Asymptotic Behavior of Smooth Solutions for

Dissipative Hyperbolic Systems with a Convex Entropy

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Hyperbolic systems of balance laws

Consider a system of balance laws with k conserved quantities,

$$\begin{aligned} \partial_t u + \partial_x F_1(w) &= 0 \\ \partial_t v + \partial_x F_2(w) &= q(w) \end{aligned} \tag{1}$$

with $w = (u, v) \in \Omega \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, and assume that there exists a strictly convex function $\mathcal{E} = \mathcal{E}(w)$ and a related entropy-flux $\mathcal{F} = \mathcal{F}(w)$, s.t. (for smooth solutions):

$$\partial_t \mathcal{E}(w) + \partial_x \mathcal{F}(w) = \mathcal{G}(w), \tag{2}$$

where

$$\mathcal{F}' = \mathcal{E}' F'(w) = \mathcal{E}' \begin{pmatrix} F_1' \\ F_2' \end{pmatrix}, \qquad \mathcal{G} = \mathcal{E}' G(w) = \mathcal{E}' \begin{pmatrix} 0 \\ q(w) \end{pmatrix}.$$

Equilibrium points: $\bar{w}$ s.t. $G(\bar{w}) = 0$. Set $\gamma = \{w \in \Omega; G(w) = 0\}$.

Definition. *The system (1) is entropy dissipative, if for every $\bar{w} \in \gamma$ and $w \in \Omega$,*

$$\mathcal{R}(w, \bar{w}) := \left(\mathcal{E}'(w) - \mathcal{E}'(\bar{w})\right) \cdot G(w) \le 0.$$

Set $W = (U, V) = \mathcal{E}'(w)$, $\Phi(W) := (\mathcal{E}')^{-1}(W)$, and rewrite (1) in the symmetric form

$$A_0(W)\partial_t W + A_1(W)\partial_x W = G(\Phi(W)) \tag{3}$$

with $A_0(W) := \Phi'(W)$ symmetric, positive definite and $A_1(W) := F'(\Phi(W))\Phi'(W)$ symmetric.

The system (3) is *strictly entropy dissipative*, if there exists a positive definite matrix $B = B(W, \bar{W}) \in \mathcal{M}^{(n-k)\times(n-k)}$ such that

$$Q(W) := q(\Phi(W)) = -D(W, \bar{W})(V - \bar{V}), \tag{4}$$

for every $W \in \mathcal{E}'(\Omega)$ and $\bar{W} = (\bar{U}, \bar{V}) \in \Gamma := \mathcal{E}'(\gamma) = \{W \in \mathcal{E}'(\Omega); G(\Phi(W)) = 0\}$.

In the following we just consider $\bar{W} = 0$ and systems like:

$$A_0(W)\partial_t W + A_1(W)\partial_x W = -\begin{matrix} 0 \\ D(W)V \end{matrix}\quad , \tag{5}$$

with D positive definite.

Kawashima condition. Consider our original system

$$\partial_t w + F'(w)\partial_x w = G(w). \tag{6}$$

Condition K. Any eigenvector of $F'(0)$ is not in the null space of $G'(0)$, which can be rewritten in entropy framework as

$$[\lambda A_0(0) + A_1(0)] \quad \begin{matrix} U \\ 0 \end{matrix} \quad \neq 0 \qquad (K)$$

Theorem 1. (Hanouzet-Natalini) *Assume that system (5) is strictly entropy dissipative and condition* **(K)** *is satisfied. Then there exists* $\delta > 0$ *such that, if* $\|W_0\|_2 \leq \delta$*, there is a unique global solution* $W = (U, V)$ *of (5), which verifies*

$$W \in C^0([0,\infty); H^2(\mathbb{R})) \cap C^1([0,\infty); H^1(\mathbb{R})),$$

and

$$\sup_{0\leq t<+\infty} \|W(t)\|_2^2 + \int_0^{+\infty} \|\partial_x U(\tau)\|_1^2 + \|V(\tau)\|_2^2 \; d\tau \leq C(\delta)\|W_0\|_2^2, \tag{7}$$

where $C(\delta)$ *is a positive constant.*

In multiD the estimate is in H^s, with s sufficiently large (Yong).

The linearized problem. The system of balance law (1) becomes

$$\partial_t w + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \partial_x w = - \begin{pmatrix} 0 & 0 \\ D_1 & D_2 \end{pmatrix} w, \tag{8}$$

(H1) $\exists\ A_0$ symmetric positive such that AA_0 is symmetric and

$$A_0 = \begin{pmatrix} A_{0,11} & A_{0,12} \\ A_{0,21} & A_{0,22} \end{pmatrix}, \qquad BA_0 = - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},$$

with $D \in \mathbb{R}^{(n-k)\times(n-k)}$ positive definite;

(H2) any eigenvector of A is not in the null space of B.

Consider the projectors $Q_0 = R_0 L_0$ on the null space of B, and its complementary projector $Q_- = I - Q_0 = R_- L_-$, to which it corresponds the decomposition

$$w = A_0 \begin{pmatrix} (A_{0,11})^{-1/2} \\ 0 \end{pmatrix} w_c + \begin{pmatrix} 0 \\ ((A_0^{-1})_{22})^{-1/2} \end{pmatrix} w_{nc}, \tag{9}$$

$$w_c = \begin{bmatrix} (A_{0,11})^{-1/2} & 0 \end{bmatrix} u, \qquad w_{nc} = \begin{bmatrix} 0 & ((A_0^{-1})_{22})^{-1/2} \end{bmatrix} A_0 u. \tag{10}$$

The system (8) takes now the form

$$\begin{pmatrix} w_c \\ w_{nc} \end{pmatrix}_t + \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} w_c \\ w_{nc} \end{pmatrix}_x = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D} \end{pmatrix} \begin{pmatrix} w_c \\ w_{nc} \end{pmatrix}, \tag{11}$$

where $\tilde{A}$ is symmetric and $\tilde{D}$ is strictly negative,

$$\tilde{D} \doteq L_- \tilde{B} R_- = ((A_0^{-1})_{22})^{-1} D ((A_0^{-1})_{22})^{-1}.$$

We want to study the Green kernel $\Gamma(t,x)$ of (11),

$$\begin{array}{rcl} \partial_t \Gamma + \tilde{A} \partial_x \Gamma & = & \tilde{B}\Gamma \\ \Gamma(0,x) & = & \delta(x) I \end{array} \qquad \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D} \end{pmatrix},$$

by means of Fourier transform $\hat{\Gamma}(t,\xi)$ and perturbation analysis of the characteristic function

$$E(z) = \tilde{B} - zA.$$

We will consider the Green kernel as composed of 4 parts,

$$\Gamma(t,x) = \begin{pmatrix} \Gamma_{00}(t,x) & \Gamma_{0-}(t,x) \\ \Gamma_{-0}(t,x) & \Gamma_{--}(t,x) \end{pmatrix}.$$

For ξ small (large space scale), the reduction of $E(z)$ on the eigenspace of the 0 eigenvalue of $\tilde{B}$ is

$$-z\tilde{A}_{11} - z^2\tilde{A}_{12}\tilde{D}^{-1}\tilde{A}_{21} + \mathcal{O}(z^3),$$

and one has to consider the decomposition

$$\tilde{A}_{11} = \sum_j \ell_j r_j l_j, \qquad l_j\tilde{A}_{12}\tilde{D}^{-1}\tilde{A}_{21}r_j = \sum_k (c_{jk}I + d_{jk})p_{jk},$$

with d_{jk} nilpotent matrix. Let us denote by $g_{jk}(t,x)$ the heat kernel of

$$g_t + \ell_j g_x = (c_{jk}I + d_{jk})g_{xx}.$$

For ξ large (small space scale), $E(z) = z(\tilde{A} + \tilde{B}/z)$, one has to consider the decomposition

$$\tilde{A} = \sum_j \lambda_j R_j L_j, \qquad L_j\tilde{B}R_j = \sum_k (b_{jk}I + e_{jk})q_{jk},$$

and let $h_{jk}(t,x)$ be Green kernel of the transport system

$$h_t + \lambda_j h_x = (b_{jk}I + e_{jk})h.$$

Define the matrix valued functions

$$K(t,x) = \sum_{jk} \begin{bmatrix} r_j g_{jk}(t,x) p_{jk} l_j & -\frac{d}{dx} r_j g_{jk}(t,x) p_{jk} l_j \tilde{A}_{12} \tilde{D}^{-1} \\ -\frac{d}{dx} \tilde{D}^{-1} \tilde{A}_{21} r_j g_{jk}(t,x) p_{jk} l_j & \frac{d^2}{dx^2} \tilde{D}^{-1} \tilde{A}_{21} r_j g_{jk}(t,x) p_{jk} l_j \tilde{A}_{21} \tilde{D}^{-1} \end{bmatrix}$$

$$\mathcal{K}(t,x) = \sum_{jk} R_j (h_{jk}(t,x) q_{jk}) L_j .$$

Theorem. *The Green kernel for* (11) *is*

$$\Gamma(t,x) = K(t,x)\chi \;\; \underline{\lambda}t \le x \le \bar{\lambda}t, t \ge 1 \;\; + \mathcal{K}(t,x) + R(t,x)\chi \;\; \underline{\lambda}t \le x \le \bar{\lambda}t \;\; , \qquad (12)$$

where $\underline{\lambda}$, $\bar{\lambda}$ *are the minimal and maximal eigenvalue of* $\tilde{A}$ *and the rest* $R(t,x)$ *can be written as*

$$R(t,x) = \sum_j \frac{e^{-(x-\ell_j t)^2/ct}}{1+t} \begin{matrix} \mathcal{O}(1) & \mathcal{O}(1)(1+t)^{-1/2} \\ \mathcal{O}(1)(1+t)^{-1/2} & \mathcal{O}(1)(1+t)^{-1} \end{matrix}$$

for some constant c.

Differences with the previous result by Y. Zeng (1999):

1. finite propagation speed (hyperbolic domain);
2. Structure of the diffusive part (operators R_0 and L_0);
3. BA_0 not symmetric $\Leftrightarrow$ $\tilde{D}$ not symmetric (as in Hanouzet-Natalini (2002), Yong (2002)).

From a technical point of view, when we study the function

$$\hat{G}(t,\xi) = \exp(E(z)t) = \exp\left((\tilde{B} - z\tilde{A})t\right),$$

and we compute its inverse Fourier transform, the differences w.r.t. Y. Zeng are:

- a carefully analysis of the families of eigenvalues whose projectors do not blow up near the exceptional points $z = 0$, $z = \infty$;
- when estimating $e^{E(z)t}$, one has to deal always with matrices;
- the path of integration in the complex plane depends now on the viscosity coefficients c_{jk}, which is a complex number.

Asymptotic behavior

Consider now the original problem

$$w_t + F(w)_x = G(w) = \begin{pmatrix} 0 \\ q(w) \end{pmatrix}, \qquad w(x,0) = w_0 \tag{13}$$

We have

$$w_t + F'(0)w_x - G'(0)w = \left(F'(0)w - F(w) \right)_x - \left(G'(0)w - G(w) \right)$$

Then we can write the solution as

$$w = \Gamma(t) * w_0 + \int_0^t \Gamma(t-\tau) * \left[\left(F'(0)w - F(w) \right)_x - \left(G'(0)w - G(w) \right) \right] d\tau.$$

Since for any vector vector $(0, V) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ one has for the principal part K of the kernel Γ

$$K(t,x) \begin{pmatrix} 0 \\ V \end{pmatrix} = \sum_{jk} \frac{d}{dx} \begin{pmatrix} -r_j g_{jk}(t,x) p_{jk} l_j \tilde{A}_{12} \tilde{D}^{-1} \\ \frac{d}{dx} \tilde{D}^{-1} \tilde{A}_{21} r_j g_{jk}(t,x) p_{jk} l_j \tilde{A}_{21} \tilde{D}^{-1} \end{pmatrix},$$

also the second term in the convolution contains an x derivative, so that one may use standard L^2 estimates.

Theorem. *Let $u(t)$ be the solution to the entropy strictly dissipative system* (13), *and let $w_c(t) = L_0 w(t)$, $w_{nc}(t) = L_- w(t)$. Then, if $\|u(0)\|_{H^s}$ is bounded and small for s sufficiently large, the following decay estimates holds: for all β,*

$$\|\partial_x^\beta w_c(t)\|_{L^p} \leq C \min\Big\{1, t^{-1/2(1-1/p)-\beta/2}\Big\} \max \; \|u(0)\|_{L^1}, \|u(0)\|_{H^s} \;\;, \tag{14}$$

$$\|\partial_x^\beta w_{nc}(t)\|_{L^p} \leq C \min\Big\{1, t^{-1/2(1-1/p)-1/2-\beta/2}\Big\} \max \; \|u(0)\|_{L^1}, \|u(0)\|_{H^s} \;\;, \tag{15}$$

with $p \in [1, +\infty]$.

Remark. These decay estimates correspond to the decay of the heat kernel $\frac{1}{\sqrt{2\pi t}} e^{-x^2/4t}$, and in particular the solution to the linearized problem

$$w_t + \tilde{A} w_x = \tilde{B} w$$

satisfies (14), (15). As a consequence these estimates cannot be improved.

Remark. Observe moreover that the non conservative variables w_{nc} decays as a derivative of w_c.

Chapman-Enskog expansion

Consider now the Chapman-Enskog expansion

$$A_0(W)\partial_t W + A_1(W)\partial_x W = -\begin{pmatrix} 0 \\ D(W)V \end{pmatrix}, \quad W = (U, V)$$

$$V \sim h(U, U_x) := -D^{-1}\left((A_1)_{21} - (A_0)_{21}(A_0)_{11}^{-1}(A_1)_{11}\right)U_x$$

In the original coordinates, equilibrium at $v = h(u)$ and

$$u_t + F_1\left(u, h(u) - D^{-1}(u, h(u))\left(F_2(u, h(u))_x - Dh(u)F_1(u, h(u))_x\right)\right)_x = 0 \qquad (16)$$

The linearized form of (16) is

$$u_t + \tilde{A}_{11} u_x - \tilde{A}_{12}\tilde{D}^{-1}\tilde{A}_{21} u_{xx} = 0,$$

so that its Green kernel G is

$$\tilde{\Gamma}(t) = K_{00}(t) + \tilde{\mathcal{K}}(t) + \tilde{R}(t), \qquad K_{00}(t, x) = \sum_{jk} r_j g_{jk}(t, x) p_{jk} l_j.$$

Since the principal part of the linear Green kernel is the same (up to the finite speed of propagation), one can prove

Theorem. *If $w(t)$ is the solution to the parabolic system* (16), *then for all $\kappa \in [0, 1/2)$*

$$\|D^\beta(w_c(t) - w(t))\|_{L^p} \leq C \min\Big\{1, t^{-m/2(1-1/p)-\kappa-\beta/2}\Big\} \max \quad \|u(0)\|_{L^1}, \|u\|_{H^s} \quad ,$$

if the initial data is sufficiently small, depending on κ, and tending to 0 *as $\kappa \to 1/2$.*

Remark. At the linear level one gains exactly $t^{-1/2}$ (one derivative), but in dimension 1 the quadratic parts of F, G matter and this is way we can only prove the decay for all $k \in [0, 1/2)$.

A Glimm Functional for Relaxation

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Consider the Jin-Xin relaxation model

$$\begin{array}{rcll} F_t^- - F_x^- & = & U - A(U) & - F^- \\ F_t^+ + F_x^+ & = & U + A(U) & - F^+ \end{array} \tag{17}$$

where $A(u)$ is strictly hyperbolic with eigenvalues $|\lambda_i| < 1$, and

$$U = \frac{1}{2}(F^- + F^+) \in \mathbb{R}^n, \qquad M^-(u) = U - A(u), \quad M^+(u) = U + A(u).$$

To prove BV bounds, we follow an approach similar to vanishing viscosity:

1. decompose the derivatives f^-, f^+ of F^-, F^+ along travelling profiles,

$$f^- = \sum_i f_i^- \tilde{r}_i^-, \qquad f^+ = \sum_i f_i^+ \tilde{r}_i^+;$$

2. write the $2n \times 2n$ system (17) as n 2×2 systems

$$\begin{array}{rcl} f_{i,t}^- - f_{i,x}^- & = & -a_i^-(t,x) f_i^- + (1 - a_i^-(t,x)) f_i^+ + s_i^-(t,x) \\ f_{i,t}^+ + f_{i,x}^+ & = & a_i^-(t,x) f_i^- - (1 - a_i^-(t,x)) f_i^+ + s_i^+(t,x) \end{array} \tag{18}$$

3. estimate the sources s_i^-, s_i^+.

Center manifold. Let $U_x = v_i \tilde{r}_i(U, v_i, \sigma)$ be the center manifold for

$$-\sigma U_x + A(U)_x = U_{xx} - \sigma^2 U_{xx}$$

near the equilibrium $(U = 0, U_x = 0, \lambda_i(0))$, so that the center manifold for (17) can be written as

$$\begin{array}{l} F^- = M^-(U) - (1-\sigma^2) v_i \tilde{r}_i(U, v_i, \sigma) \\ F^- = M^-(U) - (1-\sigma^2) v_i \tilde{r}_i(U, v_i, \sigma) \end{array} \implies \begin{array}{l} f^- = (1+\sigma) v_i \tilde{r}_i(U, v_i, \sigma) \\ f^- = (1-\sigma) v_i \tilde{r}_i(U, v_i, \sigma) \end{array}$$

Define $g^- = F_t^-$, $g^+ = F_t^+$, and decompose the couple (f^-, g^-) by

$$\begin{aligned} f^- &= \sum_i f_i^- \tilde{r}_i(U, f_i^-/(1+\sigma_i^-), \sigma_i^-) = \sum_i f_i^- \tilde{r}_i^-(U, f_i^-, \sigma_i^-) \\ g^- &= \sum_i g_i^- \tilde{r}_i(U, f_i^-/(1+\sigma_i^-), \sigma_i^-) = \sum_i g_i^- \tilde{r}_i^-(U, f_i^-, \sigma_i^-) \end{aligned} \qquad (19)$$

with $\sigma_i^- = \theta_i(g_i^-/f_i^-)$. The same for the couple (f^+, g^+), with $\tilde{r}_i^+(u, f_i^+, \sigma_i^+) = \tilde{r}_i(U, f_i^+/(1-\sigma_i^+), \sigma_i^+)$, $\sigma_i^+ = \theta_i(g_i^+/f_i^+)$.

We thus have $2n$ travelling waves, n for each family of particles, and the "interaction" among these profiles occurs because of the left hand side of (18).

If we define

$$\tilde{\lambda}_i(u, v, \sigma) = \langle \tilde{r}_i(u, v, \sigma), DA(u)\tilde{r}_i(u, v_i, \sigma)\rangle,$$

one ends up with the system

$$\begin{cases} f^-_{i,t} - f^-_{i,x} = -\frac{1+\tilde{\lambda}^-_i}{2} f^-_i + \frac{1-\tilde{\lambda}^-_i}{2} f^+_i + s^-_i(t, x) \\ f^+_{i,t} + f^+_{i,x} = \frac{1+\tilde{\lambda}^-_i}{2} f^-_i - \frac{1-\tilde{\lambda}^-_i}{2} f^+_i + s^+_i(t, x) \end{cases} \tag{20}$$

$$\begin{cases} g^-_{i,t} - g^-_{i,x} = -\frac{1+\tilde{\lambda}^-_i}{2} g^-_i + \frac{1-\tilde{\lambda}^-_i}{2} g^+_i + r^-_i(t, x) \\ g^+_{i,t} + g^+_{i,x} = -\frac{1+\tilde{\lambda}^-_i}{2} g^-_i + \frac{1-\tilde{\lambda}^-_i}{2} g^+_i + r^+_i(t, x) \end{cases} \tag{21}$$

Among other terms, the source $s^\pm$, $r^\pm$ contains the interaction term

$$f^-_i g^+_t - f^+_t g^-_i = f^-_i f^+_i \;\; \sigma^+_i - \sigma^-_i \;\; , \tag{22}$$

where the last equality holds for speeds close to $\lambda_i(0)$.

We want to show that (22) corresponds to an interaction term, to which we can associate a Glimm functional: we consider this as the kinetic interpretation of the Glimm interaction functional for waves of the same family.

For simplicity we will set $\tilde{\lambda}_i = 0$ in the following analysis.

The interaction functional

For a piecewise constant solution u of the scalar equation

$$u_t + f(u)_x = 0,$$

we consider the interaction functional $Q(u)$ defined as (outside the interacting points)

$$Q(u) = \sum_{\text{jumps } i,j} |\delta_i||\delta_j||\sigma_i - \sigma_j|, \qquad \delta_i \text{ strength}, \sigma_i \text{ speed of the jump.}$$

This functional can be extended to the parabolic equation

$$u_t + f(u)_x = u_{xx},$$

and its "form" remains the same,

$$\begin{aligned} Q(u) &= \iint_{\mathbb{R}^2} u_t(t,x)u_x(t,y) - u_t(t,y)u_x(t,x)\ dxdy \\ &= \iint_{\mathbb{R}^2} \frac{u_t(t,x)}{u_x(t,x)} - \frac{u_t(t,y)}{u_x(t,y)}\ |u_x(t,x)|dx|u_x(t,y)|dy. \end{aligned}$$

We can interpret its time derivative as the area swept by the curve $\gamma = (u_x, u_t)$.

One can give another interpretation of the interaction functional for the scalar parabolic system by considering the variable $P(t,x,y) = u_t(t,x)u_x(t,y) - u_t(t,y)u_x(t,x)$, which satisfies

$$P_t + \mathrm{div} \quad f'(u(t,x)), f'(u(t,y)) \;\; P \quad = \Delta P$$

for $t \geq 0$, $x \geq y$ and the boundary condition $P(t,x,x) = 0$.

The interaction functional $Q(P)$ is now its L^1 norm in $\{x \geq y\}$,

$$Q(P) = \iint_{x \geq y} |P(t,x,y)| dx dy,$$

and the amount of interaction is the flux of P along the boundary $\{x = y\}$,

$$\frac{d}{dt} Q(P) \leq - \int_{x=y} \nabla P \cdot (1,-1) \; dx = -2 \int_{\mathbb{R}} u_{tx} u_x - u_t u_{xx} \; dx.$$

We will show how to interpret the interaction term

$$f^- g^+ - g^- f^+$$

as a flux along a boundary. As a consequence we will be able to construct a Glimm type functional, and prove that the above term is bounded and of second order w.r.t. the L^1 norm of the components.

Consider the system (20), (21), and construct the scalar variables

$$
\begin{aligned}
P^{--}(t,x,y) &= f^{-}(t,x)g^{-}(t,y) - f^{-}(t,y)g^{-}(t,x)\\
P^{-+}(t,x,y) &= f^{+}(t,x)g^{-}(t,y) - f^{-}(t,y)g^{+}(t,x)\\
P^{+-}(t,x,y) &= f^{-}(t,x)g^{+}(t,y) - f^{+}(t,y)g^{-}(t,x)\\
P^{++}(t,x,y) &= f^{+}(t,x)g^{+}(t,y) - f^{+}(t,y)g^{+}(t,x)
\end{aligned}
$$

which satisfy the system

$$
\left\{
\begin{array}{ccc}
P_t^{--} + \mathrm{div}((-1,-1)P^{--}) & = & (P^{+-} + P^{-+})/2 - P^{--}\\
P_t^{-+} + \mathrm{div}((-1,1)P^{-+}) & = & (P^{--} + P^{++})/2 - P^{-+}\\
P_t^{+-} + \mathrm{div}((1,-1)P^{+-}) & = & (P^{--} + P^{++})/2 - P^{+-}\\
P_t^{++} + \mathrm{div}((1,1)P^{++}) & = & (P^{+-} + P^{-+})/2 - P^{++}
\end{array}
\right.
\tag{23}
$$

for $x \geq y$ and the boundary conditions

$$
P^{-+}(t,x,x) + P^{+-}(t,x,x) = 0, \qquad P^{++}(t,x,x) = P^{--}(t,x,x) = 0.
$$

We may read the boundary conditions as follows: a particle P^{-+} hits the boundary and bounce back as P^{+-} but with opposite sign. We are interested in an estimate of the number of particles colliding with the boundary $\{x = y\}$.

To prove that the average numbers of collision with the boundary is finite if the initial number of particles is finite (note that this is quadratic w.r.t. the L^1 norm of f, g)

$$Q(P) = \iint_{x \geq y} |P^{--}| + |P^{+-}| + |P^{-+}| + |P^{++}| \ dxdy < +\infty,$$

we consider the system for P in $\mathbb{R}^2$ and an initial data of the form

$$P^{+-} = -P^{-+} = \delta(x, y), \qquad P^{++} = P^{--} = 0.$$

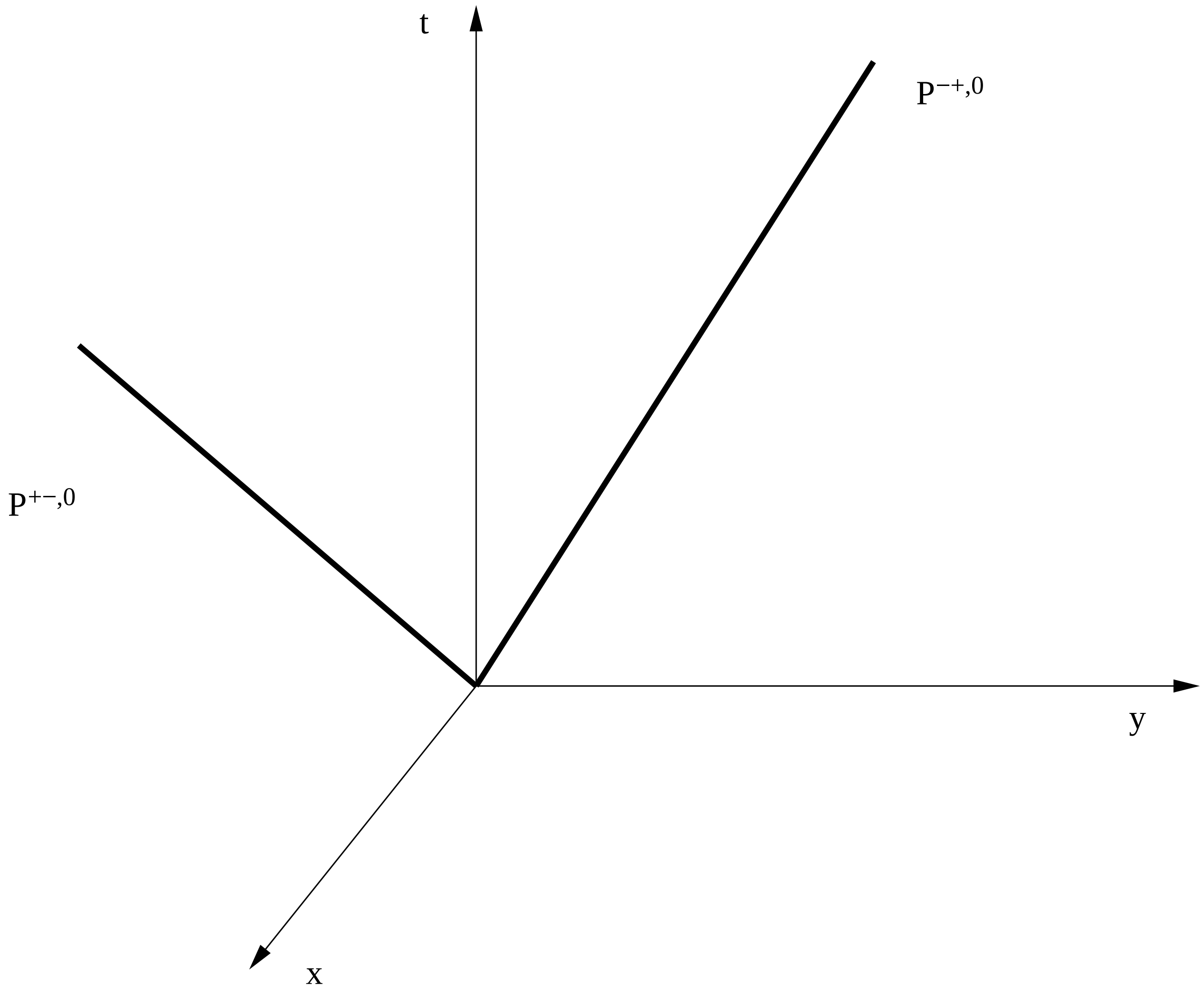
The solution will be constructed as the sum of the solutions of the cascade of systems:

$$P_t^{-+,0} + \mathrm{div}((-1,1)P^{-+,0}) = -P^{-+,0}, \qquad P_t^{+-,0} + \mathrm{div}((1,-1)P^{+-,0}) = -P^{+-,0}$$

$$\begin{array}{rcl} P_t^{--,1} + \mathrm{div}((-1,-1)P^{--,1}) & = & (P^{+-,0} + P^{-+,0})/2 - P^{--,1} \\ P_t^{++,1} + \mathrm{div}((1,1)P^{++,1}) & = & (P^{+-,0} + P^{-+,0})/2 - P^{++,1} \end{array}$$

$$\begin{array}{rcl} P_t^{-+,2} + \mathrm{div}\ (-1,1) \cdot P^{-+,2} & = & \frac{1}{2}(P^{--,1} + P^{++,1}) - P^{-+,2} \\ P_t^{+-,2} + \mathrm{div}\ (1,-1) \cdot P^{+-,2} & = & \frac{1}{2}(P^{--,1} + P^{++,1}) - P^{+-,2} \end{array}$$

The remaining terms are left as source terms for system (23).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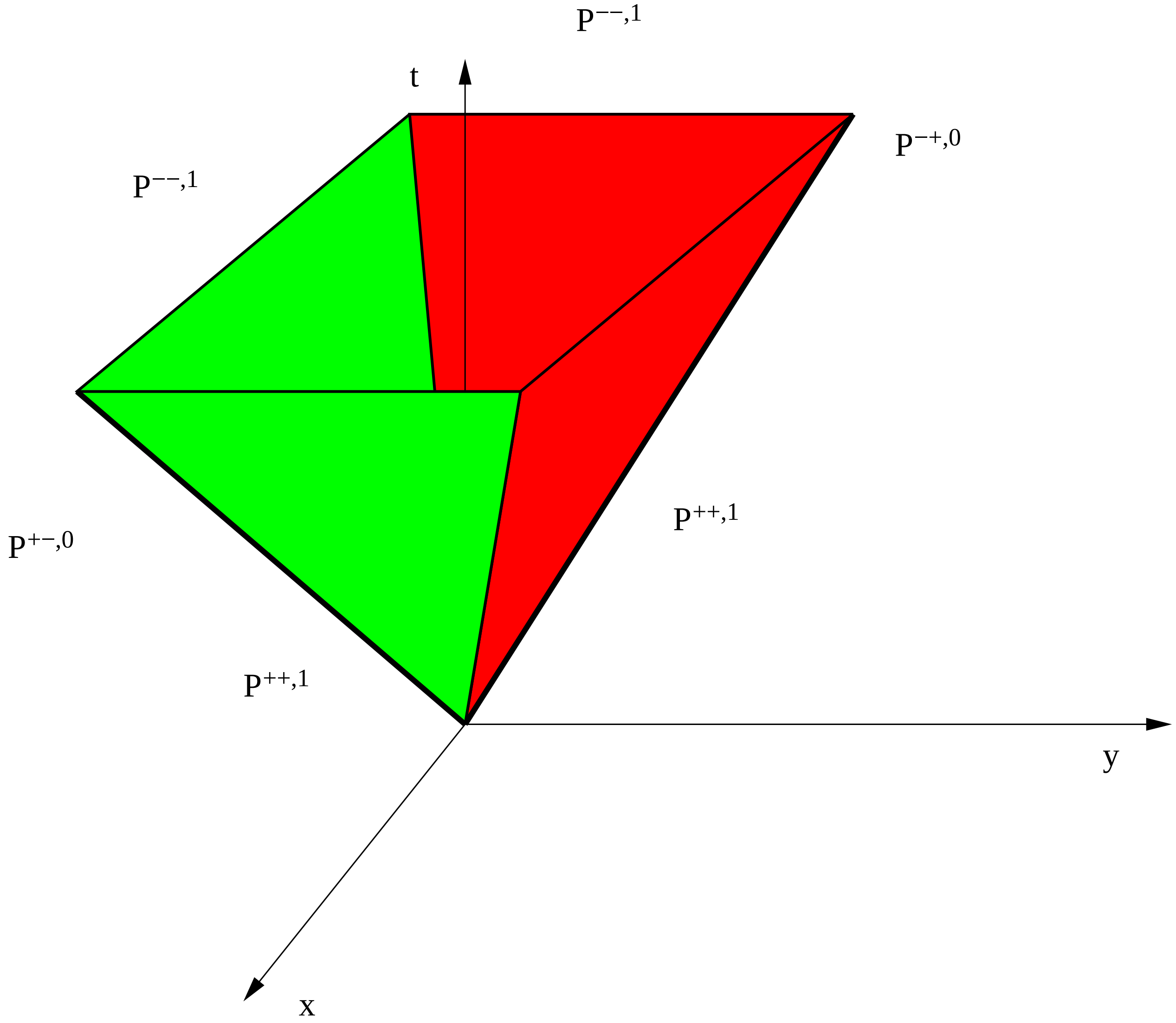
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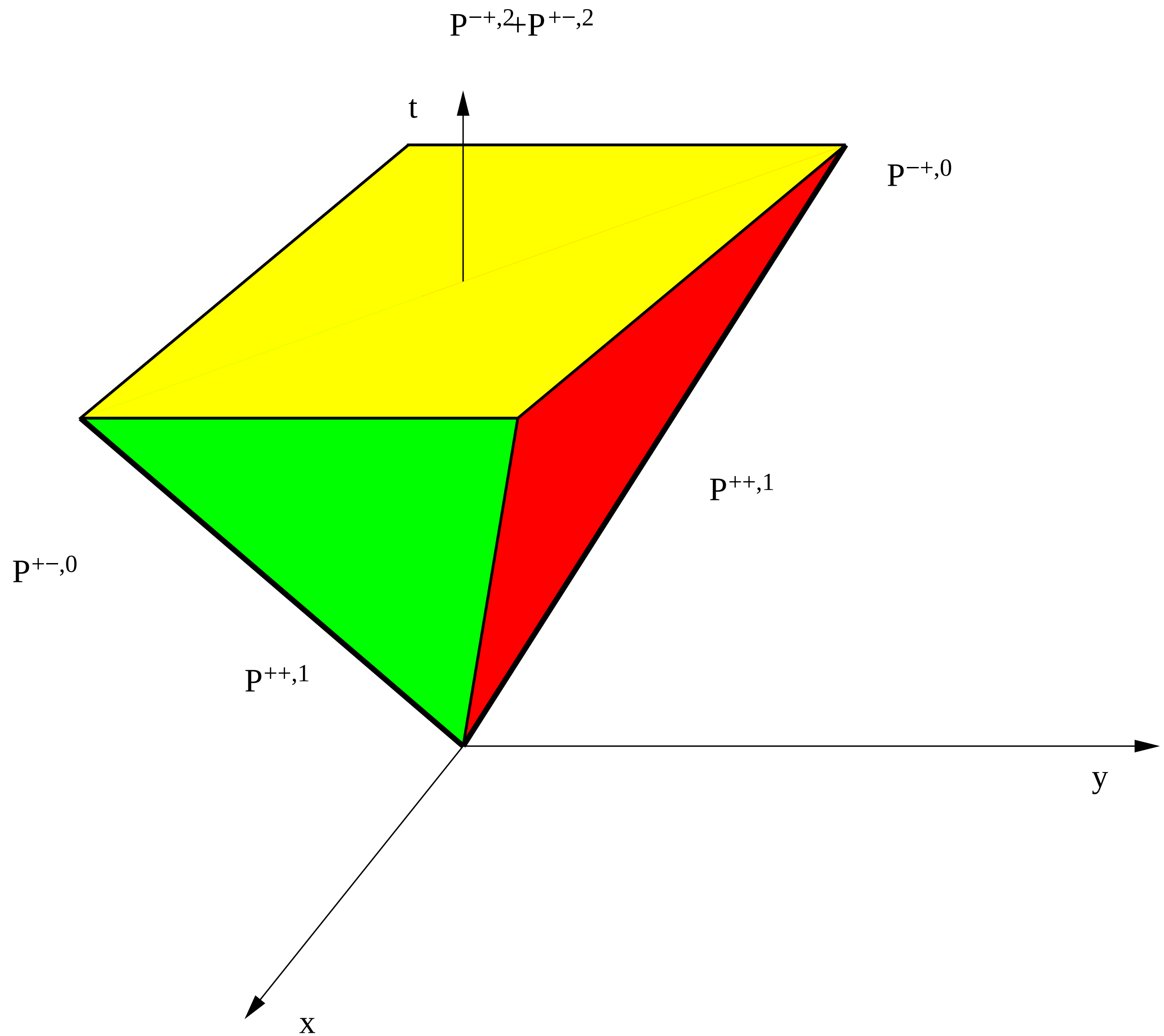
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x

The solution to the second equation is

$$P^{-+,2} = \frac{1}{16}e^{-t}\chi\Big\{|x|, |y| \leq 2t\Big\}, \qquad P^{+-,2} = -\frac{1}{16}e^{-t}\chi\Big\{|x|, |y| \leq 2t\Big\}$$

and the crossing due to this solution is

$$\frac{1}{4\sqrt{2}}\int_0^{+\infty} te^{-t}dt = \frac{1}{4\sqrt{2}}.$$

Due to symmetry, the total mass disappearing is thus

$$\frac{1}{2}\int_0^{+\infty} t^2e^{-t}dt = 1.$$

We thus obtain that the total crossing is less than.

$$2 + \frac{1}{2\sqrt{2}} \quad Q(u)$$

Remark. Observe that the amount of interaction is non local in time.