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Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory

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CONTENTS

Introduction

1. Soliton lattices and the Whitham equation.

The term *soliton lattices* is frequently used to denote solutions periodic (and quasi-periodic) in x and t of non-linear evolution partial differential equations

$\varphi_t = K(\varphi, \varphi_x, \ldots, \varphi^{(n)})$, where the function $\varphi(x, t)$ has the form

$$\varphi(x, t) = \Phi(kx + \omega t + \tau^0, u^1, \ldots, u^N); \tag{1}$$

here $\Phi(\tau_1, \ldots, \tau_m, u^1, \ldots, u^N)$ is a function which is 2π-periodic in all the variables τ_j and depends on N parameters $u^1, \ldots, u^N$, and the m-vectors k and ω are expressed in terms of $u^1, \ldots, u^N$. For each value of the parameters u^j = const, (1) represents the so-called "m-phase" exact solutions of the original system $\varphi_t = K(\varphi, \varphi_x, \ldots, \varphi^{(n)})$, where

$$\tau_j = k_j x + \omega_j t + \tau_j^0, \quad u^q = u_0^q, \quad j = 1, \ldots, m, \quad q = 1, \ldots, N. \tag{2}$$

In the theory of soliton systems integrable by the inverse scattering method, vast families of solutions of the form (1) are known. These solutions, discovered and studied in 1974–75 [48], [30], [23], [35], [84] are called "finite-zone", "periodic and quasi-periodic analogues of multi-soliton solutions" in view of some of their remarkable mathematical connections with the theory of finite-zone periodic operators and the fact that for some values of the parameters u^q they degenerate into soliton ($m = 1$) or multi-soliton ($m > 1$) solutions. In soliton theory, general complex solutions of the form (1) are called "algebraic-geometric" solutions, since they can be expressed in terms of theta-functions of Riemann surfaces and can be constructed using methods of algebraic geometry (see the survey papers [5], [24], [26], [38], [70] and the monograph [57]). Of course, for $m = 1$, solutions of the form (1) are found, as a rule, by elementary methods, and some of them were known in the 19th century (for example, the cnoidal periodic solutions of the Korteweg-de Vries (KdV) equation were obtained in 1895, while for the Sine–Gordon equation $\varphi_{tt} - \varphi_{xx} = \sin\varphi$ they were obtained even earlier, the moment they were first written down).

Let $X = \varepsilon x$, $T = \varepsilon t$ where ε is a small parameter.

Definition. A *weakly deformed soliton lattice* is a function of the form (1) for any t = const, in which the quantities $(k, \omega, u^1, \ldots, u^N)$ are smooth functions of the variable X, that is, they are "slowly varying" as x varies.

In his 1965 papers [93], [94], Whitham formulated and, for certain evolutionary systems, verified the following assertion in the case $m = 1$:

let a function of the form (1), in which the parameters are smooth functions of X and T, be the principal term of the asymptotic expansion in ε of the evolution equation $\varphi_t = K(\varphi, \varphi_X, \ldots, \varphi^{(n)})$. The phase of the function (1) should be written in the form $S(X, T)/\varepsilon = \tau$, where

$$k_j = \partial S_j/\partial X, \quad \omega_j = \partial S_j/\partial T, \tag{3}$$
$$\varphi = \Phi(S/\varepsilon, u^1, \ldots, u^N), \quad S = (S_1, \ldots, S_m).$$

(Thus, the equations $k_{jT} = \omega_{jX}$ hold by definition.) It is claimed that the parameters $u^q(X, T)$ satisfy the quasi-linear first order system (relations (3) are a part of equations (4)):

$$\frac{\partial u^q}{\partial T} = v^q_p(u)\frac{\partial u^p}{\partial X}, \tag{4}$$

which resemble the equations of hydrodynamics of a compressible fluid. These are Riemann type equations or "systems of hydrodynamic type" in our terminology. We shall call equations (4) *the Whitham equations*, or *equations of hydrodynamics of weakly deformed soliton lattices.* Sometimes they are called *equations of slow modulation of parameters.*

Later, these problems were studied by Luke [89], Maslov [43], Ablowitz and Benney [63], Hayes [80], Whitham [58], and Gurevich and Pitaevskii [14], [15]. The aspects discussed include the sufficiency of equations (4) for the construction of asymptotic solutions in the case $m = 1$, their explicit form in some important particular cases, and generalizations to the multi-phase case $m > 1$ (although at that time finite-zone solutions were not yet known and the discussion was not sufficiently explicit). Applications to physical problems in dispersive hydrodynamics were found in [14], [15]. Equations (4) for non-degenerate Lagrangian systems (all in the case $m = 1$) were derived in [93].

The theory of multi-phase systems began to develop rapidly only after the formulation in 1974–75 of the above-mentioned theory of finite-zone (algebraic-geometric) solutions of integrable soliton systems, which actually made it possible to consider multi-phase analogues of Whitham's equations (4) in the case $m > 1$ (see the papers [19], [73]). In [73], [75], [76] Flaschka, McLaughlin and others derived equations (4) from the theory of Riemann surfaces used in the construction of finite-zone solutions and obtained a number of useful generalizations of Whitham's results for the case $m > 1$ which are applicable to the well known integrable soliton systems (KdV, SG), where

$$\begin{aligned}
&\text{KdV}: && \varphi_t = 6\varphi\varphi_x - \varphi_{xxx},\\
&\text{NS}_\pm: && i\varphi_t = -\varphi_{xx} \pm |\varphi|^2\varphi,\\
&\text{SG}: && \varphi_{tt} - \varphi_{xx} = \sin\varphi \ (\text{or} = \sinh\varphi).
\end{aligned}$$

In particular, these authors showed that for KdV and SG, the Whitham equation (4) for any $m > 1$ has so-called "Riemann invariants". For NS, an analogous calculation was performed later in [51].

Definition. Riemann invariants for systems of hydrodynamic type (4) are coordinates in the u-space in which (4) is diagonal, that is, the matrix $v^q_p(u)$ is diagonal for all $u^1, \ldots, u^N$. Note that under changes of coordinates $u = u(w)$, the matrix $v^q_p(u)$ transforms as a tensor.

By classical 19th century results, for $N = 2$ Riemann invariants always exist, while for $N \geqslant 3$ this is no longer so: their existence for $N \geqslant 3$ is an indication of a substantial degeneracy of the system.

For the non-degenerate Lagrangian integrable system SG, in which $N = 2m$, these authors also obtained an analogue of results of Whitham and Hayes concerning the existence of special variables of Clebsch type J_j, τ_j, where $k_j = \tau_{jx'}$; in the (J_j, k_j) variables the equations (4) have the explicitly Hamiltonian form

$$(u) = (k_1, \ldots, k_m, J_1, \ldots, J_m),$$
$$\frac{\partial k_j}{\partial T} = \frac{\partial}{\partial X}\frac{\delta H}{\delta J_j}, \quad \frac{\partial J_j}{\partial T} = \frac{\partial}{\partial X}\frac{\delta H}{\delta k_j}$$

(or a Lagrangian form in the variables (J_j, τ_j)). However, this derivation for $m > 1$ is based on special properties of integrable systems and on the theory of Riemann surfaces, which define finite-zone solutions, unlike the more general method of derivation used by Whitham and Hayes [80], [93] for $m = 1$. An algebraic-geometric theory of the action variables J_j of finite-zone Hamiltonian systems defining finite-zone solutions that are of importance in this context was developed in [9], [10], [27], [45], [74]. As indicated in [27], [45], phenomena of particular interest arise in the attempt to single out conditions of reality in the SG equation. We also note that Novikov and Veselov developed a theory of algebraic-geometric Poisson brackets for finite-dimensional systems integrable by the methods of Riemann surfaces; this theory clarifies which Hamiltonian properties the celebrated integrable systems of classical mechanics and geometry, such as those of Jacobi, Clebsch, Kovalevskaya, von Neumann, and others, have in common with present-day integrable systems arising in the theory of solitons in the process of determining finite-zone solutions [27], [45]. We shall not give details of this theory in this survey.

II. A general survey of the authors' results of 1982–88.

We shall now review the results, due to the authors and to their colleagues in Moscow, concerning the general theory of Hamiltonian systems of hydrodynamic type and the hydrodynamics of weakly deformed soliton lattices, that were obtained in 1982–1988. As already mentioned, systems of hydrodynamic type have, by definition, the form (4). This form is invariant under local changes of coordinates in the u-space,

$$u = u(w), \tag{5}$$
$$v_p^q(u) \to v_p^q(u(w))\frac{\partial u^p}{\partial w^{p'}}\frac{\partial w^{q'}}{\partial u^q} = v_{p'}^{q'}(w).$$

Riemann invariants (if they exist) are coordinates $u^1, \ldots, u^N$, such that the matrix $v_p^q(u)$ is diagonal:

$$v_p^q(u) = v^q(u)\delta_p^q. \tag{6}$$

A *functional of hydrodynamic type* $I[u(x)]$ is a quantity whose density is independent of derivatives,

$$I[u] = \int j(u)dx. \tag{7}$$

Definition. Poisson brackets of hydrodynamic type are local Poisson brackets of the form

$$\{u^q(x), u^p(y)\} = g^{qp}(u(x))\,\delta'(x-y) + b_s^{qp}(u(x))\,u_x^s\delta(x-y). \tag{8}$$

Here g^{qp} and b_s^{qp} are smooth functions in local coordinates on the u-space, which is a finite-dimensional manifold M. For the moment, the expression (8) is to be interpreted formally. With brackets of hydronamic type, Hamiltonians of hydrodynamic type generate equations of hydrodynamic type of the form (4).

More precisely, this means that the Poisson bracket of any two functionals $I_1[u]$, $I_2[u]$ has the form

$$\{I_1, I_2\} = \int dx\left(\frac{\delta I_1}{\delta u^q(x)}\, A^{qp}\, \frac{\delta I_2}{\delta u^p(x)}\right), \tag{9}$$

where

$$A = (A^{qp}) = \left(g^{qp}(u)\frac{d}{dx} + b_s^{qp}(u)\,u_x^s\right), \tag{10}$$

and d/dx represents the total derivative with respect to x. Hamiltonian systems with Hamiltonian H have the form

$$\frac{\partial u^q}{\partial t} = A^{qp}\frac{\delta H}{\delta u^p(x)}. \tag{11}$$

However, formula (8) is more convenient. Boundary conditions in this problem are not taken into account. The Poisson bracket must be skew-symmetric and must satisfy the Leibniz and Jacobi identities:

$$\{uv, w\} = u\{v, w\} + v\{u, w\} \quad \text{(Leibniz)}, \tag{12}$$

$$\{\{u, v\}, w\} + \{\{w, u\}, v\} + \{\{v, w\}, u\} = 0 \quad \text{(Jacobi)}. \tag{12'}$$

The form (8) of the bracket is invariant under local changes of coordinates (5).

Theorem [28]. *Let* $\det g^{qp}(u) \neq 0$ *and* $b_s^{qp}(u) = -g^{ql}(u)\,\Gamma_{ls}^p(u)$. *The formula* (8) *defines a Poisson bracket with all the necessary properties if and only if:* a) *the quantity* $g^{qp}(u)$ *transforms as a tensor under local coordinate changes* $u(w)$, *while the quantities* $\Gamma_{ls}^p(u)$ *transform as connection components* (*Christoffel symbols*); b) *the tensor* $g^{qp}(u)$ *is symmetric and defines a pseudo-Riemannian metric on the u-space* M; c) *the curvature and torsion of the connection* $(\Gamma_{ls}^p(u))$ *are equal to zero and it is compatible with the metric* $g^{qp}(u)$.

Corollary. *There exist local coordinates in the u-space, such that the tensor $g^{qp}(u)$ is constant, $g^{qp}(u) = g_0^{qp}$, and $\Gamma_s^{pq}(u) \equiv 0$. The only local invariant of the Poisson bracket is the signature of the metric $g^{qp}(u)$ (see the authors' paper* [28]).

Example. Let $N = 2m$. The Hamiltonian equations (4) in Clebsch variables correspond to a metric of signature (m, m),

$$g^{qp} = g_0^{qp} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b_s^{qp} \equiv 0. \tag{13}$$

Other examples will be given in the body of the paper.

If the Hamiltonian is a quantity of hydrodynamic type, then the corresponding Hamiltonian system of hydrodynamic type has the form

$$\frac{\partial u^q}{\partial t} = A^{qp} \frac{\delta H}{\delta u^p(x)} = v_p^q(u)\, u_x^p,$$

where

$$v_p^q(u) = g^{ql}\nabla_l \nabla_p h(u), \qquad H = \int h(u)\, dx. \tag{14}$$

In particular, taking into account the fact that curvature and torsion are zero, the matrix $v_{sp}(u) = g_{sq} v_p^q(u)$ is symmetric.

Let the original evolutionary system $\varphi_t = K\ (\varphi, \varphi_x, \ldots, \varphi^{(n)})$ be Hamiltonian with a local Hamiltonian and with respect to some local translation-invariant Poisson bracket $\{\varphi\ (x), \varphi\ \ (y)\}_0$. It is only in this case that the authors can justify Whitham's assertion (see above) and construct the equations of hydrodynamics of soliton lattices (see §6). In this case, the authors stated and proved in 1983 [28] a "principle of conservation of Hamiltonian structure" under averaging, that is, under the passage from the original system to the system (4). We require the existence of N independent integrals I_q with local densities in involution with respect to the original bracket:

$$I_q = \int P_q(\varphi, \varphi_x, \ldots)\, dx, \qquad \{I_q, I_p\}_0 = 0, \tag{15}$$

such that the parameters u^q in solutions (1) can be chosen as values of integrals on these solutions (where a bar denotes the average over a torus),

$$I_q = u^q = \bar{P}_q. \tag{16}$$

The principle of conservation of Hamiltonian structure states that system (4) is well defined for the parameters (16) and is Hamiltonian with respect to the new Poisson bracket of hydrodynamic type, where from the original bracket $\{\cdot, \cdot\}_0$ and the integrals I_p we can explicitly compute a matrix $\gamma^{qp}(u)$ such that

$$g^{qp}(u) = \gamma^{qp} + \gamma^{pq}, \quad b_s^{qp} = \partial\gamma^{qp}/\partial u^s,$$
$$\{u^q(x), u^p(y)\} = \{\gamma^{qp}(u(y)) + \gamma^{pq}(u(x))\}\, \delta'(x-y). \tag{17}$$

Definition [28]. A Poisson bracket of hydrodynamic type defined by (17) is said to be *Liouville*, and the corresponding coordinates (u^q) are said to be *Liouville* (since they come from averaged densities of local field integrals of the original system that are in involution).

Liouville brackets of hydrodynamic type were studied in [3], [60]. In particular, all brackets that are linear in the fields and lead to infinite-dimensional Lie algebras, the analogues of the Virasoro algebra, are Liouville [60]. Recently, the authors noticed that from our method of derivation there follows a stronger property of brackets of hydrodynamic type obtained under averaging for equations of hydrodynamics of soliton lattices, which was missed in previous studies. The properties given below (especially for integrable systems) should serve, we conjecture, as a basis for a simple classification of the Poisson brackets thus arising.

Definition. A Liouville bracket of hydrodynamic type in coordinates (u^q) is called *strongly Liouville* if this property is preserved under the following operations:

a) affine changes of variables $u^i = A^i_j\tilde{u}^j + a^i$ (this is always true);

b) restriction of the tensor γ^{ij} to any subspace $\{\tilde{u}^{i_1}, \ldots, \tilde{u}^{i_k}\}$ linearly spanned by some of the coordinates after any affine change of coordinates.

Condition b) is a very severe restriction in the case $N \geqslant 3$.

Theorem. *The Poisson bracket obtained by the principle of conservation of Hamiltonian structure in coordinates* (u^q) *for the system* (4) *is strongly Liouville in these coordinates.*

It is of interest that the Poisson bracket of hydrodynamics of a compressible fluid in (p, ρ, s)-coordinates is also strongly Liouville (see §2).

In the averaging of integrable systems (KdV, SG, NS), the situation is as follows: there are infinitely many quantities (u_0^j), the averaged densities of local basis involutive integrals. If system (4) has N components, then any N independent quantities of the form

$$u^j = c^j_k u^k_0, \quad j = 1, \ldots, N, \qquad u^k_0 = h_k(u^1, \ldots, u^N)$$

give rise to a system of coordinates in which the bracket is strongly Liouville, while $u_0^k = h_k(u^1, \ldots, u^N)$ are densities of involutive integrals of hydrodynamic type. Densities of the Hamiltonian (and of local integrals in general) of the averaged system are always obtained by the trivial operation of averaging the densities of the original integrals. Only the Poisson bracket arises in a non-trivial way from averaging (see [28], [47] and §6 below). These results are new even for the simplest KdV equation and in the case $m = 1$, that is, in the classical Whitham case. Here $N = 3$ and the signature of the metric is (2, 1). Thus, equations (4) of hydrodynamics of soliton lattices have the following two properties: a) they are Hamiltonian (a 1983 result of the authors); b) they have Riemann invariants (Whitham, 1974, for $m = 1$, Flaschka and McLaughlin, 1979, for $m > 1$). In 1983 Novikov conjectured that Hamiltonian systems of hydrodynamic type having Riemann invariants are integrable. This conjecture was soon proved by Tsarev in his Ph.D. thesis (see [61], [62] and the survey paper [47]). He constructed a differential-geometric theory of diagonal Hamiltonian systems of hydrodynamic type and some natural "semi-Hamiltonian" generalizations. These results are described in §4. Here we state the main result of Tsarev's dissertation [62].

1. Let $v_p^q = v^q\delta_p^q$ and $I = \int j\,(u)dx$ be an integral of hydrodynamic type of a Hamiltonian system of hydrodynamic type with Hamiltonian $H = \int h\,(u)dx$. Then the metric g^{qp} is diagonal, $g^{qp} = g^q\delta_p^q$, though it is not constant, and the Hamiltonian system generated by the integral I is also diagonal with matrix $w_p^q = w^q\delta_p^q = g^{ql}\nabla_l\nabla_p j\,(u)$. Let us construct equations (18) for the functions $u^p(x, t)$:

$$w^q\,(u) = v^q\,(u)t + x. \tag{18}$$

Theorem. 1) *The solution $u^q(x, t)$ of the equations* (18) *satisfies the original system* (4):

$$u_t^q = v^q\,(u)u_x^q, \quad q = 1, \ldots, N. \tag{4'}$$

(*no summation*).

2) *For any germ of smooth functions $\{u^q\,(x, 0)\}$, there exists a density $j(u)$ (locally) such that, by the recipe* (18), *the density $j(u)$ generates a solution $u(x, t)$ of the system* (4′). *The determination of the densities $j(u)$ is reduced to solving a system of Pfaffian type.*

3) *On the set of monotone functions $u^q(x)$, the Hamiltonian system* (4′) *is integrable in the sense of Liouville.*

In a certain sense, these theorems complete the local differential geometry of one-dimensional systems of hydrodynamic type that are simultaneously Hamiltonian and diagonal. An incisive study of some particular systems was performed by Pavlov [49], [50]. Differential geometry of more general one- and multi-dimensional Poisson brackets is an interesting field for further inquiry, some results in which can be found in the present paper (see §2, 3). The theory of difference analogues of brackets of hydrodynamic type was constructed by Dubrovin in [22] (see §3).

However, the general differential-geometric integrability theorems of Tsarev do not help much in the study of particular systems of hydrodynamic type, such as KdV, which generate hydrodynamics of soliton lattices. Actually, we can effectively construct only very special integrals of hydrodynamic type generated by averaging over finite-zone tori of the well-known Kruskal integrals I_n and of linear combinations of them:

$$I = \sum_{n=0}^{n=L} c_n I_n = \int j\,(\varphi, \varphi_x, \ldots)\,dx.$$

Averaged densities $\overline{j(u)}$ generate, by Tsarev's procedure (see above), some solutions of (4). Following Novikov, these are called "averaged finite-zone" solutions (see [47]). What solutions are there for the special Kruskal integrals I_n? The answer was found by Krichever [39]: the integrals I_n generate special self-similar solutions of the Whitham system (4) in the case of KdV. For the most important case $m = 1$, the Whitham system (4) for KdV in the Riemann invariants $u^i = r_i(x, t)$ has the form (see §7)

$$\partial r_i/\partial t = v_i(r_1, r_2, r_3)\, \partial r_i/\partial x,$$

the original KdV equation is written in the form $\varphi_t + \varphi\varphi_x + \varphi_{xxx} = 0$, and its solution has the form

$$\varphi(x, t) = 2as^{-2}\,\mathrm{dn}^2\left[\left(\frac{6a}{s^2}\right)^{1/2}(x - vt).\, s\right] + \delta, \tag{19}$$

where

$$a = r_2 - r_1, \quad s^2 = \frac{r_2 - r_1}{r_3 - r_1}, \quad \delta = r_2 + r_1 - r_3. \tag{19'}$$

Here $r_3 \geqslant r_2 \geqslant r_1$, $v_3 \geqslant v_2 \geqslant v_1$. If $r_3 = r_2 > r_1$, then (19) degenerates into a soliton; if $r_3 > r_2 = r_1$, then (19) becomes a constant. Self-similar solutions of (19) for all γ are:

$$r_i(x, t) = t^{\gamma} R_i(xt^{-1-\gamma}).$$

Self-similar solutions with $\gamma = 0$ and $\gamma = 1/2$ first arose in the work of Gurevich and Pitaevskii [14], [15] in the description of asymptotics as $t \to \infty$ in the two following problems:

1. decomposition of a step function ($\gamma = 0$); here $r_3 = 1$, $r_1 = 0$;
2. the dispersive analogue of a shock-wave ($\gamma = 1/2$). (See the monograph [57], p. 261; a rigorous justification for $\gamma = 0$ in the framework of KdV theory was later discussed in [67]. For a discussion of these questions see §8 of this survey.) While in the case $\gamma = 0$ the Gurevich–Pitaevskii (GP) solution is easily found, in the case $\gamma = 1/2$ this is a non-trivial and quite special self-similar solution, whose existence these authors established numerically. For functions $R_i(z)$ (in [57] they were denoted by $l_i(z)$) it has the form shown in Fig. 1. Outside $\Delta = [z_-, z_+]$ the functions $r_1(x, t) = u(xt^{-3/2} > z_+)$ and $r_3(x, t) = u(xt^{-3/2} < z_-)$ are cubic solutions of the equation $u_t + uu_x = 0$ of the form $x = ut - u^3$. Inside Δ, the triple (r_1, r_2, r_3) is a one-phase self-similar solution of the Whitham equations with $\gamma = 1/2$. The inverse function $z(R)$ is single-valued and C^1-smooth. At the point z_- we have C^2-smoothness, while at z_+ the smoothness does not exceed $C^{2-\varepsilon}$ for $\varepsilon > 0$. By the construction in [14], there arises another possible singularity at the point $z^* \in \Delta$, where $r_2(z) = 0$. It was computed numerically in [14] that $z_- \approx -1.41$, $z_+ \approx 0.117$, $z^* \approx -1.11$. Refinements of numerical computations made plausible the suggestion that $z_- = -\sqrt{2}$ (see [1], where it was conjectured that the GP self-similar solution with $\gamma = 1/2$ can be determined analytically).

The general setting and a numerical study of the evolution of multi-valued functions with graph as in Fig. 1 and with special asymptotics at singular points, where $\Delta = \Delta(t)$ for KdV and $m = 1$, is provided in [1] and in [2] under the additional condition of small viscosity (see §§8, 9).

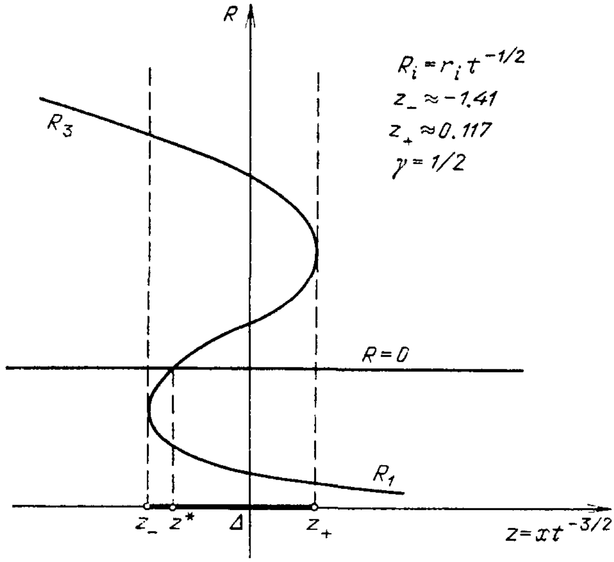

Fig. 1

For zero viscosity, the numerical conclusion of [1] is that the evolution of multi-valued functions with special asymptotics near singular edges $[r_2 = r_3]$ and $[r_1 = r_2]$ is locally well defined. For initial conditions that are C^1-close to the GP solution with $\gamma = 1/2$, this evolution is defined for infinite time and the solution converges asymptotically to the GP solution as $t \to \infty$. However, if the initial condition is not sufficiently C^1-close to the GP solution, the development of singularities in finite t is possible. In this case one must pass to the Whitham equation for $m > 1$, increasing the degree of multi-valuedness. This process has not been studied.

For non-zero viscosity, the numerical conclusion of [2] is that if for a given initial profile the evolution is defined for $t \to \infty$, the solution converges asymptotically to the stationary solution obtained in [2], [16]. Of course at present there are no rigorous proofs of all these numerical results. According to the ideological framework of the authors of [1], [2], the entire evolution, including the growth in the degree of multi-valuedness, must be described in terms of the theory of first-order systems, that is, of systems of hydrodynamic type, which is not the approach of the series of papers of Lax and others [83], [85]-[87], [91], [92], where a solution of the initial KdV with small dispersion is assumed given for $t > 0$, while equations (4) serve only as the limiting description of this solution; evolution in the framework of theory of systems of hydrodynamic type has not been studied at all.

Recently, Krichever found an algebraic geometric method of constructing exact solutions of the Whitham systems for integrable systems, and Potemin implemented Krichever's algorithm to determine the GP self-similar solution for $\gamma = 1/2$ (see [39], [53]). It turned out that this solution is analytic outside z_-, z_+ and that, moreover, we have the exact equality

$$z_- = -\sqrt{2}, \quad z_+ = \sqrt{10/27}.$$

In addition, the paper of Krichever [39] contains a number of results concerning the construction of Whitham systems for spatially two-dimensional KdV (that is, for KP), for which there are no local integrals. These results are discussed in §7.

Chapter I

HAMILTONIAN THEORY OF SYSTEMS OF HYDRODYNAMIC TYPE

§1. General properties of Poisson brackets

Let us first review the definition of the usual (finite-dimensional) Poisson bracket (more detailed information is to be found in [25], [46], [71]). Let M be an N-dimensional manifold, which we shall call the *phase space*. A *Poisson bracket* is an operation on the space of smooth functions on M, $f, g \mapsto \{f, g\}$, which has the following properties (together with the usual multiplication of functions):

1) bilinearity

$$\{\lambda f + \mu g, h\} = \lambda \{f, g\} + \mu \{g, h\},$$
$$\{f, \lambda g + \mu h\} = \lambda \{f, g\} + \mu \{f, h\}, \quad \lambda, \mu = \text{const};$$

2) skew-symmetry

$$\{g, f\} = -\{f, g\};$$

3) Jacobi identity

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0;$$

4) Leibniz identity

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

Hence, in particular, it follows that the Poisson bracket $\{f, g\}$ as a function of each of the arguments f, g is defined by a linear first-order differential operator.

In (local) coordinates $y^1, \ldots, y^N$ on M, a Poisson bracket is defined by a skew-symmetric tensor of type (2, 0),

$$h^{ij}(y) = \{y^i, y^j\}, \qquad i, j = 1, \ldots, N. \tag{1}$$

$h^{ij}(y)$ is a tensor, as follows from the Leibniz identity, from which it can also be seen that for all smooth functions $f(y)$, $g(y)$ the Poisson bracket is calculated from the formula

$$\{f, g\} = h^{ij}(y) \frac{\partial f(y)}{\partial y^i} \frac{\partial g(y)}{\partial y^j} \tag{2}$$

(here and in what follows, summation with respect to repeated indices is assumed). The Jacobi identity imposes the following restrictions on the tensor $h^{ij}(y)$:

$$\{\{y^i, y^j\}, y^k\} + \{\{y^k, y^i\}, y^j\} + \{\{y^j, y^k\}, y^i\} \equiv \frac{\partial h^{ij}}{\partial y^s} h^{sk} + \frac{\partial h^{ki}}{\partial y^s} h^{sj} + \frac{\partial h^{jk}}{\partial y^s} h^{si} = 0, \quad i, j, k = 1, \ldots, N. \tag{3}$$

(The left hand sides of this system of relations form a tensor of third rank, which is called the *Schouten bracket* $[h, h]$, see [12].)

In the non-degenerate case $\det(h^{ij}) \neq 0$, conditions (3) are equivalent to the following: the inverse matrix $(h_{ij}) = (h^{ij})^{-1}$ defines on M a symplectic structure, that is, the 2-form $\Omega = h_{ij} dy^i \wedge dy^j$ is non-degenerate and closed, $d\Omega = 0$. A manifold with a non-degenerate bracket is called *symplectic*.

A Poisson bracket allows us to define Hamiltonian equations on M; these have the form

$$\frac{d}{dt} y^i = \{y^i, H(y)\}, \qquad i = 1, \ldots, N, \tag{4}$$

where $H(y)$ is called the *Hamiltonian* of the system (4). The integrals $F = F(y)$ of the system (4) are defined by the condition

$$\{F, H\} = 0.$$

If $F(y)$ is an integral of the system (4), then the Hamiltonian system

$$\frac{dy^i}{ds} = \{y^i, F(y)\}, \qquad i = 1, \ldots, N,$$

commutes with the system (4), $\frac{d}{ds} \frac{d}{dt} y^i = \frac{d}{dt} \frac{d}{ds} y^i$.

Example 1. Constant brackets, in which h^{ij} is any constant skew-symmetric matrix, arose from Lagrangian systems. The Jacobi identity obviously holds. In the non-degenerate case $\det(h^{ij}) \neq 0$, $N = 2n$, it is convenient to choose canonical coordinates $(y^1, \ldots, y^N) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$ so that

$$\{x^i, p_j\} = \delta^i_j, \quad \{x^i, x^j\} = \{p_i, p_j\} = 0, \qquad i, j = 1, \ldots, n. \tag{5}$$

The Euler-Lagrange equations

$$\delta \int L(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)})\, dt = 0 \tag{6}$$

of one-dimensional variational problems can be written in the form (4) with constant brackets (5) if we set

$$\begin{cases} x^1 = x, & p_1 = \dfrac{\partial L}{\partial \dot{x}} - \left(\dfrac{\partial L}{\partial \ddot{x}}\right)^{\cdot} + \ldots + (-1)^{n-1}\left(\dfrac{\partial L}{\partial x^{(n)}}\right)^{(n-1)}; \\ x^2 = \dot{x}, & p_2 = \dfrac{\partial L}{\partial \ddot{x}} - \left(\dfrac{\partial L}{\partial x^{(3)}}\right)^{\cdot} + \ldots + (-1)^{n-2}\left(\dfrac{\partial}{\partial x^{(n)}}\right)^{(n-2)}; \ldots; \\ x^n = x^{(n-1)}, & p_n = \dfrac{\partial L}{\partial x^{(n)}} \end{cases} \tag{7}$$

(it is assumed that the Lagrangian is such that from equations (7) the numbers $x^1, x^2, \ldots, x^{(2n-1)}$ can be expressed in terms of $x^1, p_1, \ldots, x^n, p_n$) (see [31]). The Hamiltonian has the form

$$H(x, p) = \Sigma p_i x^{(i)} - L. \tag{7'}$$

By the classical Darboux lemma, non-degenerate Poisson brackets can be reduced to constant ones locally by smooth changes of coordinates. In the degenerate case, when $\det(h^{ij}) \equiv 0$ and the rank of h^{ij} is constant, there exists (at least locally) a full set of functions $f(y)$ such that $\{f, g\} = 0$ for any function $g(y)$. Such functions comprise the annihilator of the Poisson bracket. On their common level surface the Poisson bracket is no longer degenerate. Globally, a fibration arises here.

Brackets of the form (5) arise globally on cotangent bundles to manifolds, $M = T^*(Q)$, where p_i are the momenta and x^i are the coordinates on Q.

A Poisson bracket in a "magnetic field" is determined by a closed 2-form Ω on Q via the equations

$$\{x^i, p_j\} = \delta^i_j, \quad \{x^i, x^j\} = 0, \quad \{p_i, p_j\} = \Omega_{ij}(x). \tag{8}$$

Let $\Omega = dA$ (a vector-potential). Locally we can introduce canonical coordinates

$$\tilde{p}_i = p_i - A_i(x), \qquad \tilde{x}^i = x^i. \tag{9}$$

The global obstruction is the cohomology class of the form Ω.

Systems that are Hamiltonian with respect to the bracket (8) in a magnetic field are often reduced by an inverse Legendre transformation to Lagrangian systems on the manifold Q with multi-valued action potential ([46])

$$S[x] = \int_{x(t)} L(x, \dot{x})\, dt + \int_{x(t)} A_i\, dx^i. \tag{10}$$

Though the formula (10) is not well defined globally (since $\Omega = dA$ is not an exact form), the quantity δS is nonetheless a well defined closed l-form on the space of curves $x(t)$. The Dirac monopole is of this type and so are some systems of classical mechanics (the spinning top and others) after the exclusion of "cyclic variables". Field-theoretic analogues of Lagrangians of the form (10) are also important. The relevant references can be found in [46].

Example 2. The linear brackets

$$h^{ij}(y) = c_k^{ij} y^k, \qquad c_k^{ij} = \text{const}. \tag{11}$$

In this case, the linear functions on M form a Lie algebra relative to the operation $\{\cdot, \cdot\}$. Therefore $L = M^*$ (the dual space) is a Lie algebra with structure constants c_k^{ij}. The bracket (11) on the space dual to the Lie algebra is called a *Lie–Poisson bracket*. In general this bracket is degenerate. It becomes non-degenerate on the orbits of the co-adjoint representation Ad^* of the corresponding Lie group.

In the linear non-degenerate case

$$h^{ij}(y) = c_k^{ij} y^k + c_0^{ij}, \quad c_k^{ij} = \text{const}, \quad c_0^{ij} = \text{const}, \tag{12}$$

the quantities $c_0^{ij} = -c_0^{ji}$ form a (two-dimensional) cocycle on the Lie algebra L with structure constants $c_k{}^{ij}$. This means that

$$c_s^{ij} c_0^{sk} + c_s^{ki} c_0^{sj} + c_s^{jk} c_0^{si} = 0. \tag{13}$$

Functions of the form $f(y) = a_i y^i + b$ then form a one-dimensional central extension of the Lie algebra L by the cocycle c_0^{ij}. The cocycle c_0^{ij} is cohomologous to zero if it has the form

$$c_0^{ij} = c_k^{ij} y_0^k$$

for some collection $y_0^1, \ldots, y_0^N$. For such cocycles the bracket (12) reduces to a linear homogeneous one by the translation $y \mapsto y + y_0$.

Example 3. The linearized Yang–Baxter equation and quadratic Poisson brackets. Let $r = (r_{kl}^{ij})$ be the so-called classical r-matrix that satisfies the linearized Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{14}$$

as well as the unitarity condition

$$r_{lk}^{ji} = -r_{kl}^{ij}. \tag{15}$$

Here the matrices r_{12}, r_{13}, r_{23} have the form

$$(r_{12})_{j_1 j_2 j_3}^{i_1 i_2 i_3} = r_{j_1 j_2}^{i_1 i_2} \delta_{j_3}^{i_3} \text{ and so on.}$$

Corresponding to such an r-matrix on the space of matrices with coordinates t^i_j, there are quadratic Poisson brackets:

$$\{t^i_k, t^j_l\} = r^{ij}_{ab} t^a_k t^b_l - t^i_a t^j_b r^{ab}_{kl}, \tag{16}$$

that arise as the quasi-classical limit of commutativity relations in quantum groups [21]. The brackets (16) are frequently written in the following symbolic form:

$$\{T \otimes, T\} = [r, T \otimes T], \tag{17}$$

where $T = (t^i_j)$. The quadratic Poisson brackets

$$\{T \otimes, T\}_R = r\, T \otimes T, \quad \{T \otimes, T\}_L = T \otimes T\, r. \tag{18}$$

are also defined. For each of the brackets (18), skew-symmetry together with the Jacobi identity is equivalent to the linearized Yang–Baxter equation and the unitarity condition for the matrix r. The algebraic nature of the brackets (18) was elucidated by Drinfel'd [20]: the brackets (18) are, respectively, right- and left-invariant Poisson brackets on the full linear group, while the bracket (17) defines on the full linear group a Lie–Poisson group structure (see [20] and §3 of this paper). If the r-matrix is contained in $\mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of some Lie group G, then (17), (18) define Poisson brackets on the Lie group G.

Let us move on now to infinite-dimensional examples of phase spaces and to field-theoretic Poisson brackets. The phase space now consists of smooth vector-functions $u = (u^1(x), \ldots, u^N(x))$, and $x = (x^1, \ldots, x^d)$ is one of the indices in the formulae. By definition, the integral $\int (\ldots)' d^d x$ of a full derivative (divergence) is equal to zero in the formal theory of field-theoretic Poisson brackets. We may assume, for example, that x runs through a closed manifold or that the functions are simple.

A Poisson bracket is defined for a class of functionals on the fields $u^i(x)$. It is convenient to define it on "point" functionals concentrated on one of the local fields u, v, w at a point and on their derivatives when these are defined. The class of local fields is defined as follows. Let $v(x) = f(u)$ be any function on M. Then $f(u(x))$, $\partial_x f(u(x))$, $\partial_x^2 f(u(x))$, ..., $\partial_x^k f(u(x))$ and any rational or even analytic function of a finite number of these symbols is a local field. The Leibniz and bilinearity identities in the continuous case, with the sum being replaced by an integral, have the form

$$\{u^i(x), u^j(y)\} = h^{ij}(x, y), \quad i, j = 1, \ldots, N, \tag{19}$$

$$\{u(x)\, v(y), w(z)\} = u(x)\{v(y), w(z)\} + v(y)\{u(x), w(z)\}, \tag{20}$$

$$\{\int v(x)\, d^d x, w(y)\} = \int \{v(x), w(y)\}\, d^d x, \tag{21}$$

where the "tensor" $h^{ij}(x, y)$ has, in addition to the usual indices i, j, the continuous indices x, y. The bracket (19) is extended to general functionals $I[u]$, $J[u]$, ... by

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \frac{\delta J}{\delta u^j(y)} h^{ij}(x, y)\, d^dx\, d^dy, \tag{22}$$

which is similar to (2). Here the variational derivatives $\frac{\delta I}{\delta u^i(x)}$ are defined by the equality

$$I[u + \delta u] - I[u] = \int \frac{\delta I}{\delta u^i(x)} \delta u^i(x)\, d^dx + o(\delta u). \tag{23}$$

We refer to [31] for a derivation of (23) for local field functionals of the form

$$I[u] = \int P(x, u(x), u^{(1)}(x), \ldots, u^{(k)}(x))\, d^dx, \tag{24}$$

where by $u^{(1)}(x), \ldots, u^{(k)}(x)$ we denote collections of partial derivatives of vector-functions or orders 1, ..., k, and where P is a polynomial (or an analytic function) of the variables $u, u^{(1)}, \ldots, u^{(k)}$, called the *density of the functional* $I[u]$ (see [34]). We remind the reader that the variational derivative of local functionals is written down as an Euler-Lagrange operator

$$\frac{\delta I}{\delta u^i(x)} = \frac{\partial P}{\partial u} - \frac{\partial}{\partial x^\alpha} \frac{\partial P}{\partial u^i_\alpha} + \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \frac{\partial P}{\partial u^i_{\alpha\beta}} - \ldots, \tag{25}$$

where

$$u^i_\alpha = \frac{\partial u^i}{\partial x^\alpha}, \quad u^i_{\alpha\beta} = \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta}, \ldots .$$

It is natural to single out a class of local field-theoretic brackets having the form

$$\{u^i(x), u^j(y)\} = \sum_{|k| \leqslant K} B^{ij}_k (x, u(x), u^{(1)}(x), \ldots, u^{(n_k)}(x))\, \partial^k_x \delta(x - y), \quad i, j = 1, \ldots, N, \tag{26}$$

where $k = (k_1, \ldots, k_d)$ is a multi-index, $\partial^k_x = \left(\frac{\partial}{\partial x^1}\right)^{k_1} \ldots \left(\frac{\partial}{\partial x^d}\right)^{k_d}$, $|k| = k_1 + \ldots + k_d$, and K is some number (the order of the bracket). In this formula, $\delta(x-y)$ is the delta function; its derivatives are the formal symbols defined by

$$\int f(y)\, \delta^{(k)}(x - y)\, d^dy = \partial^k_x f(x), \quad k \geqslant 0. \tag{27}$$

A bracket is called *translation invariant* if none of the B^{ij}_k is explicitly dependent on x. If B^{ij}_k is independent of $u, u^{(1)}, \ldots, u^{(k)}$, then it is a constant bracket on the space of fields.

Let us introduce the operator

$$(28)\qquad A^{ij} = \sum_{|k|\leq K} B_k^{ij}(x)\,\partial_x^k, \qquad i, j = 1, \ldots, N.$$

(For brevity, we shall denote $B_k^{ij}(x, u(x), u^{(1)}(x), \ldots)$ by $B_k^{ij}(x)$.) Then formula (22) for the Poisson bracket of smooth functionals $I[u]$, $J[u]$ has the form

$$(29)\qquad \{I, J\} = \int \frac{\delta I}{\delta u^i(x)} A^{ij} \frac{\delta J}{\delta u^j(x)}\, d^d x.$$

The skew-symmetry condition has the form

$$(30)\qquad (A^{ij})^* = -A^{ji}, \qquad i, j = 1, \ldots, N,$$

where

$$(31)\qquad (A^{ij})^* = \sum_{|k|\leq K} (-\partial_x)^k B_k^{ij}.$$

The Jacobi identity has the form

$$(32)\quad \{\{u^i(x), u^j(y)\}, u^k(z)\} + \{\{u^k(z), u^i(x)\}, u^j(y)\} + \{\{u^j(y), u^k(z)\}, u^i(x)\} \equiv$$
$$\equiv \frac{\partial B_p^{ij}(x)}{\partial u^{s(q)}}\, \partial_x^p \delta(x-y)\, \partial_x^q (B_r^{sk}(x)\, \partial_x^r \delta(x-z)) + \ldots = 0, \quad i, j, k = 1, \ldots, N.$$

The dots on the right-hand side stand for terms obtained by the cyclic permutation $i \to j \to k$, $x \to y \to z$. The equality (32) is to be understood in the sense of generalized functions of x, y, z being zero. Since the generalized function on the right-hand side of (32) has finite order and support on the diagonal $x = y = z$, (32) is equivalent to a finite system of quadratic relations involving the coefficients B_k^{ij} and their derivatives with respect to x and $u^{s(q)}$. We shall not give here the explicit form of this system (see below for systems of hydrodynamic type and also [11], [12], [65]). Let us observe that a sufficient collection of relations is obtained if the Jacobi identity is verified only for linear functionals of the form

$$I[u] = \int a_i(x)\, u^i(x)\, d^d x$$

for arbitrary functions $a_1(x), \ldots, a_N(x)$.

Hamiltonian systems corresponding to the Poisson brackets (26) have, by definition, the form

$$(33)\qquad u_t^i(x) = \{u^i(x), H\} \equiv A^{ij} \frac{\delta H}{\delta u^j(x)}, \qquad i = 1, \ldots, N,$$

where $H = H[u]$ is a Hamiltonian and the operator A^{ij} is of the form (28). If the Hamiltonian is a local field functional of the form (24), then the Hamiltonian system (33) is an evolutionary system of partial differential equations.

Example 4. For multi-dimensional variational problems, the Euler-Lagrange equations

$$(34) \qquad \delta \iint dt\, d^d x\, L\,(u,\, u_t,\, u_x) = 0$$

are written in Hamiltonian form (33) in the space of fields $y^i\,(x) = u^i\,(x)$, $y^{n+i}\,(x) = p_i\,(x)$, $i = 1, \ldots, n$, where

$$(35) \qquad p_i = \frac{\partial L}{\partial u_t^i}\,, \qquad H = \int d^d x\,(p_i u^i - L),$$

$$(35') \quad \{u^i\,(x), p_j\,(y)\} = \delta\,(x - y)\,\delta_j^i, \quad \text{or} \quad \{y^i\,(x), y^j\,(y)\} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta\,(x - y).$$

Example 5. If M is an N-dimensional phase space with the bracket $\{u^i,\, u^j\} = h^{ij}(u)$ then for any d-dimensional manifold X in the space of fields $u^i(x) \in M$, $x \in X$ there arises the *ultra-local Poisson bracket*

$$(36) \qquad \{u^i\,(x),\, u^j\,(y)\} = h^{ij}\,(u\,(x))\,\delta\,(x - y).$$

In order that the bracket (36) be well defined under changes of coordinates in X, the fields $u^i(x)$ must transform as d-forms. A choice of canonical coordinates on M (if this is possible) brings the bracket (36) into the Lagrangian form (35′).

Example 6. Let us give another example (due to Gardner, Zakharov, and Faddeev) of constant Poisson brackets. Let $N = d = 1$ (one spatial and one field variable). Let us set

$$(37) \qquad \{u\,(x),\, u\,(y)\} = \delta'\,(x - y).$$

The skew-symmetry and the Jacobi identity are obvious. Hamiltonian systems have the form

$$(38) \qquad u_t\,(x) = \{u\,(x), H\} = \frac{d}{dx}\left(\frac{\delta H}{\delta u\,(x)}\right).$$

In particular, for

$$(39) \qquad H = \int \left(\frac{u'^2}{2} + u^3\right) dx$$

we obtain the Korteweg-de Vries (KdV) equation

$$u_t = 6uu' - u'''.$$

Let us note that the bracket (37) is degenerate. Its annihilator has the form

$$(40) \qquad I_0\,[u] = \int u\,(x)\,dx.$$

A more general bracket of this type is given by the formula

$$(41) \qquad \{u^i\,(x),\, u^j\,(y)\} = g_0^{ij}\delta'\,(x - y),$$

where g_0^{ij} is a constant non-degenerate symmetric matrix. If

$$g_0^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{42}$$

then the coordinates u^i, $i = 1, \ldots, n, n+1, \ldots, 2n$ ($N = 2n$) can be divided into two parts. Let $u^i = q^i$, $u^{n+i} = dp_i/dx$, $i = 1, \ldots, n$. Then we have

$$\{q^i(x), p_j(y)\} = \delta(x-y)\,\delta^i_j. \tag{43}$$

Therefore variables of the form (43) are called *Clebsch-type variables*. If the rank of g_0^{ij} is odd, or if the signature of the matrix g_0^{ij} does not have the form (n, n), the Clebsch variables do not exist. Clebsch type variables arose in the 19th century in the context of bringing equations into Lagrangian form.

Example 7. The simplest case of linear field-theoretic Poisson brackets is related to the algebra of currents. Let the fields $u^1(x), \ldots, u^N(x)$ be d-forms in $x \in X$, where $d = \dim X$, and let them take values in the space L^* dual to the Lie algebra L with structure constants (c_k^{ij}). Thus the phase space is formally dual to the infinite-dimensional Lie algebra L^X of the group of currents G^X, where G is a Lie group with Lie algebra L, X is the space of variables $x^1, \ldots, x^d$, and G^X denotes the group of smooth mappings $X \to G$. The Poisson bracket has the form

$$\{u^i(x), u^j(y)\} = c_k^{ij} u^k(x)\,\delta(x-y). \tag{44}$$

Let us assume that the Lie algebra L has an invariant scalar product $g^{ij} = g^{ji}$, that is,

$$c_s^{ij} g^{sk} = -c_s^{ik} g^{sj}, \qquad i, j, k = 1, \ldots, N. \tag{45}$$

Then in the spatially one-dimensional case ($d = 1$) there is a central one-dimensional extension of the Lie algebra of currents that corresponds to the following non-homogeneous Poisson bracket:

$$\{u^i(x), u^j(y)\}_g = g^{ij}\delta'(x-y) + c_k^{ij} u^k(x)\,\delta(x-y). \tag{46}$$

This extension of the current algebra plays an important role in conformal field theories. The Lie algebra corresponding to the linear brackets (46) is called the Kac–Moody algebra [81].

Example 8. Corresponding to the Lie algebra $L(d)$ of vector fields in a d-dimensional space there is a linear Poisson bracket of the form

$$\{p_i(x), p_j(y)\} = p_j(x)\,\partial_i\delta(x-y) + p_i(y)\,\partial_j\delta(x-y), \qquad i, j = 1, \ldots, d, \tag{47}$$

where the $p_i(x)$ are densities of covectors (having the structure of a 1-form multiplied by a volume element). Indeed, if a, $b \in L(d)$ are two linear functionals on the fields $p_i(x)$ defined by vector fields $a^i(x)$, $b^i(x)$,

$$a = a\,[p] = \int a^i\,(x)\,p_i\,(x)\,d^dx, \quad b = b\,[p] = \int b^i\,(x)\,p_i\,(x)\,d^dx,$$

then their Poisson bracket (47) is again a linear functional,

$$\{a, b\} = c = \int c^i\,(x)\,p_i\,(x)\,d^dx,$$

where the vector field c is the commutator of the fields a and b,

$$c^i = a^k \partial_k b^i - b^k \partial_k a^i.$$

In the one-dimensional case, where

$$\{p\,(x), p\,(y)\} = [p\,(x) + p\,(y)]\,\delta'\,(x - y), \tag{48}$$

this bracket is reduced to a constant one (37) by the change of variables $p = u^2/2$. The well-known one-dimensional central extension of the Lie algebra $L(1)$, defined by the Gel'fand–Fuks cocycle (the Virasoro algebra for the case $x \in S^1$), corresponds to the following non-homogeneous bracket:

$$\{p\,(x), p\,(y)\}_1 = c\delta'''\,(x - y) + [p\,(x) + p\,(y)]\,\delta'\,(x - y), \tag{49}$$

where c is a constant. In the theory of integrable systems, this bracket is called the *Leonard–Magri bracket* [90]. The KdV equation $p_t = 6pp' - p'''$ turns out to be Hamiltonian with respect to the bracket (49) as well. For $c = 1/2$ the Hamiltonian has the form

$$H_1 = \int dx\, p^2/2. \tag{50}$$

Let $L_0(d) \subset L(d)$ be the Lie subalgebra of divergence-free vector fields $a = (a^i)$, $\partial_i a^i = 0$. The conjugate space $L_0(d)^*$ is realized as the factor space modulo the gradients $p_i = \partial_i \varphi$,

$$L_0\,(d)^* = L\,(d)^*/(\partial_i \varphi).$$

Hamiltonian systems on $L_0(d)^*$ are conveniently written on $L(d)^*$ in the form

$$p_{i_t}\,(x) = \{p_i\,(x), H\} + \partial_i \varphi, \quad i = 1, \ldots, d. \tag{51}$$

For $H = \int d^dx \Sigma p_i^2/2\rho$, equations (51) are just the Euler equations of an ideal incompressible fluid [46].

The Lie algebra $L(d)$ of vector fields in a d-dimensional space has natural extensions that are necessary for the description of different types of compressible fluids with "frozen-in" tensor fields. It is convenient to describe this extension using the language of fields. In addition to the fields $p_i(x)$ that are dual to vector spaces, let us introduce some other fields $T_{(s)}(x)$ that are tensors of weight j_s on the x-space. For the usual compressible fluid it is necessary to introduce $\rho(x)$, the density of mass (a d-form) and $s(x)$, the entropy density (also a d-form) (and in magnetohydrodynamics, the magnetic feld). The Poisson bracket of all the fields $T_{(s)}$ is identically zero,

$$\{T_{(s')}(x), T_{(s'')}(y)\} = 0.$$

The Poisson bracket $\{p_i(x), T_{(s)}(y)\}$ must be such that for every vector field X the Hamiltonian $H = \int X^i p_i d^d x$ generates a one-parameter group of diffeomorphisms defined by the vector field $X^i(x)$. This means that the Poisson bracket $\{T_{(s)}(x), H\}$ must coincide with the Lie derivative of the field $T_{(s)}$ along X. Examples are discussed in the survey paper [46]. The case of superfluids, where the bracket is more complicated, is also of interest.

§2. Hamiltonian formalism of systems of hydrodynamic type and Riemannian geometry

Definition 1. A (homogeneous) *system of hydrodynamic type* is an equation of the form

$$(1) \qquad u_t^i = v_j^{i\alpha}(u)\, u_\alpha^j, \quad i = 1, \ldots, N, \quad \alpha = 1, \ldots, d,$$

where $u_\alpha^j \equiv \partial u^j / \partial x^\alpha$. (At this stage we do not impose a hyperbolicity condition on the system (1).)

It was Riemann who noticed that the theory of systems of the form (1) is the theory of tensors. Indeed, under invertible smooth changes of field variables of the form $u^i \mapsto w^i$, where

$$(2) \qquad u^i = u^i(w^1, \ldots, w^N), \quad i = 1, \ldots, N,$$

the coefficients $v_j^{i\alpha}$ in (1) transform for each α according to the tensor law

$$(3) \qquad v_j^{i\alpha}(u) \to v_q^{p\alpha}(w) = \frac{\partial w^p}{\partial u^i} v_j^{i\alpha}(u(w)) \frac{\partial u^j}{\partial w^q}.$$

Let us denote by M^N the space (or the manifold) in which the fields $u^1(x, t), \ldots, u^N(x, t)$ take their values for each x, t. Then (2) can be regarded as a change of coordinates on M^N; by (3), the coefficients $v_j^{i\alpha}$ of (1) form for each α a tensor of the type (1, 1) (an affinor) on M^N.

Let us review the simplest facts from the theory of affinors. Let all the eigenvalues $v^1 = \lambda^1, \ldots, v^N = \lambda^N$ of the matrix (v_j^i) (we are considering the spatially one-dimensional case) of (1) be real and distinct (that is, (1) is hyperbolic). Is it possible to reduce (1), using the transformations (2), to the diagonal form

$$(4) \qquad w_t^i = v^i(w)\, w_x^i, \quad i = 1, \ldots, N$$

(no summation over i!)? If it is, then the variables $w^1, \ldots, w^N$ are called the *Riemann invariants* for (1), while the coefficients $v^1(w), \ldots, v^N(w)$ are called the corresponding *characteristic velocities*. For $N = 2$ it is always possible locally to reduce to Riemann invariants, while for $N \geqslant 3$ this is not so in general. This is also true in the case of complex eigenvalues, if complex changes of coordinates (2) are allowed.

In the course of studying Hamiltonian systems of hydrodynamic type, which we shall presently define, there arises a richer geometry, first discovered in the authors' paper [28].

Definition 2. a) A *Poisson bracket of hydrodynamic type* is defined by the formula

$$\{u^i(x), u^j(y)\} = g^{ij\alpha}(u(x))\,\delta_\alpha(x-y) + b_k^{ij\alpha}(u)\,u_\alpha^k\delta(x-y), \tag{5}$$

where $g^{ij\alpha}(u)$, $b_k^{ij\alpha}(u)$ are certain functions, $i, j, k = 1, \ldots, N$, $\alpha = 1, \ldots, d$.

b) *Functionals of hydrodynamic type* have the form

$$H[u] = \int h(u)\, d^dx, \tag{6}$$

where the density $h(u)$ is independent of the derivatives u_α, $u_{\alpha\beta}$.

c) *Hamiltonian systems of hydrodynamic type* have the form

$$u_t^i(x) = \{u^i(x), H\} \equiv \left(g^{ij\alpha}(u)\frac{\partial^2 h(u)}{\partial u^j\partial u^k} + b_k^{ij\alpha}(u)\frac{\partial h(u)}{\partial u^j}\right)u_\alpha^k, \quad i = 1, \ldots d, \tag{7}$$

where $\{\cdot, \cdot\}$ is a bracket of hydrodynamic type (5), while the Hamiltonian $H = H[u]$ is a functional of hydrodynamic type (6).

Let us consider first the spatially one-dimensional case $d = 1$, omitting the index α. The following simple but important proposition holds.

Proposition 1. a) *The class* (5) *of Poisson brackets of hydrodynamic type is invariant with respect to changes of the field variables of the form* (2): $u^i \mapsto v^i(u)$.

b) *Under these changes of variables, the coefficients* $g^{ij}(u)$ *transform as tensors of type* (0, 2), *that is,*

$$g^{pq}(v) = \frac{\partial v^p}{\partial u^i}\frac{\partial v^q}{\partial u^j}g^{ij}(u(v)), \qquad p, q = 1, \ldots, N. \tag{8}$$

c) *Let us assume that the matrix* $(g^{ij}(u))$ *is non-degenerate and define the quantities* $\Gamma_{jk}^i(u)$ *by the equality*

$$b_k^{ij}(u) = -g^{is}(u)\,\Gamma_{sk}^j(u), \qquad i, j, k = 1, \ldots, N. \tag{9}$$

Under the changes of variables (2), *the quantities* $\Gamma_{jk}^i(u)$ *transform as the components of the differential-geometric connection (Christoffel symbols), that is,*

$$\Gamma_{qr}^p(v) = \frac{\partial v^p}{\partial u^i}\frac{\partial u^j}{\partial v^q}\frac{\partial u^k}{\partial v^r}\Gamma_{jk}^i(u) + \frac{\partial v^p}{\partial u^i}\frac{\partial^2 u^i}{\partial v^q\partial v^r}. \tag{10}$$

Proof. Let us use the Leibniz identity. We obtain

$$\{v^p(u(x)), v^q(u(y))\} = \frac{\partial v^p}{\partial u^i}(x)\frac{\partial v^q}{\partial u^j}(y)\{u^i(x), u^j(y)\}.$$

Hence, it follows from the obvious identity

$$(11)\qquad f(y)\,\delta'(x-y) = f(x)\,\delta'(x-y) + f'(x)\,\delta(x-y)$$

that formulae (8), (10) hold. The assertion is proved.

Poisson brackets of hydrodynamic type for which $\det(g^{ij}) \neq 0$ are called *non-degenerate*. By the above, the condition of non-degeneracy is invariant under changes of variables (2). In what follows we shall only consider non-degenerate brackets.

***Theorem* 1.** *In the non-degenerate case* $\det(g^{ij}) \neq 0$, (5) *defines a Poisson bracket if and only if the tensor* g^{ij} *is symmetric, that is, if it defines a pseudo-Riemannian metric (with upper indices) on the space* M^N, *and the connection* Γ^i_{jk} *of the form* (9) *is compatible with the metric* g^{ij} *and has zero curvature and torsion. Therefore, there exist local coordinates* $v^i = v^i(u^1, \ldots, u^N)$, $i = 1, \ldots, N$ *such that* $g^{ij} = \text{const}$, $b^{ij}_k = 0$. *In these coordinates the Poisson bracket* (5) *is constant*:

$$(12)\qquad \{v^i(x), v^j(y)\} = g_0^{ij}\delta'(x-y), \qquad g_0^{ij} = g_0^{ji} = \text{const}.$$

A complete local invariant of the Poisson bracket (5) *is the signature of the pseudo-Euclidean metric* g^{ij}.

Proof. The symmetry condition for the metric, $g^{ij} = g^{ji}$, together with the compatibility conditions of the connection (9) with the metric, that is,

$$\nabla_k g^{ij} \equiv \frac{\partial g^{ij}}{\partial u^k} + \Gamma^i_{sk} g^{sj} + \Gamma^j_{sk} g^{is} \equiv \frac{\partial g^{ij}}{\partial u^k} - b^{ij}_k - b^{ji}_k = 0,$$

follow immediately from the skew-symmetry of the Poisson bracket because of the relations $\delta'(y-x) = -\delta'(x-y)$, $\delta(y-x) = \delta(x-y)$ and (11). To prove that the curvature and torsion are zero, we shall use the Jacobi identity. Let

$$(13)\qquad J^{ijk}(x, y, z) = \{\{u^i(x), u^j(y)\}, u^k(z)\} + \ldots$$

be the left-hand side of the Jacobi identity (compare with (1.3)(1) above). A generalized function being zero for all i, j, k is equivalent to

$$(14)\qquad \iiint dx\,dy\,dz\,p_i(x)\,q_j(y)\,r_k(z)\,J^{ijk}(x, y, z) = 0$$

for any "good" vector functions p, q, r. This integral can be reduced to a single integral

$$\int dx \sum_{\sigma,\tau=0}^{2} A^{ijk}_{\sigma\tau} p_i q_j^{(\sigma)} r_k^{(\tau)} = 0,$$

(1)Here and in what follows, a reference of the form (1.3) means formula (3) of §1.

where the coefficients $A^{ijk}_{\sigma\tau}$ are independent of p, q, r. We thus obtain a system of relations which is equivalent to the Jacobi identity:

$$A^{ijk}_{\sigma\tau} = 0, \quad 1 \leqslant i < j < k \leqslant N, \quad 0 \leqslant \sigma, \quad \tau \leqslant 2. \tag{15}$$

Let us write down the explicit form of these relations. In order to simplify them, we shall use the compatibility condition of the connection with the metric, written in the following form:

$$g^{ij}_{,k} = b^{ij}_k + b^{ji}_k; \tag{16}$$

here and in what follows we use abbreviated notation such as

$$g^{ij}_{,k} \equiv \frac{\partial g^{ij}}{\partial u^k}.$$

We have:

$$A^{ijk}_{02} \equiv b^{ij}_s g^{sk} - b^{kj}_s g^{si} = 0. \tag{17}$$

This is the symmetry condition for the connection (9). Moreover,

$$A^{ijk}_{00} \equiv B^{ijk}_t (u) u^t_{xx} + C^{ijk}_{st} (u) u^s_x u^t_x = 0,$$

where

$$B^{ijk}_t = (b^{jk}_{s,t} - b^{jk}_{t,s}) g^{si} + b^{ij}_s b^{sk}_t - b^{ik}_s b^{sj}_t = -g^{is} g^{jq} R^k_{qts}. \tag{18}$$

Therefore, the curvature is zero. This proves the necessity of the conditions of the theorem.

To prove sufficiency, there is no need to write down explicitly the remaining equations (15). Indeed, by a change of coordinates $u^i \to v^i = v^i(u)$ the Poisson bracket can be reduced to a constant one. For that bracket the Jacobi identity is obvious. This completes the proof.

Remark 1. In the case when $\det (g^{ij}) \equiv 0$ and g^{ij} has locally constant rank $r < N$, we can choose local coordinates in such a way that $g^{ij} = 0$ for $i > r$ or $j > r$. This follows from (15). We shall not consider here the classification of degenerate Poisson brackets [13].

Remark 2. The coordinates in which the Poisson bracket (5) can be reduced to the constant form (12) do not, as a rule, have a physical meaning. In a number of problems, other natural classes of coordinates arise. In particular, coordinates $u^1, \ldots, u^N$ are called *Liouville* if the metric g^{ij} and the connection b^{ij}_k have the form

$$g^{ij}(u) = \gamma^{ij}(u) + \gamma^{ji}(u), \quad b^{ij}_k(u) = \frac{\partial \gamma^{ji}(u)}{\partial u^k}, \tag{19}$$

where $\gamma^{ij}(u)$ is some matrix. In these coordinates the Poisson bracket (5) has the form

$$\{u^i(x), u^j(y)\} = [\gamma^{ij}(u(y)) + \gamma^{ji}(u(x))]\, \delta'(x - y) \tag{20}$$

(see Examples 1, 2 below and also Chapter II). For Liouville coordinates the functionals

$$U^i = \int u^i dx, \qquad i = 1, \dots, N, \tag{21}$$

commute pairwise, $\{U^i, U^j\} = 0$.

A Liouville bracket of hydrodynamic type in coordinates $u^1, \dots, u^N$ is called *strongly Liouville* if the property of being Liouville is preserved under the following operations:

a) affine changes of coordinates $\tilde{u}^i = A^i_j \tilde{u}^j + a^i$ (this is always true);

b) restrictions of the tensor γ^{ij} to any subspace $\{\tilde{u}^{i_1}, \dots, \tilde{u}^{i_k}\}$ spanned by a subset of the coordinates after any affine change.

Example 1. The Hamiltonian formalism of one-dimensional classical hydrodynamics is provided by Poisson brackets of the form (20) with $N = 3$ in coordinates $u^1 = p$ (momentum density), $u^2 = \rho$ (mass density), and $u^3 = s$ (entropy density). Here

$$\gamma^{ij} = \begin{pmatrix} p & 0 & 0 \\ \rho & 0 & 0 \\ s & 0 & 0 \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 2p & \rho & s \\ \rho & 0 & 0 \\ s & 0 & 0 \end{pmatrix}. \tag{22}$$

The Hamiltonian has the form

$$H = \int \left[\frac{p^2}{2\rho} + \varepsilon(\rho, s) \right] dx, \tag{23}$$

where $\varepsilon(\rho, s)$ is the energy density. It is not hard to verify that the bracket (22) is strongly Liouville.

In the barotropic case the entropy drops out, $s = \text{const}$; in the variables p, ρ the metric g^{ij} is non-degenerate. It is interesting to observe that in physical coordinates p, ρ, s the Poisson bracket of one-dimensional hydrodynamics is strongly Liouville.

Example 2. A one-dimensional relativistic fluid. Here $N = 2$, as we have only two fields, $u^1 = p$ (momentum density), $u^2 = \varepsilon$ (energy density). The equations of motion have the form

$$\frac{\partial T^{ij}}{\partial x^j} = 0 \Leftrightarrow \varepsilon_t + p_x = 0, \qquad p_t + (\varepsilon - 2q)_x = 0 \tag{24}$$

(let the speed of light $c = 1$), where T^{ij} is the energy momentum tensor

$$(T^{ij}) = \begin{pmatrix} \varepsilon & p \\ p & \varepsilon - 2q \end{pmatrix}, \tag{25}$$

where $2q = \mathscr{E} - \mathscr{P}$ is the trace of this tensor in the Minkowski metric, $\mathscr{P}$ is the pressure and $\mathscr{E}$ is the energy density in the travelling coordinate frame, in which the tensor T^i_j is diagonal and has the form

$$T^i_j = \begin{pmatrix} \mathscr{E} & 0 \\ 0 & -\mathscr{P} \end{pmatrix}, \quad T^i_i = \mathscr{E} - \mathscr{P} = 2q. \tag{26}$$

The equations of motion (24) are made into a closed system by a state equation, which is a relation between components of the tensor T^{ij}. Since we require Lorentzian invariance, this relation involves only the invariants of the tensor T^{ij}, $\Phi(\mathscr{E}, \mathscr{P}) = 0$. The Poisson bracket has a Liouville form, where

$$\gamma^{ij} = \begin{pmatrix} p & \varepsilon \\ \varepsilon - 2q & p \end{pmatrix}. \tag{27}$$

The Hamiltonian has the form $H = \int \varepsilon\, dx$. The corresponding metric g^{ij} has signature $(+, -)$.

Example 3. The Benney equations (in the case of finitely many layers, see [32], [66]) have the form

$$\left.\begin{aligned} u_t^i + u^i u_x^i + \Big(\sum_{i=1}^n \eta^i\Big)_x &= 0 \\ \eta_t^i + (u^i \eta^i)_x &= 0 \end{aligned}\right\}, \qquad i = 1, \ldots, n. \tag{28}$$

The Poisson bracket in the variables $u^1, \ldots, u^n, \eta^1, \ldots, \eta^n$ is constant,

$$\begin{aligned} \{u^i(x), \eta^j(y)\} &= \delta^{ij}\delta'(x-y), \\ \{\eta^i(x), \eta^j(y)\} &= \{u^i(x), u^j(y)\} = 0. \end{aligned} \tag{29}$$

The Hamiltonian has the form

$$H = \int \Big[\frac{1}{2}\sum_{i=1}^n \eta^i u^{i2} + \frac{1}{2}\Big(\sum_{i=1}^n \eta^i\Big)^2\Big]\, dx. \tag{30}$$

In the case of infinitely many layers ($n = \infty$) the system of Benney equations can also be written in terms of "moments" [41], [66]

$$\begin{gathered} A_n(x) = \sum_i (u^i)(x)^n \eta^i(x), \\ A_{n,t} + A_{n+1,x} + nA_{n-1}A_{0,x} = 0, \qquad n \geqslant 0. \end{gathered} \tag{31}$$

In the variables $A_n(x)$ the bracket (29) is linear [42]:

$$\{A_n(x), A_m(y)\} = [nA_{n+m-1}(x) + mA_{n+m-1}(y)]\,\delta'(x-y). \tag{31$'$}$$

The Hamiltonian has the form $H = (A_2 + A_0)/2$.

Let us now derive in a more explicit form the conditions under which a system of hydrodynamic type is Hamiltonian with respect to a non-degenerate Poisson bracket [62]. Let us make the preliminary observation that the system $u_t^i(x) = \{u^i(x), H\}$ with Hamiltonian $H = \int h(u)\, dx$ and brackets of the form (5), can be written in the form

$$u_t^i = v_j^i(u)\, u_x^j, \quad v_j^i(u) = \nabla^i \nabla_j h(u). \tag{32}$$

Here ∇_j is the covariant differentiation operator, and the operator ∇^i is obtained by raising indices, $\nabla^i = g^{is}\nabla_s$. The operators ∇^i, ∇_j commute by Theorem 1.

***Proposition* 2.** *The system $u_t^i = v_j^i(u)u_x^j$ is Hamiltonian if and only if there exists a non-degenerate metric $g^{ij}(u)$ of zero curvature, such that*

$$g_{ik}v_j^k = g_{jk}v_i^k, \tag{33}$$

$$\nabla_i v_j^k = \nabla_j v_i^k, \tag{34}$$

where ∇_i is the covariant differentiation generated by the metric g^{ij}.

The proof follows at once from the formulae (32) and Theorem 1.

Let us discuss the question of uniquely reconstructing the metric g^{ij} from the coefficients $v_j^i(u)$ of a Hamiltonian system of hydrodynamic type for $N \geqslant 3$. Let us denote by $\lambda_\alpha(u)$, $\alpha = 1, \ldots, N$ the (possibly complex) eigenvalues of the matrix $v_j^i(u)$. (In hydrodynamic systems these quantities have the meaning of velocities. Therefore when we move on to applications we shall denote them by v.) Let us assume that they are all distinct. We denote the corresponding basis of eigenvectors by $e_\alpha = e_\alpha(u)$. Let us define the coefficient $c_{\alpha\beta}^\gamma(u)$ by

$$[e_\alpha, e_\beta] = c_{\alpha\beta}^\gamma e_\gamma. \tag{35}$$

Let us assume further that all the coefficients $c_{\alpha\beta}^\gamma$ with unequal α, β, γ are different from zero. We shall call the matrix $v_j^i(u)$ of the Hamiltonian system (1) for which the two previous propositions hold a Hamiltonian matrix in general position.

***Proposition* 3** [62]. *For $N \geqslant 3$, given a Hamiltonian matrix $v_j^i(u)$ in general position, we can reconstruct the corresponding non-degenerate metric $g^{ij}(u)$ with zero curvature uniquely up to multiplication by constants.*

Proof. It follows from (33) that in the basis e_α the metric g^{ij} is diagonal. Let us normalize the (complex) eigenvectors e_α in such a way that in this basis the metric has the form $g^{\alpha\beta} = \delta^{\alpha\beta}$. In this basis, the relation (34) is rewritten in the form

$$\partial_\alpha\lambda_\beta\delta_\beta^\gamma - \partial_\beta\lambda_\alpha\delta_\alpha^\gamma + (\Gamma_{\alpha\beta}^\gamma - \Gamma_{\beta\alpha}^\gamma)\lambda_\gamma + \Gamma_{\beta\alpha}^\gamma(\lambda_\beta - \lambda_\alpha) = 0 \tag{36}$$

(for the duration of this proof, there is no summation over repeated indices!). Here ∂_α is the operator of differentiation in the direction of e_α, and the connection coefficients $\Gamma_{\alpha\beta}^\gamma$ are defined by the equalities

$$\nabla_{e_\beta} e_\alpha = \sum \Gamma_{\alpha\beta}^\gamma e_\gamma$$

(see [31], Part 1, §30). Since $\Gamma_{\alpha\beta}^\gamma - \Gamma_{\beta\alpha}^\gamma = c_{\alpha\beta}^\gamma$ (ibid), for pairwise distinct α, β, γ we obtain

$$c_{\beta\alpha}^\gamma = -\Gamma_{\beta\alpha}^\gamma \frac{\lambda_\beta - \lambda_\alpha}{\lambda_\gamma - \lambda_\alpha}. \tag{37}$$

For $\gamma = \beta \neq \alpha$ it follows from (36), (37) that

$$c^{\gamma}_{\gamma\alpha} = \frac{a_\alpha \lambda_\gamma}{\lambda_\alpha - \lambda_\gamma}. \tag{38}$$

The compatibility condition of the connection with the metric in our basis takes the form $\Gamma^{\alpha}_{\beta\gamma} = -\Gamma^{\beta}_{\alpha\gamma}$. Therefore from (37) we obtain

$$c^{\gamma}_{\beta\alpha} = -c^{\beta}_{\gamma\alpha}\left(\frac{\lambda_\gamma - \lambda_\alpha}{\lambda_\beta - \lambda_\alpha}\right)^2. \tag{39}$$

The expression (39) is only true in a normalized basis of eigenvectors e_α. Let us prove that, without normalization, knowing the eigenvectors $e_\alpha(u)$ and the eigenvalues $\lambda_\alpha(u)$, we can find normalizing coefficients $k_\alpha(u)$ uniquely up to multiplication by units such that

$$e_\alpha = k_\alpha e_\alpha,$$

that is, the metric can be reconstructed from the coefficients of the original system. Indeed, let

$$[e_\alpha, e_\beta] = c^{\gamma}_{\alpha\beta} e_\gamma.$$

Then for pairwise distinct α, β, γ we have

$$c^{\gamma}_{\alpha\beta} = c^{\gamma}_{\alpha\beta}\frac{k_\alpha k_\beta}{k_\gamma}.$$

Formula (39) takes the form

$$c^{\gamma}_{\alpha\beta}\frac{k_\alpha k_\beta}{k_\gamma} = -c^{\alpha}_{\gamma\beta}\frac{k_\gamma k_\beta}{k_\alpha}\left(\frac{\lambda_\gamma - \lambda_\beta}{\lambda_\alpha - \lambda_\beta}\right)^2, \tag{40}$$

whence

$$\left(\frac{k_\alpha}{k_\beta}\right)^2 = \frac{c^{\alpha}_{\beta\gamma}}{c^{\gamma}_{\alpha\beta}}\left(\frac{\lambda_\gamma - \lambda_\beta}{\lambda_\alpha - \lambda_\beta}\right)^2. \tag{41}$$

From these relations for different choices of α, β, γ we obtain

$$\begin{cases} k^2_\alpha = c^{\alpha}_{\beta\gamma}(\lambda_\beta - \lambda_\gamma)^2 k_{\alpha\beta\gamma}, \\ k^2_\beta = c^{\beta}_{\gamma\alpha}(\lambda_\gamma - \lambda_\alpha)^2 k_{\alpha\beta\gamma}, \\ k^2_\gamma = c^{\gamma}_{\alpha\beta}(\lambda_\alpha - \lambda_\beta)^2 k_{\alpha\beta\gamma}, \end{cases} \tag{42}$$

where $k_{\alpha\beta\gamma} = k_{\alpha\beta\gamma}(u)$ is a coefficient that depends on the choice of the triplet of indices α, β, γ. From (38) we obtain

$$c^{\alpha}_{\alpha\beta} = k_\beta \frac{\bar\partial_\beta \lambda_\alpha}{\lambda_\beta - \lambda_\alpha} = c^{\alpha}_{\alpha\beta} k_\beta - \frac{k_\beta}{k_\alpha}\bar\partial_\beta k_\alpha, \tag{43}$$

where $\bar\partial_\beta = \partial_{e_\beta}$, whence we have the following expression for the derivatives of $\ln k_{\alpha\beta\gamma}$ in the direction of e_β:

$$\bar\partial_\beta \ln k_{\alpha\beta\gamma} = -\frac{\bar\partial_\beta \lambda_\alpha}{\lambda_\beta - \lambda_\alpha} + c^{\alpha}_{\alpha\beta} + \bar\partial_\beta \ln\left(c^{\alpha}_{\beta\gamma}(\lambda_\beta - \lambda_\gamma)^2\right). \tag{44}$$

If $N = 3$, then the statement of the proposition follows from (42) and (44). In the case of dimension $N > 3$, all the coefficients $k_{\alpha'\beta'\gamma'}$ have to be expressed in terms of one of these coefficients $k_{\alpha\beta\gamma}$ by comparing the expressions for k_α^2 for different triples α', β', γ'. From this and (44) we obtain all the derivatives of $\ln k_{\alpha\beta\gamma}$ and therefore determine all the k_α^2 up to multiplication by a single constant. The proposition is proved.

It is clear that the method of proof allows us to obtain effective conditions for a system to be a Hamiltonian system of hydrodynamic type.

Let us state another interesting property of one-dimensional Hamiltonian systems of hydrodynamic type [59].

***Proposition* 4.** *For any admissible changes of the independent variables*

$$(45)\quad t \to t' = a_{00}t + a_{01}x, \quad x \to x' = a_{10}t + a_{11}x, \quad \det(a_{ij}) \neq 0,$$

a one-dimensional Hamiltonian system of hydrodynamic type (with a non-degenerate bracket) is transformed into a Hamiltonian system of hydrodynamic type.

Explanation: a change of variables (45) is called *admissible* if it transforms a system of the form (1) into another system of the same form (soluble with respect to u_t^i). For a proof see [59].

Before we move on to multi-dimensional brackets, let us consider specifically the case of one-dimensional Poisson brackets of hydrodynamic type, which are linear in the field variables [3]:

$$(46)\qquad g^{ij}(u) = g_k^{ij}u^k + g_0^{ij},$$

where $b_k^{ij}, g_k^{ij} = b_k^{ij} + b_k^{ji}, g_0^{ij}$ are constants, and

$$(47)\quad \{u^i(x), u^j(y)\} = (g_k^{ij}u^k(x) + g_0^{ij})\,\delta'(x-y) + b_k^{ij}u_x^k(x)\,\delta(x-y).$$

In this case the coordinates u^k are Liouville and

$$(48)\qquad \gamma^{ij} = b_k^{ij}u^k + b_0^{ij}, \quad \text{where} \quad g_0^{ij} = b_0^{ij} + b_0^{ji}.$$

The theory of such brackets is the same as the theory of local translation invariant Lie algebras, which are a generalization of Lie algebras of vector fields on the real line and on a circle. For their parametrization, it is convenient to introduce an N-dimensional algebra B with basis $e^1, \ldots, e^N$ and structure constants b_k^{ij}:

$$(49)\qquad e^ie^j = b_k^{ij}e^k.$$

On the algebra B we define a symmetric scalar product $(\cdot, \cdot)_0$ by setting

$$(50)\qquad (e^i, e^j)_0 = g_0^{ij}.$$

Proposition 5 [3], [47]. 1) *The expression* (47) *defines a Poisson bracket if and only if the algebra B satisfies the identities.*

$$(ab)\,c - a\,(bc) = (ba)\,c - b\,(ac), \tag{51}$$

$$(ab)\,c = (ac)\,b, \tag{52}$$

where the right multiplication operators are symmetric with respect to the scalar product $(\cdot,\ \cdot)_0$:

$$(ab,\ c)_0 = (ac,\ b)_0. \tag{53}$$

2) *In the space L_B of vector functions in x with values in the algebra B, the operation*

$$[p, q] = q'p - p'q, \qquad ' = \frac{d}{dx}, \tag{54}$$

(*taking products of vector functions in the sense of multiplication in B*) *defines a Lie algebra structure.*

3) *The Lie algebra of linear functionals on the fields u^i with respect to the bracket* (47) *for $g_0^{ij} = 0$ coincides with L_B; for $g_0^{ij} \neq 0$ it is a one-dimensional central extension of the algebra L_B by the cocycle*

$$\langle p,\ q\rangle = \int (p,\ q')_0\, dx. \tag{55}$$

The proof follows from the relations (18), (17).

The relation (53) holds also for the scalar product defined by the matrix $g^{ij}(u) = g_k^{ij}u^k + g_0^{ij}$ for all u^k. The Lie algebra L_B is called *non-degenerate* if the scalar product $g^{ij}(u)$ is non-degenerate for almost all u^k. In this case, the finite-dimensional algebra B with scalar product $g^{ij}(u)$ is called *quasi-Frobenius.* If B is a commutative algebra with identity, then we have a classical Frobenius algebra. As shown in [34], every finite-dimensional algebra B with the properties (51), (52) has a non-trivial ideal with zero multiplication. Therefore all such algebras B are constructed by successive extensions of associative commutative algebras determined by the generalized cocycles of [3]. Extensions of the Lie algebras L_B in this class are given by extensions of the algebra B in the class of algebras satisfying (51), (52). If $0 \to I \to A \to B \to 0$ is an exact sequence in the class of such algebras and multiplication in the ideal I is trivial ($I^2 = 0$), then the extension A is defined by a 2-cochain α on the algebra B with values in I such that

a) $\quad d \sim d + \delta h,\ \delta h\,(b_1,\ b_2) = h\,(b_1,\ b_2) + b_1 h\,(b_2) - h\,(b_1)\,b_2,$

b) $\quad \delta d\,(b_1,\ b_2,\ b_3) - \delta d\,(b_2,\ b_1,\ b_3) = 0,$

$$\delta d\,(b_1,\ b_2,\ b_3) - \delta d\,(b_1,\ b_3,\ b_2) - d\,(b_1b_2 - b_2b_1,\ b_3) - \\ - [d\,(b_1,\ b_2) - d\,(b_1,\ b_2)]\,b_3 = 0,$$

where

$$\delta d\,(b_1,\ b_2,\ b_3) = d\,(b_1,\ b_2b_3) - d\,(b_1b_2,\ b_3) + b_1 d\,(b_2,\ b_3) - d\,(b_1,\ b_2)\,b_3.$$

Central R-extensions of the Lie algebra L_B are defined by cocycles of the form

$$\chi(p, q) = \int \gamma^{ij}_{(m)} p_i q_j^{(m)} \, dx,$$

where $x \in S^1$, $m \leqslant 3$, and $\gamma^{ij}_{(m)} = (-1)^{m-1}\gamma^{ji}_{(m)}$ is a constant matrix. According to a conjecture of the present authors, for a wide class of algebras B there are no other cocycles. If $\gamma^{ij}_{(3)} = \gamma^{ji}_{(3)}$ is a non-degenerate form on B, then the algebra B is commutative and Frobenius. For other examples of cocycles, see [3].

Let us move on now to the spatially multi-dimensional case. Here we have a linear bundle of metrics and of connections that are compatible with them: for any changes of the spatial variables $x^\alpha \mapsto c^\alpha_\beta x^\beta$, $\alpha = 1, \dots, d$, $\det(c^\alpha_\beta) = 1$, the metrics $g^{ij\alpha}$ and the connections $b^{ij\alpha}_k$ transform as components of a vector. We call the bundle of metrics $g^{ij\alpha}$ (and the corresponding bracket (5)) *strongly non-degenerate* if for some collection of constants c_α, the metric $c_\alpha g^{ij\alpha}$ is non-degenerate.

Theorem 2. *A strongly non-degenerate multi-dimensional Poisson bracket of hydrodynamic type* (5) *can be reduced to constant form for* $N = 1$ *and to linear form*

$$g^{ij\alpha}(u) = g^{ij\alpha}_k u^k + g^{ij\alpha}_0, \qquad \alpha = 1, \dots, d, \tag{56}$$

for $N \geqslant 2$, *where the coefficients* $g^{ij\alpha}_k = b^{ij\alpha}_k + b^{ji\alpha}_k$, $g^{ij\alpha}_0$, $b^{ij\alpha}_k$ *are constant.*

The proof is similar to the proof of Theorem 1, but more technical.

We see that all the multi-dimensional Poisson brackets of hydrodynamic type are defined by some local translation-invariant Lie algebras of vector functions of m variables by analogy with Proposition 5. We shall not discuss the properties of bundles of quasi-Frobenius algebras that arise here.

Example 1 [44]. For $N = d = 2$. the Poisson bracket (5) is reducible either to constant form or to the form (1.47) (with $d = 2$), that is, it is generated by the Lie algebra of vector fields on the plane.

Example 2. In the two-dimensional N-component case ($d = 2$), in coordinates in which the metric g^{ij1} is constant, the connection b^{ij2}_k is also constant and the corresponding metric $g^{ij2}(u)$ is linear: $g^{ij2}(u) = g^{ij2}_k u^k + g^{ij2}_0$. The structure constants b^{ij2}_k define an N-dimensional quasi-Frobenius algebra B with invariant scalar products $(e^i, e^j)_1 = g^{ij1}$ and $(e^i, e^j)_2 = g^{ij2}_0$, which satisfy (53) as well as the additional relation

$$(ab, c)_1 + (ca, b)_1 + (bc, a)_1 = 0. \tag{57}$$

If the metric g^{ij1} is positive definite, then this is a zero algebra and the Poisson bracket is constant in these coordinates (the same is true for all $d \geqslant 2$). This is easily proved by simultaneously reducing all commuting selfadjoint (in the metric $(\cdot, \cdot)_1$) operators $b^j = (b_k^{ij2})$ to diagonal form. There are non-trivial examples for indefinite metrics g^{ij1} [44].

Remark 1. Let us denote by $\mathcal{H}$ the algebra generated by functionals of hydrodynamic type with respect to the bracket (5). The following question is of interest: under what conditions on the metrics $g^{ij\alpha}$ and on the connections $b_k^{ij\alpha}$ will $\mathcal{H}$ be a Lie algebra? This question is non-trivial starting with the case $N = 2$, $d \geqslant 2$, since in the case $N = 1$ we always have $\{H_1, H_2\} = 0$ for any functionals of hydrodynamic type. It turns out that in all the cases $N \neq 2$ it follows from the Jacobi identity on the subalgebra $\mathcal{H}$ that (5) is a Poisson bracket. The conditions on the metric and the connection that arise in the case $N = 2$ are less restrictive, and the Poisson brackets on the subalgebra $\mathcal{H}$ depend on functional parameters (in [29] it was erroneously claimed that from the fact that the Jacobi identity holds on the subalgebra $\mathcal{H}$ it always follows that (5) defines a Poisson bracket on all local functionals; this mistake was pointed out by Mokhov, who in [44] also refined the formulation of Theorem 2). For the explicit form of the parametrization of such brackets in the first non-trivial case $N = d = 2$, see [29].

Remark 2. Multi-dimensional brackets of hydrodynamic type that arise in the theory of averaging (see §6 below) for $d > 1$ do not, as a rule, have the property of strong non-degeneracy. However, the property of *weak non-degeneracy* holds in that case: the intersection of the kernels of all the metrics $g^{ij1}, \ldots, g^{ijd}$ is zero, while the images of all these matrices generate the whole N-dimensional space. The question of the structure of weakly non-degenerate multi-dimensional Poisson brackets is not yet resolved.

§3. Generalizations: differential-geometric Poisson brackets of higher orders, differential-geometric Poisson brackets on a lattice, and the Yang-Baxter equation

Here we shall consider the case of one spatial variable.

Definition. 1) A *homogeneous differential-geometric Poisson bracket of order n* has the form

$$(1)\qquad \{u^i(x), u^j(y)\} = \sum_{k=0}^{n} B_k^{ij}(u(x), u_x(x), u_{xx}(x), \ldots)\, \delta^{(n-k)}(x-y),$$

where the coefficients B_k^{ij} are of degree k, and by definition

$$\deg u^i = 0, \quad \deg \frac{d^s u^i}{dx^s} = s, \qquad s = 1, 2, \ldots$$

in other words,

$$(2)\quad B_0^{ij} = g^{ij}(u), \quad B_1^{ij} = b_s^{ij}(u)\, u_x^s, \quad B_2^{ij} = c_s^{ij}(u)\, u_{xx}^s + d_{st}^{ij}(u)\, u_x^s u_x^t, \ldots$$

2) A *non-homogeneous bracket* is a sum of homogeneous brackets of different orders.

The class of differential-geometric Poisson brackets is invariant under local changes of field variables $u^i \mapsto v^i(u)$, with homogeneous components transforming independently. In particular, from a constant bracket of order n,

$$\{v^i(x),\ v^j(y)\} = B_n^{ij}\delta^{(n)}(x-y), \tag{3}$$

where $B_n^{ji} = (-1)^{n-1}B_n^{ij}$ is a constant matrix, we obtain after the change of variables $u^i \to v^i(u)$ a homogeneous bracket of order n.

For arbitrary homogeneous brackets of order n, the condition of reducibility to constant form is a problem of differential geometry, which is non-trivial even under the condition of the principal term being non-degenerate: $\det g^{ij} \neq 0$, where $g^{ij} = B_0^{ij}$, $g^{ij} = (-1)^{n-1}g^{ji}$.

Example 1 [54]. Homogeneous second-order brackets. Let us assume that the skew-symmetric tensor $g^{ij} = -g^{ji}$ is non-degenerate: $\det g^{ij} \neq 0$, where the coefficients g^{ij}, as well as the coefficients b_s^{ij}, c_s^{ij}, d_{st}^{ij} that determine the homogeneous second order bracket, are defined by (2).

Proposition 1. *The connection $\Gamma_{si}^k = g_{ij}c_s^{kj}$ is symmetric, has zero curvature, and coincides with the symmetric part of the connection $\widetilde{\Gamma}_{si}^k = g_{ij}b_s^{kj}$, that is, $\widetilde{\Gamma}_{si}^k + \widetilde{\Gamma}_{is}^k = 2\Gamma_{si}^k$. The torsion tensor $T_{si}^k = \widetilde{\Gamma}_{si}^k - \widetilde{\Gamma}_{is}^k$ of the connection has the following properties*:

a) T_{ksi} *is skew-symmetric in all the indices.*

b) $d\,(g_{ij}du^i \wedge du^j) = \text{const}\ T_{ksi}du^k \wedge du^s \wedge du^i$.

Moreover, the form $\Omega = T_{ksi}du^k \wedge du^s \wedge du^i$ satisfies certain differential identities (*see* [54]).

From Proposition 1 it follows, in particular, that a homogeneous second-order bracket with $\det g^{ij} \neq 0$ reduces to the constant form $g^{ij}\delta''(x-y)$ if and only if $d\,(g_{ij}du^i \wedge du^j) = 0$. [54] gives a classification of homogeneous second-order brackets with a "non-degenerate" metric g^{ij}.

Example 2. Non-homogeneous differential-geometric brackets that are sums of brackets of first and zeroth orders have the form

$$\{u^i(x),\ u^j(y)\} = g^{ij}(u(x))\,\delta'(x-y) + [b_s^{ij}(u)\,u_x^s + h^{ij}(u)]\,\delta(x-y). \tag{4}$$

Let us assume that the metric g^{ij} is non-degenerate.

Proposition 2 [29]. *In the coordinates $v^1, \ldots, v^N$, where $g^{ij} = \text{const}$ and $b_s^{ij} = 0$, the bracket* (4) *has the following form*:

$$\{v^i(x), v^j(y)\} = g^{ij}\delta'(x-y) + [c_s^{ij}v^s + c_0^{ij}]\,\delta(x-y), \tag{5}$$

where c_s^{ij} are the structure constants of some finite-dimensional Lie algebra L with invariant scalar product g^{ij}, that is, $c_s^{ij}g^{sk} = - c_s^{ik}g^{sj}$, and c_0^{ij} is a cocycle on L.

Therefore differential-geometric Poisson brackets of order $1+0$ are generated by some Lie algebra of currents (see Example 1.7 above). Let us observe that Poisson brackets of hydrodynamic type with variable coefficients (depending on x) reduce to non-homogeneous brackets of the form (5) if changes of variable that mix dependent and independent variables $u^1, \ldots, u^N$ and x are allowed.

Let us consider now, following [22], the discrete analogue of differential-geometric Poisson brackets (only in the spatially one-dimensional case). The fields u^i, $i = 1, \ldots, N$, are defined on a one-dimensional lattice: $u^i = (u^i_n)$, $n \in \mathbb{Z}$. Differential-geometric Poisson brackets of order n_0 have the form

$$\text{(6)} \qquad \{u^i_n, u^j_m\} = h^{ij}_{m-n}(u_n, u_m), \qquad h^{ij}_k = 0 \text{ when } |k| > n_0.$$

Under local changes of coordinates at the nodes of the lattice of the form

$$\text{(7)} \qquad u^i_n \mapsto u^{i'}_n = f^i(u^1_n, \ldots, u^N_n), \qquad i = 1, \ldots, N, \quad n \in \mathbb{Z},$$

the matrices $h^{ij}_k(u, v)$ transform according to the rule

$$\text{(8)} \qquad h^{ij}_k(u, v) \mapsto h^{ij'}_k(u', v') = \frac{\partial f^i(u)}{\partial u^p}\frac{\partial f^j(v)}{\partial v^q} h^{pq}_k(u, v), \qquad |k| \leqslant n_0.$$

When $n_0 = 0$ the bracket (6) is ultra-local, that is, it reduces to a finite-dimensional Poisson bracket h^{ij}_0 on the u-space. In the rest of the cases we can assume that $n_0 = 1$. (When $n_0 > 1$ we introduce new field variables v^α_n, $\alpha = 1, \ldots, n_0 N$, by setting

$$v^{i+pN}_n = u^i_{nn_0+p}, \qquad i = 1, \ldots, N, \quad p = 0, 1, \ldots, n_0 - 1.$$

After this change of variables we obtain a first order differential-geometric bracket in the variables v^α_n.)

Thus, we shall consider only first order brackets:

$$\text{(9)} \qquad \{u^i_n, u^j_n\} = h^{ij}_0(u_n), \quad \{u^i_n, u^j_{n+1}\} = h^{ij}_1(u_n, u_{n+1}),$$
$$\{u^i_n, u^j_m\} = 0 \text{ when } |n-m| > 1.$$

It turns out that under the condition of non-degeneracy of the matrix $h^{ij}_1(u, u)$ (this condition is invariant under local changes of variables (7)), the bracket (9) is parametrized by Hamilton–Lie groups of a certain kind. Let us review the basic facts of Hamilton–Lie group theory, following the paper of Drinfel'd [20]. A Lie group G is called a *Hamilton–Lie group* if a Poisson bracket $\{,\}_0$ is defined on it in such a way that multiplication $G \times G \to G$ is a mapping of Poisson manifolds. If $L = L(G)$ is the Lie algebra of the group G, then (locally) the Hamilton–Lie structures are uniquely determined by Lie algebra structures on the dual space L^*. Here we require the Lie algebras L and L^* to be *compatible* in the following sense: if $c^{\gamma}_{\alpha\beta}$ and $f^{\alpha\beta}_{\gamma}$ are the structure constants for the Lie algebras L and L^*, respectively, then we must have the identity

$$c^{\varepsilon}_{\rho\lambda}f^{\mu\nu}_{\varepsilon} = c^{\mu}_{\varepsilon\lambda}f^{\varepsilon\nu}_{\rho} + c^{\nu}_{\varepsilon\lambda}f^{\mu\varepsilon}_{\rho} - c^{\mu}_{\varepsilon\rho}f^{\varepsilon\nu}_{\lambda} - c^{\nu}_{\varepsilon\rho}f^{\mu\varepsilon}_{\lambda} \tag{10}$$

(that is, $f^{\alpha\beta}_{\gamma}$ is a 1-cocycle on L with values in $L \otimes L$). A pair of compatible Lie algebra structures on L and L^* is called a *Lie bi-algebra* in [20]. Given a Hamilton–Lie group, a bi-algebra is constructed as follows: the commutator in the Lie algebra L^* has the form

$$[a, b]_* = \{\varphi, \psi\}_0 |_e, \qquad a = d\varphi\,|_e \in L^*,\ b = d\psi\,|_e \in L^*, \tag{11}$$

where φ, ψ are smooth functions on G, and $e \in G$ is the identity. The structure constants $f^{\alpha\beta}_{\gamma}$ of the algebra L^* are defined by the formula

$$f^{\alpha\beta}_{\gamma} = \partial_{\gamma}\eta^{\alpha\beta}_{0}\,|_e, \tag{12}$$

where the ∂_{γ} are left-invariant vector fields on G, and the Poisson bracket is given in the form

$$\{\varphi, \psi\}_0 = \eta^{\alpha\beta}_{0}\partial_{\alpha}\varphi\partial_{\beta}\psi. \tag{13}$$

The bracket (13) can be uniquely reconstructed (if G is connected and simply-connected) from the bi-algebra via the following differential equations:

$$\partial_{\lambda}\eta^{\mu\nu}_{0} = c^{\mu}_{\varepsilon\lambda}\eta^{\varepsilon\nu}_{0} + c^{\nu}_{\varepsilon\lambda}\eta^{\mu\varepsilon}_{0} + f^{\mu\nu}_{\lambda} \tag{14}$$

with initial conditions

$$\eta^{\mu\nu}_{0}\,|_e = 0. \tag{15}$$

The relation (10) is the compatibility condition for the system (14).

Let us describe now the construction of Poisson brackets of the form (9).

Let G be a Hamilton–Lie group, $(L,\ c^{\gamma}_{\alpha\beta};\ L^*,\ f^{\alpha\beta}_{\gamma})$ its Lie bi-algebra, and $h^{\alpha\beta}$ a skew-symmetric matrix such that the cohomologous cocycle

$$f'^{\alpha\beta}_{\gamma} = f^{\alpha\beta}_{\gamma} + c^{\alpha}_{\varepsilon\gamma}h^{\varepsilon\beta} + c^{\beta}_{\varepsilon\gamma}h^{\alpha\varepsilon} \tag{16}$$

also defines a Lie algebra structure on L^* (it will be automatically compatible with L). We require the following relation to hold:

$$[h, h]^{\mu\nu\lambda} = f^{\mu\nu}_{\varepsilon}h^{\varepsilon\lambda} + f^{\lambda\mu}_{\varepsilon}h^{\varepsilon\nu} + f^{\nu\lambda}_{\varepsilon}h^{\varepsilon\mu}, \tag{17}$$

where $[h, h]$ is the left hand side of the classical Yang–Baxter equation (1.14) for the r-matrix $h^{\alpha\beta}$ in the Lie algebra L. Finally, there must exist a Lie algebra homomorphism

$$r\colon (L^*,\ f^{\alpha\beta}_{\gamma}) \to (L,\ c^{\gamma}_{\alpha\beta}), \qquad r = (r^{\alpha\beta}), \tag{18}$$

such that the adjoint mapping $r^{\alpha\beta}_{*} = r^{\beta\alpha}$ defines a homomorphism of these Lie algebras:

$$r_*\colon (L^*,\ f'^{\alpha\beta}_{\gamma}) \to (L,\ c^{\gamma}_{\alpha\beta}). \tag{19}$$

A Hamilton–Lie group G for which there exist a homomorphism r and a matrix h satisfying the above conditions is called *admissible.*

Theorem 1. *An admissible Hamilton–Lie group together with corresponding matrices r, h defines a Poisson bracket of the form* (9), *where $u_n \in G$ for all $-\infty < n < \infty$, according to the following formulae*:

$$\{\varphi(u_n), \psi(u_m)\} = 0 \quad \text{for} \quad |n-m| > 1, \tag{20}$$

$$\{\varphi(u_n), \psi(u_{n+1})\} = r^{\alpha\beta}\partial_\alpha\varphi(u_n)\,\partial'_\beta\psi(u_{n+1}), \tag{20'}$$

where ∂_α, ∂'_β are left- and right-invariant vector fields on G,

$$\{\varphi(u_n), \psi(u_n)\} = \eta^{\alpha\beta}(u_n)\,\partial_\alpha\varphi(u_n)\,\partial_\beta\psi(u_n), \tag{20''}$$

where the bracket $\eta^{\alpha\beta}(u)$ on G has the form

$$\eta^{\alpha\beta}(u) = \eta_0^{\alpha\beta}(u) + Ad_{u^{-1}}h^{\alpha\beta}, \tag{21}$$

and $\eta_0^{\alpha\beta}(u)$ is determined from (14), (15). *Here φ and ψ are arbitrary smooth functions on G. All brackets of the form* (9) *are obtained in this way under the non-degeneracy condition* $\det(h_1^{ij}) \neq 0$.

Remark 1. The non-degeneracy condition $\det(h_1^{ij}) \neq 0$ is equivalent to the non-degeneracy of the matrix $r^{\alpha\beta}$, that is, (18) is an isomorphism. In this case, compatibility conditions reduce to conditions on the Lie algebra L and the scalar product $r^{\alpha\beta}$ (on L^*).

Remark 2. To an admissible Hamilton–Lie group there corresponds a whole family of matrices $r^{\alpha\beta}$, $h^{\alpha\beta}$ satisfying the required conditions, which depend on the point of the group. In particular, the matrix $r^{\alpha\beta}$ can be replaced by $h_1^{\alpha\beta}(u, u)$ for any fixed $u \in G$. The structure constants $f_\gamma^{\alpha\beta}$ and the matrix $h^{\alpha\beta}$ will change accordingly. The bracket (20)–(20″) will remain the same.

Let us indicate an important class of differential geometric Poisson brackets on the lattice which correspond to *triangular* Hamilton–Lie groups (in the sense of [21]). The matrix $r^{\alpha\beta}$ here is skew-symmetric and satisfies the classical Yang–Baxter equation on the Lie algebra of the group G. In this case, formulae (20′) and (20″) that define the bracket assume the following form:

$$\{\varphi(u_n), \psi(u_{n+1})\} = r^{\alpha\beta}\partial_\alpha\varphi(u_n)\,\partial'_\beta\psi(u_{n+1}), \tag{22}$$

$$\{\varphi(u_n), \psi(u_n)\} = -r^{\alpha\beta}[\partial_\alpha\varphi(u_n)\,\partial_\beta\psi(u_n) + \partial'_\alpha\varphi(u_n)\,\partial'_\beta\psi(u_n)]. \tag{23}$$

This bracket satisfies the non-degeneracy condition if the matrix $r^{\alpha\beta}$ is non-degenerate. The Lie algebra L is quasi-Frobenius in this case.

Example (Cherkashin). For the simplest two-dimensional non-Abelian group G we can take as the matrix r an arbitrary non-degenerate matrix. We thus obtain the following family of brackets:

$$h_1^{ij}(u,v)=\begin{pmatrix}\sigma y^2 & 0\\ 0 & y/2\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ x' & y'/2\end{pmatrix},$$

$$u=(x,\ y),\qquad v=(x',\ y'),\ \ \sigma=\pm 1,\ h_0^{ij}(u)=h_1^{ij}(u,\ u)-h_1^{ji}(u,\ u).$$

When $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\sigma & 0\\ 0 & 2\end{pmatrix}$ we obtain a second Hamiltonian structure for the Toda chain [56], [64].

§4. Riemann invariants and the Hamiltonian formalism of diagonal systems of hydrodynamic type. Novikov's conjecture. Tsarev's theorem. The generalized hodograph method

It is well known that a one-dimensional system of hydrodynamic type with two field variables $u=(u^1,\ u^2)$ can be linearized by the "hodograph transform" $x=x(u^1,\ u^2)$, $t=t(u^1,\ u^2)$. Then the system

$$\left\{\begin{aligned} u_t^1&=v_1^1(u)\,u_x^1+v_2^1(u)\,u_x^2,\\ u_t^2&=v_1^2(u)\,u_x^1+v_2^2(u)\,u_x^2\end{aligned}\right.\tag{1}$$

becomes the linear system

$$\left\{\begin{aligned} x_{u^2}&=-v_1^1(u)\,t_{u^2}+v_2^1(u)\,t_{u^1},\\ x_{u^1}&=v_1^2(u)\,t_{u^2}-v_2^2(u)\,t_{u^1}.\end{aligned}\right.\tag{2}$$

Following Tsarev, let us present a new exposition of the theory of integration of two-component systems, which is methodologically well suited for generalizations. Let us assume that the system (1) is strictly hyperbolic in some region of the space of coordinates $(u^1,\ u^2)$, that is, the matrix $(v_j^i(u))$ has two distinct real eigenvalues $v_1(u)$ and $v_2(u)$. Then, by a smooth change of variables, (1) can be locally reduced to diagonal form. In the following we assume that (1) is already diagonal in the variables u^1, u^2,

$$\left\{\begin{aligned} u_t^1&=v_1(u)\,u_x^1,\\ u_t^2&=v_2(u)\,u_x^2.\end{aligned}\right.\tag{3}$$

We shall use the abbreviated notation $\partial_i=\partial/\partial u^i$, $i=1,\ 2$.

Proposition 1. *Let $w_1(u)$, $w_2(u)$ be the solution of the system*

$$\frac{\partial_2 w_1}{w_2-w_1}=\frac{\partial_2 v_1}{v_2-v_1},\qquad \frac{\partial_1 w_2}{w_2-w_1}=\frac{\partial_1 v_2}{v_2-v_1}.\tag{4}$$

Then: 1) *the functions $u^1=u^1(x,\ t)$, $u^2=u^2(x,\ t)$ defined by the system*

$$\left\{\begin{aligned} w_1(u^1,u^2)&=v_1(u^1,u^2)\,t+x,\\ w_2(u^1,u^2)&=v_2(u^1,u^2)\,t+x,\end{aligned}\right.\tag{5}$$

are solutions of (3), *and every smooth solution of system* (3) *is locally obtainable in this way*; 2) *the system of hydrodynamic type*

$$(6)\qquad \begin{cases} u^1_\tau = w_1(u)\,u^1_x, \\ u^2_\tau = w_2(u)\,u^2_x \end{cases}$$

defines the "symmetry" of the system (3), *that is*, $u^i_{t\tau} = u^i_{\tau t}$, *and all symmetries in the class of systems of hydrodynamic type are obtainable in this way.*

Proof. The system obtained from (3) by the hodograph transformation has the form

$$(7)\qquad \begin{cases} \partial_2 x + v_1(u)\,\partial_2 t = 0, \\ \partial_1 x + v_2(u)\,\partial_1 t = 0. \end{cases}$$

Let us write it in the following way:

$$(8)\qquad \begin{cases} \partial_2(v_1 t + x) = t\partial_2 v_1, \\ \partial_1(v_2 t + x) = t\partial_1 v_2. \end{cases}$$

Introducing the quantities $w_i = v_i t + x$, $i = 1,2$, we obtain

$$t = \frac{w_1 - w_2}{v_1 - v_2};$$

substituting into (8), we obtain (4). Conversely, differentiating implicit functions $u^1(x, t)$, $u^2(x, t)$ of the form (5) and using (4), we obtain (3). The first half of the proposition is proved.

Now let a symmetry

$$(9)\qquad u^i_\tau = w^i_j(u)\,u^j_x, \qquad i = 1,\ 2,$$

of (3) be given. From $u^i_{t\tau} = u^i_{\tau t}$ it follows, first of all, that the matrix w^i_j commutes with the diagonal matrix $v_j\delta^i_j$. Therefore $w^i_j = w_j\delta^i_j$. The remainder of the conditions $u^i_{t\tau} = u^i_{\tau t}$ coincides with (4). The proposition is proved.

If the system (3) is Hamiltonian, then all its symmetries are generated by integrals of hydrodynamic type. Thus, the proposition proved above clarifies the connection between the classical hodograph method and integrals of two-component systems. Several authors (see [55]) have shown that the number of integrals of hydrodynamic type in this case is infinite without introducing these concepts explicitly.

Example 1. Taking the "trivial" solution of (4), which has the form $w_i = \alpha v_i + \beta$, where α, β are constants, we obtain the so-called "simple Riemann wave" for the system (3):

$$(10)\qquad \begin{cases} v_1(u)(t - \alpha) + x - \beta = 0, \\ v_2(u)(t - \alpha) + x - \beta = 0. \end{cases}$$

Let us note that the hodograph method cannot be used formally in this case since the mapping $(x, t) \longrightarrow (u^1, u^2)$ is degenerate.

Let us now consider multi-component systems. It turns out that the combination of the following two properties: the reducibility of (1) to diagonal form, and the conservative (Hamiltonian) nature of the system, generate increased integrability of systems of hydrodynamic type. This integrability was conjectured by Novikov and proved by Tsarev [61], [62], who proposed a generalization of the hodograph method to integrate these systems. We now move on to an exposition of these ideas.

Suppose we are given a diagonal Hamiltonian system of hydrodynamic type

$$u_t^i = v_i(u)\, u_x^i, \quad i = 1, \ldots, N, \tag{11}$$

such that all the diagonal elements are pairwise distinct (in this section there is no summation over repeated indices!). Let us denote by $g^{ij}(u)$ the corresponding metric (which we assume non-degenerate) that defines the Hamiltonian structure of the system (11).

***Lemma* 1.** *In the variables $u^1, \ldots, u^N$, in which a Hamiltonian system of hydrodynamic type is diagonal, the corresponding metric $g^{ij}(u)$ is also diagonal.*

Proof. This follows from the relation (2.33).

From the differential-geometric point of view, defining a diagonal metric of zero curvature is equivalent to defining a curvilinear orthogonal system of coordinates in a flat, Euclidean or pseudo-Euclidean, space. It is known [37] that such systems are uniquely determined by $N(N-1)/2$ functions of two variables. Conversely, if an arbitrary orthogonal system of coordinates is chosen, then corresponding to it there will be a family of diagonal Hamiltonian systems, the explicit form of which is given by the following assertion.

***Lemma* 2.** *Let $u^1, \ldots, u^N$ be orthogonal curvilinear coordinates, $g_{ij}(u) = g_i(u)\delta_{ij}$ the corresponding metric, and $\Gamma_{ij}^k(u)$ its Christoffel symbols. Then all diagonal systems of hydrodynamic type*

$$u_t^i = w_i(u)\, u_x^i, \qquad i = 1, \ldots, N, \tag{12}$$

that are Hamiltonian with respect to the bracket

$$\{u^i(x), u^j(y)\} = g_i(u(x))^{-1}\Big[\delta^{ij}\delta'(x-y) - \sum_k \Gamma_{ik}^j u_x^k \delta(x-y)\Big], \tag{13}$$

are determined by the equations

$$\partial_i w_k = \Gamma_{ki}^k (w_i - w_k), \qquad i \neq k. \tag{14}$$

All these systems commute pairwise. They are parametrized locally by functions of one variable.

Proof. Let us introduce the notation $w_j^i = w_j\delta_j^i$. By Proposition 2.2, the condition for (12) to be Hamiltonian is written in the form

$$(15)\quad 0 = \nabla_i w_j^k - \nabla_j w_i^k = \partial_i w_j^k - \partial_j w_i^k + \sum_{l=1}^{N} (\Gamma_{li}^k w_j^l - \Gamma_{ji}^l w_l^k - \Gamma_{lj}^k w_i^l + \Gamma_{ij}^l w_l^k) = \partial_i w_j \delta_j^k - \partial_j w_i \delta_i^k + \Gamma_{ij}^k (w_j - w_i)$$

(condition (2.33) is satisfied automatically). For pairwise distinct values of the indices i, j, k the relation (15) becomes an identity, since $\Gamma_{ij}^k = 0$. Thus, only relations corresponding to the case $j = k \neq i$ remain. These have the form (14). The lemma is proved.

Let us remind the reader (see (2.32) above) that the Hamiltonian $h(u)$ of (11) can be obtained from the equations

$$(16)\qquad \nabla^i \nabla_i h(u) = w_i(u), \qquad i = 1, \ldots, N.$$

It must be said that the metric that defines the Poisson bracket of a diagonalizable system is not uniquely determined (unlike Hamiltonian systems in general position: see Proposition 2.3 above). Examples will be given in §7.

Using the known differential-geometric identities $\Gamma_{ki}^k = \partial_i \ln \sqrt{|g_k|}$, which hold for an arbitrary diagonal metric $g_{ij} = g_i\,\delta_{ij}$, the following relations satisfied by the coefficients of diagonal systems of hydrodynamic type can be derived from (14):

$$(17)\qquad \partial_i\left(\frac{\partial_j w_k}{w_j - w_k}\right) = \partial_j\left(\frac{\partial_i w_k}{w_i - w_k}\right), \qquad i \neq j \quad i \neq k, \quad j \neq k.$$

Definition 1. A diagonal system of hydrodynamic type $u_t^i = w_i(u) u_x^i$, $i = 1, \ldots, N$, is called *semi-Hamiltonian* if its coefficients satisfy the relations (17).

For $N = 2$ there are no relations (17), so that every diagonal system is semi-Hamiltonian. Diagonal Hamiltonian systems are also semi-Hamiltonian by the above arguments, but the converse is not true (examples are given in [50]). It turns out that the property of being semi-Hamiltonian is sufficient for the integrability (or, more precisely, the linearizability) of systems of hydrodynamic type. Let us produce the corresponding construction.

***Theorem* 1.** *Let*

$$(18)\qquad u_t^i = v_i(u)\, u_x^i, \ i = 1, \ldots, N,$$

be a diagonal semi-Hamiltonian system of hydrodynamic type, and $w_1(u), \ldots, w_N(u)$ *an arbitrary solution of the linear system*

$$(19)\qquad \frac{\partial_i w_k}{w_i - w_k} = \frac{\partial_i v_k}{v_i - v_k}, \qquad i \neq k.$$

Then the functions $u^1(x, t), \ldots, u^N(x, t)$ determined by the system of equations

$$w_i(u) = v_i(u)\,t + x, \qquad i = 1, \ldots, N, \tag{20}$$

satisfy (18); *moreover, every smooth solution of this system is locally obtainable in this way.*

Proof. Let $u^i(x, t)$, $i = 1, \ldots, N$, be a smooth solution of the equations (18). Differentiating both sides of (20) with respect to t and x, we obtain

$$\sum \partial_k w_i u_t^k = v_i + t \sum_k \partial_k v_i u_t^k,$$

$$\sum_k \partial_k w_i u_x^k = t \sum_k \partial_k v_i u_x^k + 1,$$

or

$$\begin{cases} \sum_k (\partial_k w_i - t\partial_k v_i)\, u_t^k = v_i, \\ \sum_k (\partial_k w_i - t\partial_k v_i)\, u_x^k = 1. \end{cases} \tag{21}$$

Let us show that the matrix

$$M_{ik}(u) = \partial_k w_i - t\partial_k v_i, \tag{22}$$

diagonal for $u = u(x, t)$, is a solution of (18). Indeed, by (19) with $i \neq k$ we have

$$M_{ik} = \frac{\partial_k v_i}{v_k - v_i}\,[w_k - w_i - t\,(v_k - v_i)].$$

But on solutions $u = u(x, t)$ of (18) for $i \neq k$ we have

$$w_k - w_i = t\,(v_k - v_i),$$

whence $M_{ik} = 0$. Therefore, (21) can be written in the form

$$M_{ii}(u)\, u_t^i = v_i, \qquad M_{ii}(u)\, u_x^i = 1, \qquad i = 1, \ldots, N,$$

whence $u_t^i = v_i(u) u_x^i$, $i = 1, \ldots, N$, that is, we have obtained a solution of the original system (18). Let us observe also that it follows from the second equation that $u_x^i \neq 0$ for any smooth solution of (18).

Conversely, if we have a solution $u = u(x, t)$ of (18), and in a neighbourhood of a point (x_0, t_0) the derivatives u_x^i are not zero, we can construct a solution $w_i(u)$ of (19) for which $u^i(x, t)$ is the only solution of (20) in some neighbourhood of the point $(u^i = u^i(x_0, t_0), x_0, t_0)$. Taking $u_0^i(x) = u^i(x, t_0)$, $i = 1, \ldots, N$, to be the initial conditions in the Cauchy problem for the original system (18), we obtain from (20) the values of the original functions $w_i(u)$ on the curve $u_0^i(x)$:

$$w_i(u_0(x)) = v_i(u_0(x))\, t_0 + x, \quad i = 1, \ldots, N. \tag{23}$$

Since by assumption $u^i_{0x}(x) \neq 0$ for the indicated initial data, which are the values of the functions $w_i(u)$ on the curve $u = u_0(x)$, there is a unique solution of (19) with initial conditions (23) in a neighbourhood of that curve. Let us show that for the functions $w_i(u)$ defined in this manner (20) has a single-valued smooth solution in a neighbourhood of the point (u^i_0, x_0, t_0). Indeed, the system of equations

$$\text{(24)} \qquad \Phi(u^1, \ldots, u^n, x, t) \equiv w_i(u) - v_i(u)\, t - x = 0, \qquad i = 1, \ldots, N,$$

(with respect to the unknowns $u^1, \ldots, u^N$) is satisfied by the values (u^i_0, x_0, t_0) by construction, and at that point the Jacobian matrix $(\partial\Phi_i/\partial u^k)$ is non-degenerate:

$$\partial\Phi_i/\partial u^k = \partial_k w_i - t_0\partial_k v_i = M_{ik}, \qquad M_{ik} = 0 \quad \text{for} \quad i \neq k,$$
$$M_{ii} = \partial_i w_i - t_0 \partial_i v_i \neq 0.$$

Only the last inequality has to be justified. To this end, let us differentiate (24) with respect to x: at the point (u^i_0, x_0, t_0) we shall have $M_{ii}(u)\, u^i_{0x} - 1 = 0$, whence $\partial\Phi_i/\partial u^i = M_{ii} \neq 0$. Thus, by the implicit function theorem, (20) has a unique solution $\underline{u}(x, t)$ in a neighbourhood of the point (u^i_0, x_0, t_0), and this is a smooth function of x, t. By construction, $\underline{u}(x, t_0) = u(x, t_0)$, and since we have shown above that $\underline{u}(x, t)$ is a solution of the original system (18), it coincides with the given solution $u(x, t)$ in a neighbourhood of the point (x_0, t_0) by the uniqueness of the solution of the Cauchy problem for the system (18). The theorem is proved.

As in the case $N = 2$, it can be shown that any solution $w_i(u)$ of (19) defines a symmetry

$$\text{(25)} \qquad u^i_\tau = w_i(u)\, u^i_x, \qquad i = 1, \ldots, N,$$

of the original semi-Hamiltonian system (18), that is, (18) and (25) commute: $u^i_{t\tau} = u^i_{\tau t}$. Moreover, every first-order symmetry, that is, every system of hydrodynamic type that commutes with the original one, can be obtained in this manner.

The construction of the theorem reduces the integration of the original quasi-linear system (18) to solving the linear system (19) and computing the functions $u^1(x, t), \ldots, u^N(x, t)$ defined implicitly by (20). Thus, it is a generalization of the hodograph method to the case $N > 2$ (see Proposition 1 above). Therefore it is natural to call it the *generalized hodograph method*.

Let us make some observations on integrals of hydrodynamic type,

$$\text{(26)} \qquad I[u] = \int P(u)\, dx,$$

of diagonal systems of hydrodynamic type (omitting the proofs [62]). A semi-Hamiltonian system of the form (18) has continuously many independent integrals, parametrized locally by N functions of one variable.

The densities of these integrals are sought as solutions of the following system of simultaneous equations:

$$\partial_i\partial_j P - \frac{\partial_j v_i}{v_j - v_i}\,\partial_i P - \frac{\partial_i v_j}{v_i - v_j}\,\partial_j P = 0, \qquad i \neq j. \tag{27}$$

For Hamiltonian systems (18), these integrals correspond to commuting systems of the form (25) and constitute a complete family on the set of monotone functions [62]. At this stage, the relation of these integrals with commuting systems of hydrodynamic type in the general non-Hamiltonian (semi-Hamiltonian) case is not clear. A general theory of semi-Hamiltonian systems analogous to the theory of Hamiltonian systems has not been constructed yet. So far, they can only be defined in diagonal (Riemann) form in terms of differential-geometric relations (17). Therefore the problem of semi-Hamiltonian systems is not completely solved. Let us remind the reader that in introducing the class of semi-Hamiltonian systems Tsarev [61] was motivated by the fact that even in the case $N = 2$, $d = 1$ not all systems of hydrodynamic type are Hamiltonian, even though they are integrable by the hodograph method and diagonalizable. In some problems of chemical kinetics (see [49], [50]) there arise examples of diagonal semi-Hamiltonian non-Hamiltonian systems. Thus, it is even more important to understand what is the class of non-diagonalizable semi-Hamiltonian systems.

It can also be shown that for $N \geqslant 3$ a Hamiltonian system of the form (18) in general position (in particular, a non-diagonalizable one) has only an $(N+2)$-dimensional family of integrals of hydrodynamic type. This family is generated by the Hamiltonian, by the momentum, and by the N-dimensional annihilator of the bracket (integrals of flat coordinates in which the metric that defines the Poisson bracket (10) is constant).

Example 2. The Benney equations (see Example 2.3 above). In order to reduce this system to diagonal form, let us consider, following [32], [78], the algebraic curve defined by the equation

$$F(\lambda, \mu) = -\mu + \lambda + \sum_{i=1}^{n} \frac{\eta_i}{u^i + \lambda}. \tag{28}$$

Let (λ_p, μ_p), $p = 1, \ldots, 2n$, be the branch points of this curve (relative to the projection on the λ-plane), that is, λ_p are the roots of the equation

$$\frac{\partial F}{\partial \lambda} = 1 - \sum_{i=1}^{n} \frac{\eta_i}{(u^i + \lambda)^2} = 0 \tag{29}$$

(we shall assume that all the roots are real: this defines the region of hyperbolicity of the Benney equations). In the variables $\mu_1, \ldots, \mu_{2n}$ the Benney equations are written in the form

$$\mu_{pt} = \lambda_p(\mu_1, \ldots, \mu_{2n})\,\mu_{px}, \qquad p = 1, \ldots, 2n. \tag{30}$$

The corresponding diagonal elements of the metric g_p, $p = 1, \ldots, 2n$, are the residues of the meromorphic differential

$$\frac{d\lambda}{\partial F/\partial\lambda} \tag{31}$$

on the curve (28), computed at the branch points (an observation due to Tsarev).

Chapter II

EQUATIONS OF HYDRODYNAMICS OF SOLITON LATTICES

§5. The Bogolyubov-Whitham averaging method for field-theoretic systems and solition lattices. The results of Whitham and Hayes for Lagrangian systems

It is well known that the so-called averaging method of Bogolyubov and others has proved effective in many problems in the theory of non-linear oscillations. This method is used in the case when the unperturbed system has a certain number of cycles, exactly periodic solutions (the one-phase case), or of invariant tori, quasi-periodic solutions (the multi-phase case), that depend on several parameters. A particle in phase space close to this family of solutions will oscillate "rapidly" along the tori of this family and will drift "slowly" with the parameters; thus arises an averaged (over rapid oscillations) system of equations of drift with respect to the set of parameters on which these tori depend.

A number of classical works (see [6] for references) are devoted to the study of the first approximation to the slow drift, to estimates of the subsequent terms of the series expansion with respect to the small parameter, the ratio between the fast and the slow time scales, and the analysis of the resonant case.

In principle, various field-theoretic analogues of the averaging method are possible. The version we are discussing is not only a field-theoretic analogue of a Bogolyubov et al type averaging method, but also a non-linear analogue of the WKB method in quantum mechanics (or the eikonal method in optics). In this version the system itself is not perturbed; it has a family of exact solutions ("soliton lattices") of the form

$$\varphi(x, t) = \Phi(kx + \omega t + \tau^0, u^1, \ldots, u^N), \tag{1}$$

where $k = k(u)$, $\omega = \omega(u)$ are m-vectors, $\Phi(\tau_1, \ldots, \tau_m, u^1, \ldots, u^N)$ is a 2π-periodic function in each of the variables $\tau_1, \ldots, \tau_m$ that depends on parameters $u^1, \ldots, u^N$, and the vector $\tau^0 = (\tau_1^0, \ldots, \tau_m^0)$ is arbitrary. Solutions are sought of the original system that have the form (1) in the first approximation with respect to a natural small parameter ε equal to the ratio of the "fast" and "slow" spatio-temporal scales. Here the parameters of the solution are no longer constants, but slowly varying functions of the variables x, t, $u^i = u^i$ $(\varepsilon x, \varepsilon t)$. Under certain conditions on the family (1) of solutions of the original system, we obtain in the first approximation the so-called Whitham equations of slow modulation (equations of hydrodynamics of soliton lattices)

$$(2) \qquad u^i_T = v^i_j(u)\, u^j_X, \qquad i = 1, \ldots, N, \qquad T = \varepsilon t, \ X = \varepsilon x,$$

where the matrix $v^i_j(u)$ depends both on the original system and on the family of solutions (1). This theory originated with Whitham in the sixties (see [58], [93], [94]) and then its development was continued by Maslov (see [43]), Luke [89], Hayes [80], Ablowitz and Benney [63], Gurevich and Pitaevskii [14], [15], Flaschka, McLaughlin, and Forest (see [73]), and Dobrokhotov and Maslov [19], [69].

There are different procedures for deriving slow modulation equations, the equivalence of which has been rigorously established only in the one-phase spatially one-dimensional case. Let us describe these procedures briefly in the spatially one-dimensional case. Let an evolution system having a family of solutions of the type (1) have the form

$$(3) \qquad \varphi_t = K(\varphi, \varphi_x, \ldots, \varphi^{(n)})$$

(φ and K are vectors).

A. The non-linear analogue of the WKB method.
We look for formally asymptotic solutions of (3) in the form

$$(4) \qquad \varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \ldots,$$

where ε is a small parameter, the principal term φ_0 has the form (1) with slowly varying parameters $u^1, \ldots, u^N$, that is,

$$(5) \qquad \varphi_0(x, t) = \Phi(S(X, T)/\varepsilon, u(X, T)),$$

$X = \varepsilon x$, $T = \varepsilon t$ are the "slow" coordinates and time, and $S(X, T) = (S_1(X, T), \ldots, S_m(X, T))$ is an auxiliary smooth vector function; subsequent terms of the series (4) have the same form as (5), that is,

$$(6) \qquad \varphi_k(x, t) = \Phi_k(S(X, T)/\varepsilon, X, T), \qquad k = 1, 2, \ldots,$$

where $\Phi_k(\tau_1, \ldots, \tau_m, X, T)$ are certain functions 2π-periodic in $\tau_1, \ldots, \tau_m$, smoothly dependent on the parameters X and T. Substituting the series (4) into the system (3), we obtain the following relations:

$$(7) \qquad S_X = k(u(X, T)), \qquad S_T = \omega(u(X, T))$$

(which are obviously equivalent to weak convergence as $\varepsilon \to 0$ of the principal term $\varphi_0(x, t)$ to the exact solution of (1)[(1)] in the domain $|t| < \varepsilon^{-1}$)

[(1)]This is in the one-phase case; in the multi-phase case a more thorough analysis of resonant cases is required; see [68].

and a chain of linear equations

$$(8) \qquad \hat{L}\Phi_k = F_k, \qquad k = 1, 2, \ldots$$

(here $\Phi_k = \Phi_k(\tau, X, T)$), where the operator

$$(9) \qquad \hat{L} = \omega\partial_\tau - \frac{\partial K}{\partial\varphi} - \frac{\partial K}{\partial\varphi_x} k\partial_\tau - \frac{\partial K}{\partial\varphi_{xx}} (k\partial_\tau)^2 - \ldots$$

is the linearization of equation (3) on the solution (1), in which differentiation with respect to t, x is replaced by $\omega\partial_\tau$, $k\partial_\tau$; the residuals F_k are certain functions of $u^1, \ldots, u^N$, $\varphi_0, \ldots, \varphi_{k-1}$ and of their derivatives. The operator L acts in the space of 2π-periodic functions of the variables $\tau_1, \ldots, \tau_m$. τ and X, T enter its coefficients as parameters. Let us write down the explicit form of the first residual F_1 (we shall need it soon):

$$(10) \quad F_1 = \left\{-\partial_T + \frac{\partial K}{\partial\varphi_x}\partial_X + \frac{\partial K}{\partial\varphi_{xx}} [2(k\partial_\tau)\partial_X + (k_X\partial_\tau)] + \ldots\right\}\Phi(\tau, u(X, T)).$$

In (9), (10) the function K and its derivatives are calculated from the exact solution (1), that is, we replace φ, φ_x, ... by $\Phi(\tau, u(X, T))$, $k\partial_\tau\Phi(\tau, u(X, T))$ and so on, $k = k(u(X, T))$.

Let us observe that (7) implies the compatibility relations

$$(11) \qquad k_T = \omega_X,$$

where $k = k(u(X, T))$, $\omega = \omega(u(X, T))$ are m-vectors. These form a part of the slow modulation equations. The remaining equations for the functions $u(X, T)$ arise as solubility conditions for equation (8) with $k = 1$ in the space of 2π-periodic functions of the variables $\tau_1, \ldots, \tau_m$. For the equation to be soluble, orthogonality must hold between the first residual F_1 and the kernel of the adjoint operator $\hat{L}^*$:

$$(12) \qquad \int_0^{2\pi} \cdots \int_0^{2\pi} y_\alpha F_1 d^m\tau = 0, \qquad \alpha = 1, \ldots, n,$$

Here $y_\alpha = y_\alpha(\tau)$ are the zero modes of the operator $\hat{L}^*$ acting in the space of vector functions on an m-torus. For $m = 1$, $\hat{L}^*$ is an ordinary differential operator, so that the number of its zero modes is a priori finite (for several important examples we shall obtain these zero modes explicitly). For $m > 1$, the problem of determining the zero modes of the operator $\hat{L}^*$ is more complicated: their number changes as we pass through a resonance, see [17], [18].

Returning to the relations (11), (12), we note that they comprise a system of linear homogeneous equations with respect to the derivatives u^i_T, u^i_X (see the explicit form of the first residual F_1) with coefficients depending on u. This system can be solved with respect to $u^1_T, \ldots, u^N_T$, that is, it can be written down in the form of a system (2) of equations of hydrodynamic type, if the non-degeneracy condition

$$(13) \qquad rk\,(\partial k_a/\partial u^i) + rk\left(\int d^m\tau y_\alpha \frac{\partial \Phi}{\partial u^i}\right) = N$$

is satisfied. Thus, under this condition, the system of equations (11), (12) uniquely defines the slow modulation of the parameters $u^1, \ldots, u^N$.

Strictly speaking, the reasoning above is only applicable to the one-phase case, though the result, that is, the equations of slow modulation, is used in the multi-phase case as well (at least in the most important classes of examples). Concerning the precise formulations of the multi-phase non-linear WKB method, see [18], [69].

B. Lagrangian formulation of the averaging method (see [58]).
Let the original system have Lagrangian form

$$(14) \qquad \frac{\delta}{\delta q}\int L(q, q_x, q_t)\, dx\, dt = 0,$$

and let

$$(15) \qquad q = Q\,(kx + \omega t + \tau^0,\ u^1,\ \ldots,\ u^{2m})$$

be its family of invariant tori, $Q\,(\tau_1, \ldots, \tau_m, u^1, \ldots, u^{2m})$ a function on the m-torus depending on $2m$ parameters $u^1, \ldots, u^{2m}$, and τ^0 an arbitrary point on this torus. Let us define the averaged Lagrangian by setting

$$(16) \qquad \mathcal{L}(k, \omega, u) = (2\pi)^{-m}\int L\,(Q\,(\tau, u), kQ_\tau(\tau, u), \omega Q_\tau(\tau, u))\, d^m\tau$$

(the variables k, ω, u are considered here to be independent), where the integral is taken over the torus $0 \leqslant \tau_i \leqslant 2\pi$, $i = 1, \ldots, m$. Then the equations of slow modulation of the parameters $u = u(X, T)$ are obtained as the equations of extremals of the functional

$$(17) \qquad \mathcal{S}\,(k,\ \omega,\ u) = \int \mathcal{L}\,(k,\ \omega,\ u)\, dX\, dT$$

with the relation

$$(18) \qquad k_T = \omega_X$$

(see (11)). The explicit form of these equations is

$$(19) \qquad \partial_X \mathcal{L}_k + \partial_T \mathcal{L}_\omega = 0,$$

$$(20) \qquad \frac{\partial \mathcal{L}}{\partial u} = 0.$$

These last equations (20) give us the "dispersion relations" $k = k(u)$, $\omega = \omega(u)$ that hold for the solutions (15) (this is true under the condition of non-degeneracy of the Hessian $\mathcal{L}_{u^i u^j}$). Thus, the slow modulation equations for the functions $u^1(X, T), \ldots, u^{2m}(X, T)$ have the form (18), (19), where we have made the substitutions $k = k(u)$, $\omega = \omega(u)$. It is clear that this is a system of hydrodynamic type.

Following Hayes [80] and Whitham [58], we shall transform these equations to Hamiltonian form in Clebsch variables. For that we shall consider equations (18), (19) as equations for the vector functions $k = k(X, T)$, $\omega = \omega(X, T)$. These equations can be written in Lagrangian form by introducing the potential $S(X, T)$, where $S_X = k(u(X, T))$ $S_T = \omega(u(X, T))$ (in view of (18)) and by considering the Lagrangian $\hat{\mathscr{L}} = \hat{\mathscr{L}}(S_X, S_T)$, where

$$\hat{\mathscr{L}}(k, \omega) = \mathscr{L}(k, \omega, u(k, \omega)), \tag{21}$$

and $u = u(k, \omega)$ by the "dispersion relations" (20). Performing the Legendre transformation

$$(S, S_T) \to \left(S, J = \frac{\partial \hat{\mathscr{L}}}{\partial S_T}\right), \tag{22}$$

$$\mathscr{H} = \mathscr{H}(S_X, J) = JS_T - \hat{\mathscr{L}}(S_X, S_T), \qquad S_T = S_T(J, S_X), \tag{23}$$

we obtain the Hamiltonian form of the slow modulation equations with Hamiltonian $\mathscr{H}$ and canonical Poisson brackets

$$\{S_a(X), J_b(Y)\} = \delta_{ab}\delta(X - Y), \qquad a, b = 1, \ldots, m. \tag{24}$$

In the variables $k = S_X$, J, the canonical Poisson brackets assume the form

$$\{k_a(X), J_b(Y)\} = \delta_{ab}\delta'(X - Y). \tag{25}$$

In these variables the slow modulation equations (18), (19) are again in Hamiltonian form with the same Hamiltonian $\mathscr{H} = \mathscr{H}(k, J)$:

$$k_T = \partial_X \mathscr{H}_J, \tag{26}$$

$$J_T = \partial_X \mathscr{H}_k. \tag{26'}$$

Let us direct the reader's attention to the fact that these are Hamiltonian equations with Hamiltonian $\mathscr{H}$ of hydrodynamic type. The derivation of the Hamiltonian structure of the averaged equations given above is tied up with special Clebsch type variables that are characteristic of systems arising from non-degenerate Lagrangian equations, the Hamiltonian formalism of which is defined by the Lagrangian one in a unique way and admits a representation in canonical variables (here in order to change to these variables, half of them must be integrated with respect to X).

Let us clarify the meaning of the variables J, restricting ourselves for simplicity to the one-phase case ($m = 1$). We have

$$J = (2\pi)^{-1}\partial_\omega \oint L(Q, kQ_\tau, \omega Q_\tau)\, d\tau = (2\pi)^{-1}\oint Q_\tau \frac{\partial L}{\partial q_t}\, d\tau = (2\pi)^{-1}\oint p\, dq, \tag{27}$$

where we put $p = \partial L/\partial q_t$. Thus, J is the action variable canonically conjugate to the angle variable τ.

Let us also note that equations (19) can be obtained by the procedure of section A as conditions of orthogonality of the first residual F_1 to functions in the kernel of the operator $\hat{L}^*$ adjoint to the linearization (9), if we take the following functions in the kernel of $\hat{L}^*$: $Q_{\tau_1}, \ldots, Q_{\tau_m}$.

C. **The method of averaging conservation laws** (see [94]).
Let the evolutionary systems (3) have N local field integrals

$$(28) \qquad I_i = \int P_i(\varphi, \varphi_x, \varphi_{xx}, \ldots)\, dx, \quad i = 1, \ldots, N.$$

Let $Q_i = Q_i(\varphi, \varphi_x, \varphi_{xx}, \ldots)$ be the corresponding flux densities, that is, on solutions of the system (3) we have the relations

$$(29) \qquad \frac{\partial P_i}{\partial t} = \frac{\partial Q_i}{\partial x}, \quad i = 1, \ldots, N.$$

Let us consider the averaged quantities

$$(30) \qquad \bar{P}_i(u) = (2\pi)^{-m} \int P_i(\Phi(\tau, u), \ldots)\, d^m\tau = I_i,$$

$$(31) \qquad \bar{Q}_i(u) = (2\pi)^{-m} \int Q_i(\Phi(\tau, u), \ldots)\, d^m\tau.$$

Then the equations of slow modulation of the parameters $u^1, \ldots, u^N$ have the form

$$(32) \qquad \frac{\partial \bar{P}_i}{\partial T} = \frac{\partial \bar{Q}_i}{\partial X}, \quad i = 1, \ldots, N.$$

If $\det(\partial \bar{P}_i / \partial u^j) \neq 0$, we again obtain a system of hydrodynamic type.

How can this recipe for obtaining averaged equations be compared with the previous ones?[1] It can be shown that the gradients $\delta I_i / \delta\varphi(x)$ of the conservation laws are zero modes of the operator $\hat{L}^*$; moreover, conditions of the form (12) of orthogonality to these gradients coincide with the averaged conservation laws (32). From this it is not hard to conclude that the averaged equations are independent of the choice of the averaged conservation laws. It is clear that the form of the averaged equations is conserved under reduction of the N-dimensional family of invariant tori (1) to a smaller, $(N-q)$-dimensional family determined in (1) by fixing part of the integrals $I_{j_1}, \ldots, I_{j_q}$.

In a similar way we can formulate a recipe for deriving slow modulation equations that correspond to a small perturbation of equation (3) of the form

$$(33) \qquad \varphi_t = K(\varphi, \varphi_x, \ldots, \varphi^{(n)}) + \varepsilon K_1(\varphi, \varphi_x, \ldots).$$

In this case the quantities I_i are only approximately conserved, and for their densities we have the relations

$$(34) \qquad \frac{\partial P_i}{\partial t} = \frac{\partial Q_i}{\partial x} + \varepsilon R_i, \quad i = 1, \ldots, N,$$

[1] The following general result was obtained in this direction in [19]: if the integrals (28) are such that the difference $\partial P_i/\partial t - \partial Q_i/\partial x$ "is a multiple" of the original system, that is, if it has the form $Z_i(\varphi_t - K(\varphi, \varphi_x, \varphi_{xx}, \ldots))$, where $Z_i = Z_i(\partial_t, \partial_x)$ is a differential operator, then the relations (32) follow from the existence of the formal asymptotics (4),(5).

where $R_i = R_i(\varphi, \varphi_x, \ldots)$ are functions easily computable from P_i and K_1. Let us introduce the averaged quantities

$$\bar{R}_i(u) = (2\pi)^{-m} \int R_i(\Phi(\tau, u), \ldots)\, d^m\tau. \tag{35}$$

The slow modulation equations assume the form

$$\frac{\partial \bar{P}_i}{\partial T} = \frac{\partial \bar{Q}_i}{\partial X} + \bar{R}_i, \quad i = 1, \ldots, N. \tag{36}$$

They are obviously equivalent to a non-homogeneous system of hydrodynamic type

$$u^i_T = v^i_j(u)\, u^j_X + b^i(u), \quad i = 1, \ldots, N. \tag{37}$$

For an example, see §9 below.

§6. The Whitham equations of hydrodynamics of weakly deformed soliton lattices for Hamiltonian field-theoretic systems. The principle of conservation of the Hamiltonian structure under averaging

Let the original evolutionary system

$$\varphi_t(x) = K(\varphi, \varphi_x, \ldots) = \{\varphi(x), H[\varphi]\}, \tag{1}$$

$\varphi = (\varphi^\alpha)$, be Hamiltonian with respect to local translation-invariant field-theoretic Poisson brackets

$$\{\varphi^\alpha(x), \varphi^\beta(y)\} = \sum_{k=0}^{M} B_k^{\alpha\beta}(\varphi(x), \varphi'(x), \ldots, \varphi^{(n_k)}(x))\, \delta^{(k)}(x-y) \tag{2}$$

(we are considering now only the spatially one-dimensional case), where the Hamiltonian $H[\varphi]$ is a local field functional

$$H[\varphi] = \int h(\varphi(x), \varphi'(x), \ldots)\, dx. \tag{3}$$

Moreover, let an N-parameter family of exact quasi-periodic solutions of equations (1) be given, having the form

$$\varphi(x, t) = \Phi(kx + \omega t + \tau^0, u^1, \ldots, u^N), \quad k = k(u), \quad \omega = \omega(u), \tag{4}$$

$\Phi(\tau_1, \ldots, \tau_m, u^1, \ldots, u^N)$ being 2π-periodic in $\tau_1, \ldots, \tau_m$. Let us assume that the non-degeneracy condition

$$rk\,(\partial k_\alpha / \partial u^i) = m \tag{5}$$

holds. It turns out that under some additional assumptions, system (1) averaged according to (4), that is, the equations of slow modulation of $u^1, \ldots, u^N$, inherits the Hamiltonian structure. This means that this system is also Hamiltonian with respect to the so-called averaged Poisson brackets, which are uniquely defined by (2), (4); moreover, the averaged brackets are always of hydrodynamic type.

Let us proceed now to precise formulations. Let us assume that the system (1) has N pairwise commuting local integrals

$$(6) \qquad I_i[\varphi] = \int P_i(\varphi(x), \varphi'(x), \dots)\, dx, \qquad i = 1, \dots, N,$$

$$(6') \qquad \{I_i, I_j\} = 0.$$

One of these is the Hamiltonian, say $I_1 = H$. Let us assume also that on solutions (4) the following relations hold:

$$(7) \qquad I_i[\Phi] = u^i, \qquad i = 1, \dots, N$$

(these are conditions for the choice of parameters $u^1, \dots, u^N$). Let us also assume that the coefficients $B_k^{\alpha\beta}$ of the bracket and the densities P_i of the integrals are polynomials (or analytic functions) of $\varphi, \varphi', \dots, \varphi^L$ for some L. Let us describe the procedure of constructing an averaged bracket. Let us consider pairwise brackets of densities of the integrals (6):

$$(8) \qquad \{P_i(\varphi(x), \varphi'(x), \dots), P_j(\varphi(y), \varphi'(y), \dots)\} = \\ = \sum_k A_k^{ij}(\varphi(x), \varphi'(x), \dots)\, \delta^{(k)}(x - y), \qquad i, j = 1, \dots, N.$$

By commutation relations (6′) we have

$$(9) \qquad \int A_0^{ij} dx \equiv 0 \Leftrightarrow A_0^{ij}(\varphi(x), \varphi'(x), \dots) = \partial_x Q^{ij}(\varphi(x), \varphi'(x), \dots).$$

Let us introduce a metric $g^{ij}(u)$ and connection $b_k^{ij}(u)$ (see §2 above), by setting

$$(10) \qquad g^{ij}(u) = (2\pi)^{-m} \int A_1^{ij}(\Phi, \Phi' \dots)\, d^m\tau,$$

$$(11) \qquad b_k^{ij}(u) = \frac{\partial}{\partial u^k} (2\pi)^{-m} \int Q^{ij}(\Phi, \Phi', \dots)\, d^m\tau$$

(here $\Phi = \Phi(\tau, u)$, $\Phi' = k(u)\, \partial_\tau \Phi(\tau, u)$ and so on).

***Theorem* 1.** 1) *Under the above assumptions*

$$(12) \qquad \{u^i(X), u^j(Y)\} = g^{ij}(u(X))\, \delta'(X - Y) + \\ + b_k^{ij}(u(X))\, u_x^k \delta(X - Y), \qquad i, j = 1, \dots, N$$

defines an "averaged" Poisson bracket (of hydrodynamic type).

2) *In the coordinates $u^1, \dots, u^N$ the metric $g^{ij}(u)$ is strongly Liouville,*

$$(13) \qquad g^{ij}(u) = \gamma^{ij}(u) + \gamma^{ji}(u), \quad b_k^{ij}(u) = \frac{\partial}{\partial u^k} \gamma^{ij}(u),$$

$$(13') \qquad \gamma^{ij}(u) = (2\pi)^{-m} \int Q^{ij}(\Phi, \Phi', \dots)\, d^m\tau + \gamma_0^{ij},$$

where γ_0^{ij} is a constant skew-symmetric matrix.

3) *The equations of slow modulation of the parameters* $u^1, \ldots, u^N$ *constructed according to section C of* §5 *are Hamiltonian with respect to the bracket* (12) *with Hamiltonian*

$$(14)\qquad \mathcal{H} = \int u^1 dX = \bar{H}, \quad u^1 = (2\pi)^{-m}\int h(\Phi, \Phi', \ldots)\, d^m\tau, \qquad h = P_1.$$

Proof. Let us define a bracket, depending on a parameter ε, on a larger space of fields $\varphi(x, X)$ according to the following rule: in all the formulae (2), (8), ... we substitute

$$(15)\qquad \begin{cases} \varphi(x) \to \varphi(x, X), \\ \partial_x \to \partial_x + \varepsilon\partial_X, \\ \delta(x-y) \to \delta(x-y)\,\delta(X-Y). \end{cases}$$

In particular, we obtain

$$(16)\qquad \{\varphi^\alpha(x, X), \varphi^\beta(y, Y)\}_\varepsilon = \sum_{k=0}^{M} B_k^{\alpha\beta}(\varphi(x, X), \varphi_x(x, X) + $$
$$+ \varepsilon\varphi_X(x, X), \ldots)[\delta^{(k)}(x-y)\,\delta(X-Y) + \varepsilon k\delta^{(k-1)}(x-y)\,\delta'(X-Y) + \ldots$$
$$\ldots + \varepsilon^k\delta(x-y)\,\delta^{(k)}(X-Y)].$$

The right hand side is a polynomial (or a convergent series) in ε. The operation $\{\cdot, \cdot\}_\varepsilon$ can be extended by linearity to polynomials (or convergent power series) in ε with coefficients that are functionals of φ.

Lemma 1. *The operation* $\{\cdot, \cdot\}_\varepsilon$ *defines a Poisson bracket that depends on the parameter* ε.

Proof. Let us consider a different bracket $\{\cdot, \cdot\}_0$ on the space of fields $\varphi(x, X)$ by setting

$$\{\varphi^\alpha(x, X), \varphi^\beta(y, Y)\}_0 = \sum_{k=0}^{M} B_k^{\alpha\beta}(\varphi(x, X), \varphi_x(x, X), \ldots)\,\delta^{(k)}(x-y)\,\delta(X-Y)$$

(a "direct sum" of brackets (2)). It is clear that it has all the properties of a Poisson bracket. But $\{\cdot, \cdot\}_\varepsilon$ is obtained from $\{\cdot, \cdot\}_0$ by a linear change of variables x, X,

$$(x,\ X) \to (x,\ \varepsilon x + X).$$

The lemma is proved.

In particular, for densities of integrals (6) the bracket (16) has the form

$$(17)\qquad \{P_i(\varphi(x, X), \varphi_x(x, X) + \varepsilon\varphi_x(x, X), \ldots), P_j(\varphi(y, Y), \varphi_y(y, Y) + $$
$$+ \varepsilon\varphi_Y(y, Y), \ldots)\}_\varepsilon = \sum_k A_k^{ij}(\varphi(x, X), \varphi_x(x, X), \ldots)\,\delta^{(k)}(x-y)\,\delta(X-Y) + $$
$$+ \varepsilon\,[\sum_k \tilde{A}_k^{ij}(\varphi, \varphi_x, \varphi_X, \ldots)\,\delta^{(k)}(x-y)\,\delta(X-Y) + $$
$$+ \sum_{k\geqslant 1} kA_k^{ij}(\varphi, \varphi_x, \ldots)\,\delta^{(k-1)}(x-y)\,\delta'(X-Y)] + O(\varepsilon^2),$$

where

$$\tilde{A}_k^{ij} = \frac{\partial A_k^{ij}}{\partial\varphi_x}\varphi_X + 2\frac{\partial A_k^{ij}}{\partial\varphi_{xx}}\varphi_{xX} + \ldots, \qquad k = 0, 1, \ldots$$

Let us observe that from the relations $A_0^{ij} = \partial_x Q^{ij}$ (see (9)) it follows that

$$\bar{A}_0^{ij} = \partial_X \bar{Q}^{ij}(\varphi, \varphi_x, \varphi_{xx}, \ldots). \tag{18}$$

Let us consider the Lie subalgebra of functionals in $\varphi(x, X)$ generated by the densities of the integrals $P_i(\varphi, \varphi_x + \varepsilon\varphi_X, \ldots)$ relative to the bracket $\{\cdot, \cdot\}_1 = \varepsilon^{-1}\{\cdot, \cdot\}_\varepsilon$. By (17), (18), we shall have for functionals

$$u_\varepsilon^i(X) = \int P_i(\varphi(x, X), \varphi_x(x, X) + \varepsilon\varphi_X(x, X), \ldots)\, dx, \quad i = 1, \ldots, N, \tag{19}$$

in this subalgebra:

$$\{u_\varepsilon^i(X), u_\varepsilon^j(Y)\}_1 = \partial_X \bar{Q}^{ij}(\varphi, \varphi_x, \varphi_{xx}, \ldots)\,\delta(X - Y) + \bar{A}_1^{ij}(\varphi, \varphi_x, \varphi_{xx}, \ldots)\,\delta'(X - Y) + O(\varepsilon), \tag{20}$$

where the bar stands for an integral with respect to x. From skew-symmetry and the Jacobi identity for the bracket $\{\cdot, \cdot\}_1$ we obtain skew-symmetry and the Jacobi identity for the principal term $\partial_X \bar{Q}^{ij}\delta(X - Y) + \bar{A}_1^{ij}\delta'(X - Y)$ in (20). It remains only to observe that on functions $\varphi(x, X)$ of the form

$$\varphi(x, X) = \Phi(kx + \omega t + \tau^0, u(X)), \quad k = k(u(X)) \tag{21}$$

(t is fixed) we have

$$u_\varepsilon^i(X) = u^i(X) + O(\varepsilon),$$
$$\partial_X \bar{Q}^{ij} = b_k^{ij}(u(X))\, u_X^k,$$
$$\bar{A}_1^{ij} = g^{ij}(u(X))$$

(we have replaced averages over x by averages over the torus, using the non-degeneracy condition (5)). Therefore, passing in (20) to the limit $\varepsilon \to 0$, we obtain the bracket (12). Therefore (12) is a Poisson bracket.

The fact that the metric is Liouville in the coordinates $u^1, \ldots, u^N$ is obvious from (11). Let us prove that it is strongly Liouville. First of all, to linear changes of the parameters $u^i \to \tilde{u}^i = c_j^i u^j$ there correspond identical linear transformations of the densities P_i. The matrix $\gamma^{ij}(u)$ then transforms as a tensor. Therefore, the whole construction is invariant with respect to linear changes of coordinates. Now let $i_1, \ldots, i_p, j_1, \ldots, j_q$, $p + q = N$, be a subdivision of the set $1, \ldots, N$ into two disjoint subsets. Let us use the procedure of averaging the bracket (2) over a p-dimensional family of m-dimensional tori, identified in (4) by the equations

$$u^{j_1} = \text{const}, \ \ldots, \ u^{j_q} = \text{const}. \tag{22}$$

For the fields $u^{i_1}, \ldots, u^{i_p}$ we obtain a Liouville-Poisson bracket of hydrodynamic type defined by the matrix

$$\tilde{\gamma}^{i_k i_l} = \gamma^{i_k i_l}(u^{i_1}, \ldots, u^{i_p}, u^{j_1} = \text{const}, \ldots, u^{j_q} = \text{const}). \tag{23}$$

But that is exactly what being strongly Liouville means.

Furthermore, integrating (8) with $j = 1$ with respect to y, we obtain

$$(24)\qquad \frac{\partial P_i(\varphi(x), \ldots)}{\partial t} \equiv \{P_i(\varphi(x), \ldots), I_1\} = \frac{\partial Q^{i1}(\varphi(x), \ldots)}{\partial x}, \; i = 1, \ldots N.$$

Therefore the averaged equations have the following form:

$$(25)\qquad u_T(X) \equiv \frac{\partial}{\partial T}\bar{P}_i = \frac{\partial \bar{Q}^{i1}}{\partial X} \equiv b_k^{i1}(u(X))u_X^k = \left\{u^i(X), \int u^1(Y)\,dY\right\}.$$

The theorem is proved.

Remark 1. The solutions (4) form general invariant tori for the family of pairwise-commuting Hamiltonian systems with Hamiltonians $I_1, \ldots, I_N$:

$$(26)\qquad \frac{\partial \varphi(x)}{\partial t_k} = \{\varphi(x), I_k\}, \qquad k = 1, \ldots, N$$

(when $k = 1$ we have the original system (1)). The equations of slow modulation of the parameters $u^1, \ldots, u^N$ for these systems have the form

$$(27)\qquad u^i_{T_k} = b_s^{ik}(u)\,u_X^s, \qquad i = 1, \ldots, N, \quad T_k = \varepsilon t_k.$$

All these equations are Hamiltonian with respect to the bracket (2) with Hamiltonian $\int u^k dX$, and commute in pairs.

If the original Hamiltonian system (1) is equivalent to a Lagrangian system, then from the Whitham–Hayes theorem (see §5, section C above) it follows that the averaged bracket is non-degenerate and, in the coordinates $k_1, \ldots, k_m, J_1, \ldots, J_m$, where the J_α are action variables canonically conjugate to the angles τ_a on the tori (4), can be reduced to the constant form

$$(28)\qquad \{k_a(X), J_b(Y)\} = \delta_{ab}\delta'(X - Y).$$

The authors have analysed a number of examples, enabling them to conjecture in a more general case that if $N = 2m + k$, and among the integrals $I_1, \ldots, I_N$ exactly k integrals form the annihilator of the bracket (2), and if moreover the invariant manifold (4) is divided by the level surfaces of the annihilators

$$(29)\qquad I_{j_1} = \text{const}, \ldots, I_{j_k} = \text{const}$$

and of the "wave numbers"

$$(30)\qquad k_1 = \text{const}, \ldots, k_m = \text{const}$$

into a family of completely integrable Hamiltonian systems, then in the variables $u^{j_1}, \ldots, u^{j_k}, k_1, \ldots, k_m, J_1, \ldots, J_m$ the averaged bracket is constant.

Example 1. Let us consider the non-linear wave equation

$$(31)\qquad q_{tt} - q_{xx} + V'(q) = 0.$$

It is Hamiltonian in the variables q, $p = q_t$ with brackets

$$\{q(x),\ p(y)\} = \delta(x - y), \tag{32}$$

the rest of the brackets being zero, and with Hamiltonian

$$H = \int \left[\frac{1}{2}(p^2 + q_x^2) + V(q)\right] dx. \tag{33}$$

In addition to the energy integral $I_1 = H$, there is also the momentum integral

$$I_2 = P = \int p q_x dx \tag{34}$$

(the generator of translations). The family of one-phase (periodic) solutions has the form

$$q(x,\ t) = Q(kx + \omega t + \tau^0), \qquad Q(\tau + 2\pi) = Q(\tau), \tag{35}$$

$$(\omega^2 - k^2)^{1/2} dQ = [2(E - V(Q))]^{1/2} d\tau, \tag{35'}$$

where the constant E of integration is related to the wave number k and the frequency ω (arbitrary parameters) by the dispersion relation

$$(\omega^2 - k^2)^{1/2} \oint dQ/\sqrt{2(E - V(Q))} = 2\pi \tag{36}$$

(integration is performed over the entire domain of oscillation $V(Q) \leqslant E$). From (33), (34), together with (36), the quantities k, ω can be expressed in terms of $u^1 = I_1 = H$, $u^2 = I_2 = P$. Let us compute the averaged Poisson bracket. Setting

$$P_1(x) = \frac{1}{2}(p^2 + q_x^2) + V(q), \quad P_2(x) = p(x) q'(x), \tag{37}$$

we have

$$\{P_1(x),\ P_1(y)\} = \{P_2(x),\ P_2(y)\} = (q'p)'\,\delta(x - y) + 2q'p\delta'(x - y),$$

$$\{P_1(x),\ P_2(y)\} = (p^2 + q'^2)\,\delta'(x - y) + [(p^2 + q'^2)/2 + V(q)]'\,\delta(x - y),$$

$$\{P_2(x),\ P_1(y)\} = (p^2 + q'^2)\,\delta'(x - y) + [(p^2 + q'^2)/2 - V(q)]'\,\delta(x - y).$$

In these expressions $p = p(x)$, $q = q(x)$. Therefore the matrix $(\gamma^{ij}(u))$ that defines the bracket in Liouville coordinates $u^1 = H$, $u^2 = P$ has the form

$$(\gamma^{ij}) = \begin{bmatrix} u^2 & u^1 \\ u^1 - 2\Delta & u^2 \end{bmatrix}, \tag{38}$$

$$\Delta = \Delta(u^1,\ u^2) = \overline{V(Q)}. \tag{39}$$

The averaged equations are

$$u^1_T = u^2_X, \quad u^2_T = (u^1 - 2\Delta)_X. \tag{40}$$

They are equivalent to the equations of one-dimensional relativistic hydrodynamics

(41) $$\frac{\partial T^{ij}}{\partial X^j} = 0, \quad X^1 = T, \quad X^2 = X$$

in two-dimensional space-time ($c = 1$), where the energy-momentum tensor has the form

(42) $$(T^{ij}) = \begin{bmatrix} u^1 & -u^2 \\ -u^2 & u^1 - 2\Delta \end{bmatrix}.$$

It is obtained by averaging from the energy-momentum tensor of the original system (31):

(42') $$T^{ij} = \overline{t^{ij}}, \quad (t^{ij}) = \begin{bmatrix} P_1 & -P_2 \\ -P_2 & P_1 - 2V \end{bmatrix}.$$

Averaging the conservation law

$$\frac{\partial t^{ij}}{\partial x^j} = 0, \quad x^1 = t, \quad x^2 = x,$$

(using the procedure of section C of §5) we immediately obtain (41).

The quantity $2\Delta = \mathscr{E} - \mathscr{P}$ is the metric trace of the energy-momentum tensor in the Minkowski metric, $\mathscr{P}$ is the pressure and $\mathscr{E}$ is the energy density in the travelling coordinate frame, in which the tensor T_j^i is diagonal and has the form

(43) $$(T_j^i) = \begin{bmatrix} \mathscr{E} & 0 \\ 0 & -\mathscr{P} \end{bmatrix}, \quad T_i^i = \mathscr{E} - \mathscr{P}.$$

The state equation that completes (39) and connects the components of (T_j^i) is determined from (39). By Lorentz-invariance, these relations apply only to the invariants $\mathscr{E}$, $\mathscr{P}$ of the tensor T_j^i. Flat coordinates that reduce the bracket (36) to a constant form are k, J (by Hayes' construction; see above), where

(44) $$J = (2\pi)^{-1} \oint p\, dq = u^2/k.$$

Their brackets have the form

(45) $$\{k(X),\ J(Y)\} = \delta'(X - Y),$$

all the rest being zero. The metric $g^{ij}(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is indefinite.

Remark. The symmetry conditions $g^{is}b_s^{jk} = g^{js}b_s^{ik}$ on the connection $b_k^{ij} = \partial\gamma^{ij}/\partial u^k$ give rise to a non-trivial relation for the function $\Delta = \Delta(u^1, u^2)$:

$$u^2 \frac{\partial \Delta}{\partial u^1} + (u^1 - \Delta) \frac{\partial \Delta}{\partial u^2} = 0.$$

This defines the implicit state equation

$$(46) \qquad \Delta = f\,((u^1 - \Delta)^2 - (u^2)^2), \qquad 2\Delta = \mathscr{E} - \mathscr{P},$$

where the function f is determined by the potential V of the original equation.

The Hamiltonian structure of the Whitham equations (41) in this particular case in Clebsch type variables k, J was established for the first time in [58], [80] (using the Lagrangian structure of equation (31) and the methods of §5, part B), while the inherent isomorphism with relativistic hydrodynamics was found in [43].

Example 2. KdV type systems have the form

$$(47) \qquad \varphi_t = \partial_x \frac{\delta H}{\delta\varphi(x)}, \qquad H = \int [\varphi_x^2/2 + V(\varphi)]\,dx$$

with the Gardner-Zakharov-Faddeev bracket

$$(48) \qquad \{\varphi(x), \varphi(y)\} = \delta'(x-y).$$

There are three integrals in involution:

$$(49) \qquad I_0 = \int \varphi\,dx, \text{ the annihilator of the bracket,}$$

$$(50) \qquad I_1 = \int \varphi^2/2dx, \text{ the momentum, and}$$

$$(51) \qquad I_2 = H = \int [\varphi_x^2/2 + V(\varphi)]dx, \text{ the Hamiltonian.}$$

The family of one-phase exact solutions is given in the form $\varphi = \Phi(kx + \omega t)$, $\Phi(\tau + 2\pi) = \Phi(\tau)$ and depends on three parameters, where

$$(52) \qquad k\,d\Phi = \sqrt{2V(\Phi) - \frac{\omega}{k}\Phi^2 + a\Phi + b}\,d\tau,$$

the constants k, ω, a, b being connected by one relation

$$(53) \qquad k \oint d\Phi \Big/ \sqrt{2V(\Phi) - \frac{\omega}{k}\Phi^2 + a\Phi + b} = 2\pi$$

(the integral is taken over a whole cycle of oscillation). These three parameters can be expressed in terms of $u^0 = I_0$, $u^1 = I_1$, $u^2 = I_2$. The Poisson bracket in Liouville coordinates u^0, u^1, u^2 is defined by the matrix $(\gamma^{ij}(u))$,

$$(54) \qquad (\gamma^{ij}) = \begin{pmatrix} 1/2 & u^0 & -cu^0 - a \\ 0 & u^1 & -cu^1 - b \\ 0 & u^2 & -cu^2 - bc + a^2/2 \end{pmatrix},$$

where $c = -\omega/k$. The calculations are similar to those in the previous example.

Let us introduce the quantities

$$(55) \qquad p_+ = \overline{\varphi^2}/2 = u^1, \quad p_- = \overline{(\varphi - \overline{\varphi})^2}/2 = u^1 - (u^0)^2/2.$$

By (54), their Poisson brackets have the form

$$(56)\qquad \{p_+, p_-\} = 0,\quad \{p_\pm(X), p_\pm(Y)\} = 2p_\pm(X)\,\delta'(X-Y) + p'_\pm\delta(X-Y).$$

This shows that both variables are similar to momentum ("transport momentum" p_+ and "fluctuation momentum" p_-); thus an averaged KdV type system represents an interesting example of "two fluid" hydrodynamics.

It can be shown that the flat coordinates that reduce the bracket to constant form are k, J, $u^0 = \overline{\varphi}$, where

$$(57)\qquad J = -f_a + f_a^2/f_b,\quad f = f(a,b,c) = \frac{1}{2\pi}\oint\sqrt{2V(\Phi)+c\Phi^2+a\Phi+b}\,d\Phi.$$

The corresponding matrix (g^{ij}) in these coordinates has the form

$$(58)\qquad (g^{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Other examples will be considered in the next section.

The principle of conservation of Hamiltonian structure under averaging, which we discussed in detail for the simplest problem of modulation of initial conditions, holds also in some problems in which not only the initial conditions but also the system itself is perturbed:

$$(59)\qquad \varphi_t(X) = K(\varphi, \varphi_x, \dots) + \varepsilon K_1(\varphi, \varphi_x, \dots),$$

if the perturbation is "conservative". In this case we shall obtain non-homogeneous systems of hydrodynamic type that are Hamiltonian with respect to non-homogeneous brackets of hydrodynamic type (see §3 above). Let us make more precise what we mean by a conservative perturbation.

We shall consider conservative perturbations of a particular type generated by infinitesimal deformations of the Poisson bracket (2):

$$(60)\qquad \{\varphi^\alpha(x), \varphi^\beta(y)\}_\varepsilon = \{\varphi^\alpha(x), \varphi^\beta(y)\} + \varepsilon\{\varphi^\alpha(x), \varphi^\beta(y)\}_1,$$

where $\{\cdot, \cdot\}$ is the unperturbed Poisson bracket, and $\{\cdot, \cdot\}_1$ is the cocycle defining the deformation. Here, by definition, we require the operation (60) to be skew-symmetric and satisfy the Jacobi identity in the linear approximation in ε. The cocycle $\{\cdot, \cdot\}_1$ is also required to be local, that is, of the form

$$(61)\qquad \{\varphi^\alpha(x), \varphi^\beta(y)\}_1 = \sum_{k=0}^{M} C_k^{\alpha\beta}(\varphi(x), \varphi'(x), \dots, \varphi^{(n_k)}(x))\,\delta^{(k)}(x-y).$$

The perturbed system is of the form

$$(62)\qquad \varphi_t(x) = K(\varphi, \varphi_x, \dots) + \varepsilon K_1(\varphi, \varphi_x, \dots) \equiv \{\varphi(x), H\} + \varepsilon\{\varphi(x), H\}_1.$$

For finite-dimentional brackets, perturbations of this type were studied in [36], [88]. A particular case is that of deformations generated by infinitesimal Bäcklund transformations

$$\varphi^{\alpha}(x) \to \varphi^{\alpha}(x) + \varepsilon f^{\alpha}(\varphi(x), \varphi'(x), \ldots). \tag{63}$$

Then the cocycle $\{\cdot, \cdot\}_1$ is cohomologous to zero.(1) The corresponding perturbations (59) are obtained by substituting (63) into the unperturbed equations. In this case the construction of asymptotic solutions of the perturbed equation is reduced to the unperturbed case.

Let all the conditions of Theorem 1 hold for the unperturbed system (1), its family of solutions (4), and the integrals (6). Let the perturbation be generated by an infinitesimal deformation of a Poisson bracket, that is, it is of the form (60). Let us define the corresponding averaged (non-homogeneous) Poisson bracket. Let

$$\{P_i(\varphi(x), \varphi'(x), \ldots), P_j(\varphi(y), \varphi'(y), \ldots)\}_1 = \sum_k D_k^{ij}(\varphi(x), \varphi'(x), \ldots)\,\delta^{(k)}(x-y). \tag{64}$$

Let us set

$$h^{ij}(u) = (2\pi)^{-m}\int D_1^{ij}(\Phi, \Phi', \ldots)\,d^m\tau, \tag{65}$$

where $\Phi = \Phi(\tau, u)$, $\Phi' = k(u)\,\partial_\tau\Phi(\tau, u)$ and so on, and the quantities $g^{ij}(u)$, $b_k^{ij}(u)$ are determined from the unperturbed bracket by means of (10), (11).

Theorem 2. *Under the above assumptions, the operation*

$$\{u^i(X), u^j(Y)\} = g^{ij}(u(X))\,\delta'(X-Y) + b_k^{ij}(u(X))\,u_X^k\,\delta(X-Y) + h^{ij}(u)\,\delta(X-Y) \tag{66}$$

defines a non-homogeneous Poisson bracket of hydrodynamic type. The equations of slow modulation of the parameters $u^1, \ldots, u^N$ are Hamiltonian, with Hamiltonian $\mathcal{H} = \int u^1 dX = \overline{H}$.

The proof is similar to that of Theorem 1.

Since the equations $k_T = \omega_X$ are preserved in form under an arbitrary Hamiltonian and an arbitrary perturbation, the variables $k_1, \ldots, k_m$ lie in the annihilator of the finite-dimensional bracket h^{ij}. Therefore the Lie algebra that defines a non-homogeneous Poisson bracket of hydrodynamic type in flat coordinates (see § 3 above) has an m-dimensional centre. In a number of cases this is already sufficient for the conclusion that the bracket (66) can be reduced to a constant one.

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(1)The study of the corresponding cohomology theory was initiated in [8]. It appears that in the one-component case, the two-dimensional cohomology is zero, that is, every deformation of a bracket is generated by an infinitesimal Bäcklund transformation.

Let us now consider briefly the spatially multi-dimensional case, $x = (x^\alpha)$, $\alpha = 1, \ldots, d$. Let an evolutionary system

$$\varphi_t(x) = K(\varphi, \varphi_x, \ldots) \equiv \{\varphi(x), H\}, \tag{67}$$

be given, and let it have a family of invariant tori, that is, periodic or quasi-periodic solutions of the form

$$\varphi(x, t) = \Phi(k_\alpha x^\alpha + \omega t + \tau^0, u^1, \ldots, u^N), \tag{68}$$

where the m-vectors k_α have the form $k_\alpha(u^1, \ldots, u^N)$ and the rest of the notation is as in (2)–(4). Furthermore, let pairwise commuting integrals of the system (67),

$$I_i[\varphi] = \int P_i(\varphi(x), \varphi'(x), \ldots)\, d^d x, \quad i = 1, \ldots, N,$$

be given and assume in addition that the relations (7) hold for the family (68). In order to construct the averaged equations and determine their Hamiltonian structure, let us consider Poisson brackets for pairs of densities of these integrals:

$$\begin{aligned} &\{P_i(\varphi(x), \ldots), P_j(\varphi(y), \ldots)\}_1 = \\ &= A_0^{ij}(\varphi(x), \ldots)\,\delta(x-y) + A_1^{ij\alpha}(\varphi(x), \ldots)\,\delta_\alpha(x-y) + \ldots, \end{aligned} \tag{69}$$

where the dots denote terms containing higher derivatives of the delta function. In view of pairwise commutativity, we have

$$A_0^{ij}(\varphi(x), \ldots) = \frac{\partial}{\partial x^\alpha} Q^{ij\alpha}(\varphi(x), \ldots). \tag{70}$$

Let us define averaged brackets in the space of fields $u^1(X), \ldots, u^N(X)$, $X = (X^\alpha)$, $X^\alpha = \varepsilon x^\alpha$, by setting

$$\{u^i(X), u^j(Y)\} = g^{ij\alpha}(u(X))\,\delta_\alpha(X-Y) + b_k^{ij\alpha}(u(X))\,u_\alpha^k\,\delta(X-Y), \tag{71}$$

where $u_\alpha^k = \partial u^k/\partial X^\alpha$ and

$$g^{ij\alpha}(u) = \overline{A_1^{ij\alpha}}, \quad b_k^{ij\alpha}(u) = \frac{\partial}{\partial u^k}\overline{Q^{ij\alpha}}. \tag{72}$$

As above, it is proved that (71) is a Poisson bracket. Equations (67) averaged over the tori (68) are Hamiltonian with Hamiltonian $\overline{H} = \int \overline{H}\, d^d X$.

Let us emphasize that the local nature of conservation laws is essential in constructing the Hamiltonian formalism of averaged equations. Therefore, our approach to the description of the Hamiltonian formalism of spatially multi-dimensional system is as yet inapplicable to integrable systems of Kadomtsev–Petviashvili type, for which conservation laws are non-local. In this case, equations averaged over algebraic-geometric solutions were first obtained by Krichever in [39].

If (67) is equivalent to a Lagrangian system, then, as in the spatially one-dimensional case in the variables $k_{\alpha 1}, \ldots, k_{\alpha m}, J_1, \ldots, J_m$, $\alpha = 1, \ldots, d$, the averaged bracket reduces to constant form:

$$\{k_{\alpha a}(X), J_b(Y)\} = \delta_{ab}\delta_\alpha(X-Y), \quad \alpha = 1, \ldots, d, \quad a, b = 1, \ldots m,$$

all the other brackets being zero. In other words, the matrices $g^{ij\alpha}$ (of order $m(d+1)$) are degenerate for each α if $d > 1$ and have rank $2m$ (and this property is invariant with respect to linear changes of spatial variables). However, a *weak non-degeneracy* condition holds (see §2 above): the intersection of the kernels of all the matrices $g^{ij\alpha}$ is empty and their images generate the whole $m(d+1)$-dimensional space.

Example 3. Let us consider the multi-dimensional non-linear wave equation

$$q_{tt} - \Delta q + V'(q) = 0, \tag{73}$$

where $\Delta = \sum (\partial_\alpha)^2$ is the Laplacian in the spatial variables (x_α), $\alpha = 1, \ldots, d$. The energy integrals I_i, $i = 0, 1, \ldots, d$, have the form

$$I_0 = H \equiv \int P_0 d^d x = \int [p^2/2 + (\nabla q)^2/2 + V(q)]\, d^d x, \tag{74}$$

$$I_\alpha \equiv \int P_\alpha d^d x = \int p q_\alpha d^d x, \tag{75}$$

where

$$p = q_t, \quad \{q(x), p(y)\} = \delta(x-y), \tag{76}$$

the rest of the brackets being zero. The one-phase solutions have the form

$$q(x, t) = Q(k_\alpha x^\alpha + \omega t), \quad Q(\tau + 2\pi) = Q(\tau), \tag{77}$$

where the function Q is as in (35′) with $k^2 = \sum k_\alpha^2$, while the parameters $\omega, k_1, \ldots, k_\alpha, E$ are related by one dispersion relation that is the same as (36). The averaged brackets have the (multi-dimensional) Liouville form:

$$\{u^i(X), u^j(Y)\} = [\gamma^{ij\alpha}(u(Y)) + \gamma^{ji\alpha}(u(X))]\,\delta_\alpha(X-Y), \tag{78}$$
$$i, j = 0, 1, \ldots, d,$$

where $u^0 = I_0 = H$, $u^\alpha = I_\alpha$,

$$\gamma^{00\alpha} = u^\alpha, \quad \gamma^{0\beta\alpha} = u^0\delta^{\alpha\beta}, \tag{79}$$

$$\gamma^{\beta 0\alpha} = -u^0\delta^{\alpha\beta} + \Delta_{\alpha\beta}(u), \quad \gamma^{\beta\gamma\alpha} = u^\beta\delta^{\gamma\alpha} + u^\gamma\delta^{\beta\alpha}, \tag{79′}$$

where the indices $\alpha, \beta, \gamma = 1, 2, \ldots, d$, and the matrix $\Delta_{\alpha\beta}(u)$ is of the form

$$\Delta_{\alpha\beta}(u) = \overline{p^2 + q_\alpha q_\beta}. \tag{80}$$

The averaged equations are

$$\frac{\partial T^{ij}}{\partial X^j} = 0, \quad X^0 = T = \varepsilon t, \quad X^\alpha = \varepsilon x^\alpha, \tag{81}$$

where the energy-momentum tensor $T^{ij} = T^{ij}(u)$ is obtained by averaging the energy-momentum tensor of (73) as in Example 1 above.

Let us introduce the function $F(E)$ by setting

$$F(E) = \frac{1}{2\pi} \oint [2(E - V(Q))]^{1/2}\, dQ \tag{82}$$

(the integral is taken over a whole cycle of oscillations). Then the "dispersion relation" (36) is written as

$$\omega^2 - k^2 = F_E^{-2}. \tag{83}$$

The explicit formulae for the coordinates $u^0, \ldots, u^d$ are

$$u^0 = E + k^2 F F_E, \quad u^\alpha = \omega k_\alpha F F_E. \tag{84}$$

The energy-momentum tensor is

$$T^{00} = u^0 = E + k^2 F F_E, \quad T^{0\alpha} = -u^\alpha = -\omega k_\alpha F F_E = T^{\alpha 0}, \tag{85}$$

$$T^{\alpha\beta} = (F F_E^{-1} - E)\,\delta^{\alpha\beta} + k_\alpha k_\beta F F_E. \tag{86}$$

Therefore the tensor T^{ij} is reduced to diagonal form $T^{00} = \mathscr{E}$, $T^{\alpha\beta} = -\mathscr{P}\delta^{\alpha\beta}$ by passing to the travelling coordinate frame, where the vector $(\omega, k) \mapsto (\omega', 0)$, $\omega^2 - k^2 \mapsto \omega'^2$. We obtain the form of the state equations:

$$\mathscr{E} = E, \quad \mathscr{P} = E - F(E)\, F_E^{-1}(E). \tag{87}$$

Conclusion. The multi-dimensional wave equation (73) averaged over one-phase solutions (73) coincides with the equation of multi-dimensional relativistic hydrodynamics (compare with [43]) with state equation (87). The averaged bracket (78) is the Poisson bracket of relativistic hydrodynamics:

$$\begin{aligned}
\{u^0(X), u^0(Y)\} &= -[T^{0\alpha}(u(Y)) + T^{0\alpha}(u(X))]\,\delta_\alpha(X - Y),\\
\{u^0(X), u^\beta(Y)\} &= T^{00}(u(Y))\,\delta_\beta(X - Y) + T^{\beta\alpha}(u(Y))\,\delta_\alpha(X - Y),\\
\{u^\beta(X), u^\gamma(Y)\} &= u^\beta(Y)\,\delta_\gamma(X - Y) + u^\gamma(X)\,\delta_\beta(X - Y),\\
u^0 &= T^{00}, \quad u^\alpha = -T^{0\alpha}.
\end{aligned}$$

§7. Modulations of soliton lattices of completely integrable evolutionary systems. Krichever's method. The analytic solution of the Gurevich-Pitaevskii problem on the dispersive analogue of a shock wave

Completely integrable evolutionary systems such as KdV have a vast number of families of exact periodic and quasi-periodic solutions of the form (6.4) with any number of phases $m = 1, 2, \ldots$; these are the so-called *finite-zone* or *algebraic-geometric solutions*. These solutions are expressed in terms of theta functions of Riemann surfaces well known in classical algebraic geometry. It turns out that methods of algebraic geometry are quite well suited to the description of the hydrodynamics of small deformations of these solition lattices. Here we shall illustrate the main principles involved in applying the methods of algebraic geometry to the study of the hydrodynamics of slow deformations of soliton lattices, using KdV as an example. (For other spatially one-dimensional integrable systems, the situation is in general similar; the study of modulations of finite-zone solutions of spatially two-dimensional integrable systems was initiated by Krichever [39], [40].)

First we provide the necessary information concerning finite-zone solutions of the KdV equation (see, for example, [26]).

As is well known, the integrability of the KdV equation

$$\varphi_t = 6\varphi\varphi_x - \varphi_{xxx} \tag{1}$$

is based on its (Lax) commutation representation

$$L_t = [A, L] \Leftrightarrow [L, \partial_t - A] = 0, \tag{2}$$

where

$$L = -\partial_x^2 + \varphi, \quad A = 4\partial_x^3 - 6\varphi\partial_x - 3\varphi_x. \tag{3}$$

Finite-zone (m-zone or m-phase) solutions of the KdV equation are defined by the condition for the existence of a common eigenfunction $\psi = \psi(x, t, \lambda)$ of the commuting operators

$$L\psi = \lambda\psi, \quad (\partial_t - A)\psi = 0, \tag{4}$$

which is meromorphic (in λ) on a hyperelliptic Riemann surface Γ of genus m having the form

$$\mu^2 = R(\lambda) \equiv \prod_{i=1}^{2m+1} (\lambda - r_i), \tag{5}$$

which covers the λ-plane with a two-sheeted covering. For smooth real solutions $\varphi(x, t)$, the numbers $r_1, \ldots, r_{2m+1}$ are real and distinct. (Let $r_1 > r_2 > \ldots > r_{2m+1}$.) The functions $\varphi(x, t)$ corresponding to the surface Γ turn out to be periodic or quasi-periodic (we shall give explicit formulae later). The intervals $[r_{2m+1}, r_{2m}]$, $[r_{2m-1}, r_{2m-2}]$, $\ldots$, $[r_1, \infty)$ on the real axis are resolved zones (stability zones) in the spectrum of the operator L, while the remaining portion of the real axis forms gaps in the spectrum. If the function $\varphi(x, t)$ is periodic in x with period T_x, then the corresponding eigenfunction ψ is Bloch (in x), that is,

$$\psi(x + T_x, t, \lambda) = \exp(ip(\lambda)T_x)\,\psi(x, t, \lambda), \tag{6}$$

where the quantity $p = p(\lambda)$ is called the *quasi-momentum*. Similarly, in the case of periodicity in t with period T_t the function ψ is Bloch in t, that is,

$$\psi(x, t + T_t, \lambda) = \exp(iq(\lambda)T_t)\,\psi(x, t, \lambda), \tag{7}$$

where $q = q(\lambda)$ is the *quasi-energy*. For any finite-zone quasi-periodic solution, the quasi-momentum and quasi-energy are defined by the averaging operation

$$p(\lambda) = -i\,\overline{(\ln\psi)_x}, \quad q(\lambda) = -i\,\overline{(\ln\psi)_t}. \tag{8}$$

The analytic properties of the functions $p(\lambda)$ and $q(\lambda)$ are as follows: they are Abelian integrals (that is, $dp(\lambda)$ and $dq(\lambda)$ are Abelian differentials) on the Riemann surface Γ with poles only at the point at infinity, $\lambda = \infty$, and asymptotics of the form

$$(9)\qquad p(\lambda) \equiv \int dp(\lambda) = \int \frac{P(\lambda)}{2\sqrt{R(\lambda)}} d\lambda, \quad P(\lambda) = \lambda^m + a_1\lambda^{m-1} + \ldots + a_m,$$

$$(10)\qquad q(\lambda) \equiv \int dq(\lambda) = \int \frac{6\lambda^{m+1} + Q(\lambda)}{\sqrt{R(\lambda)}} d\lambda,$$

$$(11)\qquad Q(\lambda) = b_0\lambda^m + b_1\lambda^{m-1} + \ldots + b_m, \quad b_0 = -3\sum_{i=1}^{2m+1} r_i.$$

The coefficients $a_1, \ldots, a_m, b_1, \ldots, b_m$ are expressed uniquely in terms of $r_1, \ldots, r_{2m+1}$ by using the normalization conditions

$$(12)\qquad \int_{\lambda_{2i}}^{\lambda_{2i-1}} dp(\lambda) = \int_{\lambda_{2i}}^{\lambda_{2i-1}} dq(\lambda) = 0, \quad i = 1, \ldots, m.$$

(As Krichever observed, the following normalization condition is more general, that is, applicable to any integrable system: all the periods of the differentials $dp(\lambda)$ and $dq(\lambda)$ are real.)

Finally, let us write down the explicit theta-function formulae for the family of finite-zone (m-zone) solutions of the KdV equation defined by the parameters $r_1, \ldots, r_{2m+1}$ (that is, by the Riemann surface (5)). Let us choose a canonical basis for the cycles $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$ on the surface Γ such that the cycles $\alpha_1, \ldots, \alpha_m$ lie in Γ above the gaps $[r_2, r_1], \ldots, [r_{2m}, r_{2m-1}]$. Let $\Omega_1, \ldots, \Omega_m$ be a basis of holomorphic differentials on Γ normalized by

$$(13)\qquad \oint_{\alpha_k} \Omega_j = 2\pi\delta_{jk}, \quad j, k = 1, \ldots, m, \quad \Omega_j = \sum_{q=1}^{m} c_{jq}\lambda^{q-1} d\lambda/\sqrt{R}(\lambda).$$

The matrix of periods of the Riemann surface Γ has the form

$$(14)\qquad iB_{jk} = \oint_{\beta_k} \Omega_j, \quad j, k = 1, \ldots, m.$$

The matrix B_{jk} is symmetric, real, and positive definite. The (Riemann) theta-function is defined by its Fourier series

$$(15)\qquad \theta(\tau \mid B) = \sum_{-\infty < n_1 < \ldots < n_m < \infty} \exp\left(-\frac{1}{2}\sum_{j,k} B_{jk}n_jn_k + i\sum_j n_j\tau_j\right),$$

where $\tau = (\tau_1, \ldots, \tau_m)$. The function $\theta(\tau \mid B)$ is periodic of period 2π in each of the variables $\tau_1, \ldots, \tau_m$. Then the finite-zone solutions $\varphi(x, t)$ of the KdV equation defined by the parameters $r_1, \ldots, r_{2m+1}$ have the form

$$(16)\qquad \varphi(x, t) = -2\partial_x^2 \ln \theta(kx + \omega t + \tau \mid B) + c,$$

where $\tau = (\tau_1, \ldots, \tau_m)$ is an arbitrary real vector. The vectors of wave numbers $k = (k_1, \ldots, k_m)$ and of frequencies $\omega = (\omega_1, \ldots, \omega_m)$ are

$$k_j = \oint_{\beta_j} dp, \quad \omega_j = \oint_{\beta_j} dq, \qquad j = 1, \ldots, m, \tag{17}$$

and the constant $c = c(r_1, \ldots, r_{2m+1})$ is of the form

$$c = \sum r_i - 2 \sum_{q=1}^{m} \oint_{\alpha_q} \lambda \Omega_q.$$

Thus, the branch points $r_1, \ldots, r_{2m+1}$ of the Riemann surface Γ of the form (5) can serve to parametrize the invariant tori (16) of the KdV equation. Another set of parameters is provided by the Kruskal integrals $I_0, I_1, \ldots, I_{2m}$. It is well known that a generating function for the Kruskal integrals is the quasi-momentum $p(\lambda)$, that is,

$$p(\lambda) = -i\,\overline{(\ln \psi)_x} = \sqrt{\lambda} + \sum_{s=0}^{\infty} \frac{I_s}{(2\sqrt{\lambda})^{2s+1}}. \tag{18}$$

Lemma. *The densities of the Kruskal integrals,*

$$I_s = \overline{P_s(x)}, \qquad s = 0, 1, \ldots, \tag{19}$$

are obtained as the coefficients of the expansion as $\lambda \to \infty$ of the function

$$-i(\ln \psi)_x = \sqrt{\lambda} + \sum_{s=0}^{\infty} \frac{P_s(x)}{(2\sqrt{\lambda})^{2s+1}}; \tag{20}$$

the densities of the fluxes are obtained from the expansion of the function

$$-i(\ln \psi)_t = -iA\psi/\psi = 4(\sqrt{\lambda})^3 + \sum_{s=0}^{\infty} \frac{Q_s(x)}{(2\sqrt{\lambda})^{2s+1}}, \tag{21}$$

$$\partial_t P_s = \partial_x Q_s, \qquad s = 0, 1, \ldots \tag{22}$$

Here (modulo total derivatives)

$$\begin{gathered} P_0 = \varphi, \quad P_1 = \varphi^2/2, \quad P_2 = \varphi_x^2/2 - \varphi^3, \ldots, \\ Q_0 = 3\varphi^2, \quad Q_1 = 2\varphi^3 + \frac{3}{2}\varphi_x^2, \\ Q_2 = \frac{9}{2}\varphi^4 - \varphi_x\varphi_{xxx} + \frac{1}{2}\varphi_{xx}^2 - 3\varphi^2\varphi_{xx} + 6\varphi\varphi_x^2, \ldots \end{gathered} \tag{23}$$

Proof (Krichever). This follows from the equality of the mixed derivatives

$$[(\ln \psi)_x]_t = [(\ln \psi)_t]_x. \tag{24}$$

Expanding both sides of (24) in powers of $(\sqrt{\lambda})^{-1}$, we obtain (22). The lemma is proved.

Let us consider now a weakly deformed lattice of the form (16) in which the parameters $r_1, \ldots, r_{2m+1}$ (or, equivalently, the parameters $u^i = I_i$, $i = 0, 1, \ldots, 2m$) are slowly varying functions of x, t:

$$u^i = u^i(X, T), \quad X = \varepsilon x, \quad T = \varepsilon t, \quad i = 0, 1, \ldots, 2m. \tag{25}$$

Following the procedure of the previous section, the equations of slow modulation can be written in Hamiltonian form:

$$u^i_T(X) = \{u^i(X), \overline{\mathcal{H}}\}, \tag{26}$$

where the averaged Hamiltonian $\overline{\mathcal{H}}$ is

$$\overline{\mathcal{H}} = \int \mathcal{H}\, dX = \int u^2 dX. \tag{27}$$

The form of the bracket $\{\cdot, \cdot\}$ for $m = 1$ obtained by averaging the Gardner–Zakharov–Faddeev bracket was given in the previous section (where we set $V(\varphi) = \varphi^3$).

It is well known that the KdV equation is also Hamiltonian with respect to the Magri–Leonard bracket [90]

$$\{\varphi(x), \varphi(y)\}_{ML} = -\frac{1}{2}\delta'''(x-y) + (\varphi(x) + \varphi(y))\,\delta'(x-y), \tag{28}$$

$$\varphi_t = 6\varphi\varphi_x - \varphi_{xxx} = \{\varphi(x), I_1\}_{ML}, \text{ where } I_1 = \int \varphi^2/2dx. \tag{28$'$}$$

This gives a new Hamiltonian structure of the averaged equations

$$u^i_T(X) = \{u^i(X), \overline{I}_1\}_1, \quad \overline{I}_1 = \int u^1 dX. \tag{29}$$

The coordinates $u^0, \ldots, u^{2m}$ of the averaged Magri bracket are also Liouville (and even strongly Liouville). Setting $c = -\omega/k$, the rest of the notation being as in the previous section, for $m = 1$ the corresponding matrix $\gamma_1^{ij}(u)$ has the form

$$\gamma_1^{ij}(u) = \begin{pmatrix} 2u^0 & -cu^0 - a/2 & 10u^2 \\ 2u^1 & -cu^1 + 2b & \frac{5}{3}(-cu^2 - bc + a^2/4) \\ 2u^2 & -cu^2 - bc + a^2/4 & -(c^2 - a)u^2 + 2bc^2 - ca^2/4 + ab \end{pmatrix}. \tag{30}$$

***Theorem* 1** (Flaschka and McLaughlin [73]). *The equations of slow modulation of m-zone solutions of the KdV equation have the form*

$$\partial_T dp = \partial_X dq, \tag{31}$$

where $dp(\lambda)$, $dq(\lambda)$ are Abelian differentials of the form (9)–(12).

Proof (Krichever; the original proof of Flaschka and McLaughlin is much more involved). Let us use the procedure of averaging integrals (see §5 above) for their generating functions, that is, let us "average" the relation

$$[(\ln\psi)_x]_t = [(\ln\psi)_t]_x. \tag{32}$$

We obtain

$$\partial_T \overline{(\ln \psi)_x} = \partial_X \overline{(\ln \psi)_t}, \tag{33}$$

that is, $\partial_T p(\lambda) = \partial_X q(\lambda)$. Differentiation with respect to λ yields (31). The theorem is proved.

By expanding the relation (31) (or (33)) into a series of powers of $(\sqrt{\lambda})^{-1}$ in [73], we obtain the slow modulation equations in the form

$$\partial_T u^s = \partial_X \overline{Q}_s, \qquad s = 0, 1, \ldots, 2m, \tag{34}$$

where $\overline{Q}_s$ can be expressed in the form $\overline{Q}_s = \overline{Q}_s(u^0, \ldots, u^{2m})$. Let us also note the following important corollary.

Corollary. *The branch points* $r_1, \ldots, r_{2m+1}$ *are Riemann invariants for the slow modulation equations* (34), *that is,*

$$\partial_T r_i = v_i(r_1, \ldots, r_{2m+1})\, \partial_X r_i, \qquad i = 1, \ldots, 2m+1, \tag{35}$$

where the characteristic velocities have the form

$$v_i(r_1, \ldots, r_{2m+1}) = \left.\frac{dq(\lambda)}{dp(\lambda)}\right|_{\lambda = r_i} = 2\,\frac{6r_i^{m+1} + Q(r_i)}{P(r_i)}, \qquad i = 1, \ldots, 2m+1. \tag{36}$$

Proof. We multiply equation (31) by $(\lambda - r_i)^{3/2}$ and pass to the limit as $\lambda \mapsto r_i$. The corollary is proved.

In particular, for modulations of the one-phase solution of the KdV equation (cnoidal wave), the slow modulation equations assume the form already found by Whitham [94] and used by Gurevich and Pitaevskii [14], [15]:

$$\partial_T r_i = v_i(r_1, r_2, r_3)\, \partial_X r_i, \qquad i = 1, 2, 3, \tag{37}$$

while the characteristic velocities $v_1 < v_2 < v_3$ are expressed in terms of complete elliptic integrals by

$$\begin{cases} -v_1 = (r_1 + r_2 + r_3)/3 - \frac{2}{3}(r_2 - r_1)\dfrac{K}{K - E}, \\ -v_2 = (r_1 + r_2 + r_3)/3 - \frac{2}{3}(r_2 - r_1)\dfrac{(1 - s^2)K}{E - (1 - s^2)K}, \\ -v_3 = (r_1 + r_2 + r_3)/3 + \frac{2}{3}(r_3 - r_1)\dfrac{(1 - s^2)K}{E}, \end{cases} \tag{38}$$

where $K = K(s)$, $E = E(s)$ are complete elliptic integrals (see [4]), and $s^2 = (r_2 - r_1)/(r_3 - r_1)$, $v_3 \geqslant v_2 \geqslant v_1$, $r_3 \geqslant r_2 \geqslant r_1$.

Following Krichever, let us show how to obtain an effective procedure for integrating the equations (35). Let us consider first the averaged finite-zone solutions of these equations (see the Introduction). The origin of these solutions is as follows. Any linear combination of Kruskal integrals defines a Hamiltonian evolutionary system of the form

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial x} \frac{\delta}{\delta \varphi(x)} \sum_{j=1}^{n} c_j I_j, \tag{39}$$

which commutes with the dynamics of KdV and has invariant tori (16) in common with KdV. All such systems admit Lax representations of the form

$$L_y = \left[\sum_{j=1}^{n} c_j A_j, L\right], \tag{40}$$

where the $A_j = \varkappa_j (\partial_x)^{2j-1} + \ldots$ are ordinary differential operators, and the $\varkappa_j$ are constants. An eigenfunction $\psi = \psi(x, t, y, \lambda)$ satisfies the following equation in y:

$$\psi_y = \left(\sum_{j=1}^{n} c_j A_j\right) \psi. \tag{41}$$

The quantity

$$s(\lambda) = -i\, \overline{(\ln \psi)_y} \tag{42}$$

is an Abelian integral in Γ with a pole only at the point at infinity, $\lambda = \infty$, having the form

$$s(\lambda) = \sum_{j=1}^{n} (-1)^{j-1} c_j \varkappa_j (\sqrt{\lambda})^{2j-1} + O(1). \tag{43}$$

The unique normalization of the differential $ds(\lambda)$ is defined by equations similar to (12):

$$\int_{\lambda_{2i}}^{\lambda_{2i-1}} ds(\lambda) = 0, \quad i = 1, \ldots, m. \tag{44}$$

The y-dynamics on invariant tori are linear:

$$\tau \mapsto \tau + y\nu, \tag{45}$$

where the vector $\nu = (\nu_1, \ldots, \nu_m)$ is the vector of β-periods of the differential $ds(\lambda)$. "Averaging" the system (1) over the tori (16), we obtain a system of hydrodynamic type

$$u^i_Y = \partial_X \left(\sum_{j=1}^{n} c_j \overline{Q}^{ij}\right), \quad Y = \varepsilon y, \tag{46}$$

where the quantities Q^{ij} are defined by the formulae

$$\partial_x Q^{ij} = \{P_i(\varphi(x), \ldots), I_j\} \tag{47}$$

(see the previous section). This system commutes with the averaged KdV equation and has the same Hamiltonian structure with Hamiltonian $\sum c_j u^j$.

By analogy with Theorem 1, it is proved that the system (47) can be written in the following equivalent form:

$$\partial_Y dp = \partial_X ds, \tag{48}$$

where the differentials dp, ds on the surface Γ are defined by (9), (12), (43), (44). A consequence of this is the reducibility of the system (47) to diagonal form in the same variables $r_1, \ldots, r_{2m+1}$,

$$\partial_Y r_i = w_i (r_1, \ldots, r_{2m+1})\, \partial_X r_i, \quad i = 1, \ldots, 2m+1. \tag{49}$$

The characteristic velocities have the form

$$w_i (r_1, \ldots, r_{2m+1}) = \left.\frac{ds(\lambda)}{dp(\lambda)}\right|_{\lambda=r_i}, \quad i = 1, \ldots, 2m+1. \tag{50}$$

Thus, (35) and (49) are commuting diagonal Hamiltonian systems. By the scheme of §4, they generate an exact solution $r_i = r_i(X, T)$ of the averaged KdV equation (35), having the form

$$w_i (r_1, \ldots, r_{2m+1}) = v_i (r_1, \ldots, r_{2m+1})\, T + X, \quad i = 1, \ldots, 2m+1. \tag{51}$$

Using the explicit formulae (36), (50) for the characteristic velocities v_i, w_i, we can rewrite the relations (51) in the form

$$(X\, dp\, (\lambda) + T\, dq\, (\lambda) - ds\, (\lambda))_{\lambda=r_i} = 0, \quad i = 1, \ldots, 2m+1. \tag{52}$$

This is the most convenient analytic form for the averaged finite-zone solutions.

The relations (52) can be generalized in such a way that all the solutions of the averaged KdV are obtained. To explain the idea of this generalization, we shall give a direct proof of the fact that the quantities $r_i = r_i(X, T)$ defined by (52) satisfy the averaged KdV. Let us consider the differential

$$\Omega = X\, dp + T\, dq - ds, \tag{53}$$

and let us differentiate it with respect to X and T. We obtain

$$\Omega_X = dp + (X\, \partial_X dp + T\, \partial_X dq - \partial_X ds), \tag{54}$$

$$\Omega_T = dq + (X\, \partial_T dp + T\, \partial_T dq - \partial_T ds). \tag{55}$$

The expressions in the brackets in (54), (55) are Abelian integrals on the Riemann surface Γ; they are holomorphic in Γ except, possibly, at the branch points $r_1, \ldots, r_{2m+1}$ and the point at infinity $\lambda = \infty$. There is no pole at the point at infinity, since the principal parts of the differentials dp, ds, dq are independent of X and T. On the other hand, poles at the branch points $\lambda = r_i$ do not occur in view of (52). Thus, the differentials in the brackets in (54), (55) are holomorphic on Γ. By the normalization conditions (12), (44), their periods over α-cycles are all zero. Thus, these holomorphic differentials are themselves equal to zero. We obtain

$$\Omega_X = dp, \quad \Omega_T = dq. \tag{56}$$

The compatibility condition for these relations is just the averaged KdV equation in the form (31).

It is clear that these arguments also work in the case of more complex analytic properties of the differential $ds(\lambda)$. In particular, if we consider a suitable class of piecewise analytic differentials $ds(\lambda)$, we obtain from the relations (52) the general solution of the averaged KdV (see [39]).

Let us return to averaged finite-zone solutions. Let us consider such a solution corresponding to one, say the Lth, of the Kruskal integrals, that is, the one that is obtained from the averaged flux of the form

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial x} \frac{\delta}{\delta \varphi(x)} I_L. \tag{57}$$

The corresponding averaged finite-zone solution then has the form (52), where the differential $ds = ds_L(\lambda)$ with zero α-periods has a unique simple pole at $\lambda = \infty$ of the form

$$ds_L(\lambda) = \text{const}\; d\,(\lambda^{2L-1} + O(1)). \tag{58}$$

The system (57) averaged over the tori (16) in the variables $r_1, \ldots, r_{2m+1}$ will have the form

$$\partial_Y r_i = w_{i,L}(r_1, \ldots, r_{2m+1})\, \partial_X r_i, \quad i = 1, \ldots, 2m+1, \tag{59}$$

where

$$w_{i,L}(r_1, \ldots, r_{2m+1}) = \left.\frac{ds_L(\lambda)}{dp(\lambda)}\right|_{\lambda = r_i}, \tag{60}$$

while the corresponding averaged finite-zone solutions of the system (1) have the form

$$w_{i,L}(r_1, \ldots, r_{2m+1}) = v_i(r_1, \ldots, r_{2m+1})\, T + X, \quad i = 1, \ldots, 2m+1. \tag{61}$$

Lemma [39]. *The averaged finite-zone solution* (60), (61) *is a self-similar solution, that is,*

$$r_i(X, T) = T^{\gamma} R_i(X T^{-1-\gamma}), \tag{62}$$

with self-similarity exponent

$$\gamma = 1/(L-2). \tag{63}$$

Proof. Under expansions $\lambda \to k^2\lambda$, the quantities r_i, $dp(\lambda)$, $dq(\lambda)$, $ds_L(\lambda)$ transform in the following way:

$$\begin{cases} r_i \to k^{-2} r_i, \\ dp \to k\, dp, \\ dq \to k^3 dq, \\ ds \to k^{2L-1} ds. \end{cases} \tag{64}$$

This means precisely that (62), (63) are true. The lemma is proved.

Let us consider an important example. We shall obtain analytic formulae for the Gurevich–Pitaevskii solution that describes the dynamics of a collisionless shock wave after the "moment of toppling". After the toppling, the evolution of this dispersion analogue of a shock wave is defined in the oscillatory zone by a multi-valued (three-valued) function $r_i = r_i(x, t)$, $i = 1, 2, 3$, that satisfies the averaged equations (37). Outside the oscillatory zone $x^-(t) \leqslant x \leqslant x^+(t)$, the multi-valued solution must become the usual Riemann solution defined by the equation

$$x + 6ut = u^3. \tag{65}$$

This requirement leads to the following boundary conditions for the solutions of (37):

$$\begin{cases} r_2(x^+) = r_3(x^+), & r_1(x^+) = u(x^+), \\ r_1(x^-) = r_2(x^-), & r_3(x^-) = u(x^-). \end{cases} \tag{66}$$

For large t, the required solution converges to a self-similar solution with self-similarity exponent $\gamma = 1/2$.

In order to construct such a solution, let us take an averaged finite-zone solution of the form (61) with $L = 4$, defined by the relations

$$x + v_i t = w_i, \quad i = 1, 2, 3, \tag{67}$$

where the $w_i = w_i(r_1, r_2, r_3)$ are computed from

$$w_i = \frac{1}{35}[(3v_i - a) f_i + f], \quad f = 5a^3 - 12ab + c, \tag{68}$$

$$a = r_1 + r_2 + r_3, \quad b = r_1 r_2 + r_1 r_3 + r_2 r_3, \quad c = r_1 r_2 r_3, \quad f_i = \partial f / \partial r_i.$$

The following assertion is proved in [53].

***Theorem* 2.** *The system of equations* (67), (68) *is non-degenerate in the domain* $z^- < z < z^+$, *where* $z = xt^{-3/2}$ *is the self-similar variable, and defines there a smooth* (*and even analytic*) *solution with self-similarity exponent* $\gamma = 1/2$. *On the boundary of the domain the solutions satisfy the boundary conditions* (66). *The boundary values of the self-similar variable have the form* $z^- = -\sqrt{2}$, $z^+ = \sqrt{10/27}$.

It can also be shown that the behaviour of the solutions (67) near the boundary of the oscillatory zone is described by the formulae derived in [14] (taking into account the improvements of [16]).

§8. Evolution of the oscillatory zone in the KdV theory. Multi-valued functions in the hydrodynamics of soliton lattices. Numerical studies

Applications of hydrodynamical equations of weakly deformed soliton lattices (or of the Whitham equations of slow modulation of lattice parameters) to concrete problems of the physics of dispersive media were discussed in the work of Gurevich and Pitaevskii in 1973 in the framework of special self-similar solutions. Later, this program of investigation was developed in the work of Avilov; Novikov, and Krichever [1], [2]. They provided the mathematical statement and a numerical realization of the problem of evolution of multi-valued functions having the property that the number of solutions is different in different regions of the (x, t)-plane. In addition to the GP self-similar regimes, the need to study the evolution of multi-valued functions is indicated by the Lax–Levermore–Venakides theorems [83], [85]–[87], [91], [92] on weak limits of the KdV theory with small dispersion:

$$(1) \qquad u_t + uu_x + \varepsilon u_{xxx} = 0, \quad \varepsilon \to 0.$$

According to their theorems, there corresponds to the solution $u_\varepsilon(x, t)$ as $\varepsilon \to 0$ a decomposition of the (x, t)-plane into a number of domains (which is assumed to be finite), and in each domain the principal term of the asymptotics is described by a k-zone Whitham equation, where k is a number that depends on the domain, that is, by a $(2k+1)$-component system. In these papers, global restrictions on $u_\varepsilon(x, t)$ are imposed: either rapid decay as $|x| \to \infty$ or periodicity in x. These restrictions make it possible to use the results of the inverse scattering method for rigorous proofs, though many facts are doubtless independent of global restrictions and have a local character, as long as time is not too large. We shall not introduce the small parameter ε multiplying the dispersion term explicitly, but will assume as usual that the solutions of the KdV equation oscillate rapidly, that is, that the parameter ε arises in the solutions. Unlike Lax and others, the problem of evolution of multi-valued functions must be posed and solved in our framework independently of the KdV equation theory, on the basis of the theory of systems of hydrodynamic type (that is, of first order in the derivatives). It appears to us that this set-up by itself can be of more universal value than the description as $\varepsilon \to 0$ of the solutions of one high-order equation (in this case, the KdV).

The real need to consider the evolution of multi-valued functions in physical problems of dispersive hydrodynamics is what contrasts it sharply with the usual hydrodynamics, in which only jumps (shock waves) had to be introduced. As was noted in the recent paper [82], when the trivial equation $u_t + uu_x = 0$ is made discrete (that is, into a dispersive equation), the difference analogue of the KdV appears (see [57], Ch. 1, §7). This simple circumstance shows that the case of the GP problem 1 (see below) also occurs in the discretized equations, and that the oscillatory zone that arises here appears in hydrodynamics in the numerical computation of shock waves, since the properties of the discrete analogue of the KdV are quite similar to the properties of the continuous KdV studied here. This connection between these "numerical" oscillations with dispersion was not noticed before.

Let us consider the following two problems, following GP ([57], §4 of Chapter 4).

Problem 1. The decomposition of a step in KdV theory. What is the asymptotic behaviour for $t \gg 1$ of solutions of the KdV with initial datum a step function, or, more generally, with initial data (say, smooth and monotone in x) of the form indicated in Fig. 2? The case $u(-\infty) > u(+\infty)$ is of more interest here (see [57], §4, Chapter 4). Let $u(-\infty) = 1$, $u(+\infty) = 0$.

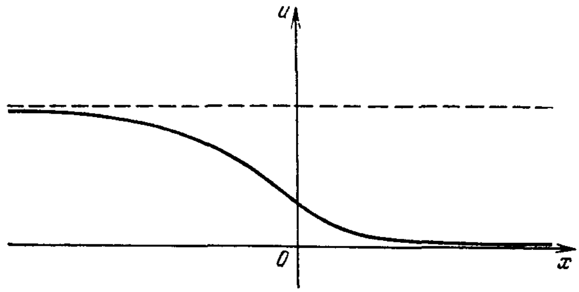

Fig. 2

Problem 2. The dispersive analogue of a shock wave. What is the asymptotic behaviour for $t \gg 1$ of solutions of KdV with initial datum $u(x)$ of the form $x = ut - u^3$ for large $|x| \to \infty$? The physical motivation for Problem 2 is as follows. Let us assume that initially the dispersion term u_{xxx} is sufficiently small and does not influence the evolution, which is described by the truncated equation $u_t + uu_x = 0$. Its general solution of the form $x = ut + P(u)$ with an arbitrary function P is such that in a large class of cases it "topples over" at some moment of time $t = 0$, that is, $u_x \to \infty$ as $t \to 0-$, $x = x_0$. In the generic case, the solution in a neighbourhood of the point $(0, x_0)$ is approximately described (up to shifts and scaling transformations) by the cubic function

$$x \approx ut - u^3. \tag{2}$$

In this neighbourhood, the dispersion term u_{xxx} in the KdV equation can no longer be neglected: it cannot be small in this domain. Taking this term into consideration changes the solution in a small neighbourhood of the point $(0, x_0)$ drastically (there is no "toppling over" in the KdV theory). An oscillatory zone appears; GP consider it to be approximately described by a one-zone Whitham equation in a small region $\Delta(t)$. Outside the region $\Delta(t)$ this oscillatory zone changes into a smooth solution of the form $x = ut - u^3$, and outside the zone the dispersive term is of no importance. Thus, Problem 2 describes a situation which is local in x, t. The scales in x and t on which the GP regime establishes itself in this small region must be small in comparison with the scales on which global asymptotics in x, t operate, as in Problem 1 for example, or in any other problem.

Therefore Problem 2 describes a universal local situation, which is called the "dispersive analogue of a shock wave". Locally the application of this structure is not entirely rigorous, since the domain of interest will contain only a finite number of oscillations, and this approximate structure will be observable only for a finite time. Used in this fashion, it constitutes intermediate time asymptotics. "Large t" in Problem 2 is much smaller than the times encountered in Problem 1.

Asymptotic states for $t \gg 1$ in Problems 1 and 2 are described, according to GP, by self-similar solutions of one-zone Whitham equations:

$$(3)\qquad r_{i_t} + v_i(r)\, r_{i_x} = 0, \quad i = 1, 2, 3, \quad v(\lambda r) = \lambda v(r).$$

It follows from the homogeneity of $v_i(r)$ that these equations admit self-similar solutions of the form

$$(4)\qquad r_i(x, t) = t^{\gamma} R_i(x t^{-1-\gamma}),$$

where γ is arbitrary, and

$$(5)\qquad a = r_2 - r_1, \quad s^2 = \frac{r_2 - r_1}{r_3 - r_1}, \quad \delta = r_2 + r_1 - r_3.$$

The average $\bar{u}$ over a cycle of oscillations is

$$(6)\qquad \bar{u} = \delta + 2aE(s)\, s^{-2} K^{-1}(s).$$

Problem 1. Here we must take $\gamma = 0$, $\tau = x/t$. We have solutions of the form

$$(7)\qquad r_1 \equiv 0, \quad r_3 \equiv 1, \quad v_2 \equiv \tau = x/t \geqslant 0, \quad s^2 = r_2 = a.$$

The function $r_2(\tau)$ is determined by the relation $v_2(s^2) = \tau$.

The oscillatory interval has the form shown in Fig. 3,

$$(8)\qquad \tau^- = -1, \quad \tau^+ = 2/3, \quad [\tau^-, \tau^+] = \Delta = \{0 \leqslant s^2 \leqslant 1\}.$$

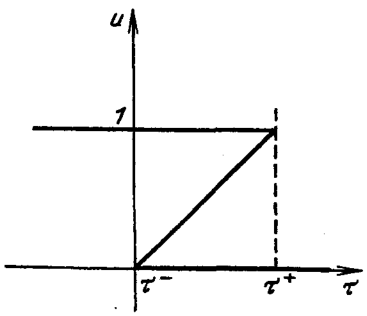

Fig. 3

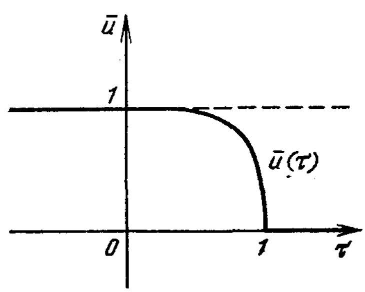

Fig. 4

The question of the rigorous justification of this asymptotic behaviour was studied in [67] using the inverse scattering method. By the scaling transformation $u \to Au$, $x \to A^{1/2}x$, $t \to A^{3/2}t$, we obtain another solution of the same problem, where $u \to A$ as $x \to -\infty$ and $u \to 0$ as $x \to +\infty$. The graph of the quantity $\bar{u}$ is of the form resembling the usual shock wave (see Fig. 4). Here we have $\bar{u}(\tau) \approx 4/\ln(|\tau^+ - \tau^-|)$ as $\tau \to \tau^+ - 0$.

Problem 2. The situation here is more complicated. For correct matching with the boundary condition we have to consider a self-similar solution with exponent $\gamma = 1/2$,

$$r_i(x, t) = t^{1/2}R_i(xt^{-3/2}), \quad z = xt^{-3/2}, \tag{9}$$

since outside the oscillatory interval Δ we must have the solution $x = ut - u^3$ of the equation $u_t + uu_x = 0$. On the boundaries of the interval Δ the function $r(x, t)$ is continuous:

$$r(x, t) = u \quad \text{outside } \Delta, \quad r(x, t) = \{r_1, r_2, r_3\} \in \Delta, \tag{10}$$

$$r^- = r_3(x^-) = u(x^-), \quad r_1^- = r_2^-, \tag{11}$$

$$r^+ = r_1(x^+) = u(x^+), \quad r_2^+ = r_3^+. \tag{11'}$$

In both Problems 1 and 2, the oscillatory zone covers the entire region between the two singular Whitham equations: at the left endpoint x^- we must have $r_2 = r_1$, that is, the one-zone solution of the KdV becomes a constant, while at the right endpoint x^+ we must have $r_2 = r_3$, that is, the one-zone solution of the KdV becomes a soliton. The oscillatory interval $\Delta = [z^-, z^+]$ must be constant in the self-similar variable z. The whole graph must be C^1-smooth, including the ends of the oscillatory zone. Let us denote $u(x, t)$ by r. We have the unique C^1-smooth function $r(x, t)$ of the form shown in Fig. 5.

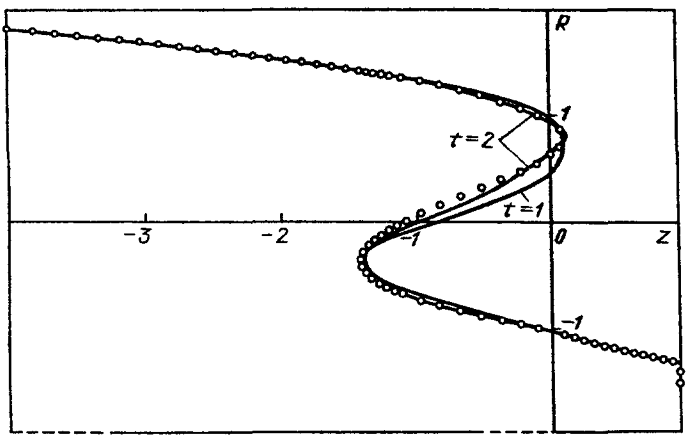

Fig. 5. Evolution of the multi-valued function $R(z, t) = t^{-1/2}r(z, t)$ $(z = xt^{-3/2})$ of Problem 2. The initial condition ($t = 1$) in the oscillatory zone corresponds to a perturbation of the self-similar solution. For $t = 2$ this distortion is significantly smaller. The self-similar solution is indicated by dots.

The existence of such a solution is a non-trivial fact. It was found approximately in [14]. The exact analytic form of this solution in the interval Δ was discovered only recently (see §7 above). In particular, it is analytic inside the interval Δ, where

$$(12) \qquad z^- = -\sqrt{2}, \quad z^+ = \sqrt{10}/27, \quad R_3 > 0, \quad R_1 < 0.$$

The function $z(r)$ is C^2-smooth in a neighbourhood of the point (z^-, r_1^-) $(s^2 = 0)$ and $C^{2-\varepsilon}$-smooth in a neighbourhood of the point (z^+, r_3^+) $(s^2 = 1)$, where $\varepsilon > 0$ is arbitrary and z depends asymptotically only on s^2 as $s \to 1$:

$$(13) \qquad c(z^+ - z) = cz'' \approx (1 - s^2)\left(\ln \frac{16}{1 - s^2} + 1/2\right), \quad c = \mathrm{const} < 0, \quad z'' < 0.$$

Without entering into details (see [57], §4 of Ch. 4), we see that here also the quantity $\bar{u}(z)$ has a form that resembles the usual shock wave (see Fig. 6). The dotted line is the function $\theta(z)$, that is, the solution $x = ut - u^3$ as $u(z)\, t^{-1/2} = \theta(z)$, $z = xt^{-3/2}$.

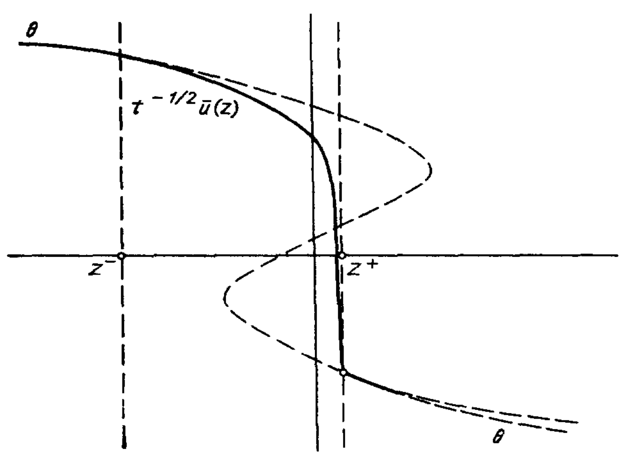

Fig. 6

Such are the self-similar regimes that describe the asymptotic behaviour of the oscillation zone for $t \gg 1$ in Problems 1 and 2 according to the hypothesis of GP. Are these regimes stable? Do they really occur asymptotically in the framework of the theory of hydrodynamic type or as asymptotics of a wide class of sufficiently general initial conditions?

The work of Avilov and Novikov [1] is devoted to the numerical solution of this problem. What is the exact mathematical setting of the problem of evolution of the multi-valued functions $r(x, t)$?

The paper [1] deals with this question numerically in the case when the multi-valued function $r(x, t)$ is single-valued in a domain $\mathbb{R} \setminus \Delta$ (where $r(x, t)$ is denoted by $u(x, t)$) and three-valued in a domain $\Delta \subset \mathbb{R}$ that depends on the time t. If Δ is an interval, then at its left endpoint x^- we must have $r_2^- = r_1^- = r^- < r_3(x^-)$, while at its right endpoint x^+ we must have $r^+ = r_3^+ = r_2^+ > r_1(x^+)$. The graph of the function $r(x, t)$ must be C^1-smooth everywhere. More precisely, we require the following to hold asymptotically (near the points x^+, r^+ and x^-, r^-):

$$x'' = [a_+ + b_+ (r - r^+)] f (1 - s^2) + O (r - r^+)^3, \tag{14}$$

$$x'' = x - x^+ < 0, \quad f(y) = y^2 \left[\ln \frac{16}{|y|} + 1/2\right], \tag{15}$$

$$x' = [a_- + b_- (r - r^-)](r - r^-)^2 + o (r - r^-)^3, \tag{16}$$

$$x' = x - x^- > 0. \tag{17}$$

It is exactly the asymptotic behaviour which, as a simple analysis shows, is compatible with singularities of the coefficients of a one-zone Whitham equation (3). The velocities v_i have limits as $x \to x^\pm$, denoted by $v_i^\pm$. For all t, the following formulae for the evolution of the ends of the boundaries of the oscillatory zone Δ are derived from the asymptotics (14)–(17):

$$dx^+/dt = \overset{+}{v_2} = \overset{+}{v_3}, \quad dr^+/dt = -| \overset{+}{r_3} - \overset{+}{r_1} |/12a_+, \tag{18}$$

$$dx^-/dt = v_1^- = v_2^-, \quad dr^-/dt = -1/2a_-. \tag{19}$$

We see from (18), (19) that the coefficients $b_\pm$ in (14), (15) above are not necessary for the determination of the evolution of the interval Δ. A certain extra smoothness in (14), (16) should not confuse anyone; the parameters $b_\pm(t)$ are very convenient in interpolation. The numerical computation uses the method of characteristics inside the zone $[x^- + \varepsilon, x^+ - \varepsilon]$, taking into consideration the diagonal structure of the Whitham equations (3). In a neighbourhood of the points $x^\pm$ we use the asymptotics (14), (16), matching them with the remaining part. At every step in time (or every several steps; this is a technical question) the new values of the coefficients $a_\pm(t)$, $b_\pm(t)$ are determined by matching them with the groups of points in the interval Δ (obtained by the method of characteristics) that are nearest to them.

The conclusions are as follows: locally in t, the evolution of multi-valued smooth functions with asymptotics (14), (16) is well defined. If the initial condition is a C^1-small perturbation of the GP regimes, then the solution is defined for all t, and as $t \to \infty$ it converges to GP regimes in both Problems 1 and 2.

It would be desirable to prove a precise mathematical theorem about the local evolution. What smoothness away from the boundary is really needed? When the method of characteristics is used, the answer to this question cannot be gleaned from numerical experiments.

For large perturbations of the initial conditions, a new "toppling over of the front" can occur in the equations of hydrodynamics of soliton lattices themselves. This necessitates increasing the degree of multi-valuedness of the function $r(x, t)$. Evolutionary processes of this kind have not yet been studied. Precise analogues of the asymptotics (14), (16) near the boundary of the new zone have to be found. In order that no toppling occurs, it is necessary (but apparently not sufficient) that the graph of $r(x, t)$ with asymptotics (14), (16) be monotone in x in every interval of single-valuedness if Δ is an interval (that is, $u_x = r_x < 0$ for $x \in \mathbb{R} \setminus \Delta$, $r_{3x} > 0$, $r_{1x} > 0$, $r_{2x} < 0$, $x \in \Delta$).

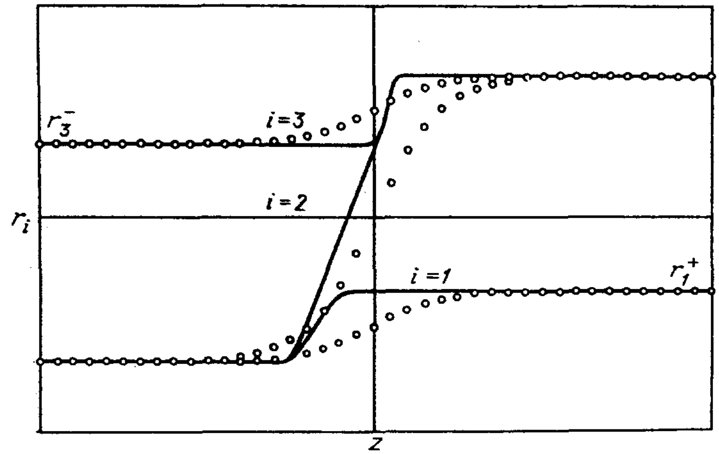

Fig. 7. Evolution of the multi-valued function $r(z, t)$ ($z = xt^{-1}$) of Problem 1. The initial condition ($t = 1$) inside the oscillatory zone corresponds to the self-similar solution of Problem 2, and outside that zone it converges to constants.

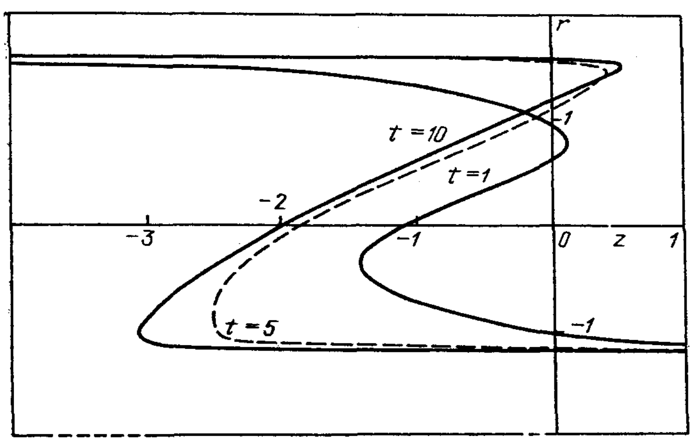

Fig. 8. Evolution of the infinite oscillatory zone in Problem 3. The functions $r_i(z, t)$ ($z = xt^{-1}$), $i = 1, 2, 3$, at the initial time $t = 1$ are indicated by dots; solid lines correspond to $t = 11$.

In the simpler Problem 3, where $\Delta = \mathbb{R}$ and $r_i \to r_i^+$ as $|x| \to \pm\infty$, for the evolution to be well defined for all $t \geqslant 0$ it is apparently necessary and sufficient that $r_{ix} > 0$ for all x, $t = 0$. If $r_1^- = r_2^- < r_3^-$ and $r_2^+ = r_3^+ > r_1^+$, then the GP asymptotic behaviour of Problem 1 is established in Problem 3 in the domain between r_1^+ and r_3^-, assuming that $r_3^+ > r_1^-$ for $\Delta = \mathbb{R}$ (see [1]). This is shown by numerical experiment. This last question could possibly be resolved by methods of the recent paper [67].

§9. Influence of small viscosity on the evolution of the oscillatory zone

Let us consider the Korteweg–de Vries–Burgers (KdVB) equation with small viscosity $\mu > 0$,

$$u_t + uu_x + u_{xxx} + \mu u_{xx} = 0. \tag{1}$$

Let us use the Bogolyubov–Whitham averaging method, using the same family of cnoidal waves (0.19) of the KdV equation. The averaging of the viscous term leads after some calculations to additional terms in the right-hand sides[(1)] of the equations of hydrodynamics of soliton lattices

$$r_{it} + v_i(r)\, r_{ix} + \mu g_i(r) = 0, \quad i = 1, 2, 3, \tag{2}$$

where

$$g_i(r) = -4(r_2 - r_1)^2 Q(s)/3\Phi_i, \quad i = 1, 2, 3, \tag{3}$$
$$0 < Q = \frac{4}{15}[(E-K)/s^4 - (-E + 3K/2)/s^2 + E - K/2],$$

$s^2 = (r_2 - r_1)/(r_3 - r_1)$, and $K = K(s)$, $E = E(s)$ are complete elliptic integrals. Let us note the properties

$$g_1 \geqslant 0, \quad g_2 \leqslant 0, \quad g_3 \leqslant 0, \quad g_i(\lambda r) = \lambda^2 g_i(r). \tag{4}$$

Let us study the question of the behaviour of the oscillatory zone in the process of decomposing a step function, as in Problem 1 of §8 above. Let $u \to A_\pm$, $x \to \pm\infty$, where $A_+ > 0$, $A_- < 0$. As in §8, let us use the one-zone three-component Whitham equation in the region $\Delta = [x^-(t), x^+(t)]$, and the trivial equation $u_t + uu_x = 0$ outside Δ. Thus, according to our hypothesis, viscosity is important only in the region Δ, where we use equation (2) instead of (8.3). As in §8, we shall describe the process by a multi-valued function $r(x, t)$, which is three-valued inside Δ and single-valued outside Δ (where $r = u$), and C^1-smooth everywhere.

To find solutions in the form $u(x - Vt)$ when $\mu \neq 0$, we write the stationary KdVB equation in the form

$$u'' = -u^2/2 + Vu + C - \mu u', \tag{5}$$

[(1)]The algebraic-geometric representation of equations of type (2) was first discussed in [75].

where the constant $C = A_+A_-/2$ of integration is such that for $\mu = 0$ the critical points of equation (5) in the (u, u') phase plane have the form

$$u' = 0, \quad u = A_\pm = V \mp \sqrt{V^2 + 2C}, \tag{6}$$
$$A_+ < 0, \quad A_- > 0, \quad 2V = A_+ + A_-.$$

When $\mu = 0$ the phase portrait is as in Fig. 9.

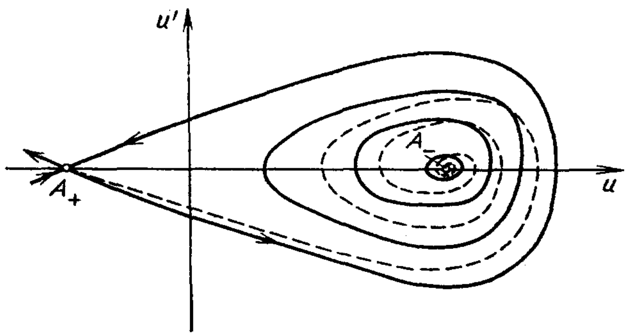

Fig. 9

Remembering the definitions of the quantities r_i, we obtain

$$3A_-A_+ = 4r_1r_2 - (r_3 - r_2 - r_1)^2, \tag{7}$$

$$A_- + A_+ = 2V = \frac{2}{3}(r_1 + r_2 + r_3). \tag{8}$$

As is clear from Fig. 9, for constant A_+, A_- and small $\mu > 0$ there is a unique solution $u(x - Vt, \mu)$ of the stationary KdVB equation such that

$$u \to A_\pm, \quad x \to \pm\infty. \tag{9}$$

In Fig. 9 it is denoted by a broken line going from the critical point A_+ to A_- for small $\mu > 0$. Therefore the averaged Whitham equation also has a stationary solution on which the quantities (7), (8) are constant. By (7), (8), such a solution can be determined by one quadrature. Its graph is given in Figure 10. In this solution the oscillatory zone Δ continues infinitely to the left. If $A_\pm = \mp 1$, then $V = 0$, and the forward front $r_2^+ = r_3^+$ is at a finite point x^+, where $r_2^+ = r_3^+ = 1/2$, while $r_1^- = r_2^- = -1/2$, where $x^- = -\infty$. We can reduce the problem to this situation by scaling and Galilean transformations

$$x \to x + Ct, \quad r_i \to r_i + D, \quad v_i \to v_i + C. \tag{10}$$

The stationary solution (see above) and the averaged equation (2) were found in [2], [16].

In [2] the evolution of multi-valued functions $r(x, t)$ in the presence of small viscosity was investigated numerically. This class of functions coincides with (8.18), (8.19) at every given moment t of time, but for the quantities $\dot{r}^{\pm}(t)$, $\dot{x}^{\pm}(t)$ we obtain

$$\dot{r}^{+} = -(r^{+} - \overset{+}{r_1})^2[(12a_{+})^{-1} + 16/45], \tag{11}$$

$$\dot{x}^{+} = v^{+} = (\overset{+}{r_1} + 2r^{+})/3, \tag{12}$$

$$\dot{r}^{-} = -1/2a_{-}, \quad \dot{x}^{-} = 2r^{-} - \overset{-}{r_3}. \tag{13}$$

We see that equations (11), (12) for $\dot{x}^{+}$, $\dot{r}^{+}$ are not the same as (8.18). Moreover, if there are terms on the right-hand side, the values of r_i are not conserved along characteristics. This leads to certain numerical complications, but the numerical scheme remains in principle the same.

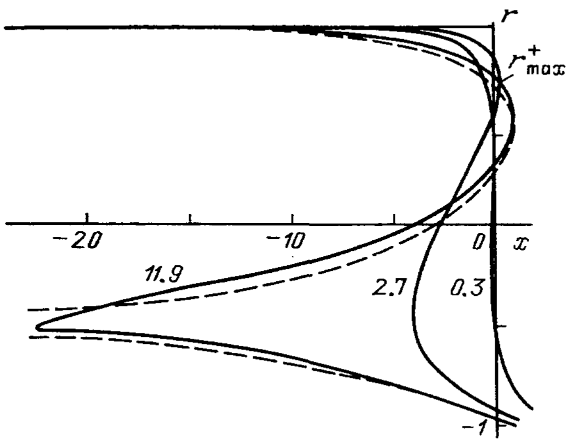

Fig. 10. Evolution of the multi-valued function $r(x, t)$ for $\mu = 0, 1$. The broken line denotes the stationary solution. Here and in Figs. 11, 13, the numbers next to the curves indicate the time.

In the computation the initial condition is taken to be the GP self-similar solution of Problem 2 in some small region $\Delta(t_0)$ at time $t_0 > 0$, and it converges to a constant as $|x| \to \infty$. Let us discuss now the applicability of our scheme at the level of rough physical estimates.

Stage 1. Let there be no oscillatory zone for $t < 0$. In order to apply the trivial equation $u_t + uu_x = 0$, it is necessary that the conditions

$$\begin{aligned} |u_{xxx}| &\ll |uu_x|, \\ |\mu u_{xx}| &\ll |uu_x| \end{aligned} \tag{14}$$

hold. Let the solution $x = ut + P(u)$ be such that $P(u)$ changes on a characteristic scale A in the variable u, we denote the characteristic scale of changes in x by B. It follows from (14) that

$$AB^{-3} \ll A^2B^{-1}, \quad \mu AB^{-2} \ll A^2B^{-1}. \tag{15}$$

Stage 2. When $t \simeq 0$ an oscillatory zone is formed close to the region of "toppling". By time t_0 it has already developed into a GP self-similar solution of Problem 2. Here, in the region $\Delta(t)$, $0 \leqslant t \leqslant t_0$, we can neglect the viscous term μu_{xx}:

$$(16) \qquad |\mu u_{xx}| \ll |u_{xxx}|, \quad |\mu u_{xx}| \ll |u u_x|$$

(in $\Delta(t)$). It follows from (16) that

$$(17) \qquad \mu (\Delta r)(\Delta x)^{-2} \ll (\Delta r)(\Delta x)^{-3}, \quad \mu (\Delta r)(\Delta x)^{-2} \ll (\Delta r)^2(\Delta x)^{-1},$$

where $\Delta x \simeq \Delta(t_0)$, $\Delta r \simeq r_3^-(t_0) - r_1^+(t_0)$.

The end of stage 2 (*beginning of stage* 3). When $t \simeq t_0$ the time derivative inside $\Delta(t_0)$ must be mostly determined by the non-viscous part, that is,

$$(18) \qquad (\Delta r)/t_0 \gg \mu\, |g_i(r)| \simeq \mu (\Delta r)^2,$$

since $g_i(\lambda r) = \lambda^2 g_i(r)$. Moreover, it is necessary that $\Delta(t_0)$ and $\Delta r(t_0)$ be small:

$$(19) \qquad \Delta r \ll A.$$

Finally, many periods of oscillation must be accommodated in the zone $\Delta(t_0)$, where

$$(20) \qquad T^{-1} = \frac{\pi}{K(s)}\left(\frac{a}{6s^2}\right)^{1/2}, \qquad T \ll \Delta(t_0)$$

and $T(\lambda r) = \lambda^{-1/2} T(r)$. Therefore (19) takes the form

$$(21) \qquad (\Delta r)^{-1/2} \ll \Delta(t_0).$$

Let us introduce "dimensionless variables"

$$(22) \qquad x = Bx', \quad u = Au', \quad t = BA^{-1}t', \quad \mu = B^{-1}\mu'.$$

In these dimensionless variables we have in the GP regime

$$(23) \qquad \Delta'(t_0') \simeq (t_0')^{3/2}, \quad (\Delta r') \simeq (t_0')^{1/2},$$

where $\Delta r = A\Delta r'$, $t_0' = BA^{-1}t_0$, $\Delta = B\Delta'$. Comparing inequalities (21)-(23), we obtain

$$(24) \qquad t_0' \ll 1, \quad AB^2 \gg 1, \quad AB^2 \gg \mu',$$

$$(25) \qquad (\mu')^{-2/3} \gg t_0' \gg (AB)^{-8/7}.$$

Conditions (24), (25) are clearly compatible. Thus, the process we have described could occur. In the new variables x', u', μ', t' we have $A_\pm = \mp 1$. The quantities A and B fall out from the Whitham equations. They have to do only with the fact that the system comes from KdVB. The quantity μ' does not have to be small in dimensionless variables. According to (24) we had to have $\mu = B^{-1}\mu'$ small and AB^2 large. Keeping this in mind, we omit the primes in what follows.

In the numerical experiment we took the function $x = tu + 3(u - \tanh^{-1} u)$ outside Δ. When $t = t_0$ the GP solution and the interval Δ have the form

$$r_i(x, t_0) = (t_0 - t_1)^{1/2} R_i(z) + r_0, \tag{26}$$

$$z^- \leqslant z \leqslant z^+, \quad z = \lambda(x - x_1)(t_0 - t_1)^{-3/2}, \tag{27}$$

by performing scaling and Galilean transformations on the original GP solution. The parameters (t_1, λ, r_0, x_1) must be chosen in such a way that the function $r(x, t_0)$ be C^1-smooth; this is the condition for matching it with $u(x^\pm)$, $u'(x^\pm)$ at the ends of the (as yet unknown) oscillatory zone $\Delta(t_0)$. Thus, the numbers of parameters and of conditions are both four, so the oscillatory zone Δ is uniquely determined locally by the outer function $u(x, t_0)$. The results of numerical experiments are shown in Figs. 10–13 for $\mu = 1$ and $\mu = 0.1$. The quantities $V(r) = (A_+ + A_-)/2$ and $A_+A_-(r)$ are taken as indicators of how close the solution is to the stationary one, where $A_+ \equiv -1, A_- \equiv +1$.

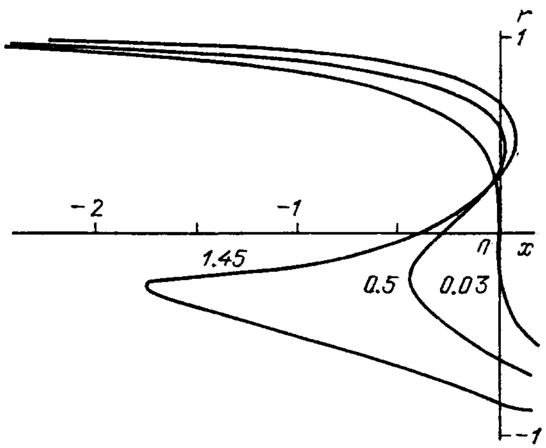

Fig. 11. Evolution of $r(x, t)$ when $\mu = 1$.

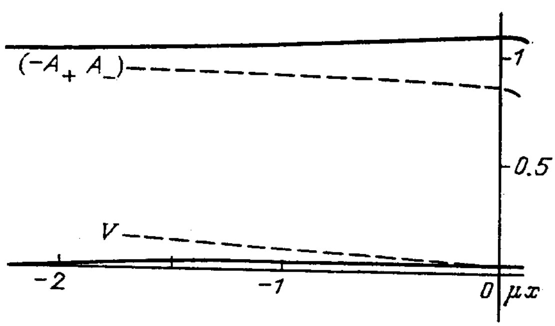

Fig. 12. Quantities that measure closeness to the stationary solution: curve 1 is $V(x)$, curve 2 is $(-A_+A_-)$ as functions of μx for $\mu = 0.1$ and $t = 11.9$. The broken curves are the corresponding curves for $\mu = 1$ and $t = 1.45$.

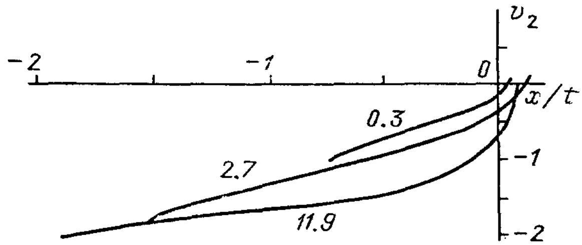

Fig. 13. Evolution of $V_2(x, t)$ for $\mu = 0.1$. The time corresponds to the maximum of $r^+(t)$ (see Fig. 10).

Conclusions. For $\mu = 0.1$ and $t = 2.7$ we see that the GP regime of Problem 1 is realized as intermediate asymptotics with $v_2 \simeq xt^{-1}$. For $\mu = 1$ this regime is never realized as an intermediate step in the dynamics. Its time of occurrence competes with the viscous term.

For all μ as $t \to \infty$ we see that the solution converges asymptotically to the stationary solution determined above. We remind the reader that at the end we changed the notation used for the variables by removing primes (see above). Therefore the new μ is what was earlier denoted by μ'. It does not have to be small.

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