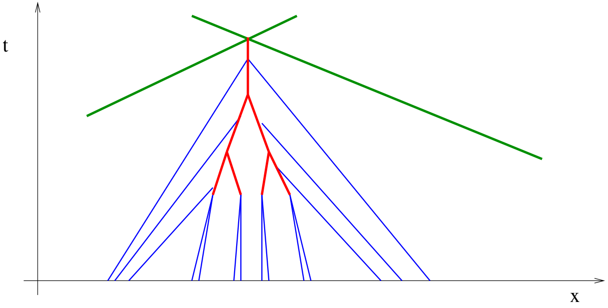
Equation for the atomic part

The fact that μ^J is a measure (signed distribution) follows from the fact that it is easy to create a jump because of nonlinearity



Equation for the atomic part

The fact that μ^J is a measure (signed distribution) follows from the fact that it is easy to create a jump because of nonlinearity, but to cancel it you have to use cancellation or interaction.



SBV regularity

The continuous part v_i^c of v_i thus satisfies

$$(v_i^c)_t + (\lambda_i v_i^c)_x = \mu_i^c, \quad \mu_i^c := \mu_i^I - \mu_i^{ICJ}.$$

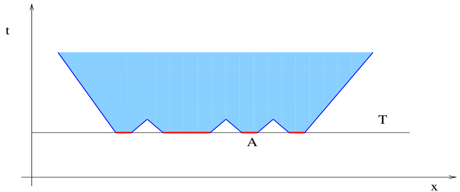
SBV regularity

The continuous part v_i^c of v_i thus satisfies

$$(v_i^c)_t + (\lambda_i v_i^c)_x = \mu_i^c, \quad \mu_i^c := \mu_i^I - \mu_i^{ICJ}.$$

As argument similar to the estimate of the decay of positive waves yields now

$$v_i^c(T, A) \geq -\frac{1}{c_0} \frac{\mathcal{L}^1(A)}{t - T} - |\mu_i^c|\Big(\text{Domain of influence of } A\Big).$$



In particular, if A is a set of measure 0 where the Cantor part is concentrated, then by taking a sequence $t_n \searrow T$ we obtain

$$|\mu_i^c|(A) > 0.$$

In particular, if A is a set of measure 0 where the Cantor part is concentrated, then by taking a sequence $t_n \searrow T$ we obtain

$$|\mu_i^c|(A) > 0.$$

Since μ_i^c is a bounded measure, then the set of times where a Cantor part appears is countable.

In particular, if A is a set of measure 0 where the Cantor part is concentrated, then by taking a sequence $t_n \searrow T$ we obtain

$$|\mu_i^c|(A) > 0.$$

Since μ_i^c is a bounded measure, then the set of times where a Cantor part appears is countable.
These times corresponds to:

In particular, if A is a set of measure 0 where the Cantor part is concentrated, then by taking a sequence $t_n \searrow T$ we obtain

$$|\mu_i^c|(A) > 0.$$

Since μ_i^c is a bounded measure, then the set of times where a Cantor part appears is countable.
These times corresponds to:

1. strong interactions among waves;

In particular, if A is a set of measure 0 where the Cantor part is concentrated, then by taking a sequence $t_n \searrow T$ we obtain

$$|\mu_i^c|(A) > 0.$$

Since μ_i^c is a bounded measure, then the set of times where a Cantor part appears is countable.
These times corresponds to:

1. strong interactions among waves;
2. generation of shock with the same strength of the Cantor part.

Extensions

- SBV regularity for fluxes with countably many inflection points [Robyr], or SBV regularity for $v_i(D\lambda_i r_i)$ [Yu]

Extensions

- SBV regularity for fluxes with countably many inflection points [Robyr], or SBV regularity for $v_i(D\lambda_i r_i)$ [Yu]
- SBV regularity for u solutions to HJ equation

$$u_t + H(t, x, \nabla u) = 0$$

with uniformly convex Hamiltonian [De Lellis, Robyr, Tonon] or with simple degeneracies [Tonon]

Extensions

- SBV regularity for fluxes with countably many inflection points [Robyr], or SBV regularity for $v_i(D\lambda_i r_i)$ [Yu]
- SBV regularity for u solutions to HJ equation

$$u_t + H(t, x, \nabla u) = 0$$

  with uniformly convex Hamiltonian [De Lellis, Robyr, Tonon] or with simple degeneracies [Tonon]
- SBV regularity for Temple class systems with source terms [Ancona-Nguyen]

Extensions

- SBV regularity for fluxes with countably many inflection points [Robyr], or SBV regularity for $v_i(D\lambda_i r_i)$ [Yu]
- SBV regularity for u solutions to HJ equation

$$u_t + H(t, x, \nabla u) = 0$$

  with uniformly convex Hamiltonian [De Lellis, Robyr, Tonon] or with simple degeneracies [Tonon]
- SBV regularity for Temple class systems with source terms [Ancona-Nguyen]
- Open: no Cantor part in the measure $\mathrm{div} d$, where d is the direction of the optimal ray for the solution

$$u_t + H(\nabla u) = 0,$$

  with H only smooth, convex.

Outline

L. Ambrosio and C. De Lellis.
A note on admissible solutions of 1d scalar conservation laws and 2d HJ equations.

F. Ancona and K.T. Nguyen.
SBV regularity for solutions to genuinely nonlinear Temple systems of balance laws.

S.B. and L. Caravenna.
SBV regularity for hyperbolic systems.

S.B. and D. Tonon.
SBV regularity for HJ equations with Hamiltonian depending on (t, x).

S.B. and D. Tonon.
SBV regularity for HJ equations with Hamiltonian depending on (t, x).

- S.B. and D. Tonon.
  SBV-like regularity for HJ equations with a convex Hamiltonian.
- S.B., C. De Lellis and R. Robyr.
  SBV regularity for HJ equations in $\mathbb{R}^n$.
- S.B. and M. Gloyer.
  Euler equation for a singular variational problem.
- R. Robyr.
  SBV regularity for entropy solutions for a class of genuinely nonlinear scalar balance laws with nonconvex flux function