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THE KADOMCEV-PETVIAŠVILI EQUATION AND THE RELATIONS BETWEEN THE PERIODS OF HOLOMORPHIC DIFFERENTIALS ON RIEMANN SURFACES

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ABSTRACT. S. P. Novikov's conjecture that the relations between theta functions that follow from the nonlinear Kadomcev-Petviašvili equation, well known in mathematical physics, characterize the Jacobian varieties of Riemann surfaces among all abelian varieties is proved in this paper, except for the possibility of superfluous components.

Bibliography: 15 titles.

§0. Introduction

A symmetric matrix $B = (B_{jk})$ with negative definite real part $\operatorname{Re} B < 0$ is called a *Riemann matrix*. For a $g \times g$ Riemann matrix $B = (B_{jk})$ one can construct a theta function of g complex variables $(z_1, \ldots, z_g) = z$, defined by a convergent Fourier series of the form

$$\theta(z) = \theta(z \mid B) = \sum_{k \in \mathbf{Z}^g} \exp\left\{\frac{1}{2}\langle Bk, k\rangle + \langle k, z\rangle\right\}. \tag{0.1}$$

In this formula the summation is taken over all integral vectors $k = (k_1, \ldots, k_g)$; the angular brackets denote the euclidian inner product: $\langle Bk, k\rangle = \sum B_{ij}k_ik_j$, $\langle k, z\rangle = \sum k_iz_i$.

We define a lattice Λ in the complex space $\mathbf{C}^g$ as consisting of the vectors of the form

$$\Lambda = \{2\pi iN + BM\}, \tag{0.2}$$

where $N = (N_1, \ldots, N_g)$ and $M = (M_1, \ldots, M_g)$ are integral vectors. Under a translation by a vector of this lattice the theta function is transformed according to the following rule:

$$\theta(z + 2\pi iN + BM) = \exp\{-\tfrac{1}{2}\langle BM, M\rangle - \langle M, z\rangle\}\theta(z). \tag{0.3}$$

We shall call Λ the *period lattice* of the theta function. The quotient space of $\mathbf{C}^g = \mathbf{R}^{2g}$ by this lattice is a $2g$-dimensional complex torus

$$T^{2g}(B) = \mathbf{C}^g / \{2\pi iN + BM\}. \tag{0.4}$$

This torus is an abelian torus (or an abelian variety). A Kähler metric on $T^{2g}(B)$ of the form

$$ds^2 = -\frac{1}{2\pi}\sum_{j,k=1}^{g} (\operatorname{Re} B)^{-1}_{jk}\, dz_j\, d\bar{z}_k \tag{0.5}$$

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is called a *Hodge metric*: in real coordinates x_k, y_k on the space $\mathbf{C}^g = \mathbf{R}^{2g}$, where

$$z_k = 2\pi i x_k + \sum_j B_{kj} y_j, \qquad k = 1,\ldots,g, \tag{0.6}$$

the imaginary part of this metric equals

$$\Omega = \frac{1}{4\pi i} \sum_{k,l} (\operatorname{Re} B)^{-1}_{k,l}\, dz_k \wedge d\bar{z}_l = \sum_{k=1}^{g} dx_k \wedge dy_k. \tag{0.7}$$

REMARK. The cohomology class of the imaginary part of a Hodge metric on an abelian variety (i.e. on an algebraic torus) is called a *polarization* of the variety. A polarization is called *principal* if this class has the form (0.7). Thus the torus $T^{2g}(B)$ constructed from a Riemann matrix is a principally polarized abelian variety; and all such varieties are gotten in this way.

The totality $\mathbf{H}_g$ of all $g \times g$ Riemann matrices B is called the *(left) Siegel half-plane*. By the *Siegel modular group* G_g we mean the group of all integral symplectic $2g \times 2g$ matrices $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ factored by the subgroup $\{\pm 1\}$:

$$G_g = \operatorname{Sp}(g, \mathbf{Z})/\{\pm 1\}. \tag{0.8}$$

A matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ belongs to $\operatorname{Sp}(g, \mathbf{Z})$ if α, β, γ and δ are integral $g \times g$ matrices and

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\alpha^t & \gamma^t\\ \beta^t & \delta^t\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{0.8$'$}$$

(the superscript t denotes the transpose). The modular group G_g acts on the half-plane $\mathbf{H}_g$ by the formula

$$B \mapsto B' = 2\pi i(\alpha B + 2\pi i \beta)(\gamma B + 2\pi i \delta)^{-1}. \tag{0.9}$$

Riemann matrices B and B' related by a transformation from the modular group are called *equivalent*. They define unique abelian tori $T^{2g}(B) = T^{2g}(B')$ with a unique polarization. It is not hard to see that inequivalent Riemann matrices B and B' give rise to different tori. Thus the quotient space

$$\mathbf{H}_g/G_g = \mathbf{A}_g \tag{0.10}$$

is the moduli space of principally polarized abelian varieties (of complex dimension g). This space is an irreducible algebraic variety of (complex) dimension $g(g+1)/2$ (after a suitable compactification; see [14]). Note that under transformations of the form (0.9) the theta function transforms according to the rule

$$\theta(z' \mid B') = \kappa \exp\left\{-\frac{1}{2} \sum_{i \leqslant j} z_i z_j \frac{\partial \ln \det M}{\partial B_{ij}} + \sum_i l_i z_i\right\} \theta(z + \lambda \mid B),$$

where $M = (\gamma B + 2\pi i \delta)^{-1}$ and $z' = 2\pi i M z$; the explicit form of the multiplier κ (independent of z) and of the quantities l_i and λ is not essential for us.

Let Γ be a compact Riemann surface of genus g (a smooth algebraic curve). We choose a canonical basis of cycles $a_1,\ldots,a_g$, $b_1,\ldots,b_g$ in the integral cohomology group $H_1(\Gamma, \mathbf{Z})$ with intersection matrix of the form $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. To this basis there corresponds a basis of the holomorphic differentials $\omega_1,\ldots,\omega_g$ normalized by the conditions

$$\oint_{a_j} \omega_k = 2\pi i \delta_{jk}. \tag{0.11}$$

Then the b-periods of these differentials define a Riemann matrix

$$B_{jk} = \oint_{b_j} \omega_k. \tag{0.12}$$

The corresponding theta function $\theta(z) = \theta(z \mid B)$ is called *the theta function of the Riemann surface* Γ, and the corresponding abelian torus $T^{2g}(B)$ is called the *Jacobian variety* or the *Jacobian* of this Riemann surface. The passage to another canonical basis of the cycles is given by an integral symplectic transformation; the Riemann matrix B of the Riemann surface Γ is transformed to an equivalent one, and the Jacobian generally does not change.

Let $\mathbf{R}_g$ be the moduli space of Riemann surfaces of genus $g \geqslant 2$. This space is an irreducible variety of dimension $3g - 3$. There is defined a period mapping

$$\mathbf{R}_g \overset{\mathbf{B}}{\to} \mathbf{A}_g; \qquad \Gamma \mapsto \mathbf{B}(\Gamma), \tag{0.13}$$

which assigns to a curve Γ the equivalence class of its Riemann matrix (see [15]). The classical Torelli theorem asserts that this mapping is an embedding. The proof of this theorem is not effective, since it requires a knowledge of the "theta divisor" $\{\theta(z) = 0\}$ (see [1]). The problem of finding the equations characterizing the image of the period mapping (0.13) is the *Riemann-Schottky problem*. In other words, this problem is contained in the following: to find the equations imposed on a Riemann matrix (B_{jk}) that characterize the period matrices of Riemann surfaces. The Riemann-Schottky problem is nontrivial for $g \geqslant 4$. For $g = 4$ the equation characterizing the period matrices was found by Schottky [3] (see also [4] and [5], which indicate generalizations of the Schottky relations to higher genus). Schottky's method was essentially advanced by Farkas and Rauch [6]. In that paper a method was presented that allows one to get a large number of explicit identities on theta functions by using the theory of the Prym variety. However, the question of the completeness of this system of identities, insofar as the author knows, was not investigated (it is not even clear whether the dimension is right). Finally, another approach to the Riemann-Schottky problem was found by Andreotti and Mayer [7]; to solve this problem they used information about the singularities of the θ-divisor (for Jacobian varieties of Riemann surfaces these singularities have codimension 4). The Andreotti-Mayer equations (we shall state them at the end of the paper) give the right dimension, but the process of writing them down is very ineffective. Moreover, it is known that the solutions of these equations have superfluous components in $\mathbf{A}_g$ (not corresponding to period matrices)—see [13].

In this paper we obtain explicit formulas for inverting the period mapping (0.13), i.e. for recovering the Riemann surface from its Jacobian (in other words, from its period matrix (B_{jk})). Moreover, the Riemann-Schottky problem of characterizing the image of the period mapping (0.13) is solved precisely up to components.

To solve these classical algebro-geometric problems the author, following ideas of S. P. Novikov, uses an important equation of mathematical physics, the Kadomcev-Petviašvili equation (KP):

$$3u_{yy} = (4u_t - 6uu_x - u_{xxx})_x. \tag{0.14}$$

I. M. Kričever found a wide class of exact solutions of this equation, each of which is constructed by starting from an arbitrary smooth algebraic curve Γ (see [9]). These solutions have the form

$$u(x, y, t) = 2\partial_x^2 \ln \theta(Ux + Vy + Wt + z_0); \tag{0.15}$$

here $\theta(z) = \theta(z \mid B)$ is the theta function of the curve Γ; the vectors U, V and W are given in a quite complicated way by Γ and a point P on it (see below, (1.6)); and z_0 is an arbitrary vector.

We now take an arbitrary Riemann matrix B and some vectors U, V and W, and try to satisfy the KP equation using (0.15), where $\theta(z) = \theta(z \mid B)$. After substituting we get a system of relations on the variables U, V, W and the matrix B:

$$\mathrm{KP}(U, V, W, B) = 0 \tag{0.16}$$

(the explicit form of this equation will be given below; see Theorem 1). S. P. Novikov has announced the conjecture that solving the system (0.16) for the matrix B will yield the period matrices of Riemann surfaces and these only; that is, we have a solution of the Riemann-Schottky problem. It has also turned out that if one knows that B is the period matrix of a Riemann surface Γ, then solving (0.16) for U, V and W yields explicit equations for this Riemann surface.

§1. Formulation of the main results

We introduce 2^g "second-order theta functions"

$$\hat{\theta}[n](z) = \sum_{k \in \mathbf{Z}^g} \exp\{\langle B(k+n), k+n\rangle + \langle k+n, z\rangle\}, \tag{1.1}$$

where the "characteristic" $n \in \frac{1}{2}(\mathbf{Z}_2)^g$; that is, all the coordinates of the vector $n = (n_1, \ldots, n_g)$ are 0 or $\frac{1}{2}$. The quantities $\hat{\theta}[n](0)$ (depending on the Riemann matrix B), as well as the values of arbitrary products (of even order) at zero—$\hat{\theta}_{ij}[n](0)$, $\hat{\theta}_{ijkl}[n](0), \ldots$—are called *theta constants*. We agree to omit the zero argument of a theta constant; that is, $\hat{\theta}[n] \equiv \hat{\theta}[n](0)$, $\hat{\theta}_{ij}[n] \equiv \hat{\theta}_{ij}[n](0)$, etc.

THEOREM 1. *A function of the form* (0.15) *is a solution of the KP equation* (0.14) *if and only if the following system of relations holds between the vectors U, V, W, the Riemann matrix B, and an additional constant d:*

$$\begin{aligned} f[n] &= f[n](U, V, W, d, B) \\ &= \sum_{i,j,k,l} U_iU_jU_kU_l\hat{\theta}_{ijkl}[n] + \sum_{i,j}\left(\frac{3}{4}V_iV_j - U_iW_j\right)\hat{\theta}_{ij}[n] + d\hat{\theta}[n] = 0, \end{aligned} \tag{1.2}$$

where $n \in \frac{1}{2}(\mathbf{Z}_2)^g$.

For the proof of this theorem see the survey [11] (Chapter 4).

We introduce the variety $\mathbf{X}_g$ whose points are sequences (U, V, W, d, B), where $0 \neq U, V, W \in \mathbf{C}^g$, $d \in \mathbf{C}$ and $B \in \mathbf{H}_g$, factored by the action of the following groups:

$$\begin{gathered} U \mapsto \lambda U, \qquad V \mapsto \pm(\lambda^2 V + 2\alpha\lambda U), \\ W \mapsto \lambda^3 W + 3\lambda^2\alpha V + 3\lambda\alpha^2 U, \qquad d \mapsto \lambda^4 d, \qquad B \mapsto B \end{gathered} \tag{1.3}$$

(λ and α are arbitrary constants, $\lambda \neq 0$), and

$$\begin{gathered} U \mapsto 2\pi i M U, \quad \text{where } M = (\gamma B + 2\pi i \delta)^{-1}, \\ V \mapsto 2\pi i M V, \\ W \mapsto 2\pi i M W + \pi i M U\{U, U\}, \quad \text{where } \{X, Y\} = X^t M \gamma Y, \\ d \mapsto d + \tfrac{3}{8}\{V, V\} - \tfrac{1}{2}\{U, W\} - \tfrac{3}{4}\{U, U\}^2, \\ B \mapsto 2\pi i(\alpha B + 2\pi i \beta)(\gamma B + 2\pi i \delta)^{-1}. \end{gathered} \tag{1.4}$$

Here the matrix $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{Sp}(g, \mathbf{Z})$ defines a transformation in the modular group G_g (see above). The variety $\mathbf{X}_g$ (after a suitable compactification) is algebraic.

THEOREM 2. *The system* (1.2) *characterizes an algebraic subvariety* $\mathbf{Y}_g$ *of* $\mathbf{X}_g$.

This immediately follows from the easily checked invariance of $\mathbf{Y}_g$ under the transformations (1.3) and (1.4). The invariance of the system under the transformations (1.3) is evident. Under the action (1.4) of the modular group the system (1.2) is invariant only under the action of some subgroup of G_g of finite index, not in general. But the set of zeros $\mathbf{Y}_g$ of this system is invariant under the action of G_g. Actually, this is connected with the fact that the construction of Kričever's solutions does not depend on the choice of a canonical basis of the cycles.

We have introduced the space $\mathbf{R}_g$ parametrizing the Riemann surfaces of genus $g \geqslant 2$—the variety of moduli of Riemann surfaces. By $\tilde{\mathbf{R}}_g$ we shall denote the natural bundle over $\mathbf{R}_g$ whose fiber over a given point is the corresponding Riemann surface. The dimension of this variety is $3g - 2$. The period mapping (0.13) evidently extends to a mapping

$$\tilde{\mathbf{R}}_g \overset{\mathbf{B}}{\to} \mathbf{H}_g/G_g. \tag{1.5}$$

Let $\mathbf{B}(\tilde{\mathbf{R}}_g)$ be the graph of this mapping. According to Kričever's construction (see the Introduction), one can define an embedding of this graph into the variety $\mathbf{Y}_g$ of solutions of the basic system (1.2). This embedding is given by the following formulas: if Γ is a Riemann surface with a canonical basis for the cycles a_i and b_j, $\omega_1, \ldots, \omega_g$ is a basis of the differentials on Γ, and $P \in \Gamma$ is a fixed point, then B is the period matrix (0.12) of the surface Γ, and

$$\begin{gathered} U_i = -\omega_i(P)/dz, \qquad V_i = -\omega_i'(P)/dz, \\ W_i = -\tfrac{1}{2}\omega_i''(P)/dz - \tfrac{1}{2}c(P)U \end{gathered} \tag{1.6}$$

(the prime denotes the derivative with respect to a fixed local parameter z in the neighborhood of P), where $c(P)$ is a projective connection on Γ (see [2]) of the form

$$3c(P) = \frac{\Sigma\,\omega_i''(P)\theta_i(\zeta)}{\Sigma\,\omega_i(P)\theta_i(\zeta)} - \frac{3}{2}\left[\frac{\Sigma\,\omega_i'(P)\theta_i(\zeta)}{\Sigma\,\omega_i(P)\theta_i(\zeta)}\right]^2 + \frac{3}{2}\left[\frac{\theta_{xx}(\zeta)}{\theta_x(\zeta)}\right]^2 - 2\,\frac{\theta_{xxx}(\zeta)}{\theta_x(\zeta)} \tag{1.7}$$
$$(\theta_x(\zeta) = \Sigma\,U_i\theta_i(\zeta), \ldots);$$

here ζ is an arbitrary nonsingular point of the theta divisor; that is, $\theta(\zeta) = 0$ and grad $\theta(\zeta) \neq 0$, and $d = d(P)$ is a quadratic differential on Γ of the form

$$8d(P) = -(\ln\theta(\xi))_{xxxx} - 6[(\ln\theta(\xi))_{xx}]^2 + 4(\ln\theta(\xi))_{xt} - 3(\ln\theta(\xi))_{yy} \tag{1.8}$$

(ξ is an arbitrary vector). It is easily checked that this mapping is well defined (it is independent of the choice of the basis of cycles and of the local parameter z). In fact (see [11]), the graph $\mathbf{B}(\tilde{\mathbf{R}}_g)$ is contained in an open subset $\mathbf{X}_g^0 \subset \mathbf{X}_g$ characterized by the following *nondegeneracy condition*:

$$\operatorname{rank}\big(\hat{\theta}_{ij}[n], \hat{\theta}[n]\big) = \frac{g(g+1)}{2} + 1. \tag{1.9}$$

(The matrix occurring in this condition consists of theta constants and has size $[\frac{1}{2}g(g+1) + 1] \times 2^g$.)

We can now give a precise algebro-geometric formulation of the main conjecture.

CONJECTURE (S. P. NOVIKOV). *The intersection of the set of zeros $\mathbf{Y}_g$ of the system* (1.2) *with the domain $\mathbf{X}_g^0$ coincides with the graph $\mathbf{B}(\tilde{\mathbf{R}}_g)$ of the period mapping* (1.5).

(It is necessary to take the intersection with $\mathbf{X}_g^0$ to omit the "trivial" solutions of (1.2); they correspond to the direct products of Jacobians on an arbitrary abelian variety.)

For $g = 2$ and 3, where no conditions on the Riemann matrix arise, this conjecture was proved by the author in [10], and this has allowed us to solve the problem of making effective the formulas for solving the KP equation and the equations connected with it (for a proof, see [11]). For $g > 3$ one has the following weaker assertion:

THEOREM 3. *The irreducible component of the variety $\mathbf{Y}_g$ containing the graph $\mathbf{B}(\tilde{\mathbf{R}}_g)$ coincides with it. In other words, Novikov's conjecture holds except for the possibility that the variety of solutions of the system* (1.2) *may have superfluous components.*

The proof of this theorem will be given in the next section (the idea of the proof was published in [11]).

§2. Proof of the main theorem

Because all the constructions are algebraic, it suffices to compute the dimension of the component $\mathbf{Y}_g^0$ of the variety $\mathbf{Y}_g$ of zeros of the system (1.2) that contains the graph $\mathbf{B}(\tilde{\mathbf{R}}_g)$ and show that this is $3g - 2$. We shall show that the local coordinates at a general point of $\mathbf{Y}_g^0$ can be taken to be the variables $U_1, \dots, U_g, V_1, \dots, V_g$ (defined up to transformations $U \mapsto \lambda U$ and $V \mapsto \pm(\lambda^2 V + 2\lambda\alpha U)$) and $\varepsilon_i = \exp B_{ii}$ $(i = 1, \dots, g)$. For this we must show that the rank of the Jacobian matrix

$$\left(\frac{\partial f[n]}{\partial d}, \frac{\partial f[n]}{\partial W_i}, \frac{\partial f[n]}{\partial B_{ij}}\right) \tag{2.1}$$

is $\frac{1}{2}g(g+1) + 1$. Here $f[n]$ denotes the left-hand members of the system (1.2), and in the matrix (2.1) the arguments U, V, W, d and B lie on the graph $\mathbf{B}(\tilde{\mathbf{R}}_g)$, i.e. have the form (1.6)–(1.8) for some Riemann surface Γ and point P on it. We shall deform this surface Γ, moving its a-cycles. Then the quantities $\varepsilon_i = \exp B_{ii}$ will converge to zero (see [2]). The limiting subvariety $\{\varepsilon_1 = \cdots = \varepsilon_g = 0\}$ is a so-called "Enriques curve"; that is, a rational curve with g double points. The basic idea of the proof of Theorem 3 is to solve (1.2) in the neighborhood of the subvariety $\{\varepsilon = 0\}$ "by the theory of perturbations".(1) From all

(1) For genus $g = 2$ and the Korteweg-deVries equation, similar computations, according to the theory of perturbations, were first made by S. Ju. Dobrohotov—see [12].

the basic systems $f[n]=0$ we take only $\frac{1}{2}g(g+1)+1$ equations corresponding to the following characteristics:

1) $n=0$;

2) $n=n_{(p)}=\frac{1}{2}e_p$, where $e_p=(0,\dots,1,\dots,0)$ is the pth basis vector (with a unit in the pth place); and

3) $n=n_{(p,q)}=\frac{1}{2}(e_p+e_q), p\neq q$.

The functions $\hat{\theta}[n](z\mid B)$ are analytic (in ε) in the neighborhood of the subvariety $\{\varepsilon=0\}$ after a simple renormalization. For the characteristics listed above the explicit form of these renormalized functions as $\varepsilon\to 0$ is

$$\hat{\theta}\,[0](z)=1+2\sum_{i=1}^{g}\varepsilon_i\cosh z_i+\cdots;$$

$$\frac{1}{2\sqrt[4]{\varepsilon_p}}\hat{\theta}\,[n_{(p)}](z)$$

$$=\cosh\frac{z_p}{2}+\sum_{i\neq p}\varepsilon_i\left\{\zeta_{pi}\cosh\left(z_i+\frac{z_p}{2}\right)+\frac{1}{\zeta_{pi}}\cosh\left(z_i-\frac{z_p}{2}\right)\right\}+\cdots; \qquad (2.2)$$

where we have introduced the notation $\zeta_{ij}=\exp B_{ij}$ for $i\neq j$;

$$\frac{\sqrt{\zeta_{pq}}}{2\sqrt[4]{\varepsilon_p\varepsilon_q}}\hat{\theta}\,[n_{(p,q)}](z)=\zeta_{pq}\cosh\frac{z_p+z_q}{2}+\cosh\frac{z_p-z_q}{2}$$

$$+\sum_{i\neq p,q}\varepsilon_i\left\{\zeta_{pq}\zeta_{pi}\zeta_{qi}\cosh\left(z_i+\frac{z_p+z_q}{2}\right)+\frac{\zeta_{pq}}{\zeta_{pi}\zeta_{qi}}\cosh\left(z_i-\frac{z_p+z_q}{2}\right)\right.$$

$$\left.+\frac{\zeta_{qi}}{\zeta_{pi}}\cosh\left(z_i+\frac{z_q-z_p}{2}\right)+\frac{\zeta_{pi}}{\zeta_{qi}}\cosh\left(z_i+\frac{z_p-z_q}{2}\right)\right\}+\cdots. \qquad (2.3)$$

In these formulas the dots always denote the terms of higher degree than ε. For these characteristics we shall correspondingly renormalize the functions $f[n]$ (without changing the notation for them). We seek the unknowns d, W_i and $\zeta_{ij}=\exp B_{ij}$ as power series in

$$d=d^{(0)}+d^{(1)}+\cdots,\quad W_i=W_i^{(0)}+W_i^{(1)}+\cdots,\quad \zeta_{ij}=\zeta_{ij}^{(0)}+\zeta_{ij}^{(1)}+\cdots, \qquad (2.4)$$

where the upper index $(k)=(0),(1),\dots$ denotes in the decomposition of the respective quantity the component of degree k in ε (the "kth approximation"). Substituting the decompositions (2.2) and (2.3) into the corresponding equations $f[n]=0$ of the system (1.2) and putting $\varepsilon_1=\cdots=\varepsilon_g=0$, we get the equations for the zeroth approximation. The equation $f[0]=0$ yields

$$d^{(0)}=0. \qquad (2.5)$$

Further, we require that all coordinates of U be nonzero. Then the equation $f[n_{(p)}]=0$ together with (2.5) yields

$$W_p^{(0)}=\frac{1}{4}U_p^3+\frac{3}{4}\frac{V_p^2}{U_p}. \qquad (2.6)$$

We now replace U and V by new variable vectors X and Y, putting

$$U_i = X_i - Y_i, \quad V_i = X_i^2 - Y_i^2 \qquad (i = 1,\dots,g). \tag{2.7}$$

Then (2.6) can be rewritten as

$$W_p^{(0)} = X_p^3 - Y_p^3. \tag{2.8}$$

Finally, from the equation $f[n_{(p,q)}] = 0$ we find that

$$\zeta_{pq}^{(0)} = \frac{(X_p - X_q)(Y_q - Y_p)}{(X_q - Y_p)(X_p - Y_q)} \tag{2.9}$$

(assuming the quantities X_i and Y_j are pairwise distinct). Thus (1.2) is solved in zeroth approximation. The variables $X_1,\dots,X_g$, $Y_1,\dots,Y_g$, defined up to transformations of the form

$$\begin{aligned} X_i &\mapsto \lambda X_i, & Y_i &\mapsto \lambda Y_i, \\ X_i &\mapsto X_i + \alpha, & Y_i &\mapsto Y_i + \alpha, \end{aligned} \tag{2.10}$$

are coordinates on the intersection $\mathbf{Y}_g \cap \{\varepsilon = 0\}$. This shows the validity of Theorem 3 in first approximation (for Enriques curves).

One now easily computes the Jacobian matrix (2.1) for $\varepsilon = 0$ with the change $\partial/\partial B_{ij} \to \partial/\partial\zeta_{ij}$. This matrix has the form

$$\begin{array}{r} \\ n = 0 \\ n = n_{(p)} \\ n = n_{(p,q)} \end{array} \begin{array}{c} \begin{array}{ccc} \dfrac{\partial f[n]}{\partial d} & \dfrac{\partial f[n]}{\partial W_i} & \dfrac{\partial f[n]}{\partial \zeta_{ij}} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ * & -\delta_{ip} & 0 \\ * & * & \delta_{ip}\delta_{jq}U_pU_q(X_q - Y_p)(X_p - Y_q) \end{pmatrix} \end{array} \tag{2.11}$$

where the asterisks denote terms inessential for us. Evidently, this matrix has maximal rank. From this and the complex-analytic implicit function theorem we get a solution of (1.2) in the form

$$U_i = X_i - Y_i, \quad V_i = X_i^2 - Y_i^2, \tag{2.12}$$

$$d = d(X, Y, \varepsilon), \quad W_i = W_i(X, Y, \varepsilon), \quad \zeta_{ij} = \zeta_{ij}(X, Y, \varepsilon), \tag{2.13}$$

where (2.13) are single-valued analytic functions of ε (for small ε). The coefficients of the expansions of these functions as power series in ε can be found by recursively substituting (2.4) into (1.2) (with the same characteristics as above). It is clear that these expansions are invariant under the transformations (2.10). We get precisely a $(3g - 2)$-dimensional family of solutions. This means that the family of solutions coincides with $\mathbf{B}(\tilde{\mathbf{R}}_g)$, since they have the same dimension and are irreducible. Theorem 3 is proved.

§3. Concluding remarks

1. The component $\mathbf{Y}_g^0$ of the totality $\mathbf{Y}_g$ of solutions of (1.2) that is found in the main theorem (and coincides with $\mathbf{B}(\tilde{\mathbf{R}}_g)$) can be given by a part of the equations of (1.2). It suffices to take the equations $f[n] = 0$ for those characteristics $n^1,\dots,n^r$ ($r = g(g+1)/2 + 1$) which number the lines of a nonzero minor of the matrix (1.9). For example, for the matrix (B_{jk}) with small $\varepsilon_i = \exp B_{ii}$ such characteristics are $n = 0$, $n = n_{(p)}$ ($p = 1,\dots,g$) and $n = n_{(p,q)}$ ($p, q = 1,\dots,g$; $p < q$) in the notation of the preceding section. It is interesting to note that, eliminating the variables U, V, W and d, among which there are $3g-1$ independent ones, from the system

$$f[n^1]=0,\ldots,\quad f[n^r]=0 \qquad \left(r=\frac{g(g+1)}{2}+1\right), \tag{3.1}$$

we get just

$$\frac{g(g+1)}{2}+1-(3g-1)=\frac{(g-1)(g-4)}{2}$$

relations on the Riemann matrix (B_{jk}), one less than the number of such relations needed, namely $\frac{1}{2}(g-2)(g-3)$. The structure of the variety $\mathbf{B}(\tilde{\mathbf{R}}_g)=\mathbf{Y}_g^0$ explains the form of the missing relation: if (B_{jk}) is the period matrix of some Riemann surface Γ, then its preimage $\mathbf{B}^{-1}(B_{jk})$ is this Riemann surface itself rather than isolated points (as would follow from the simple combination of system (1.2)). We shall explain how to obtain the required number of relations on theta constants in the first nontrivial example, $g=4$. Let $n^1,\ldots,n^{11}$ be the number of lines of a nonzero minor of (1.9):

$$\det\left(\hat{\theta}_{ij}[n],\hat{\theta}\,[n]\right)\neq 0, \qquad (1\leqslant i\leqslant j\leqslant 4;\, n=n^1,\ldots,n^{11}). \tag{3.2}$$

We denote the inverse matrix by

$$\left(a_{ij}[n],a[n]\right) \qquad (1\leqslant i\leqslant j\leqslant 4;\, n=n^1,\ldots,n^{11}). \tag{3.3}$$

From (1.2) with $n=n^1,\ldots,n^{11}$ we then get

$$\frac{3}{4}V_p^2-U_pW_p=-Q_{pp}(U), \qquad p=1,\ldots,4, \tag{3.4}$$

$$\frac{3}{2}V_pV_q-U_pW_q-U_qW_p=-Q_{pq}(U), \qquad 1\leqslant p<q\leqslant 4, \tag{3.5}$$

where the polynomials $Q_{pq}(U)$ have the form

$$Q_{pq}(U)=\sum_{i,j,k,l=1}^{4}U_iU_jU_kU_l\left(\sum_{s=1}^{11}a_{pq}[n^s]\hat{\theta}_{ijkl}[n^s]\right). \tag{3.6}$$

The W variables are eliminated at once. After simple transformations we get

$$U_pV_q-U_qV_p=\frac{2i}{\sqrt{3}}\sqrt{P_{pq}(U)} \qquad (1\leqslant p<q\leqslant 4), \tag{3.7}$$

where the polynomials $P_{pq}(U)$ have the form

$$P_{pq}(U)=U_p^2Q_{qq}(U)-U_pU_qQ_{pq}(U)+U_q^2Q_{pp}(U). \tag{3.8}$$

The compatibility conditions of system (3.7) have the form

$$\begin{aligned}
&U_2\sqrt{P_{13}(U)}-U_3\sqrt{P_{12}(U)}-U_1\sqrt{P_{23}(U)}=0,\\
&U_2\sqrt{P_{14}(U)}-U_4\sqrt{P_{12}(U)}-U_1\sqrt{P_{24}(U)}=0,\\
&U_3\sqrt{P_{14}(U)}-U_4\sqrt{P_{13}(U)}-U_1\sqrt{P_{34}(U)}=0.
\end{aligned} \tag{3.9}$$

The vector $U=(U_1,U_2,U_3,U_4)$ can be normalized so that $U_1=1$. Finding U_2 and U_3 from the first two equations of (3.9) and substituting them into the third equation gives us a relation $P(U_4)=0$, where $P(U_4)$ is a polynomial. Equating any of its coefficients to zero gives us the desired relation on theta constants. Evidently, the required number of relations are gotten in just the same way in the general case $g > 4$.

2. The procedure indicated in the proof of the main theorem for constructing solutions of (1.2) for the component $\mathbf{Y}_g^0$ as a function of the coordinates $\{X, Y, \varepsilon\}$ also works on subvarieties in $\mathbf{B}(\tilde{\mathbf{R}}_g)$ corresponding to hyperelliptic or trigonal (three-sheeted coverings of $\mathbf{C}P^1$) Riemann surfaces. For hyperelliptic surfaces the coordinates X and Y must be connected by relations

$$X_i + Y_i = 0 \qquad (i = 1,\dots,g), \tag{3.10}$$

$\varepsilon_1,\dots,\varepsilon_g$ being arbitrary. Indeed, by (1.6) the vector $(V_i) = (X_i^2 - Y_i^2)$ vanishes at $2g + 2$ Weierstrass points of a hyperelliptic surface of genus g. Therefore, condition (3.10) cuts out on the variety $\mathbf{Y}_g^0$ a $(2g + 2)$-sheeted covering of the totality of all hyperelliptic curves. Formulas (2.13) will then give explicit expressions for the period matrices of hyperelliptic curves. Likewise, the totality of periods of trigonal curves is characterized in $\mathbf{Y}_g^0$ by the relations

$$W_i(X, Y, \varepsilon) = X_i - Y_i, \qquad i = 1,\dots,g. \tag{3.11}$$

3. As already stated in the first remark, if (B_{jk}) is the period matrix of some Riemann surface Γ, then the solutions of the corresponding system (1.2) lying in $\mathbf{Y}_g^0$ yield the Riemann surface Γ itself. By the preceding remark, this also holds in the hyperelliptic and trigonal cases. Combining this with the Noether-Enriques theorem on canonical curves (see [1], as well as Chapter 4 in [11]) allows us to state the following

ASSERTION. *If (B_{jk}) is the period matrix of a general hyperelliptic Riemann surface Γ, then the projection of the solutions of the corresponding system* (1.2) *into the space $\mathbf{C}P_U^{g-1}$ with homogeneous coordinates $(U_1:\dots:U_g)$ is the image of the canonical embedding of the curve Γ.*

Thus, if (B_{jk}) is the period matrix of Γ, then eliminating V, W and d from (1.2) yields canonical equations for the surface Γ. For the hyperelliptic case it suffices to eliminate just W and d.

4. At the present time one knows many important nonlinear equations that are integrated by theta functions (see [8], [9], and also the survey [11]). The method developed by the author in [10] and [11] allows one to infer a series of useful relations among theta functions from these nonlinear equations.([2]) Here we mention one simple example: the two-dimensional Toda lattice, which I. M. Kričever (see the Appendix to [11]) integrated by theta functions. Here we have differential-difference equations:

$$\left.\begin{aligned} \frac{\partial v_n}{\partial y} &= c_{n+1} - c_n, \\ \frac{\partial c_n}{\partial x} &= c_n(v_n - v_{n-1}) \end{aligned}\right\} \qquad (n = 0, \pm 1, \pm 2,\dots). \tag{3.12}$$

([2]) Unrelated to nonlinear equations, some of these equations appeared earlier (see [2]). Thus, for example, Corollary 2.12 in [2] is easily compared with the KP equation, and identity (39) with the Toda lattice.

The solutions of this system are parametrized by an arbitrary Riemann surface Γ and a pair of points P^+ and P^- on it; it has the form (the vector z_0 being arbitrary)

$$c_n = \frac{\theta(xU^+ + yU^- + (n+1)\Delta + z_0)\theta(xU^+ + yU^- + (n-1)\Delta + z_0)}{\varepsilon^2\theta^2(xU^+ + yU^- + n\Delta + z_0)}$$
$$v_n = \frac{\partial}{\partial x}\ln\frac{\theta(xU^+ + yU^- + (n+1)\Delta + z_0)}{\theta(xU^+ + yU^- + n\Delta + z_0)}. \tag{3.13}$$

Here $\Delta = (\Delta_1,\dots,\Delta_g)$, $U^\pm = (U_1^\pm,\dots,U_g^\pm)$ and

$$\Delta_i = \int_{P^-}^{P^+}\omega_i, \qquad U_i^\pm = -\frac{\omega_i(P^\pm)}{dz_\pm}, \tag{3.14}$$

where $\omega_1,\dots,\omega_g$ are a basis of the holomorphic differentials on Γ with the normalization (0.11), $z_\pm$ are local parameters in the neighborhoods of P^+ and P^-, and ε^2 is a constant depending on P^+ and P^-. Substituting these formulas into (3.12) yields, after some simple transformations,

$$\frac{\theta(z_0+\Delta)\theta(z_0-\Delta)}{\varepsilon^2\theta^2(z_0)} = a + \sum_{i,j=1}^{g} U_i^+U_j^-(\ln\theta)_{ij}(z_0) \tag{3.15}$$

(a is a new constant). Applying the addition formula (see [11]) gives the following system of relations, equivalent to (3.15):

$$\frac{1}{\varepsilon^2}\hat\theta[n](2\Delta) = a\hat\theta[n] + 2\sum_{i,j}U_i^+U_j^-\hat\theta_{ij}[n], \tag{3.16}$$

where $n \in \frac{1}{2}(\mathbf{Z}_2)^g$. This system is transcendental with respect to the quantity Δ. It can be redeemed from transcendence as follows. Consider the g-dimensional subvariety $K^g(B)$ of the space $\mathbf{C}P^N$ $(N = 2^g - 1)$ with homogeneous coordinates $\lambda[n]$, $n \in \frac{1}{2}(\mathbf{Z}_2)^g$, with parametric representation

$$\lambda[n] = \hat\theta[n](2\Delta), \tag{3.17}$$

where $\Delta \in \mathbf{C}^g$ is an arbitrary vector, $n \in \frac{1}{2}(\mathbf{Z}_2)^g$. This is a Kummer variety; the coefficients of its equations in $\mathbf{C}P^N$ can be expressed by means of theta constants (see [7]). Let

$$g_k(\lambda[n]) = 0, \qquad k = 1,\dots,N-g, \tag{3.18}$$

be these equations. Then from (3.16) we obtain

$$g_k\Big(a\hat\theta[n] + 2\sum_{i,j}U_i^+U_j^-\hat\theta_{ij}[n]\Big) = 0, \tag{3.19}$$

i.e. the totality of vectors of the form

$$\lambda[n] = a\hat\theta[n] + \sum_{i,j}U_i^+U_j^-\hat\theta_{ij}[n] \tag{3.20}$$

has nonempty intersection with the surface $K^g(B)$. Eliminating the quantities a and U^+, U^- from (3.20), we can obtain the desired relations on theta constants. It is interesting to note the analogy between the relations we have gotten and the Andreotti-Mayer relations

[7], which have the following form: *the totality of vectors* $\lambda[n]$ *in* $\mathbf{C}P^N$ *that satisfy the equations*

$$\sum_n \lambda[n]\hat{\theta}[n] = 0, \qquad \sum_n \lambda[n]\hat{\theta}_{ij}[n] = 0, \quad i, j = 1, \ldots, g, \tag{3.21}$$

has nonempty intersection with the Kummer variety.

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