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Weakly deformed soliton lattices

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Introduction

Soliton lattices are solutions of nonlinear evolution equations

$$\phi_t = K(\phi, \phi_x, \dots, \phi^{(n)}) \tag{0.1}$$

periodic (and quasiperiodic) in the variables x and t, $\phi = (\phi^\alpha)$, $K = (K^\alpha)$ are vector-valued functions (we consider here only (1+1)-equations). These solutions have the form

$$\phi(x,t) = \Phi(kx + \omega t + \tau^0; u^1, \dots, u^N) \tag{0.2}$$

where $\Phi = \Phi(\tau_1, \dots, \tau_m; u^1, \dots, u^N)$ is a vector-valued function 2π- periodic in each variable $\tau_1, \dots, \tau_m$, $k = (k_1, \dots, k_m)$ and $\omega = (\omega_1, \dots, \omega_m)$ are m-vectors of *wave numbers* and *frequencies*. These vectors do depend on the parameters $u = (u^1, \dots, u^N)$; $k = k(u), \omega = \omega(u)$. $\tau^0 = (\tau_1^0, \dots, \tau_m^0)$ are arbitrary numbers. For fixed values of the parameters $u = (u^1, \dots, u^N)$ the formula (0.2) gives the so-called exact m-phase oscillating solutions of the evolution equations (0.1).

For linear systems the one-phase oscillating solutions of (0.1) are standard exponentials

$$\phi(x,t) = ue^{i(kx+\omega t)} \Leftrightarrow \Phi(\tau; u) = ue^{i\tau}. \tag{0.3}$$

Here the wave-number k and frequency ω do not depend on the amplitude u (but they obey a suitable dispersion relation $f(k,\omega) = 0$).

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Multiphase solutions are linear superposition of the exponentials (0.3),

$$\phi(x,t) = \sum_{a=1}^{m} u^a e^{i(k_a x + \omega_a t + \tau_a^0)}.$$

One-phase oscillating solutions of nonlinear evolution equations (0.1) can be found very often explicity. E.g., for nonlinear Klein-Gordon equation

$$q_{tt} - q_{xx} + V'(q) = 0, \tag{0.4}$$

($V(q)$ is the potential) the two-dimensional family of one-phase solutions has the form $q(x,t) = Q(kx + \omega t; E)$,

$$\sqrt{\omega^2 - k^2} \int \frac{dQ}{\sqrt{2(E - V(Q)}} = d\tau. \tag{0.5}$$

The *dispersion relation* for the parameters ω, k, E has the form

$$\frac{\sqrt{\omega^2 - k^2}}{2\pi} \oint \frac{dQ}{\sqrt{2(E - V(Q)}} = 1 \tag{0.6}$$

where the integral is defined over the whole period of oscillation $V(Q) \leq E$.

The existence of multiphase oscillating solutions of nonlinear evolution systems (0.1) is a feature of integrable systems. (These solutions can be regarded as nonlinear superpositions of one-phase solutions). In the theory of integrable systems rich families of multiphase oscillating solutions - soliton lattices (0.2) - are well-known. They were found and investigated in the middle of seventies [1] - [7]. Usually they are called finite-gap solutions because of its very remarkable relations with the spectral theory of finite-gap linear differential operators with periodic and almost periodic coefficients. They are periodic and quasiperiodic analogues of multisoliton solutions because they do reduce to solitons (for $m = 1$) or to multisoliton solutions (for $m > 1$) for special values of the parameters $u^1, \ldots, u^N$. In the theory of solitons the general complex solutions of the form (0.2) are called algebraic-geometry solutions because they are given in terms of θ-functions of Riemann surfaces and they can be constructed using algebraic geometry techniques.

Now let me introduce weakly deformed solitons lattices. Let ϵ be a small parameter and $X = \epsilon x, T = \epsilon t$ slow variables.

Definition.- The weakly-deformed soliton lattice is a function of the form (0.2) (for any fixed t) such that the parameters $u^1, \ldots, u^N$ are smooth functions of the slow variable $X = \epsilon x$.

That means that the parameters $u^1, \ldots, u^N$ are slow varying functions of the x. The Whitham hypothesis (1965 [21]; see also [13] - [20]) (for $m = 1$) claims that the $\epsilon t = T$-dependence of parameters $u^1(X.T), \ldots, u^N(X,T)$ of a weakly-deformed soliton lattice is uniquely defined from a system of the form

$$u^i_T = v^i_j(u) u^j_X, \qquad i = 1 \ldots, N \tag{0.7}$$

(summation over repeated indices will be assumed). The matrix of coefficients $(v^i_j(u))$ depends on the initial system (0.1) and its family of exact solutions (0.2).

More precisely, if the function

$$\begin{aligned} &\phi_0(x,t) = \Phi(\epsilon^{-1} S(X,T); u(X,T)), \\ &S = (S_1, \ldots, S_m), \qquad X = \epsilon x, T = \epsilon t, \\ &\frac{\partial S_a}{\partial X} = k_a(u(X,T)), \qquad \frac{\partial S_a}{\partial T} = \omega_a(u(X,T)) \end{aligned} \tag{0.8}$$

is a leading term of the asymptotic oscillating solution of the system (0.1) having the form

$$\phi(x,t) = \phi_0(x,t) + \epsilon \phi_1(x,t) + \epsilon^2 \phi_2(x,t) + \ldots, \tag{0.9}$$

then the equation (0.7) holds for the parameters $u^1(X,T), \ldots, u^N(X,T)$. Therefore the system (0.7) describes the evolution of weak deformations of the soliton lattice (0.2). From the formal point of view the equations (0.7) look like equations of hydrodynamic type. Up to now all physical applications of the theory of weakly deformed soliton lattices for nonlinear evolution equations belong to the field of dispersive waves in nonlinear media without energy dissipation (see [8] -[21]). The plan of my lectures is:

1) Theory of integrable systems of hydrodynamic type (0.7).
2) Their Hamiltonian formalism i.e. theory of Poisson brackets of hydrodynamic type. It turns out to be extremely connected with Riemannian geometry. Generalisations of the Poisson brackets of hydrodinamic type for multidimensional spaces are also related to the theory of some infinite-dimensional Lie algebras. The space discretization of the Poisson brackets of hydrodynamic type is described in terms of r-matrices.
3) Applications of algebraic geometry to effective integration of the integrable equations of hydrodynamic type.

The lectures are based on the works of Novikov, Dubrovin, Krichever, Tsarev [22] - [31].

1. Poisson brackets of hydrodynamic type.

It is very remarkable that the class of systems of hydrodynamic type (SHT)

$$u^i_t = v^i_j(u)u^j_x \tag{1.1}$$

is invariant under local transformations of coordinates in the target space: $u \to w(u)$

$$v^i_j(u) \to v^{i'}_{j'}(w) = v^i_j(u(w))\frac{\partial w^{i'}}{\partial u^i}\frac{\partial u^j}{\partial w^{j'}}. \tag{1.2}$$

Riemannian invariants (if they exist) are coordinates in which the matrix v^i_j becomes diagonal, $v^i_j = v_j\delta_{ij}$ (then v_j are characteristic velocities). All the Hamiltonians we need are functionals of hydrodynamic type

$$H[u] = \int h(u)dx \tag{1.3}$$

(the density $h(u)$ does not depend on derivatives $u_x, u_{xx}, \ldots$). Poisson brackets of hydrodynamic type (PBHT) have the form

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + b^{ij}_k(u(x))u^k_x(x)\delta(x-y) \tag{1.4}$$

($\delta(x)$ is the standard Dirac delta-function). A Hamiltonian of hydrodynamic type generates via PBHT Hamiltonian SHT

$$u^i_t(x) = \{u^i(x), H\} \equiv \left(g^{ik}(u)\frac{\partial^2 h(u)}{\partial u^k \partial u^j} + b^{ik}_j(u)\frac{\partial h(u)}{\partial u^k}\right)u^j_x. \tag{1.5}$$

Example 1. Classic compressible fluid (see e.g. [32]).

We have here three field-theoretical variables

$$\begin{aligned} u^1(x) &\equiv p(x) && \text{(momentum density)}\\ u^2(x) &\equiv \rho(x) && \text{(mass density)}\\ u^3(x) &\equiv s(x) && \text{(entropy density).}\end{aligned}$$

The PBHT have the following form:

$$\begin{aligned} \{p(x), p(y)\} &= 2p(x)\delta'(x-y) + p_x(x)\delta(x-y)\\ \{p(x), \rho(y)\} &= \rho(x)\delta'(x-y)\\ \{p(x), s(y)\} &= s(x)\delta'(x-y)\\ \{\rho(x), \rho(y)\} &= \{s(x), s(y)\} = \{\rho(x), s(y)\} = 0. \end{aligned} \tag{1.6}$$

The Hamiltonian has the form

$$H = \int [\frac{p^2}{2\rho} + \mathcal{E}(\rho, s)]dx \tag{1.7}$$

where $\mathcal{E}(\rho, s)$ is the energy density.

Example 2. Relativistic fluid.

The equations of motion have the form

$$\frac{\partial T^{ij}}{\partial x^j} = 0, \qquad i,j = 1, x^0 = t, x^1 = x. \tag{1.8}$$

Here

$$(T^{ij}) = \begin{pmatrix} \epsilon & -p \\ -p & \epsilon - 2q \end{pmatrix}$$

is the energy-momentum tensor and $2q = T_i^i$. The unique constraint of the T^{ij} tensor is the state equation

$$f(\mathcal{E}, \mathcal{P}) = 0 \tag{1.9}$$

where $\mathcal{E}, \mathcal{P}$ are, respectively, the enery density and the pressure in the comoving frame (then

$$T^{ij} = \begin{pmatrix} \mathcal{E} & 0 \\ 0 & -\mathcal{P} \end{pmatrix}$$

in the comoving frame, and its trace $T_i^i = 2q = \mathcal{E} - \mathcal{P}$). We have here two field-theoretical variables $u^1 = \epsilon$ (energy density), $u^2 = \rho$ (momentum density) and PBHT of the form

$$\begin{aligned} \{p(x), p(y)\} &= \{\epsilon(x), \epsilon(y)\} = 2p(x)\delta'(x-y) + p_x(x)\delta(x-y) \\ \{p(x), \epsilon(y)\} &= 2(\epsilon(x) - q(x))\delta'(x-y) + (\epsilon - 2q)_x\delta(x-y). \end{aligned} \tag{1.10}$$

The equations of motion (1.8) have Hamiltonian form with the functional $H = \int \epsilon dx$ as Hamiltonian.

The general structure of one-dimensional PBHT is given by the following theorem [22].

<u>*Theorem 1.*</u>

1) Under local transformations of the field-theoretical variables of the form $u^i(x) \to w^i(x) = w^i(u(x))$ (i.e. changes of coordinates $u^i \to w^i = w^i(u)$ in the target space) the coefficients $g^{ij}(u)$ of the PBHT (1.4) transform like a metric tensor (with two superscripts) and the coefficients $b_k^{ij}(u)$ transform like an affine connection (more precisely, for a non-degenerate metric, $\det(g^{ij}) \neq 0$, one can define the Christoffel symbols

Γ^i_{jk} via the formula $b^{ij}_k = -g^{is}\Gamma^j_{sk}$ and the coefficients Γ^i_{jk} transform like coefficients of an affine connection).

2) In the nondegenerate case, $\det(g^{ij}) \neq 0$, the PBHT (1.4) satisfy skew-symmetry and the Jacobi identity iff the metric g^{ij} is symetric, $g^{ij} = g^{ji}$, and the connection $b^{ij}_k = -g^{is}\Gamma^j_{sk}$ is the Levi-Civita connection of the metric g^{ij} (i.e. it is symmetric $\Gamma^i_{jk} = \Gamma^i_{kj}$ and compatible with the metric g^{ij}: $\nabla_k g^{ij} = 0$) and the curvature of this connection vanishes.

Corollary

1) The unique (local) invariant of the PBHT (1.4) with a nondegenerate metric, $\det(g^{ij}) \neq 0$, is the signature of this metric (i.e. the number of positive and negative coefficients in its canonical form)
2) There exists *flat* local coordinates $w^i = w^i(u)$ in the target space such that the PBHT (1.4) have in these coordinates a constant form

$$\{w^i(x), w^j(y)\} = \tilde{g}^{ij}\delta'(x-y), \qquad \tilde{g}^{ij} = \tilde{g}^{ji} = \text{cte}. \tag{1.11}$$

Nevertheless it turns out that, usually, the physical coordinates are not flat coordinates. In the example 1 the three-dimensional metric g^{ij} has the form

$$g^{ij} = \begin{pmatrix} 2p & \rho & s \\ \rho & 0 & 0 \\ s & 0 & 0 \end{pmatrix}. \tag{1.12}$$

It is identically degenerate (the theory of PBHT) with degenerate metric was partially investigated in [33]). But for barotropic fluids, $s = $ cte., we have only two field-theoretic variables $u^1 = p, u^2 = \rho$ and the corresponding metric

$$g^{ij} = \begin{pmatrix} 2p & \rho \\ \rho & 0 \end{pmatrix}$$

is nondegenerate. Its signature is (1.1).

For a relativistic fluid (see example 2 above) the two-dimensional metric

$$g^{ij} = 2\begin{pmatrix} p & \epsilon - q \\ \epsilon - q & p \end{pmatrix}$$

is non degenerate and has a signature (1,1).

The proof of Theorem 1 is straightforward (see [31]). This Theorem establishes a very important relation between the theory of Hamiltonian SHT and differential geometry[1]. In particular this relation is very fruitful in the theory of integrability of SHT (see section 3 below).

A necessary and sufficient condition for a SHT (1.1) to be Hamiltonian is given by the following proposition [28] .

Lemma. A SHT (1.1) is a Hamiltonian system iff there exists a metric $g^{ij}(u)$ of zero curvature on the target space such that

$$g_{ik}v_j^k = g_{jk}v_i^k \tag{1.13}$$

$$\nabla_i v_j^k = \nabla_j v_i^k \tag{1.14}$$

Here ∇_i denotes the covariant derivative associated with the metric $(g^{ij}) = (g_{ij})^{-1}$.

The proof is almost evident if one rewrites the Hamiltonian SHT (1.5) with the Hamiltonian (1.3) in the following manner:

$$u_t^i = v_j^i(u)u_x^j = g^{ij}\nabla_k\nabla_j h(u)u_x^j. \tag{1.15}$$

It can be shown [28] that for generic Hamiltonian SHT the PBHT (i.e. the metric g^{ij}) is uniquely defined from the relations (1.13), (1.14).

Some generalizations.

1) Non-homogeneous systems of hydrodynamic type

$$u_t^i = v_j^i(u)u_x^j + f^i(u) \tag{1.16}$$

require non-homogeneous PBHT having the following form:

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + b_k^{ij}(u(x)u_x^k(x)\delta(x-y) + h^{ij}(u)\delta(x-y). \tag{1.17}$$

Here h^{ij} defines (finite-dimensional) Poisson brackets on the target space.

Theorem 2. [23] Under the assumption of non-degeneracy, $\det(g^{ij}) \neq 0$, in flat coordinates $w^i = w^i(u)$ the non-homogeneous PBHT (1.17) have the form

$$\{w^i(x), w^j(y)\} = \tilde{g}^{ij}\delta'(x-y) + (c_k^{ij}w^k + c_0^{ij})\delta(x-y) \tag{1.18}$$

[1] The general notion of differential geometry Poisson brackets was introduced in [23] (see also [31]).

where $\tilde{g}^{ij}, c_k^{ij}, c_0^{ij}$ are some constants. More precisely, c_k^{ij} are structure constants of a Lie algebra $\mathcal{G}$ with invariant inner product $\tilde{g}^{ij}$ and c_0^{ij} is a 2-cocycle on this Lie algebra.

On the *dual space* of linear functionals of the form $\int a_i(x)w^i(x)dx + b$ the PB (1.18) defines a structure of a Lie algebra. This is a well-known Kac-Moody Lie algebra.

2) Multidimensional space case. The corresponding PBHT have the form

$$\begin{gathered}\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta_\alpha(x-y) + b_k^{ij}(u(x)u_\alpha^k\delta(x-y) \\ x = (x^\alpha), \quad \alpha = 1, \dots, d, \quad u_\alpha^k \equiv \partial u^k/\partial x^\alpha.\end{gathered} \tag{1.19}$$

Here we have a family of metrics $g^{ij\alpha}$ whose symmetric metric connections $b_k^{ij\alpha}$ have zero curvature, $\alpha = 1, \dots, d$. But the simultaneous reduction of the PBHT (1.19) to a constant form is in general impossible. The following proposition was proved in [23].

<u>*Theorem 3*</u>.- In the non-degenerate case, $\det(g^{ij\alpha}) \neq 0$, for $d \geq 2$ the PBHT (1.19) in some coordinates $w^9 = w^i(u)$ can be reduced to a linear form

$$\{w^i(x), w^j(y)\} = (g_k^{ij\alpha}w^k(x) + g_0^{ij\alpha})\delta_\alpha(x-y) + b_k^{ij\alpha}w_\alpha^k\delta(x-y) \tag{1.20}$$

where $g_k^{ij\alpha}, g_0^{ij\alpha}, b_k^{ij\alpha}$ are some constant coefficients. (They obey a complicated system of equations which shall not be discussed here - see [31]).

We see that multidimensional PBHT can be reduced to a non-homogeneous PBHT linear form. That means that these PBHT are the so-called Lie-Poisson brackets for some (infinite-dimensional) Lie-algebra. In other words, the dual space of linear functionals (let $g_0^{ij\alpha} \equiv 0$) has a natural Lie algebra structure :

$$\begin{gathered}\{A, B\} = C, \qquad A = \int A_i(x)w^i(x)d^dx, \qquad B = \int B_i(x)w^i(x)d^dx \\ C = \int C_i(x)w^i(x)d^dx, \qquad C_k = A_i b_k^{ji\alpha}\partial_\alpha B^j - B_i b_k^{ji\alpha}\partial_\alpha A^j.\end{gathered} \tag{1.21}$$

The class of PBHT linear in field variables is interesting even in the case of a one-dimensional space $d = 1$. the corresponding infinite-dimensional Lie algebras and their central extensions of Virasoro type were investigated in [30][34].

3) Space discretization (only 1-dimensional case).

It can be shown that the correct space discretization of PBHT has the following form:

$$\begin{gathered}\{u_n^i, u_m^j\} = h_{m-n}^{ij}(u_n, u_m), \qquad n, m \in Z, \\ h_k^{ij} = 0 \qquad for \qquad |k| > 1.\end{gathered} \tag{1.22}$$

Under changes of coordinates in the target space $u^i \to z^i(u)$ the coefficients h_k^{ij} transform in the following way:

$$h_k^{ij}(u,v) \to \frac{\partial z^i(u)}{\partial u^p}\frac{\partial z^j(v)}{\partial v^q} h_k^{pq}(u,v). \tag{1.23}$$

The standard procedure of taking the continuous limit $u_n^i = u^i(n\epsilon)$, $\epsilon \to 0$, realizes the PB (1.22) as a family of non-local PB of the form

$$\{u^i(x), u^j(y)\} = \sum_k h_k^{ij}(u(x), u(y))\delta(x - y + k\epsilon). \tag{1.24}$$

It turns out [29] that the PB of the form (1.22) for non-degenerate h_k^{ij}), ($\det(h_1^{ij}(u,u) \neq 0)$, can be parametrized by Lie-Poisson groups (i.e. semi-classic limits of quantum groups). We give here explicit formulae only for the case of PB related to the so-called triangular Lie-Poisson groups (see [35]). Let $r = (r^{\alpha\beta}) \in \mathcal{G} \otimes \mathcal{G}$ be a solution of the classical Yang-Baxter equation for the Lie algebra $\mathcal{G}$

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1.25}$$

(r_{12}, r_{13}, r_{23} are defined in the following way: if $r = \sum a_i \otimes b_i$ then $r_{12} = \sum a_i \otimes b_i \otimes 1$, $r_{13} = \sum a_i \otimes 1 \otimes b_i$, etc.) The target space for the PB (1.12) in this case coincide with the Lie group G with Lie algebra $\mathcal{G}$. The PB is of the form

$$\begin{aligned}
&\{\varphi(u_n), \psi(u_{n+1})\} = -r^{\alpha\beta}\partial_\alpha\varphi(u_n)\partial'_\beta\psi(u_{n+1}),\\
&\{\varphi(u_n), \psi(u_n)\} = r^{\alpha\beta}[\partial_\alpha\varphi(u_n)\partial_\beta\psi(u_n) + \partial'_\alpha\varphi(u_n)\partial'_\beta\psi(u_n)],\\
&\{\varphi(u_n), \psi(u_m)\} = 0 \qquad for \qquad |m-n| > 1.
\end{aligned} \tag{1.26}$$

Here φ, ψ are arbitrary smooth functions on the group G, and ∂_α ∂'_α left and right-invariant vector-fields on the group G. Some special cases of PB of the form (1.26) (for $\dim G = 2$) can be found in [36].

2. Integrability of SHT

The integrability of two-component SHT

$$u_t^i = v_j^i(u)u_x^j, \qquad i = 1\ldots, N \tag{2.1}$$

($N = 2$) is well-known: the system (2.1) can be linearized using the so-called hodograph transformation $(u^1, u^2) \leftrightarrow (x, t)$. This method can not be generalized for $N \geq 3$. So we shall consider the integrability of SHT (2.1) only for $N \geq 3$.

The main contribution in the theory of integrability of SHT was made by Novikov. His hypothesis (1983) was that a Hamiltonian SHT is integrable if it can be diagonalized (i.e. there are Riemann invariants for it). This hypothesis was proved by Tsarev [28] (see also [31]). He also found a generalization of the *hodograph method* for diagonal Hamiltonian SHT of the form

$$u_t^i = v_i(u)u_x^i, \qquad i = 1\dots, N \tag{2.2}$$

(in this paragraph there is no summation over repeated indices !). That means that the SHT is reduced to Riemann invariants. The main results are in the following theorems.

Theorem 4. Diagonal Hamiltonian SHT are completely integrable on the subspace of monotonic functions $u^1(x), \dots, u^N(x)$.

Theorem 5.- Let $w^1(u), ...w^N(u)$ be arbitrary solutions of the following linear system:

$$\partial_i w_j = \frac{\partial_i v_j}{v_i - v_j}(w_i - w_j) \qquad (i \neq j) \tag{2.3}$$

(here $\partial_i \equiv \partial/\partial u^i$). Then, the functions $u^1(x,t), \dots, u^N(x,t)$ given implicitly by the system

$$w^i(u) = v_i(u)t + x, \qquad i = 1\dots, N, \tag{2.4}$$

are solutions of the diagonal Hamiltonian SHT (2.2). Any set of solutions $u^1(x,t), \dots, u^N(x,t)$ such that $u_x^i \neq 0$ can be locally obtained in such a way.

Let me briefly comment on these theorems. What does *complete integrability* of a diagonal Hamiltonian SHT mean? It means that a suitable infinite-dimensional analogue of Liouville Theorem holds for such systems. First of all it can be shown that there is a sufficiently rich family of conservation laws of this system with pairwise commuting integrals of motion. All of them have a hydrodynamic form $I[u] = \int P(u)dx$. And the words *sufficiently rich* mean that any tangent vector to a level surface of all these commuting integrals of motion can be represented as a linear combinations of their skew-gradients. Algorithms for finding integrals of motion for diagonal Hamiltonian SHT will be given below.

Let us proceed now to the proof of Theorem 5. The crucial point is the following proposition.

Lemma. The metric g^{ij} of the PBHT associated to diagonal SHT with pairwise different characteristic velocities $v_i \neq v_j$ in the coordinates $u^1, \dots, u^N$ is also diagonal : $g^{ij} = g^i\delta^{ij}$

Proof. It is obvious from the symmetry (1.13). Using the second symmetry (1.14) it can be shown that the system (2.3) is compatible (i.e. its general solution depends on N arbitrary functions of one variable). We omit the details. It is more important to emphasize that any solution of the system (2.3) defines a commuting flow of the form

$$u_s^i = w_i(u)u_x^i, \qquad i = 1\dots, N \tag{2.5}$$

for the initial SHT (2.2):

$$(u_s^i)_t = (u_t^i)_s.$$

We see that these commuting flows also are diagonal SHT. It can be shown that they are Hamiltonian SHT whose Hamiltonians were found in the previous Theorem 4. It also can be shown that (2.3), (2.5) is a general description of commuting flows for (2.2). To achieve the proof of Theorem 5 one needs to check up that the solutions of the system (2.4) satisfy (2.2). This proof is straightforward.

From the previous lemma and Theorem 1 we can get a very important conclusion: the theory of integrable systems of hydrodynamic type is closely related to the classical theory of *N-orthogonal systems of curvilinear coordinates.* Every N-orthogonal system of coordinates (i.e. diagonal metric of zero curvature -not necessarily positive definite) gives rise to a family of integrable SHT, their integrals and commuting flows. I shall give now the explicit algorithm.

Let

$$ds^2 = \sum_{i=1}^{N} \sigma_i \; h_i^2(u)(du^i)^2 \tag{2.6}$$

be a diagonal N-dimensional metric, $\sigma_i = \pm 1$, (not necessarily having zero curvature). Let us introduce the so-called Darboux coefficients (or rotation coefficients)

$$\beta_{ij} = \frac{\partial_j h_i}{h_j}, \qquad i \neq j. \tag{2.7}$$

From the definition (2.7) we also have :

$$\partial_j h_i = \beta_{ij} h_j. \tag{2.8}$$

The compatibility equations of (2.8) have the following form:

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj}, \qquad k \neq i, j. \tag{2.9}$$

Then, in order to define characteristic velocities $w_1(u), \ldots, w_N(u)$ of commuting flows of the form (2.5) one needs to solve the system

$$\partial_j \overline{w_i} = \beta_{ji} \overline{w_j} \tag{2.10a}$$

$$\overline{w_i} \equiv h_i w_i. \tag{2.10b}$$

and similarly the solutions of the system

$$\partial_j p_i = \beta_{ji} p_j \tag{2.11a}$$

$$p_i = h_i^{-1} \partial_i P \tag{2.11b}$$

are needed for the definition of the densities of the integrals $I = \int P(u)dx$. The flows (2.5), (2.10a), (2.10b) will be a Hamiltonian SHT iff the metric (2.6) has zero curvature.

Remark 1.- If the curvature of the diagonal metric (2.6) is not zero, the diagonal SHT (2.5), (2.10a), (2.10b) is not a Hamiltonian system. But it has a rich family of commuting flows (2.5), (2.10a), (2.10b) and integrals (2.11a), (2.11b). It can be shown that such systems can also be linearized using the techniques of the proof of Theorem 5. (In [28] such SHT were called semi-Hamiltonian systems).

Remark 2.- For the so-called Egoroff metrics the rotation coefficients satisfy one more equation

$$\sum_k \partial_k \beta_{ij} = 0. \tag{2.12}$$

Such metrics have zero curvature if the rotation coefficients satisfy the condition

$$\beta_{ji} = \sigma_i \sigma_j \beta_{ij}. \tag{2.13}$$

The system of the non-linear equation (2.9), (2.13) is well-known in the theory of solitons [42]. It can be obtained from the system of equations of motion of the *N-dimensional rigid body* (see details in [7]). The most important application of this observation is the construction of self-similar solutions of SHT using the modern theory of Painleve-type equations. It should be noted that the Hamiltonian structure of the equations of motion of weakly deformed soliton lattices for integrable equations is defined by Ergoroff metrics (see below).

It is not trivial to complete the program of Theorem 5 for interesting cases of SHT.

Example. Benny equations [37].

They describe one dimensional waves in weakly non-linear stratified fluids without dissipation:

$$\left.\begin{aligned} u_{i_t} + u_i u_{i_x} + (\sum_{i=1}^{n} \eta_i)_x &= 0 \\ \eta_{i_t} + (u_i \eta_i)_x &= 0 \end{aligned}\right\} \qquad i = 1, \ldots, n. \tag{2.14}$$

Here $N = 2n$ and the PBHT in the variables u_i, η_i have constant form

$$\{u_i(x), \eta_j(y)\} = \delta_{ij}\delta'(x - y), \tag{2.15}$$

while the other PB vanish. The Hamiltonian is

$$H = \tfrac{1}{2} \int [\sum \eta_i u_i^2 + (\sum \eta_i)^2] dx. \tag{2.16}$$

In order too find Riemann invariants of Benny equations let us consider the algebraic curve

$$f(\lambda, \mu) = \lambda + \sum_{i=1}^{n} \frac{\eta_i}{u_i + \lambda} - \mu = 0. \tag{2.17}$$

Let $(\lambda_p, \mu_p), p = 1, \ldots, 2n$ be the branching points of this curve in the λ-plane. That means that $\lambda_1, \ldots, \lambda_{2p}$ are roots of the equation

$$\frac{\partial F}{\partial \lambda} \equiv 1 - \sum \frac{\eta_i}{(u_i + \lambda)^2} = 0. \tag{2.18}$$

It turns out [38] that $\mu_1, \ldots, \mu_{2n}$ are Riemann invariants for Benny equations, i.e.

$$\mu_{p_t} = \lambda_p(\mu_1, \ldots, \mu_{2p} \mu_{p_x}), \qquad p = 1, \ldots, 2n. \tag{2.19}$$

For diagonal entries g_{pp} of the associated metric defining a PBHT in μ_p-coordinates, the following formula holds [2]:

$$g_{pp} = \operatorname*{Res}_{\lambda=\lambda_0} \left(\tfrac{\partial F}{\partial \lambda} \right)^{-1}. \tag{2.20}$$

The general solution of equations (2.3) for commuting flows is not known up to now.

[2] Tsarev; private communication

3. Soliton latticces and average of Poisson brackets.

Let us define here the procedure [22] of averaging any field-theoretic PB over invariant tori.

We have:

1) A local field-theoretic PB of the form

$$\{\phi^\alpha(x), \phi^\beta(y)\}_\phi = \sum_s B_s^{\alpha\beta}(\phi(x), \phi'(x), \ldots)\ \delta^{(s)}(x-y). \tag{3.1}$$

(The sum is finite and its coefficients depend on a finite number of derivatives).

2) N pairwise commuting local field-theoretical functionals $I^1[\phi], \ldots I^N[\phi]$:

$$I^i[\phi] = \int P^i(\phi, \phi', \ldots)dx, \qquad \{I^i, I^j\}_\phi = 0. \tag{3.2}$$

3) A N-dimensional family of m-dimensional tori which are invariant for the flows of the Hamiltonian $I^i[\phi]$:

$$\phi(x) = \Phi(kx + \tau^0; u^1, \ldots, u^N). \tag{3.3}$$

$\Phi = \Phi(\tau_1, \ldots, \tau_m; u^1, \ldots, u^N)$ is a 2π-periodic in each $\tau_a, k = (k_1, \ldots, k_m) = k(u)$ and τ^0 is an arbitrary m-vector . The parameters $u^1, \ldots, u^N$, are chosen in such a manner that

$$I^i[\phi(x) = \Phi(kx + \tau^0; u)] = u^i, \qquad i = 1, \ldots, N. \tag{3.4}$$

Then we define a PBHT on the phase space of the fields $u^1(X), \ldots, u^N(X)$ of the form

$$\{u^i(X), u^j(Y)\}_u = g^{ij}(u(X))\delta'(X-Y) + b_k^{ij}(u)u_X^k(x)\delta(X-Y) \tag{3.5}$$

-the so-called averaged over invariant tori (3.3) PB (3.1). Let us define the coefficients A_s^{ij} via the following formula:

$$\{P^i(\phi(x), \phi'(x), \ldots), P^i(\phi(y), \phi'(y), \ldots)\}_\phi = \sum_s A_s^{ij}(\phi(x), \phi'(x), \ldots)\delta^{(s)}(x-y). \tag{3.6}$$

From commutativity $\{I^i, I^j\}_\phi = 0$ we have

$$A_0^{ij}(\phi(x), \phi'(x), \ldots) = \partial_x Q^{ij}(\phi, \ldots). \tag{3.7}$$

Let us define for any polynomial $P(\phi(y), \phi'(y), \ldots)$ the averaged function

$$\overline{P} \equiv \overline{P}(u) = (2\pi)^{-m} \int P(\Phi(\tau; u), (k\partial_\tau)\Phi(\tau; u), \ldots)d^m\tau \tag{3.8}$$

(averaging over invariant tori (3.3)). Finally we define the coefficients of averaged PBHT (3.5) via the following formulae:

$$g^{ij}(u) = \overline{A_1^{ij}}, \qquad b_k^{ij}(u) = \frac{\partial}{\partial u^k}\overline{Q^{ij}}. \tag{3.9}$$

Theorem 6.- Under the conditions given above the formulae (3.5) with g^{ij} and b_k^{ij} given by (3.9) define PBHT.

This Theorem was found in [22]; for its proof see ref. [31].

Theorem 7.- Let us now assume that the $\{\ ,\ \}_\phi$ -PB are nondegenerate, $N = 2m$, and that the invariant tori (3.3) are Liouville tori. It means that there exist action variables $J_1, \dots, J_m$ which are canonically conjugated to angular variables $\tau_1, \dots, \tau_m$. Then:

1) The metric $g^{ij}(u)$ of the form (3.9) is nondegerate.
2) The variables $k_1, \dots, k_m, J_1, \dots, J_m$ give flat coordinates for $\{\ ,\ \}_u$ - PBHT:

$$\begin{aligned} \{k_i(X), J_j(Y)\}_u &= \delta_{ij}\delta'(X - Y), \\ \{k_i(X), k_j(Y)\}_u &= \{J_i(X), J_j(Y)\}_u = 0 \end{aligned} \tag{3.10}$$

For Lagrangian system of evolution equations this Theorem can be deduced from Hayes results [18]. For the case of degenerate PB (3.1) the formulation of the Theorem 7 is slightly changed -see [22].

Let us now consider a Hamiltonian system whose Hamiltonian is one of the integrals I^i (e.g. $H = I^1$).

$$\phi_t(x) = \{H, \phi(x)\}_\phi.) \tag{3.11}$$

It has a family of solutions of a form

$$\phi(x,t) = \Phi(kx + \omega t + \tau^0; u^1, \dots, u^N), \qquad \omega = \omega(u) \tag{3.12}$$

-soliton lattices or m-phase oscillating solution. Let us look for slow deformations of these solutions as an asymptotic (in a small parameter ϵ) solution of the form

$$\phi_\epsilon(x,t) = \Phi(\epsilon^{-1}S(u(X,T)); u^1(X,T), \dots, u^N(X,T)) + \mathcal{O}(\epsilon). \tag{3.13}$$

Here $X = \epsilon x, T = \epsilon t, S = (S_1, \dots, S_m) = S(u(X,T))$,

$$S_X = k(u(X,T)), S_T = (\omega(u(X,T)). \tag{3.14}$$

(The weak limit with $\epsilon \to 0$ of the leading term for $|\epsilon x| << 1, |\epsilon t| << 1$ coincides with the exact solution: it is clear from the expansion $S(X,T) = S_0 + \epsilon kx + \epsilon\omega t + \ldots \Rightarrow$

$$\Phi(\frac{S_0}{\epsilon} + kx + \omega t + \ldots; u) \to, \Phi(kx + \omega t; u).$$

Theorem 8. [22] The dependence of the parameters $u^1, \ldots, u^N$ on the slow variables can be defined from the following Hamiltonian SHT:

$$u^i_T(X) = \{\widehat{H}, u^i(X)\}_u, \qquad \widehat{H} = \int H(u)dX. \tag{3.15}$$

(Here $H(u) = u^1 = I[\phi]$ in our coordinates).

Example.- The nonlinear Klein-Gordon equation (0.4) is a Hamiltonian system with standard PB

$$\{q(x), p(y)\} = \delta(x-y) \qquad p = q_t \tag{3.16}$$

and Hamiltonian

$$H = \int [\tfrac{1}{2}(p^2 + q_x^2) + V(q)]dx. \tag{3.17}$$

Here we have two commuting integrals

$$u^1 = I^1 = H, \qquad u^2 = I^2 = P = \int pq_x dx. \tag{3.18}$$

The one-phase soliton lattice has the form given by (0.5) and (0.6). The metric of the averaged PB has the form

$$(g^{ij}) = 2\begin{pmatrix} u^2 & u^1 - \Delta \\ u^1 - \Delta & u^2 \end{pmatrix}, \qquad \Delta = \overline{V(q)}. \tag{3.19}$$

The Hamiltonian equations (0.5) and (0.6) of weak deformations of soliton lattice coincide with the equations (1.8) of the relativistic fluid . The state equation (1.9) has the following form:

$$\mathcal{P}(\mathcal{E}) = \mathcal{E} - F(\mathcal{E})F'(\mathcal{E})^{-1}, \qquad F(E) = \frac{1}{2\pi}\oint \sqrt{2(E - V(Q))}dQ. \tag{3.20}$$

(The relation of the averaged non-linear Klein-Gordon equation with relativistic hydrodynamic was observed by Maslov [14] without any discussion of the Hamiltonian structure).

Other examples of averaged PBHT can be found in [31].

It is very important to notice that in *physical* coordinates $u^1, \ldots, u^N$ the averaged PBHT (3.10), have non-constant coefficients (3.9). But these coordinates have another differential geometry description. We say that $u^1, \ldots, u^N$ are Liouville coordinates for the PBHT (3.10) if there exists a matrix $\gamma^{ij}(u)$ that

$$g^{ij}(u) = \gamma^{ij}(u) + \gamma^{ji}(u), \qquad b^{ij}_k(u) = \frac{\partial \gamma^{ij}(u)}{\partial u^k}. \tag{3.21}$$

They are strongly - Liouville coordinates if these properties still hold after any combination of the following procedures:

1) affine transformations $u^i \to \overline{u}^i = a^i_j u^j + b^j$.

2) restriction to any part of coordinates: if $(i_1, \ldots, i_p; j_1, \ldots, j_p)$ is any partition of the set $(1, 2, ..., N)(p + q = N)$, then the matrix

$$\tilde{\gamma}^{kl}(u^{i_1}, \ldots, u^{i_p}) = \gamma^{i_k i_l}(u)|_{u^{j_1}, \ldots, u^{j_p} = fixed} \tag{3.22}$$

again defines via formulae of the type (3.21) a PBHT on the space of the fields $u^{i_1}(X), \ldots, u^{i_p}(X)$.

It was observed in [31] that according with the construction of Theorem 6 the coordinates $u^1, \ldots, u^N$ (3.4) are strongly Liouville coordinates for the averaged PBHT (3.10), with coefficients (3.9). It imposes a very strong restriction on the geometry induced on the space of parameters of invariant tori (3.3).

4. Deformations of soliton lattices for integrable equations

Integrable equations of the theory of solitons have rich families of soliton lattices of the form (3.3). Such periodic and quasi-periodic solutions can be found and investigated using methods of algebraic geometry of Riemann surfaces. It turns out [24][39] that algebraic geometry is also very efficient to describe weak deformations of these solutions.

I shall explain the main idea for the example of the Korteweg de Vries (KdV) equation

$$\phi_t = 6\phi\phi_x - \phi_{xxx}. \tag{4.1}$$

Its m-phase solutions $(m = 1, 2, \ldots)$ are defined by the following construction.

1. Parameters $r_1 < r_2 < \ldots < r_{2m+1}$. It is useful to represent them as branching points of the two-sheeted Riemann surface

$$\mu^2 = \prod_{i=1}^{2m+1} (\lambda - r_i). \tag{4.2}$$

2. We need two Abelian differentials on the surface (4.2)

$$dp = \frac{P(\lambda)}{2\sqrt{\prod(\lambda - r_i)}} d\lambda, \tag{4.3a}$$

$$P(\lambda) \equiv P(\lambda; r_1, \dots, r_{2m+1}) = \lambda^m + a_1\lambda^{m-1} + \dots + a_m \tag{4.3b}$$

$$dq = \frac{Q(\lambda)}{2\sqrt{\prod(\lambda - r_i)}} d\lambda, \tag{4.4a}$$

$$Q(\lambda) \equiv Q(\lambda; r_1, \dots, r_{2m+1}) = 12\ \lambda^{m+1} + b_0\lambda^m + \dots + b_m, \qquad b_0 = -6\sum r_i \tag{4.4b}$$

Here the coefficients $a_j = a_j(r_1, \dots, r_{2m+1}), b_j = b_j(r_1, \dots, r_{2m+1})$ are uniquely defined by the parameters $r_1, \dots, r_{2m+1}$ from the normalization conditions

$$\int_{r_{2k}}^{r_{2k+1}} dp = \int_{r_{2k}}^{r_{2k+1}} dq = 0, \qquad k = 1 \dots, m. \tag{4.5}$$

3. The $(2m+1)$−dimensional family of invariant m-tori of the KdV equation is given via the following formula:

$$\phi(x,t) = -2\partial_x^2 \log\theta(kx + \omega t + \tau^0; r_1, \dots, r_{2m+1}) + c. \tag{4.6}$$

Here $\theta(\tau; r_1, \dots, r_{2m+1})$ is the theta-function of the Riemann surface (4.2) (see definition, e.g., in [40]). Vectors $k = (r_1, \dots, r_{2m+1})$, and $\omega = \omega(r_1, \dots, r_{2m+1})$ are given via the formulae

$$k_a = 2\int_{r_1}^{r_{2a}} dp, \qquad \omega = 2\int_{r_1}^{r_{2a}} dq, \qquad a = 1, \dots, m. \tag{4.7}$$

The constant c has the form [42]

$$c = \sum r_i - 2\sum_{a=1}^{m} \int_{r_{2a}}^{r_{2a+1}} \lambda\Omega_a. \tag{4.8}$$

The solutions (4.6) reduce to multisoliton solutions in the limiting case $r_{2i} \to r_{2i-1}$, $i = 1, \dots, m$.

4. The KdV equation has an infinite series of conservation laws (*Kruskal integrals*)

$$I^0 = \int \phi dx, \qquad I^1 = \frac{1}{2}\int \phi^2 dx, \qquad I^2 = H = \int [\tfrac{1}{2}\phi_x^2 + \phi^3] dx, \dots \tag{4.9}$$

They commute under Gardner-Zakharov-Faddeev PB

$$\{\phi(x), \phi(y)\} = \delta'(x - y). \tag{4.10}$$

The integral I^2 is a Hamiltonian of the KdV equation.

5. The equations of weak deformations of the solutions (4.6) can be represented in the following form [39].

$$\partial_T dp(\lambda) = \partial_X dq(\lambda) \tag{4.11}$$

(for an elementary proof see ref. [31]). Here λ is an arbitrary parameter. From (4.11) it follows that Riemann invariants for the equation of weak deformations are:

$$r_{i_T} = v_i(r_i; r_1, \dots, r_{2m+1}) r_{i_X}, \tag{4.12a}$$

$$v_i = \frac{Q(r_i; r_1, \dots, r_{2m+1})}{P(r_i; r_1, \dots, r_{2m+1})}, \qquad i = 1, \dots, 2m+1. \tag{4.12b}$$

6. Let

$$dr = R(\lambda; r_1, \dots, r_{2m+1}) d\lambda \tag{4.13}$$

be an arbitrary piecewise-analytic differential on the Riemann surface with a prescribed jump on the real axis. This differential is uniquely defined by this jump and by the parameters $(r_1, \dots, r_{2m+1})$ from the normalization conditions of the type (4.5).

Theorem 9. [24] The general solution $r_i = r_i(X,T)$, $i = 1, \dots, 2m+1$, of the equations (4.12)of weak deformations of soliton lattices (4.6) is given implicitly by the following formula

$$(X dp + T dq - dr)_{\lambda = r_i} = 0, \qquad i = 1, \dots, 2m+1. \tag{4.14}$$

(The jump of the differential dr is a functional parameter of the general solution).

Sketch of proof. Let us consider the differential

$$\Omega = X dp + T dq - dr \tag{4.15}$$

and its derivatives

$$\partial_T \Omega = dq + (X \partial_T dp + T \partial_T dq - \partial_T dr) \equiv dq + \Omega_1. \tag{4.16}$$

$$\partial_X \Omega = dp + (X \partial_X dp + T \partial_X dq - \partial_X dr) \equiv dp + \Omega_2. \tag{4.17}$$

Let us show that the differentials Ω_1 and Ω_2 vanish. They have no jump on the real axis because the jump of dr is fixed (i.e. it does not depend on X, T). Neither they have singularities at infinity $\lambda = \infty$ (because the differentials dp, dq have fixed singularities of the form

$$\left.\begin{aligned} dp &= \left(\frac{1}{2\sqrt{\lambda}} + \mathcal{O}(\lambda^{-3/2})\right) d\lambda, \\ dq &= 6\left(\sqrt{\lambda} + \mathcal{O}(\lambda^{-3/2})\right) d\lambda, \end{aligned}\right\} \qquad \lambda \to \infty. \tag{4.18}$$

They have no singularities at the branching points $\lambda = r_i$. It holds from (4.14). Hence the differentials Ω_1 and Ω_2 are everywhere holomorphic on the Riemann surface (4.2). But they have zero periods of the type (4.5) and therefore they vanish (see any textbook on Riemann surfaces).

The compatibility condition of these equations coincides with (4.11).

We have shown that the solution of (4.14) satisfies (4.11). In order to prove the converse assertion we need to construct a piecewise-analytic differential dr starting from an arbitrary solution of equations (4.11). Let us construct the differential Ω via the formula

$$\Omega = \int (dq)\, dT = \int (dp)\, dX. \tag{4.19}$$

This is possible because of (4.11). The differential is analytic outside the real axis and it has a jump on the trajectories of the branching points $r_i = r_i(X,T)$. It is easy to show that this differential has the form (4.15) and satisfies (4.14). The Theorem is then proved.

Analytic solutions of the Gurevitch-Pitayevskii problem [10] -[42] about the dispersive analogue of shock-waves were obtained in ref.[41] by using this Theorem.

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