# Integrable systems in topological field theory

B. Dubrovin *

*INFN, Sezione di Napoli * *, and Dipartimento di Scienze Fisiche, Università di Napoli,
Mostra d'Oltremare, pad. 19, 80125 Naples, Italy*

Integrability of the system of PDE for dependence on coupling parameters of the (tree-level) primary partition function in massive topological field theories, being imposed by the associativity of the perturbed primary chiral algebra, is proved. In the conformal case it is shown that all the topological field theories are classified as solutions of a universal high-order Painlevé-type equation. Another integrable hierarchy (of systems of hydrodynamic type) is shown to describe coupling to gravity of the matter sector of any topological field theory. Different multicritical models with the given structure of primary correlators are identified with particular self-similar solutions of the hierarchy. The partition function of any of the models is calculated as the corresponding $\tau$-function of the hierarchy.

## 1. Introduction

The recent progress in low-dimensional QFT essentially was related with the application [1] of the machinery of integrable systems of KdV-type to the study of matrix models. An alternative approach to two-dimensional gravity was suggested by Witten [2] basing on the ideas of topological field theory.

Topological field theories are solvable models without local, propagating degrees of freedom [3]. After the identification [2] of one-matrix models with the topological theory of two dimensional gravity a lot of remarkable links of two-dimensional gravity and topological field theories were found [4–6]. It was shown [5] that the multi-matrix models can be identified with some matter systems coupled to topological gravity. These matter systems proved to be [7] twisted versions of the $N = 2$ minimal superconformal models [8]. Deep relations of these models to catastrophe theory were studied in ref. [9]. Correlation functions in minimal topological field theories were calculated by Dijkgraaf et al. [10] using the Landau–Ginzburg (LG) potentials machinery [11]. (This calculation was analyzed from the point of view of catastrophe theory in ref. [12].) The observation of ref. [10]

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* On leave of absence from Dept. Mech. & Math., Moscow State University, Moscow 119899, Russia.
* * E-mail: DUBROVIN@NA.INFN.IT

concerning the relation of topological $A_n$ minimal models to the generalized KdV-hierarchy (or the Gelfand–Dikii (GD) hierarchy) was elucidated by Krichever [13]. He showed that the calculation of ref. [10] has a natural interpretation as a part of the theory of the so-called dispersionless Lax hierarchy (the genus-zero semi-classical limit of the Gelfand–Dikii hierarchy). He introduced also $\tau$-functions of the dispersionless hierarchy and showed that the tree-level partition function of the minimal model (before coupling to gravity) coincides with the $\tau$-function of a particular solution of a part of the dispersionless Lax hierarchy. So topological string theories can be considered as a semi-classical approximation of ordinary string theories. The indications of refs. [2,4,5,10] about equivalence (at tree-level) of topological string theory and ordinary string theory (in $d < 1$) can be interpreted therefore like statements about exactness (in some sense) of the semi-classical approximation of the KdV hierarchy.

The construction of ref. [13] was generalized in ref. [14] and, independently, in refs. [15,16] using the genus-$g$ semiclassical approximation of the GD hierarchy (so-called Whitham-averaged GD hierarchy [17–20]). This gives [14] a multicut solution of the loop equations [21]. Also the corresponding "partition function" was proved [14] to satisfy a truncated version of the Virasoro constraints that were obtained in [22] for the partition function of two-dimensional gravity. The genus-$g$ variant of the LG model of ref. [10] (here $g$ is the "genus" of the LG superpotential) was constructed in ref. [16] using the differential geometry of moduli space [15] and the hamiltonian formulation [20,23] of the Whitham-averaging procedure. The tree-level primary chiral algebra and the partition function of this model were also calculated in refs. [15,16]. In sect. 5 (see below) I argue that these models can be obtained from minimal models as a result of "phase transition".

The main aim of this paper is to construct an "inverse spectral transform" for any 2D topological QFT (at tree-level). It is shown that two types of integrable systems are hidden in a topological field theory. The first one coincides with the equations of associativity of the primary chiral algebra (before coupling to gravity). The second integrable system determines the dependence of correlators on the coupling parameters (including all the descendant couplings).

An outline of the results of the paper is as follows. I will be considering (only at tree-level) a general 2D topological field theory [2,3,10] with $N$ primary fields $\phi_1, \ldots, \phi_N$. The symbol $\langle \phi_{\alpha_1} \phi_{\alpha_2} \ldots \rangle$ will denote the genus-zero correlation functions. These do not depend on positions of operators $\phi_{\alpha_1}, \phi_{\alpha_2}, \ldots$ The double-point correlators

$$\langle \phi_\alpha \phi_\beta \rangle = \eta_{\alpha\beta} = \eta_{\beta\alpha} \tag{1.1}$$

determine a non-degenerate scalar product on the space of primaries. The triple correlators

$$c_{\alpha\beta\gamma} = \langle \phi_\alpha \phi_\beta \phi_\gamma \rangle \tag{1.2}$$

determines the structure of the operator algebra (or primary chiral algebra) of the model,

$$\phi_\alpha \cdot \phi_\beta = c_{\alpha\beta}{}^\gamma \phi_\gamma, \qquad c_{\alpha\beta}{}^\gamma = \eta^{\gamma\epsilon} c_{\alpha\beta\epsilon}, \qquad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}, \qquad (1.3)$$

$$\langle \ldots \phi_\alpha \phi_\beta \ldots \rangle = c_{\alpha\beta}{}^\gamma \langle \ldots \phi_\gamma \ldots \rangle. \qquad (1.4)$$

This is a commutative associative algebra A of dimension $N$ with a unity $\phi_1$

$$c_{1\alpha\beta} = \eta_{\alpha\beta}, \qquad c_1{}^\alpha{}_\beta = \delta^\alpha_\beta. \qquad (1.5)$$

The symmetry of the tensor $c_{\alpha\beta\gamma}$ is equivalent to invariance of the scalar product (1.1) on A:

$$\langle ab, c \rangle = \langle a, bc \rangle, \qquad \text{for } a, b, c \in A. \qquad (1.6)$$

I recall that such algebras A are called Fröbenius algebras.

In fact one has to consider an $N$-parametric family of primary operator algebras $A = A(t)$, $t = (t^1, \ldots, t^N)$, of the form

$$c_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}(t), \qquad \eta_{\alpha\beta} = \text{const}. \qquad (1.7)$$

satisfying all the previous conditions. As was shown in ref. [10], the perturbed correlators (1.2) can be represented in the form

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t), \qquad (1.8)$$

$\partial_\alpha = \partial/\partial t^\alpha$, where the function $F(t)$ is the primary free-energy. The conditions of associativity of the perturbed primary chiral algebra give a system of non-linear PDE for the free energy $F(t)$. I will call this system the Witten–Dijkgraaf–E. Verlinde–H. Verlinde (WDVV) equations since the idea to classify topological theories solving differential equations of associativity of primary operator algebra seems to belong to Witten [2] (general topological models with two primaries were studied in ref. [2] using this approach).

Formal integrability of the WDVV equations follows from the "commutation representations": the compatibility condition of the linear system for the vector function $\xi$

$$\partial_\alpha \xi_\beta = z c_\alpha{}^\gamma{}_\beta \xi_\gamma, \qquad (1.9a)$$

with the constraint of symmetry of the tensor

$$c_{\alpha\beta\gamma} = \eta_{\gamma\epsilon} c_{\alpha\beta}{}^\epsilon, \qquad (1.9b)$$

and the normalization

$$c_1{}^\gamma{}_\beta = \delta_\beta^\gamma, \tag{1.9c}$$

are equivalent to the WDVV equations. Here $z$ is a spectral parameter. In this paper an inverse spectral transform for the WDVV system is constructed with a genericity assumption for the perturbed Fröbenius algebra $A(t)$. I assume that (locally) $A(t)$ has no nilpotents * (or, equivalently, it can be decomposed into a direct sum of 1-dimensional Fröbenius algebras). Identically indecomposable deformations of Fröbenius algebras will be considered in next publications (note that in topological sigma-models [2,5] where $A(t)$ is the "quantized" cohomology ring of the target space, the deformation $A(t)$ is indecomposable – see sect. 5, example 2). Under this decomposability assumption the WDVV equations proved to be gauge equivalent (in the sense of ref. [24]) to the integrable system

$$\partial_k \gamma_{ij}(u) = \gamma_{ik}(u)\gamma_{kj}(u) \quad i, j, k \text{ distinct,}$$

$$\sum_{k=1}^{N} \partial_k \gamma_{ij}(u) = 0,$$

$$\gamma_{ji}(u) = \gamma_{ij}(u) \quad \text{defined only for } i \neq j. \tag{1.10}$$

Here $u = (u^1, \ldots, u^N)$ are new coordinates on the coupling space, $u = u(t)$, $\partial_i = \partial/\partial u^i$. They are defined by the following representation of the perturbed structure constants

$$c_{\alpha\beta}{}^\gamma(t) = \sum_{i=1}^{N} \frac{\partial t^\gamma}{\partial u^i} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial u^i}{\partial t^\beta}, \tag{1.11}$$

(proof of the existence of such coordinates is the crucial step in the construction of gauge equivalence).

The functions $\gamma_{ij}(u)$ are expressed via components of the metric $\eta_{\alpha\beta}$ in the coordinates $u^1, \ldots, u^N$

$$g_{ij}(u) = \eta_{\alpha\beta} \frac{\partial t^\alpha}{\partial u^i} \frac{\partial t^\beta}{\partial u^j} \equiv g_{ii}(u)\delta_{ij}, \tag{1.12a}$$

$$\gamma_{ij}(u) = \frac{\partial_j \sqrt{g_{ii}(u)}}{\sqrt{g_{jj}(u)}} = \gamma_{ji}(u), \quad i \neq j. \tag{1.12b}$$

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* These topological theories can be called massive ones. I am thankful to S. Cecotti for explanation of this point.

The system (1.10) is nothing but the conditions of vanishing of the curvature of the metric (1.12).

The system (1.12) is familiar in soliton theory [26–29]. It is equivalent to the $(1 + 1)$-evolutionary system of $N(N - 1)/2$ PDEs (and dependence on $u^1, \ldots, u^N$ is determined by some particular isospectral symmetries of the system) being equivalent [26] to the so-called pure imaginary reduction of the $N$-wave interaction system (for $N \geqslant 3$; for $N = 2$ (1.10) is a linear system). The inverse spectral transform for (1.10) can be constructed in a standard way [26,28,29]. The corresponding solution of the WDVV is determined by a solution of (1.10) with an $N$-dimensional ambiguity (by solving the linear problem giving the commutation representation for (1.10)). These results are represented in sect. 2. Interpretation of the linear operators (1.9) as an affine connection on the coupling space (depending on $z$) was very useful in the proofs and calculations of sects. 2 and 3.

In fact self-similar solutions of the WDVV equations are of special importance since they describe perturbed chiral ring of topological conformal field theories. In the decomposable case they are classified by solutions of the similarity reduction of the system (1.10)

$$\gamma_{ij}(cu) = c^{-1}\gamma_{ij}(u). \tag{1.12}$$

This reduction is a system of ODEs of order $N(N - 1)/2$ (nonlinear for $N > 2$). For $N = 3$ this system was proved to be equivalent [30] to a particular case of the Painlevé-VI equation. For any $N \geqslant 3$ the isomonodromic deformation method [31] is developed in sect. 4 for (1.10), (1.12). The scaling dimensions of the model are shown to coincide with the monodromy indices in $z = 0$ of the corresponding linear ODE in $z$ (with rational coefficients). All the primary correlators are expressed in quadratures via these "high-order Painlevé transcendents".

Let us consider now coupling of the model to gravity [2,3,5,10]. Here one has an infinite number of fields $\sigma_q(\phi_\alpha)$, $q = 0, 1, \ldots$, where the fields $\sigma_0(\phi_\alpha)$ can be identified with the primaries $\phi_\alpha$ and $\sigma_q(\phi_\alpha)$ for $q \geqslant 1$ are called the gravitational descendants of $\phi_\alpha$. The operator $\sigma_0(\phi_1)$ usually is denoted by $\mathscr{P}$ and is called as the puncture operator of the model. The tree-level correlators of these operators depend on an infinite family of coupling parameters $T^{\alpha,q}$ ("descendant couplings") in such a way that

$$\langle \sigma_{q_1}(\phi_{\alpha_1})\sigma_{q_2}(\phi_{\alpha_2})\ldots \rangle = \frac{\partial}{\partial T^{\alpha_1,q_1}} \frac{\partial}{\partial T^{\alpha_2,q_2}} \ldots \log Z(T). \tag{1.13}$$

Here $Z(T)$ is the partition function of the topological model. The $N$-dimensional subspace $T^{\alpha,q} = 0$, $q \geqslant 1$ (the couplings $T^{\alpha,0}$ are arbitrary) in the phase space of all couplings is called the small phase space. For the correlators on the small phase space one should have

$$\langle \mathscr{P}\phi_\alpha \rangle = t_\alpha = \eta_{\alpha\beta}t^\beta = \eta_{\alpha\beta}T^{\beta,0}, \tag{1.15}$$

$$\langle \phi_\alpha \phi_\beta \phi_\gamma \rangle = c_{\alpha\beta\gamma}(t). \tag{1.16}$$

Correlators on the complete phase space are determined by the recursion relations

$$\langle \sigma_p(\phi_\alpha)\phi_A\phi_B \rangle = \langle \sigma_{p-1}(\phi_\alpha)\phi_\lambda \rangle \eta^{\lambda\mu} \langle \phi_\mu \phi_A \phi_B \rangle, \tag{1.17}$$

for any operators $\phi_A = \sigma_q(\phi_\beta)$, $\phi_B = \sigma_r(\phi_\gamma)$ of the model, and their dependence on $T$ is determined by the string equation

$$\sum_{p \geqslant 1, \alpha} T^{\alpha,p} \partial_{T^{\alpha,p-1}} \partial_{T^{\mu,0}} F(T) + T_{\mu,0} = t_\mu \quad \mu = 1, \ldots, N. \tag{1.18}$$

The main observation of sect. 3 is that this procedure of "switching on topological gravity" has a natural interpretation in the theory of an integrable hierarchy of PDEs of the form

$$\partial_{T^{\alpha,p}} t^\gamma = c_{(\alpha,p)}{}^\gamma{}_\beta(t) \partial_X t^\beta \tag{1.19}$$

(system of hydrodynamic type) being constructed by the primary operator algebra. The idea of describing (at tree-level) coupling to topological gravity of sigma models using a hamiltonian hierarchy of the form (1.19), (1.21), was proposed by Witten [34]. I show in sect. 3 how one can effectively construct such a hierarchy for any solution of the WDVV equations. This provides an additional argument for WDVV equations being the fundamental equations of 2D TFT. Here

$$c_{(\alpha,0)}{}^\gamma{}_\beta = c_\alpha{}^\gamma{}_\beta(t), \tag{1.20}$$

$X$ is the "cosmological constant", the eqs. (1.19) for $p \geqslant 1$ are constructed using an appropriate recursion operator. The systems (1.19) are hamiltonian systems on the loop space $\mathscr{L}M$ (where M is the coupling space) w.r.t. the Poisson bracket

$$\{t^\alpha(X), t^\beta(Y)\} = \eta^{\alpha\beta}\delta'(X-Y). \tag{1.21}$$

Their hamiltonians $H_{\alpha,p}$ have the form

$$H_{\alpha,p} = \int \langle \mathscr{P}\sigma_{p+1}(\phi_\alpha) \rangle \, \mathrm{d}X. \tag{1.22}$$

The densities of the hamiltonians (as functions of $t$) are determined from the linear system (1.9) in the form

$$x_\alpha(t, z) = \sum_{p=0}^{\infty} \langle \mathscr{P}\sigma_p(\phi_\alpha) \rangle z^p, \quad \partial_\beta x_\alpha = \xi_\beta \quad \text{obeys (1.9)}, \quad x_\alpha(t, 0) = t_\alpha. \tag{1.23}$$

The double correlators (again as functions of $t$) have the form

$$(z + w) \sum z^p w^q \langle \sigma_p(\phi_\alpha) \sigma_q(\phi_\beta) \rangle = \langle \nabla x_\alpha(t, z), \nabla x_\beta(t, w) \rangle - \eta_{\alpha\beta} \quad (1.24)$$

(here $\nabla = (\eta^{\alpha\beta} \partial_\beta)$ is the gradient; the scalar product of gradients via the scalar product $\eta_{\alpha\beta}$). The recursion relations (1.17) proved to be a consequence of the recursion procedure of constructing the hierarchy (1.19).

The last step is in determining the dependence of the special amplitudes $t_\alpha = \langle \mathcal{P} \phi_\alpha \rangle$ on the coupling parameters $T^{\alpha, p}$. This dependence is given as a particular solution of the hierarchy (1.19). This solution can be specified as the unique solution of (1.19) being defined for any $T^{1,1}$ for sufficiently small $T^{\alpha, p}$ (for $(\alpha, p) \neq (1, 1)$). The $\tau$-function of this particular solution proved to coincide with the partition function of the topological field theory. The "generalized hodograph transform" [25] for solving integrable systems of hydrodynamic type (being represented in variational form [26]) immediately gives the string equations (1.18).

In the conformal case the dependence $t = t(T)$ is given also by a particular self-similar solution of the hierarchy (1.19). For the solution $\sigma_1(\phi_1)$ is the marginal operator (adding of it to the action does not change the scaling dimensions). Other self-similar solutions of (1.19) (with other $\sigma_p(\phi_a)$ as marginals), that can be constructed using the idea of refs. [5,10] also are of importance in the theory. They give dependence of the correlators on the coupling parameters in different multi-critical models [5,10]. Note that the importance of self-similar solutions of the Whitham hierarchy in dispersive hydrodynamics * was realized many years ago [32] (see also refs. [20,28]). Generic solution of the hierarchy (1.19) can be considered, therefore, as an interpolation between different multicritical models.

I hope also that the hydrodynamic nature of the hierarchy (1.19) (together with some ideas of dispersive hydrodynamics [20,28,32]) might be instructive in studying of global properties of correlators in topological conformal field theories (see the end of sect. 5 for the discussion of a sort of "phase transitions" in multicritical models).

So the hierarchy (1.19) in topological field theory plays a role similar to the role of KdV-type hierarchies do in QFT basing on matrix models. It is interesting to find conditions (in terms of the primary operator algebra) providing that the hierarchy (1.19) can be obtained as a semiclassical limit (or can be obtained by the averaging procedure) of some integrable system of KdV-type. A variant of such specifications using so-called "strong Liouvillean property" [20] of the hamiltonian formalism of the averaged systems is conjectured in sect. 3.

In this paper I did not consider the recursion relations determining the high-genus correlators. Also it seems very interesting to find an appropriate variant

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* It is interesting that the numerical solution of the string equations in the matrix model with the criticality $k = 3$ has an amazing similarity to the dispersionless shock-wave [20,28,32]. I am grateful to S.P. Novikov for paying my attention to this point.

of the "truncated Virasoro constraints" of ref. [14] in general topological field theory. I hope to do it in forthcoming publications.

## 2. Geometry of primary operator algebra in topological field theory

Let A be an $N$-dimensional commutative associative algebra over $\mathbb{C}$ with a unity $e$. It is called Fröbenius algebra if a non-degenerate $\mathbb{C}$-bilinear invariant scalar product $\langle \ , \ \rangle$ on A exists:

$$\langle ab, c \rangle = \langle a, bc \rangle \quad a, b, c \in A. \tag{2.1}$$

*Remark.*   Let us define a linear functional on A by the formula

$$\omega_e(a) = \langle e, a \rangle. \tag{2.2}$$

Then the invariant scalar product $\langle \ , \ \rangle$ can be written in the form

$$\langle a, b \rangle = \omega_e(ab). \tag{2.3}$$

And for any linear functional $\omega \in A^*$ the scalar product

$$\langle a, b \rangle_\omega = \omega(ab) \tag{2.4}$$

is invariant. It is non-degenerate for generic $\omega$. Any invariant $\mathbb{C}$-bilinear form on A can be represented in the form (2.4).

Let $e_\alpha$, $\alpha = 1, \ldots, N$ be any basis in A such that $e_1 = e$. Let

$$\langle e_\alpha, e_\beta \rangle = \eta_{\alpha\beta} \tag{2.5a}$$

$$e_\alpha e_\beta = c_\alpha{}^\gamma{}_\beta e_\gamma \tag{2.5b}$$

(summation over repeated indices is assumed).

The matrix $\eta_{\alpha\beta}$ and the structure constants $c_\alpha{}^\gamma{}_\beta$ satisfy the following conditions:

$$\eta_{\beta\alpha} = \eta_{\alpha\beta}, \qquad \det(\eta_{\alpha\beta}) \neq 0, \tag{2.6a}$$

$$c_\alpha{}^\epsilon{}_\beta c_\epsilon{}^\delta{}_\gamma = c_\alpha{}^\delta{}_\epsilon c_\beta{}^\epsilon{}_\gamma \tag{2.6b}$$

(associativity),

$$c_1{}^\alpha{}_\beta = \delta_\beta^\alpha \tag{2.6c}$$

(normalization $e_1 = e$),

$$c_{\alpha\beta\gamma} = c_{\alpha}{}^{\epsilon}{}_{\beta}\eta_{\epsilon\gamma} = c_{\beta\alpha\gamma} = c_{\alpha\gamma\beta} \tag{2.6d}$$

(commutativity and invariance of the scalar product).

The operators $T_\alpha$,

$$(T_\alpha)^\beta{}_\gamma = c_\alpha{}^\beta{}_\gamma,$$

of (left) multiplication form an exact $N$-dimensional representation of A. The Fröbenius algebra is called *decomposable* (semisimple) if the operator

$$T = x^\alpha T_\alpha$$

has simple eigenvalues for generic $x^1, \ldots, x^N$. A decomposable Fröbenius algebra A is isomorphic to a direct sum of one-dimensional Fröbenius algebras

$$f_i f_j = \delta_{ij} f_i, \qquad \langle f_i, f_j \rangle = \eta_i \delta_{ij}. \tag{2.7}$$

Decomposability of a Fröbenius algebra is an open property.

Deformation of a Fröbenius algebra is a $k$-parametric family $c_\alpha{}^\gamma{}_\beta(t)$, $\eta_{\alpha\beta}(t)$, $t = (t^1, \ldots, t^k)$, satisfying eqs. (2.6).

*Definition 2.1.* An $N$-parametric deformation of an $N$-dimensional Fröbenius algebra A is called *potential deformation* if

$$\partial_\gamma \eta_{\alpha\beta} \equiv 0, \tag{2.8a}$$

$\partial_\gamma = \partial/\partial t^\gamma$,

$$c_1{}^\alpha{}_\beta \equiv \delta^\alpha_\beta, \tag{2.8b}$$

and a potential function $F(t)$ exists such that

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t). \tag{2.8c}$$

Note that

$$\partial_1 \partial_\alpha \partial_\beta F(t) = \eta_{\alpha\beta} \tag{2.8d}$$

(this follows from eqs. (2.8c), (2.6c)).

The problems of classification of potential deformations of Fröbenius algebras is equivalent to a complicated system of non-linear PDEs for the potential function $F(t)$ (the primary free-energy) being obtained by substitution (2.8c), (2.8d) to (2.6) (in fact, only equations (2.6b) of associativity are non-trivial). This system was discussed first for $N = 2$ by Witten [2], and the conditions (2.8) were obtained by

Dijkgraaf, E. Verlinde and H. Verlinde, so I'll call it the Witten–Dijkgraaf–
E. Verlinde–H. Verlinde (WDVV) equations. The aim of this section is to
construct "inverse spectral transform" for this system.

Let us construct a representation of the WDVV equations in a form of
compatibility conditions of a over-determined linear system.

*Proposition 2.1.* Let $c_\alpha{}^\gamma{}_\beta(t)$ be a family of functions of $t = (t^1, \ldots, t^N)$ satisfy-
ing eqs. (2.6c) and (2.6d) for some constant $(\eta_{\alpha\beta})$. Then $(c_\alpha{}^\gamma{}_\beta(t), \eta_{\alpha\beta})$ is a potential
deformation of a Fröbenius algebra iff the following linear system depending on a
spectral parameter $z$ is compatible.

$$\partial_\alpha \xi_\beta = z c_\alpha{}^\gamma{}_\beta \xi_\gamma \quad \alpha, \beta = 1, \ldots, N. \tag{2.9}$$

*Proof.* The compatibility $\partial_{\alpha'}\partial_{\alpha''} = \partial_{\alpha''}\partial_{\alpha'}$ of the system (1.9) is equivalent to the
equations

$$c_{\alpha'}{}^\gamma{}_\beta c_{\alpha''}{}^\epsilon{}_\gamma = c_{\alpha''}{}^\gamma{}_\beta c_{\alpha'}{}^\epsilon{}_\gamma, \tag{2.10a}$$

$$\partial_{\alpha''} c_{\alpha'}{}^\gamma{}_\beta = \partial_{\alpha'} c_{\alpha''}{}^\gamma{}_\beta. \tag{2.10b}$$

The first one together with the symmetry (2.6d) implies the associativity. The same
symmetry and the second equation provide existence of a potential $F(t)$. The
proposition is thus proved.

Unfortunately I do not know how one can use the commutative representation
(2.9) for integration of the WDVV equations. What I'm going to do is to construct
a gauge equivalence of the linear problem (2.9) to a more familiar in the theory of
integrable systems "commutative representation" (i.e. to construct a gauge equiva-
lence [24] of the WDVV equations to a more familiar integrable system).

It turns out that a geometric interpretation of the linear system (2.9) will be very
useful in constructing such a gauge equivalence (it also will be very useful in sect. 3
in calculation of all the correlators in the topological field theory with given
primary chiral algebra).

Let M be the space of (complex) parameters $t = (t^1, \ldots, t^N)$ (the coupling
space) of a deformation $(c_\alpha{}^\gamma{}_\beta(t), \eta_{\alpha\beta})$. Let us introduce a multiplication on the
fibers of the tangent bundle TM by the formula

$$\partial_\alpha \cdot \partial_\beta \mid_t = c_\alpha{}^\gamma{}_\beta(t) \partial_\gamma. \tag{2.11}$$

$\partial_\alpha = \partial/\partial t^\alpha$. This provides in the space of all vector fields on M a structure of a
commutative associative algebra over the ring $\mathscr{F}(M)$ of functions on M. The unity
vector field $\partial = \partial_1$ is specified on M. Also a metric

$$ds^2 = \eta_{\alpha\beta} \, dt^\alpha \, dt^\beta \tag{2.12}$$

is determined on M. In other words, a structure of a Fröbenius $\mathscr{F}(\mathrm{M})$-algebra is specified on the space of vector fields Vect (M).

This point of view is instructive to give a coordinate free reformulation of the main problem.

Let M be an $N$-dimensional Riemann [*] manifold M with a metric $ds^2$ and with a structure algebra with unity $\partial$ over $\mathscr{F}(\mathrm{M})$ in the space of vector fields

$$(X \cdot Y \mid_t)^k = c_{ij}^k(t) X^i(t) Y^j(t), \qquad (2.13)$$

such that the scalar product $ds^2$ is invariant (see (2.1)) with respect to this multiplication. Let $\nabla_X Y$ be the Levi-Civita covariant derivative for the metric $ds^2$. Let $\tilde{\nabla}$ be a new covariant derivative depending on a parameter $z$ of the form

$$\tilde{\nabla}_X Y = \nabla_X Y + z X \cdot Y \qquad (2.14a)$$

or, equivalently, on 1-forms $\omega$

$$\tilde{\nabla}_X \omega(Y) = \nabla_X \omega(Y) - z\omega(X \cdot Y). \qquad (2.14b)$$

*Definition 2.2.* M is a Fröbenius manifold if: (1) the connection $\tilde{\nabla}$ is symmetric and has zero curvature for any $z$; (2) the unity vector field $\partial$ is constant with respect to $\nabla$ (i.e. $\nabla_X \partial = 0$ for any $X$).

*Proposition 2.2.* Any solution of the WDVV equations determines a Fröbenius manifold via the formulae (2.10), (2.11). Conversely, the metric $ds^2$ on a Fröbenius manifold is flat. In the corresponding flat coordinates $t^\alpha$ the metric and the multiplication have the form (2.12), (2.11) where $(c_\alpha{}^\gamma{}_\beta(t), \eta_{\alpha\beta})$ is a potential deformation of a Fröbenius algebra (i.e. it determines a solution of the WDVV equations).

*Proof.* Symmetry of $\tilde{\nabla}$ is equivalent to commutativity of the multiplication of vector fields. The metric $ds^2$ is flat since vanishing of the curvature of $\nabla = \tilde{\nabla} \mid_{z=0}$. In the flat coordinates $t^\alpha$ for $ds^2$ vanishing of the curvature of $\tilde{\nabla}$ needs as (2.10). The proposition is proved.

Let us explain the differential-geometric sense of the "free energy" $F(t)$. The family (2.14) of the flat symmetric connections $\tilde{\nabla}$ depending on the parameter $z$ generates a deformation of the $N$-dimensional space with the metric $ds^2$. It turns out that the displacement vector of the deformation coincides with the gradient of the function $F(t)$. So the strain tensor of the deformation coincides with the hessian of $F(t)$.

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[*] I mean that $ds^2$ is a non-degenerate quadratic form on TM (not necessary positive definite). For complex manifold M $ds^2$ is a complex quadratic form.

More precisely, let

$$x^\alpha = x^\alpha(t, z) = t^\alpha + z v^\alpha(t) + \mathrm{O}(z^2) \tag{2.15}$$

be flat coordinates for the connection $\tilde{\nabla}$. They are specified by the condition of vanishing of the covariant hessian $\tilde{\nabla}_\alpha \tilde{\nabla}_\beta x = 0$, or, equivalently, by the system

$$\partial_\alpha \partial_\beta x^\gamma = z c_\alpha{}^\epsilon{}_\beta \partial_\epsilon x^\gamma. \tag{2.16}$$

The (infinitesimal) displacement vector $v^\alpha(t)$ is determined uniquely up to a transformation of the form

$$v^\alpha(t) \mapsto v^\alpha(t) + T^\alpha{}_\beta t^\beta$$

for any constant matrix $(T^\alpha{}_\beta)$.

*Proposition 2.3.* The gradient of the "free energy" $F(t)$ coincides with the displacement vector

$$\eta^{\alpha\mu}\partial_\mu F(t) = v^\alpha(t). \tag{2.17}$$

*Proof.* The equation (2.16) for the vector $v$ reads

$$\partial_\alpha \partial_\beta v^\gamma = c_\alpha{}^\gamma{}_\beta.$$

This proves (2.17).  □

The flat coordinates $x^\alpha(t, z)$ will be useful also in sect. 3. Note that gradients $\xi_\beta = \partial_\beta x^\alpha$ of these flat coordinates obey the system (2.9).

Let us return to investigation of the WDVV equation. The main idea of it is in choosing a special coordinate system on Fröbenius manifold M in which the multiplication (2.13) of vector fields is determined by constant structure constants (but the metric tensor is not constant). This I can do under additional assumptions on the deformed Fröbenius algebra. From here on in this section I will consider only deformations of *decomposable* Fröbenius algebras (it is sufficient to assume decomposability of $c_\alpha{}^\gamma{}_\beta(t)$ for some fixed $t$).

We now arrive at our first main lemma

*Lemma 2.1.* For any potential deformation of a decomposable Fröbenius algebra canonical local coordinates $u^i = u^i(t)$, $i = 1,\ldots,N$ exist such that

$$\partial_i \cdot \partial_j = \delta_{ij} \cdot \partial_i, \tag{2.18a}$$

$$\partial_i = \partial/\partial u^i. \tag{2.18b}$$

*Proof.* Due to openness of the decomposability property locally $N$ linearly independent smooth vector fields $\partial_1, \ldots, \partial_N$ exist satisfying eq. (2.18a) (the idempotents of the algebra). Let the commutators of the fields have the form

$$\left[ \partial_i, \partial_j \right] = f_{ij}^k \partial_k$$

for some functions $f_{ij}^k$ on M. The Christoffel symbols for the connection $\nabla$ are determined by the formula

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k \, .$$

Vanishing of the Riemann curvature tensor for the connection $\tilde{\nabla}$ gives

$$\Gamma_{kj}^l \delta_i^l + \Gamma_{ki}^l \delta_{kj} - \Gamma_{ki}^l \delta_j^l - \Gamma_{kj}^l \delta_{ki} = f_{ij}^l \delta_k^l \, . \tag{2.19}$$

For $l = k$ this gives $f_{ij}^k = 0$. Lemma 2.1 is proved.  □

Now let us pay attention at the invariant scalar product $ds^2$. What are the features of it in the canonical coordinates $u^1, \ldots, u^N$?

To explain these properties of $ds^2$ I have to give some not well-known constructions of classical theory of curvilinear orthogonal coordinates. I recall that a diagonal metric

$$ds^2 = \sum_{i=1}^{N} g_{ii}(u)(du^i)^2 \tag{2.20}$$

determines curvilinear orthogonal co-ordinates in some euclidean space iff the curvature of it vanishes. The metric (2.20) is called *Egoroff metric* [30] (see also refs. [25], [26]) if the rotation coefficients

$$\gamma_{ij}(u) = \frac{\partial_j \sqrt{g_{ii}(u)}}{\sqrt{g_{jj}(u)}} \quad i \neq j, \tag{2.21}$$

satisfy the symmetry condition

$$\gamma_{ji}(u) = \gamma_{ij}(u). \tag{2.22}$$

Equivalently, a potential $V = V(u)$ exists such that

$$g_{ii}(u) = \partial_i V(u) \quad i = 1, \ldots, N. \tag{2.23}$$

Vanishing of the curvature of the Egoroff metric (2.20) is equivalent to the following system

$$\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj} \quad \text{for distinct } i, j, k, \tag{2.24a}$$

$$\partial \gamma_{ij} = 0 \quad i \neq j, \tag{2.24b}$$

where

$$\partial = \sum_{i=1}^{N} \partial_i. \tag{2.25}$$

It is easy to see that (2.24) can be reduced to a $(1 + 1)$-PDE, i.e. a solution $\gamma_{ij}(u) = \gamma_{ji}(u)$ is specified uniquely by fixation of $N(N - 1)/2$ functions of one variable.

It was shown in ref. [26] that (2.24) is an integrable system (without the symmetry $\gamma_{ji} = \gamma_{ij}$ it was studied in [27]). It is equivalent to the compatibility conditions of the system

$$\partial_j \psi_i = \gamma_{ij} \psi_j \quad i \neq j, \tag{2.26a}$$

$$\partial \psi_i = z \psi_i, \tag{2.26b}$$

$z$ is a spectral parameter. The relation of (2.24) to the $N$-waves interaction system is explained in ref. [26].

The Egoroff zero-curvature metric (2.20) is called $\partial$-*invariant* if

$$\partial g_{ii}(u) = 0 \quad i = 1, \ldots, N. \tag{2.27}$$

It can be specified uniquely by its rotation coefficients and by $N$ arbitrary constants via solving the linear system (2.26) for $z = 0$. The same is true for the corresponding flat coordinates. More precisely, let us consider the linear system

$$\partial_j \psi_i = \gamma_{ij} \psi_j \quad i \neq j, \tag{2.28a}$$

$$\partial \psi_i = 0, \tag{2.28b}$$

for some solution $\gamma_{ij}(u) = \gamma_{ji}(u)$ of eqs. (2.24). It is easy to see that (2.28) is equivalent to a linear system of $N$ linear ODEs of first order.

As follows from (2.21) and (2.27) $\psi_i = \sqrt{g_{ii}}$ is a solution of (2.28). Conversely, any solution $\psi_{i1}$ of (2.28) determines a $\partial$-invariant Egoroff metric with the same rotation coefficients by the formula

$$g_{ii} = (\psi_{i1})^2. \tag{2.29}$$

Let $\psi_{i1}(u), \ldots, \psi_{iN}(u)$ be a basis in the space of solutions of (2.28). Note that the scalar product

$$\eta_{\alpha\beta} = \sum_{i=1}^{N} \psi_{i\alpha}(u)\psi_{i\beta}(u) \tag{2.30}$$

is non-degenerate and does not depend on $u$. Then the flat coordinates $t^1, \ldots, t^N$ are determined by quadratures from the system

$$\partial_i t^\alpha = \psi_{i1}\psi_i^\alpha \equiv \sqrt{g_{ii}}\,\psi_i^\alpha, \tag{2.31a}$$

where

$$\psi_i^\alpha = \eta^{\alpha\beta}\psi_{i\beta}, \qquad \left(\eta^{\alpha\beta}\right) = \left(\eta_{\alpha\beta}\right)^{-1}. \tag{2.31b}$$

Note that

$$t_1 = \eta_{1\alpha}t^\alpha = V \tag{2.32}$$

is the potential (2.23) of the metric.

*Lemma 2.2.* The invariant metric $ds^2$ on a Fröbenius manifold M in the canonical coordinates $u^1, \ldots, u^N$ is a $\partial$-invariant Egoroff metric of zero curvature.
   *Proof.* Let $\omega$ be a 1-form on M of the form

$$\omega(X) = \langle \partial, X \rangle. \tag{2.33}$$

Here $\partial$ is the unity vector field. It has the form (2.25) in the canonical coordinates. From (2.3) and (2.18) one has

$$ds^2 = \sum_{i=1}^{N} \omega(\partial_i)(du^i)^2.$$

This metric has zero curvature since it is constant in the coordinates $t^\alpha$. The potential (2.23) for the metric $g_i(u) = \omega(\partial_i)$ exists since the 1-form

$$\omega = \omega(\partial_i)\,du^i = \eta_{1\alpha}\,dt^\alpha$$

is closed. From the covariant constancy of the vector field $\partial$ it follows that

$$\sum_{k=1}^{N} \Gamma_{ik}^j = 0 \quad \text{for any } i, j.$$

From this and from identity $\nabla_k g_{ij} \equiv 0$ the $\partial$-invariance (2.27) follows. The lemma is proved.

*Theorem 2.1.* Any solution of the WDVV equations (in the decomposable case) is determined by a solution $\gamma_{ij}(u) = \gamma_{ji}(u)$ of the integrable system (2.24) and by $N$ arbitrary constants by the formulae (2.29)–(2.31) and

$$c_{\alpha\beta\gamma}(t) = \sum_{i=1}^{N} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}. \tag{2.34}$$

*Proof.* Almost everything was proved in the two lemmas. One needs to verify eq. (2.19) (for $f_{ij}^k = 0$) for any Egoroff metric. This can be done straightforward. To prove the formula (2.34) one can use that, by definition (2.18),

$$c_{\alpha}{}^{\gamma}{}_{\beta}(t) = \sum_{i=1}^{N} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial u^i}{\partial t^\beta} \frac{\partial t^\gamma}{\partial u^i}. \tag{2.35}$$

The orthogonality conditions

$$\frac{\partial t^\gamma}{\partial u^i} = g_{ii}\frac{\partial u^i}{\partial t^\mu}\eta^{\mu\gamma}, \ i = 1,\dots,N \tag{2.36}$$

together with (2.31) and (2.35) give (2.34). The theorem is proved.   □

*Corollary.* Any solution of the WDVV equation depends on $N(N-1)/2$ arbitrary functions of one variable and also on $N$ arbitrary constants.

The following statement explains in what sense the system (2.24) is gauge equivalent to the WDVV equations (cf. ref. [24]).

*Proposition 2.4.* The transformation

$$\xi_\alpha = \sum_{i=1}^{N} \sqrt{g_{ii}} \frac{\partial u^i}{\partial t^\alpha}\psi_i, \tag{2.37a}$$

or, equivalently,

$$\psi_i = g_{ii}^{-1/2}\frac{\partial t^\alpha}{\partial u^i}\xi_\alpha, \tag{2.37b}$$

transforms any solution $\psi_i$ of (2.26) to a solution $\xi_\alpha$ of eq. (2.9) and vice versa. The proof is straightforward.

*Remark 2.1.* Existence of the canonical diagonal coordinates $u^1,\dots,u^N$ can be proved even without the normalization (2.8b). The metric $g_{ii}(u)$ is specified in the form (2.29) by the rotation coefficients and by a solution $\psi_i(u)$ of the system (2.26a). General solution of (2.26a) depends on $N$ arbitrary functions of one variable. If a global assumption on the behaviour of the functions $g_{ii}(u)$ is imposed then one can represent $\sqrt{g_{ii}(u)}$ as a linear combination of eigenfunctions of the

spectral problem (2.26). For the deformations of Fröbenius algebras having been constructed in the appendix of ref. [15] (for the case of a decomposable Fröbenius algebra) all the rotation coefficients $\gamma_{ij}$ vanish identically. The normalization condition (2.8b) in this example is not fulfilled.

I end this section with a discussion of potential deformations of indecomposable Fröbenius algebras. One can consider a particular case of deformations for which a coordinate system $u^1, \ldots, u^N$ in the coupling space M exists such that the multiplication (2.11) in these coordinates has constant structure coefficients

$$\partial_i \cdot \partial_j = c_{ij}{}^k \partial_k, \tag{2.38a}$$

$$\partial_i = \partial/\partial u^i, \tag{2.38b}$$

$$c_{ij}{}^k = \text{const.} \tag{2.38c}$$

(Probably, this is the general case. But this still should be proved.) Here $c_{ij}{}^k$ are the structure constants of a fixed Fröbenius algebra $A_0$. The invariant scalar product (2.12) of the deformation in these coordinates has the form

$$ds^2 = c_{ij}{}^k \omega_k(u) \, du^i \, du^j. \tag{2.39}$$

This follows from (2.3). The 1-form

$$\omega = \omega_k(u) \, du^k \tag{2.40a}$$

is defined as

$$\omega_k(u) = \langle \partial, \partial_k \rangle, \tag{2.40b}$$

where $\partial$ is the unity of the deformation (cf. the proof of lemma 2.2). The 1-form $\omega$ is closed

$$d\omega = 0. \tag{2.40c}$$

Vanishing of the curvature of the metric (2.39), (2.40) reads as a system of PDE for the functions $\omega_k(u)$. It depends on the Fröbenius algebra $A_0$ as on the parameter. For the decomposable algebra $A_0$ this system is equivalent to (2.24).

It would be interesting to construct IST for this zero curvature system for any Fröbenius algebra $A_0$. Another interesting problem is to prove that vanishing of the curvature of (2.39), (2.40) provides existence of a potential $F$ for the deformation (such that $c_{ijk} = c_{ij}{}^p c_{pk}{}^q \omega_q(u) = \nabla_i \nabla_j \nabla_k F(u)$). This will give an extension of the theorem 2.1 to deformations of arbitrary Fröbenius algebras (i.e. identically indecomposable). As was noted in sect. 1, this might be of use for classification of topological sigma models. I hope to do this in forthcoming publications.

## 3. Integrable hamiltonian hierarchies of hydrodynamic type, their solutions, τ-functions and topological amplitudes

Let us fix a solution of the WDVV equations. In other words, let us assume that the dependence of all the (tree-level) primary correlators on the special amplitudes

$$t_\alpha = \langle \mathscr{P}\phi_\alpha \rangle \quad \alpha = 1,\ldots, N,$$

is given on the "small phase space" $T^{\alpha,p} = 0$, $p > 0$ (i.e. on the coupling space M) of a model of topological field theory (I recall that $t_\alpha = \eta_{\alpha\beta}T^{\beta,0}$ for $T^{\alpha,p} = 0$, $p > 0$). How can one calculate all the tree-level correlators of the model on the whole phase space with arbitrary couplings $T^{\alpha,p}$? Here I'll show that the dependence of the special correlators $\langle \mathscr{P}\phi_\alpha \rangle$ on the coupling constants $T^{\beta,p}$ is determined by a hierarchy of integrable hamiltonian PDE systems with M as the targets space. Following the idea of ref. [13] I define the τ-function of the hierarchy. The particular solution of the hierarchy is specified for which τ coincides with the tree-level partition function of the model of the topological field theory. The genus-zero recursion relations of ref. [2] for the correlators of the model are identified with the recursion operator of the hierarchy. And "generalized hodograph transform" of ref. [25] (being represented in the variational form of ref. [26]) for solving the hierarchy proves to coincide with the string equations [2,10] (or "pre-string" equations in the terminology of ref. [2]).

As will be shown in sect. 5 for the topological $A_n$ minimal model [10] my hierarchy coincides with the dispersionless Lax–Gelfand–Dikii hierarchy (essentially it follows from ref. [13]). And for the model of ref. [16] it coincides with the Whitham-type hierarchy being obtained by averaging over $g$-gap solutions of the Lax–Gelfand–Dikii hierarchy.

I start with recalling some ideas from the hamiltonian theory [20,23] of systems of hydrodynamic type.

Let M be any manifold and $v^1,\ldots,v^N$ any local coordinates on M. I recall that the formula

$$\{v^a(X), v^b(Y)\} = g^{ab}(v(X))\delta'(X - Y) + b_c^{ab}(v)v_X^c\delta(X - Y), \quad (3.1)$$

determines a Poisson bracket on the loop space $^*$ $\mathscr{L}$M of smooth functions $v^a(X)$, $X \in S^1$ (Poisson brackets of hydrodynamic type) iff the tensor

$$g^{ab}(v) = g^{ba}(v) \tag{3.2}$$

---

$^*$ Components of the loop space $\mathscr{L}$M are numerated by conjugate classes of the fundamental group $\pi_1(M)$. Here only formal theory of Poisson brackets is considered. All the statements are proved for the component of $\mathscr{L}$M consisting of loops of trivial homotopy class.

determines a flat metric on M (the matrix $(g^{ab}(v))$ is assumed to be non-degenerate), and the coefficients $b_c^{ab}(v)$ can be represented in the form

$$b_c^{ab}(v) = -g^{ad}(v)\Gamma_{dc}^b(v),\qquad(3.3)$$

where $\Gamma_{dc}^b(v)$ are the Christoffel symbols of the Levi-Civita connection $\nabla_i$ for the metric

$$ds^2 = g_{ab}(v)\,dv^a\,dv^b,\qquad (g_{ab}(v)) = (g^{ab}(v))^{-1}\qquad(3.4)$$

(see ref. [20]). As in ref. [20] I shall consider as hamiltonians only

$$H = \int h(v(X))\,dX,\qquad(3.5)$$

$\int = \int_0^{2\pi}$, ("functionals of hydrodynamic type"); the density $h = h(v)$ does not depend on derivatives. So any function $h(v)$ on M determines a hamiltonian system on $\mathscr{L}$M

$$\partial_\tau v^a(X) = \left\{v^a(X),\,\int h(v(Y))\,dY\right\} = w_b^a(v)\partial_X v^b,\qquad(3.6a)$$

$$w_b^a(v) = \nabla^a\nabla_b h(v)\qquad(3.6b)$$

(a hamiltonian system of hydrodynamic type).

The class of hamiltonian systems of hydrodynamic type is invariant under changes of coordinates on M. The following three types of coordinate systems are of special use in the theory of hamiltonian systems of hydrodynamic type (see ref. [20] for details).

(i) Flat coordinates $t^\alpha$, $\alpha = 1,\ldots,N$

$$ds^2 = \eta_{\alpha\beta}\,dt^\alpha\,dt^\beta,\qquad (\eta_{\alpha\beta}) = \text{const.},\qquad(3.7a)$$

$$\{t^\alpha(X), t^\beta(Y)\} = \eta^{\alpha\beta}\delta'(X-Y),\qquad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}.\qquad(3.7b)$$

The functionals

$$\int t^\alpha(X)\,dX\quad \alpha = 1,\ldots,N,\qquad(3.8)$$

span the annihilator of $\{\ ,\ \}$. The hamiltonian system (3.6) in flat coordinates has the form

$$\partial_\tau t^\alpha = \eta^{\alpha\beta}\partial_\beta\partial_\gamma h(t)\partial_X t^\gamma,\qquad(3.9)$$

$\partial_\beta = \partial/\partial t^\beta$.

(ii) Curvilinear orthogonal coordinates $u^i$

$$\mathrm{d}s^2 = \sum_{i=1}^{N} g_{ii}(u)(\mathrm{d}u^i)^2.$$  (3.10)

These are of special importance in the theory of integrability of systems of hydrodynamic type ([25], see also ref. [20]). A system of orthogonal coordinates $u^1, \ldots, u^N$ specifies a maximal lagrangian subspace $\mathscr{H} \subset \mathrm{Funct}\,(\mathscr{L}\mathrm{M})$ of functionals of hydrodynamic type (3.5). The densities $h(u) \in \mathscr{H}$ are specified by the condition of diagonality of the covariant hessian

$$\nabla_i \nabla_j h(u) = 0 \quad \text{for } i \neq j.$$  (3.11)

Equivalently, the vector-function

$$\psi_i^h = g_{ii}^{-1/2} \partial_i h,$$  (3.12)

$\partial_i = \partial/\partial u^i$, satisfies the system (cf. (2.26a))

$$\partial_j \psi_i^h = \gamma_{ji} \psi_j^h \quad \text{for } i \neq j.$$  (3.13)

Here $\gamma_{ij}$ are the rotation coefficients of the metric (3.10). The corresponding commuting hamiltonian systems (3.6) are diagonal,

$$\partial_\tau u^i(X) = \left\{ u^i(X), \int h(u)\,\mathrm{d}Y \right\} = w^i(u)\partial_X u^i(X),$$  (3.14)

$i = 1, \ldots, N$, $h \in \mathscr{H}$, so the variables $u^1, \ldots, u^N$ are the common Riemann invariants for them (and $w^1(u), \ldots, w^N(u)$ are corresponding characteristic speeds). All the systems (3.14) are integrable [25] (the algorithm of integration in the form of ref. [26] will be given below).

(iii) "Physical" (or Liouville) coordinates $v^a$ in which the Poisson bracket (3.6) have the form

$$\{v^a(X), v^b(Y)\} = \left[ q^{ab}(v(Y)) + q^{ba}(v(X)) \right]\delta'(X - Y),$$  (3.15)

for some matrix $q^{ab}(u)$. So the metric $g^{ab}$ and the connection $b_c^{ab}$ can be represented in the form

$$g^{ab}(v) = q^{ab}(v) + q^{ba}(v),$$  (3.16a)

$$b_c^{ab}(v) = \partial q^{ab}(v)/\partial v^c.$$  (3.16b)

These mean that the functionals

$$\int v^a(X) \, dX \quad a = 1, \dots, N, \tag{3.17}$$

commute pairwise. So they are the standard dependent variables in the equations of hydrodynamics of an ideal fluid (possibly, with inner degrees of freedom). Also the hamiltonian averaging procedure of [20,23] always provides a Liouville coordinate system for the averaged system. The particular case of linear functions $q^{ab}(v)$ is of special interest due to its relation to vector analogues of the Virasoro algebra [34]. It can be shown (see below) that the string amplitudes

$$v_\alpha = \langle \mathscr{P} \sigma_{k_\alpha}(\phi_\alpha) \rangle \tag{3.18}$$

for any $k_1, \dots, k_N \geq 0$ (here $\mathscr{P}$ is the puncture operator) are physical coordinates on the coupling space. (I recall that the flat coordinates on M are the amplitudes $t_\alpha = \langle \mathscr{P} \phi_\alpha \rangle$.)

Let us fix a solution of the WDVV equations (i.e. a potential deformation $c_\alpha{}^\gamma{}_\beta(t)$, $\eta_{\alpha\beta}$ of a Fröbenius algebra). The metric

$$ds^2 = \eta_{\alpha\beta} \, dt^\alpha \, dt^\beta \tag{3.19}$$

determines a Poisson bracket (3.7). Let us consider a family of systems of hydrodynamic type

$$\partial_{T^\alpha} t^\beta = c_\alpha{}^\beta{}_\gamma(t) \partial_X t^\gamma, \ \alpha = 1, \dots, N. \tag{3.20}$$

(Note that $\partial_{T^1} t^\beta = \partial_X t^\beta$. So $T^1$ can be identified with $X$.)

This system can be rewritten in an elegant coordinate-free way using the multiplication (2.11) of vector-fields on the coupling space M: a map $f: R^2_{X,T^\alpha} \to M$ enjoys the equation (3.20) if

$$\partial_{T^\alpha} f = \partial_\alpha \cdot \partial_X f \tag{3.21}$$

(we consider here $\partial_{T^\alpha} f(X, T^\alpha)$ and $\partial_X f(X, T^\alpha)$ as vector-fields on M).

*Proposition 3.1.* The systems (3.20) commute pairwise, and they are hamiltonian with respect to the Poisson bracket (3.7) with the (density of) hamiltonians being equal to $F_\alpha(t)$ respectively. Conversely, if the systems (3.20) commute pairwise and are hamiltonian system w.r.t. the Poisson bracket (3.7) and their hamiltonians are derivatives of a function $F(t)$, then $F(t)$ obeys the WDVV equations and $(c_\alpha{}^\beta{}_\gamma(t), \eta_{\alpha\beta})$ is the correspondent potential deformation.

*Proof.* The "direct" part of the proposition can be verified straightforwardly from (3.9). Conversely, if the system (3.20) is a hamiltonian one with the hamiltonian

$$H_\alpha = \int F_\alpha \, \mathrm{d}X, \quad F_\alpha = \partial_\alpha F(t) \tag{3.22}$$

then

$$c_{\alpha \ \gamma}^{\ \beta} = \eta^{\beta\lambda} F_{\alpha\gamma\lambda} \tag{3.23}$$

so

$$c_{\alpha\beta\gamma} = \eta_{\gamma\epsilon} c_{\alpha \ \beta}^{\ \epsilon} \tag{3.24}$$

is a symmetric tensor. The condition of commutativity of the flows (3.20) is equivalent to the associativity condition for the structure constants (3.23). The proposition is proved.

Note that the functional $\int F_1 \mathrm{d}X = \frac{1}{2} \int \eta_{\alpha\beta} t^\alpha t^\beta \mathrm{d}X$ generates the spatial translations $\partial_{T^1} = \partial_X$. The proposition provides another reformulation of the WDVV equations: a function $F(t)$ on a space with a metric $\mathrm{d}s^2 = \eta_{\alpha\beta} \mathrm{d}t^\alpha \mathrm{d}t^\beta$ satisfies the WDVV equations iff the Legendre transform

$$t^\alpha \mapsto v_\alpha(t) = \partial_\alpha F(t) \tag{3.25}$$

provides Liouville coordinates for the metric $\mathrm{d}s^2$, and, particularly, $F_1(t)$ is the momentum density.

*Remark 3.1.* It will be shown below that the form (3.25) for the transform from the flat coordinates to Liouville coordinates provides existence of an infinite number of conservation laws of eqs. (3.20). Therefore, this provides existence of a rich family of non-linear changes of coordinates conserving the Liouville form (3.16) of the Poisson bracket (a priori, (3.16) admits only affine transformations of the coordinates $v$). Existence of such non-linear transformations is a feature of hierarchies of hydrodynamic type being obtained from hamiltonian hierarchies of KdV-type by the averaging procedure [20]. Another feature ("strong Liouville property" [20]) of the hamiltonian formalism of the averaged systems is compatible with the Liouville form (3.16) of the averaged Poisson bracket with the restriction onto any affine subspace (the affine structure here is determined by the physical coordinates!). This does not hold for any solution of the WDVV equations. Explicit forms of the constraint on $F$ will be given in the forthcoming publication.

The topological counterparts of the matrix-type model seem to have this feature. This might give a non-trivial procedure of restriction of a solution of the WDVV equation to some subspaces in the coupling space (affine subspaces in the physical coordinates (3.25)).

Let us construct conservation laws for the system (3.20).

I am going to show that the flat coordinates for the connection (2.14) are the generating functions for the conservation laws we need. This gives also a recursion operator for the conservation laws.

*Proposition 3.2.* For any $\alpha = 1, \ldots, N$ formal series

$$x^\alpha = x^\alpha(t, z) = \sum_{p=0}^\infty h^{\alpha,p}(t) z^p, \qquad h^{\alpha,0} = t^\alpha, \tag{3.26}$$

exist such that

$$\partial_\beta \partial_\gamma x^\alpha = z c_\beta{}^\epsilon{}_\gamma \partial_\epsilon x^\alpha. \tag{3.27}$$

*Proof.* For the coefficients $h^{\alpha,p}(t)$ the system (3.27) gives the recursion relations

$$\partial_\beta \partial_\gamma h^{\alpha,p+1}(t) = c_\beta{}^\epsilon{}_\gamma \partial_\epsilon h^{\alpha,p}(t) \quad p = 0, 1, \ldots \tag{3.28}$$

Solvability of them follows from the conditions

$$c_\gamma{}^\epsilon{}_\beta = c_\beta{}^\epsilon{}_\gamma, \quad \partial_\lambda c_\beta{}^\epsilon{}_\gamma = \partial_\beta c_\lambda{}^\epsilon{}_\gamma.$$

The proposition is proved.  □

For calculations it is convenient to rewrite eq. (3.27) in the form

$$\partial_\beta (\partial_\gamma x^\alpha) = z (\partial_\beta \cdot \partial_\gamma) x^\alpha \tag{3.27'}$$

(the multiplication (2.11) is used in the r.h.s.).

It is more convenient to use linear combinations of the functions (3.26)

$$x_\alpha(t, z) = \eta_{\alpha\beta} x^\beta(t, z) = \sum_{p=0}^\infty h_{\alpha,p}(t) z^p, \tag{3.29a}$$

$$h_{\alpha,p}(t) = \eta_{\alpha\beta} h^{\beta,p}(t). \tag{3.29b}$$

I recall (see sect. 2) that

$$h_{\alpha,1}(t) = F_\alpha(t) \equiv \partial_\alpha F(t). \tag{3.30}$$

Particularly,

$$h_{1,1}(t) = \tfrac{1}{2} \eta_{\alpha\beta} t^\alpha t^\beta. \tag{3.31}$$

The solution $x_\alpha(t, z)$ can be normalized in such a way that

$$\langle \nabla x_\alpha(t, z), \nabla x_\beta(t, -z) \rangle = \eta_{\alpha\beta}, \tag{3.32}$$

$$\partial_1 x_\alpha(t, z) = z x_\alpha(t, z) + \eta_{1\alpha}. \tag{3.33}$$

Here $\nabla$ means the gradient and $\langle \ , \ \rangle$ the scalar product w.r.t. $(\eta_{\alpha\beta})$

$$\nabla^\lambda x_\alpha(t, z) = \eta^{\lambda\mu} \partial_\mu x_\alpha(t, z), \tag{3.34}$$

$$\langle \nabla x_\alpha(t, z), \nabla x_\beta(t, w) \rangle = \eta_{\lambda\mu} \nabla^\lambda x_\alpha(t, z) \nabla^\mu x_\beta(t, w) = \eta^{\lambda\mu} \partial_\lambda x_\alpha(t, z) \partial_\mu x_\beta(t, w). \tag{3.35}$$

The ambiguity in the above definiton of the functions $x_\alpha(t, z)$ has the form

$$x_\alpha(t, z) \rightarrow T_\alpha^\lambda(z) x_\lambda(t, z) \tag{3.36}$$

where $T_\alpha^\lambda(z)$ are any power series in $z$ with constant coefficients satisfying the conditions

$$T_\alpha^\lambda(0) = \delta_\alpha^\lambda \tag{3.37}$$

$$\eta_{\lambda\mu} T_\alpha^\lambda(z) T_\beta^\mu(-z) = \eta_{\alpha\beta}. \tag{3.38}$$

Expressions of the functions $h_{\alpha,p}(t)$ via eigenfunctions of the gauge-equivalent linear problem (2.26) will be given in appendix A.

The gradients of the functions $h_{\alpha,p}(t)$ obey different bilinear identities that will be used in the following. These can be summarized in the following "generating identity":

$$\nabla \langle \nabla x_\alpha(t, z), \nabla x_\beta(t, w) \rangle = (z + w) \nabla x_\alpha(t, z) \cdot \nabla x_\beta(t, w). \tag{3.39}$$

Here the product (2.11) is used in the r.h.s.

It is interesting that the commutators of the gradient vector fields also can be expressed via the same multiplication

$$\left[ \nabla x_\alpha(t, z), \nabla x_\beta(t, w) \right] = (w - z) \nabla x_\alpha(t, z) \cdot \nabla x_\beta(t, w). \tag{3.40}$$

*Proposition 3.3.* For any solution $t = t(T^1, \ldots, T^N, X)$ of the system (3.20) the following identities hold:

$$\partial_{T^a} x_\beta(t, z) = \partial_X \left[ z^{-1} \left( \partial_\alpha x_\beta(t, z) - \eta_{\alpha\beta} \right) \right]. \tag{3.41}$$

Conversely, eq. (3.41) implies (3.20).

*Proof.* For the l.h.s. of (3.41) one has

$$\partial_{T^\alpha} x_\beta(t; z) = \partial_\epsilon x_\beta(t; z) c_{\alpha}{}^\epsilon{}_\gamma \partial_X t^\gamma. \tag{3.42}$$

Since the functions $x_\beta(t; z)$ satisfy the system (3.27), eq. (3.42) can be rewritten as

$$\partial_{T^\alpha} x_\beta(t; z) = z^{-1} \partial_\gamma \partial_\alpha x_\beta(t; z) \partial_X t^\gamma.$$

The normalization condition

$$\partial_\alpha x_\beta(t; 0) = \eta_{\alpha\beta}$$

completes the proof. □

The representation (3.42) of the equations (3.20) is of the Flaschka–Forest–Mc-Laughlin (FFM) type [18] (but it does not coincide with the FFM representation even for the original case of the Whitham equations!).

*Corollary.* The functions

$$h_{\beta,p}(t) = \operatorname*{res}_{z=0} z^{-p-1} x_\beta(t; z) \quad p = 0, 1, \ldots \tag{3.43}$$

for any $\beta = 1, \ldots, N$, $p = 0, 1, \ldots$, are densities of conservation laws for the equations (3.20):

$$\partial_{T^\alpha} h_{\beta,p}(t) = \partial_X f_{\alpha\beta,p}(t), \tag{3.44a}$$

$$f_{\alpha\beta,p}(t) = \operatorname*{res}_{z=0} z^{-p-2} \partial_\alpha x_\beta(t, z). \tag{3.44b}$$

Commutativity of these conservation laws w.r.t. the Poisson bracket (3.7) follows from the following statement.

*Proposition 3.4.* The Poisson brackets (3.7) of the functionals $x_\alpha(t(X), z)$ have the following Liouville form

$$\{x_\alpha(t(X), z_1), x_\beta(t(Y), z_2)\} = \left[ q_{\alpha\beta}(t(Y); z_1, z_2) + q_{\beta\alpha}(t(X); z_2, z_1) \right]$$

$$\times \delta'(X - Y), \tag{3.45a}$$

where

$$q_{\alpha\beta}(t; z_1; z_2) = \frac{z_2}{z_1 + z_2} \langle \nabla x_\alpha(t, z_1), \nabla x_\beta(t, z_2) \rangle. \tag{3.45b}$$

*Proof.* For the derivatives of $q_{\alpha\beta}(t; z_1, z_2)$ one has from eq. (3.39)

$$\nabla q_{\alpha\beta}(t; z_1, z_2) = z_2 \nabla x_\alpha(t, z_1) \cdot \nabla x_\beta(t, z_2).$$

The l.h.s. of eq. (3.45a) has the form

$$\{x_\alpha(t(X), z_1), x_\beta(t(Y), z_2)\} = \langle \nabla x_\alpha(t(X), z_1), \nabla x_\beta(t(Y), z_2)\rangle \delta'(X - Y)$$

$$= \langle \nabla x_\alpha(t(X)), z_1), \nabla x_\beta(t(X), z_2)\rangle \delta'(X - Y)$$

$$+ z_2 \langle \nabla x_\alpha(t, z_1) \cdot \nabla x_\beta(t, z_2), \partial_X t\rangle \delta(X - Y).$$

This completes the proof.   □

*Corollary.*   Functionals with the densities $h_{\alpha,p}(t)$ commute pairwise.
The commuting hamiltonians

$$H_{\alpha,p} = \int h_{\alpha,p+1}(t(X))\, \mathrm{d}X \quad \alpha = 1,\dots,N, \quad p = -1, 0, 1,\dots, \quad (3.46)$$

generate a hierarchy of commuting hamiltonian systems of hydrodynamic type

$$\partial_{T^{\alpha,p}} t^\beta(X) = \{t^\beta(X), H_{\alpha,p}\} = c_{(\alpha,p)\gamma}{}^\beta \partial_X t^\gamma \quad p = 0, 1, \dots \quad (3.47a)$$

(the functionals $H_{\alpha,-1}$ span the annihilator of the Poisson bracket (3.7)) where

$$c_{(\alpha,p)\gamma}{}^\beta = \eta^{\alpha\lambda}\partial_\lambda\partial_\gamma h_{\alpha,p+1} = \eta^{\mu\lambda}\partial_\lambda h_{\alpha,p} c_\mu{}^\beta{}_\gamma. \quad (3.47b)$$

So the system (3.47) can be rewritten like (3.21) using the multiplication by the gradient vector field $\nabla h_{\alpha,p}$

$$\partial_{T^{\alpha,p}} f = \nabla h_{\alpha,p} \cdot \partial_X f \quad (3.47c)$$

(this is an equation for a map $f: \mathbb{R}^2_{X,T^{\alpha,p}} \to M$ solving the system (3.47)). For $p = 0$ one has

$$c_{(\alpha,0)\gamma}{}^\beta = c_\alpha{}^\beta{}_\gamma. \quad (3.48)$$

So the system (3.47) coincides with (3.20) for $p = 0$, $T^{(\alpha,0)} \equiv T^\alpha$. The formula (3.47b) gives a recursion procedure for constructing the system (3.47) on the basis of the system (3.20)

$$\partial_{T^{\alpha,p}} = \eta^{\mu\lambda}\partial_\lambda h_{\alpha,p}\partial_{T^\mu}. \quad (3.49)$$

For $p = 1$ eqs. (3.47) read

$$\partial_{T^{\alpha,1}} t^\beta = c^{\mu\beta}{}_\gamma F_{\mu\alpha}\partial_X t^\gamma. \quad (3.50)$$

The particular value $\alpha = 1$ is of special importance

$$\partial_{T^{1,1}} t^\beta = c_{\lambda}{}^{\beta}{}_{\gamma} t^{\lambda} \partial_X t^{\gamma}, \tag{3.51}$$

the density of the corresponding hamiltonian $H_{1,1}$ equals

$$h_{1,2} = F_\alpha t^\alpha - 2F. \tag{3.52}$$

For the generating function $x_\beta(t, z)$ for the densities of the conservation laws (3.43) one obtains

$$\partial_{T^{\alpha,p}} x_\beta(t, z) = \partial_X \operatorname*{res}_{w=0} w^{-p-1} z^{-1} \langle \nabla x_\alpha(t, w), \nabla x_\beta(t, z) \rangle. \tag{3.53}$$

This provides a FFM-type representation of the complete hierarchy (3.47).

Commutativity of the systems (3.47) provides the commutativity of the operators $c_{(\alpha,p)\gamma}{}^{\beta}$,

$$c_{(\alpha,p)\mu}{}^{\lambda} c_{(\beta,q)\nu}{}^{\mu} = c_{(\beta,q)\mu}{}^{\lambda} c_{(\alpha,p)\nu}{}^{\mu}. \tag{3.54}$$

All these operators are symmetric with respect to the scalar product $(\eta_{\alpha\beta})$.

In the decomposable case the commutativity (3.54) implies diagonality of the systems (3.47) in the canonical diagonal coordinates $u^1, \ldots, u^N$.

It can be proved (see appendix A) that in this case linear combinations of the functionals $H_{\alpha,p}$ form a complete set of conservation laws of the system (3.20) (or, of the hierarchy (3.47)). In other words, they span the canonical lagrangian subspace $\mathcal{H}$ (see above). It would be interesting to investigate completeness of the conservation laws $H_{\alpha,p}$ for identically indecomposable deformation.

Let us proceed to construction of solutions of the hierarchy (3.47). It turns out that, in some sense, it has only one solution. The others formally can be obtained by shifts along the $T^{\alpha,p}$-axis. At least it can be proved using ref. [25] for the decomposable case. To construct it, let us use the obvious scaling group of symmetries of the systems (3.47)

$$X \mapsto cX, \qquad T^{\alpha,p} \mapsto cT^{\alpha,p}, \qquad t^\beta \mapsto t^\beta. \tag{3.55}$$

Let us denote by $T = (T^{\alpha,p})$ the infinite vector with the coordinates $T^{\alpha,p}$.

*Proposition 3.5.* The hierarchy (3.47) in the domain

$$T^{1,1} = \epsilon, \quad X, T^{\alpha,p} = o(\epsilon) \quad \text{for} \quad (\alpha, p) \neq (1, 1), \quad \epsilon \to 0, \tag{3.56}$$

has a non-constant solution $t^\beta = t^\beta(X, T)$ being invariant with respect to the scaling transformations (3.55). It can be found in implicit form from the "variational principle"

$$\partial\big[\Phi_T(t) + Xt_1\big] = 0, \tag{3.57}$$

where

$$\Phi_T(t) = \sum_{\alpha,p} T^{\alpha,p} h_{\alpha,p}(t). \tag{3.58}$$

*Proof.*  For scaling invariant solutions of (3.47) one has

$$\Big(X\delta_\gamma^\beta + \sum c_{(\alpha,p)\gamma}{}^\beta(t)\Big)\partial_X t^\gamma = 0 \quad \beta = 1, \dots, N.$$

Using (3.47b), this system can be represented in the form (3.57). In the domain (3.56) one has

$$\partial_\mu\partial_\nu\big[\Phi_T(t) + Xt_1\big] = \epsilon\eta_{\mu\nu} + \mathrm{o}(\epsilon). \tag{3.59}$$

Hence the solution $t = t(X, T)$ locally is unique. Therefore it satisfies (3.47). The proposition is proved.  □

The variable $X$ can be omitted in the solution (3.57) (since it can be restored by a shift $T^{1,0} \mapsto T^{1,0} + X$).

Let us construct the "$\tau$-function" of the hierarchy (3.47) (cf. refs. [13,14,16]) for the particular solution (3.57). It is defined by the formula

$$\log \tau_0(T) = \tfrac{1}{2} \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} (z + w)^{-1}$$

$$\times \sum z^{-r-1} w^{-s-1} T^{\lambda,r} T^{\mu,s}\big[\langle \nabla x_\lambda(t, z), \nabla x_\mu(t, w)\rangle - \eta_{\lambda\mu}\rangle\big]\Big|_{t=t(T)}. \tag{3.60}$$

Here the functions $x_\alpha(t, z)$ are assumed to obey the normalization (3.32).

*Proposition 3.6.*  The $\tau$-function (3.60) satisfies the following equations:

$$\partial_{T^{\alpha,p}}\partial_{T^{\beta,q}} \log \tau_0(T) = \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} (z + w)^{-1} z^{-p-1} w^{-q-1}\langle \nabla x_\alpha(t, z), \nabla x_\beta(t, w)\rangle. \tag{3.61}$$

*Proof.*  The equations (3.57) can be represented in the form

$$\bigg[\operatorname*{res}_{z=0} \sum z^{-r-1} T^{\lambda,r} \nabla x_\lambda(t, z)\bigg]_{t=t(T)} = 0.$$

Using (3.39), one has

$$
\partial_{T^{\alpha,p}} \log \tau_0 = \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} z^{-p-1}(z+w)^{-1} \sum w^{-s-1} T^{\mu,s}
$$

$$
\times \left[ \langle \nabla x_\alpha(t,z), \nabla x_\mu(t,w) \rangle - \eta_{\alpha\mu} \right]
$$

$$
+ \tfrac{1}{2} \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} \sum z^{-r-1} w^{-s-1} T^{\lambda,r} T^{\mu,s} \langle \nabla x_\lambda(t,z) \cdot \nabla x_\mu(t,w), \partial_{T^{\alpha,p}} t \rangle
$$

$$
= \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} z^{-p-1}(z+w)^{-1}
$$

$$
\times \sum w^{-s-1} T^{\mu,s} \left[ \langle \nabla x_\alpha(t,z), \nabla x_\mu(t,w) \rangle - \eta_{\alpha\mu} \right].
$$

The derivative of this expression w.r.t. $T^{\beta,q}$ can be calculated in similar way. The proposition is proved. $\square$

Let us introduce a notation for the generating function of the second derivatives of $\log \tau_0$

$$
V_{\alpha\beta}(t; z, w) = (z+w)^{-1} \left[ \langle \nabla x_\alpha(t,z), \nabla x_\beta(t,w) \rangle - \eta_{\alpha\beta} \right]
$$

$$
= \sum_{p,q} V_{(\alpha,p),(\beta,q)}(t) z^p w^q. \tag{3.62}
$$

The proposition implies

$$
\partial_{T^{\alpha,p}} \partial_{T^{\beta,q}} \log \tau_0 = V_{(\alpha,p),(\beta,q)}. \tag{3.63}
$$

Note that the Poisson brackets (3.7) of the functionals $h_{\alpha,p}(t(X))$ have the form

$$
\{ h_{\alpha,p}(t(X)), h_{\beta,q}(t(Y)) \} = \left[ V_{(\alpha,p),(\beta,q-1)}(t(Y)) + V_{(\beta,p),(\alpha,q-1)}(t(X)) \right]
$$

$$
\times \delta'(X-Y). \tag{3.64}
$$

For the coefficients $V_{(\alpha,p),(\beta,q)}$ one has formulae

$$
V_{(\alpha,0),(\beta,0)} = F_{\alpha\beta}, \tag{3.65a}
$$

$$
V_{(\alpha,p),(1,0)} = h_{\alpha,p}, \tag{3.65b}
$$

$$
V_{(\alpha,p),(\beta,0)} = \partial_\beta h_{\alpha,p+1}, \tag{3.65c}
$$

$$
V_{(\alpha,p),(1,1)} = (t^\lambda \partial_\lambda - 1) h_{\alpha,p+1}, \tag{3.65d}
$$

$$V_{(\alpha,0),(1,1)} = F_{\alpha\lambda}t^\lambda - F_\alpha, \tag{3.65e}$$

$$V_{(1,1),(1,1)} = F_{\lambda\mu}t^\lambda t^\mu - 2F_\lambda t^\lambda + 2F. \tag{3.65f}$$

Also one has an identity

$$\operatorname*{res}_{w=0} w^{-1}V_{\alpha 1}(t; z, w) = x_\alpha(t, z). \tag{3.66}$$

The recurrence relation (3.49) reads

$$\partial_{T^{\alpha,p}} = \eta^{\mu\lambda}\partial_{T^{\alpha,p-1}}\partial_{T^{\lambda,0}} \log \tau_0 \, \partial_{T^{\mu,0}}. \tag{3.67}$$

Particularly, one obtains recurrence relation for derivatives of the $\tau$-functions $\tau = \tau_0$

$$\partial_{T^{\alpha,p}}\partial_{T^{\beta,q}}\partial_{T^{\gamma,r}} \log \tau = (\partial_{T^{\alpha,p-1}}\partial_{T^{\lambda,0}} \log \tau)\eta^{\lambda\mu}\partial_{T^{\mu,0}}\partial_{T^{\beta,q}}\partial_{T^{\gamma,r}} \log \tau. \tag{3.68}$$

The identities

$$\partial_\mu\left(\sum T^{\alpha,p}h_{\alpha,p}\right) = 0 \quad \mu = 1,\dots,N,$$

for finding the solution (3.57) read

$$\sum_{p\geqslant 1} T^{\alpha,p}\partial_{T^{\alpha,p-1}}\partial_{T^{\mu,0}} \log \tau_0 + T_{\mu,0} = 0. \tag{3.69}$$

Also the $\tau$-function satisfies the Euler identity

$$\sum T^{\alpha,p}\partial_{T^{\alpha,p}} \log \tau_0 = 2 \log \tau_0. \tag{3.70}$$

Let us introduce explicitly the shift

$$T^{1,1} \mapsto T^{1,1} + 1, \tag{3.71}$$

into the solution (3.57). Thus the functions $t^\beta(T)$ are specified now by the system

$$\nabla\hat{\Phi}_T(t) = 0, \tag{3.72a}$$

$$\hat{\Phi}_T(t) = \Phi_T(t) - \tfrac{1}{2}\eta_{\alpha\beta}t^\alpha t^\beta. \tag{3.72b}$$

They are well-defined for sufficiently small $T$. The solution $t = t(T)$ can be considered as the fixed point of the gradient map $t \mapsto \nabla\Phi_T(t)$,

$$\nabla\Phi_T(t) = t. \tag{3.73}$$

For the vector fields $\partial_{T^{\alpha,p}} t$ from (3.73) and (3.28) one obtains

$$\partial_{T^{\alpha,p}} t = \frac{\nabla h_{\alpha,p}}{\partial - \sum_{q \geq 1} T^{\gamma,q} \nabla h_{\gamma,q-1}}. \tag{3.73'}$$

Since $\partial = \partial_{\alpha=1}$ is the unity for the multiplication (2.11), the denominator is an invertible vector-field in some neighbourhood of the "small phase space" $T^{\gamma,q} = 0$, $q > 0$.

The formula (3.60) for the $\tau$-function should be modified by the same shift

$$\log \tau_{1,1}(T) = \tfrac{1}{2} \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} \sum z^{-r-1} w^{-s-1} T^{\lambda,r} T^{\mu,s} V_{\lambda\mu}(t(T); z, w)$$

$$- \operatorname*{res}_{z=0} \operatorname*{res}_{w=0} w^{-2} \sum z^{-r-1} T^{\lambda,r} V_{\lambda 1}(t(T); z, w) + \tfrac{1}{2} V_{(1,1),(1,1)}(t(T)).$$

$$\tag{3.74}$$

The identity (3.63) (but not (3.70)) is still valid for this $\tau$-function.

Let us consider the restriction of the solution (3.72) onto the $N$-dimensional vectors

$$T_0 = (T^{1,0}, \ldots, T^{N,0}, 0, \ldots). \tag{3.75}$$

For these vectors one has

$$\hat{\Phi}_{T_0}(t) = \eta_{\alpha\beta} T^{\alpha,0} t^{\beta} - \tfrac{1}{2} \eta_{\alpha\beta} t^{\alpha} t^{\beta}. \tag{3.76}$$

This gives an obvious solution

$$T^{\alpha,0} = t^{\alpha} \quad \alpha = 1, \ldots, N, \tag{3.77}$$

of the system (3.20). So a solution $t = t(T)$ of eqs. (3.72) can be specified in such a way that

$$t^{\alpha}(T) \approx T^{\alpha,0} \quad \text{for small} \quad T - T_0. \tag{3.78}$$

Further in this section I shall use only this solution of eqs. (3.72).

*Proposition 3.7.* The $\tau$-function (3.74) restricted onto the subspace (3.75) coincides with the primary partition function

$$\log \tau_{1,1}(T_0) = F(T_0). \tag{3.79}$$

*Proof.* This follows from the formulae (3.65).

This proposition can be considered as a hint to consider the $\tau$-function (3.74) for any $T$ as the (tree-level) partition function of a model of topological field

theory coupled to topological gravity with given primary operator algebra. The additional arguments are provided by eqs. (3.68) and (3.69). Eq. (3.68) still is valid for the $\tau$-function (3.74). It can be considered as Witten's recursion relation [2] for the tree-level correlators in topological field theory

$$\langle \sigma_p(\phi_\alpha)\phi_A\phi_B\rangle = \langle \sigma_{p-1}(\phi_\alpha)\phi_\lambda\rangle\eta^{\lambda\mu}\langle\phi_\mu\phi_A\phi_B\rangle, \tag{3.80}$$

where

$$\phi_A = \sigma_q(\phi_\beta), \qquad \phi_B = \sigma_r(\phi_\gamma),$$

and the correlators of the descendants $\sigma_p(\phi_\alpha)$ are defined by the derivatives of $\log \tau$

$$\langle \sigma_p(\phi_\alpha)\phi_\lambda\rangle = \partial_{T^{\alpha,p}}\partial_{T^{\lambda,0}}\log \tau_{1,1}, \tag{3.81a}$$

$$\langle \sigma_p(\phi_\alpha)\sigma_q(\phi_\beta)\sigma_r(\phi_\gamma)\rangle = \partial_{T^{\alpha,p}}\partial_{T^{\beta,q}}\partial_{T^{\gamma,r}}\log \tau_{1,1}. \tag{3.81b}$$

So the variables $T^{\alpha,p}$ can be considered as the descendant couplings. Note that for the puncture operator $\mathscr{P} = \sigma_0(\phi_1)$ the correlators $\langle\mathscr{P}\phi_\alpha\rangle$ have the form

$$\langle\mathscr{P}\phi_\alpha\rangle = \partial_{T^{1,0}}\partial_{T^{\alpha,0}}\log \tau \equiv t_\alpha \tag{3.82}$$

(the identity for any $T$, cf. ref. [10]). So $\langle\mathscr{P}\phi_\alpha\rangle$ are the flat coordinates on the coupling space.

The identity (3.69) after the shift (3.71) reads

$$\sum_{p\geqslant 1} T^{\alpha,p}\partial_{T^{\alpha,p-1}}\partial_{T^{\mu,0}}\log \tau_{1,1} + T_{\mu,0} = \partial_{T^{1,0}}\partial_{T^{\mu,0}}\log \tau_{1,1} \equiv t_\mu, \tag{3.83a}$$

or, after integration

$$\sum_{p\geqslant 1} T^{\alpha,p}\partial_{T^{\alpha,p-1}}\log \tau_{1,1} + \tfrac{1}{2}\eta_{\alpha\beta}T^{\alpha,0}T^{\beta,0} = \partial_{T^{1,0}}\log \tau_{1,1}. \tag{3.83b}$$

(vanishing of the integration constant can be proved by reducing to $T_0$ and using (3.49)).

This is nothing but the string equation [2,10].

*Trivial example.* Let us consider trivial deformation $c_{\alpha\beta\gamma} = $ const. of an $N$-dimensional Fröbenius algebra A. Let $e_1 = e,\ldots, e_N$ be a basis of the algebra,

$$e_\alpha e_\beta = c_\alpha{}^\gamma{}_\beta e_\gamma.$$

The solutions $x_\alpha(t, z)$ of eq. (3.27) have the form

$$x_\alpha(t, z) = z^{-1}\langle e_\alpha, \exp zt - e\rangle, \tag{3.84}$$

where

$$t = t^\lambda e_\lambda \in A. \tag{3.85}$$

The function (3.62) has the form

$$V_{\alpha\beta}(t; z, w) = (z + w)^{-1}\left[\langle e_\alpha e_\beta, \exp(z + w)t\rangle - \langle e_\alpha, e_\beta\rangle\right]. \tag{3.86}$$

The physical coordinates are

$$v_\alpha \equiv \partial_\alpha F = \tfrac{1}{2}\langle e_\alpha, (t)^2\rangle. \tag{3.87}$$

Their Poisson brackets (3.7) are linear (the so-called Lie–Poisson brackets)

$$\{v_\alpha(X), v_\beta(Y)\} = \left[c_{\alpha\,\beta}^{\;\epsilon}v_\epsilon(Y) + c_{\beta\,\alpha}^{\;\epsilon}v_\epsilon(X)\right]\delta'(X - Y) \tag{3.88}$$

(see (3.64)). Therefore linear functionals of $v_\alpha(X)$ form an infinite-dimensional Lie algebra. For $N = 1$ it coincides with the Lie algebra of one-dimensional vector fields (i.e. the zero charge Virasoro algebra). For $N > 1$ the Lie algebras being dual to (3.88) were studied in ref. [34] (also the quadratic transforms of the type (3.87) were used for reduction the Poisson bracket (3.88) to a constant form). Also some non-commutative and non-associative generalization of Fröbenius algebras was proposed in ref. [34]. This also gives rise to linear Poisson brackets of the form (3.88) (but $c_{\beta\,\alpha}^{\;\epsilon} \neq c_{\alpha\,\beta}^{\;\epsilon}$). It would be interesting to investigate possible relations of these non-associative analogues of Fröbenius algebras to topological field theory.

The hierarchy (3.47) has the form

$$\partial_{T^{\alpha,p}}t = \frac{1}{p!}e_\alpha(t)^p\partial_X t. \tag{3.89}$$

Let us introduce vectors

$$T^p = T^{\alpha,p}e_\alpha \in A \quad p = 0, 1, \ldots \tag{3.90}$$

The $\tau$-function (3.74) has the form

$$\log \tau = \tfrac{1}{6}\langle e, (t)^3\rangle - \sum_p \frac{\langle T^p, (t)^{p+2}\rangle}{(p+2)p!}$$

$$+ \tfrac{1}{2}\sum_{p,q} \frac{\langle T^p T^q, (t)^{p+q+1}\rangle}{(p+q+1)p!q!}. \tag{3.91}$$

Here the dependence $t = t(T)$ is determined by the "fixed-point equation"

$$G(t) = t, \tag{3.92a}$$

where

$$G(t) = \sum_{p=0}^{\infty} T^p \frac{(t)^p}{p!}. \tag{3.92b}$$

The solution has the well-known form

$$t = G(G(G(\ldots))) \tag{3.93}$$

(infinite number of iterations).

Note that for the $T^{\alpha,p}$-derivatives can be found from (3.92) in the form

$$\partial_{T^{\alpha,p}} t = \frac{e_\alpha(t)^p / p!}{e - \sum_{s \geqslant 1} T^s (t)^{s-1} / (s-1)!} \tag{3.94}$$

(the denominator for small $T$ is an invertible element of the Fröbenius algebra). The formulae (3.86) and (3.91)–(3.94) complete the solution of the topological model with constant primary correlators. So

$$\langle \sigma_p(\phi_\alpha) \sigma_q(\phi_\beta) \rangle = \frac{\langle e_\alpha e_\beta, (t)^{p+q+1} \rangle}{(p+q+1) p! q!}, \tag{3.95}$$

$$\langle \sigma_p(\phi_\alpha) \sigma_q(\phi_\beta) \sigma_r(\phi_\gamma) \rangle = \frac{1}{p! q! r!} \langle e_\alpha e_\beta e_\gamma, \frac{(t)^{p+q+r}}{e - \sum_{s \geqslant 1} T^s (t)^{s-1} / (s-1)!} \rangle, \tag{3.96}$$

etc. For $N = 1$ the formula (3.95), (3.96) coincide with the "pure gravity model" [2] (up to normalization of the coupling constants). For $N > 1$, particularly, one obtains the correlators for the $K_3$ model [5]. In this case $N = 24$, the Fröbenius algebra is the cohomology algebra of a generic $K_3$ surface. So it has generators $P$, $Q_1, \ldots, Q_{22}$, $R$ in dimensions 0, 2 and 4 respectively. The multiplication has the form

$$Q_i Q_j = \eta_{ij} R, \qquad P \text{ is the unity}, \qquad Q_i R = R^2 = 0. \tag{3.97}$$

Here $(\eta_{ij})$ is a non-degenerate symmetric matrix. The scalar product (the intersection number) has the form

$$\eta_{PR} = 1, \qquad \eta_{Q_i Q_j} = \eta_{ij}. \tag{3.98}$$

This is not a decomposable case, so the formulae (3.91)–(3.94) cannot be decoupled into a sum of $N$ one-dimensional entries.

Examples of non-constant $c_{\alpha\beta\gamma}(t)$ will be considered in sect. 5.

## 4. Self-similar solutions of the WDVV equations

Let us look for solutions of the WDVV equations self-similar with respect to some scaling transformations

$$t^{\alpha} \mapsto k^{1-q_{\alpha}}t^{\alpha} \quad \alpha = 1, \dots, N, \tag{4.1a}$$

$$ds^2 = \eta_{\alpha\beta}\, dt^{\alpha}\, dt^{\beta} \mapsto k^{2-d}\, ds^2, \tag{4.1b}$$

$$c_{\alpha\beta\gamma} \mapsto k^{q_{\alpha}+q_{\beta}+q_{\gamma}-l}c_{\alpha\beta\gamma}, \tag{4.1c}$$

for some $q_1 = 0, q_2, \dots, q_N, d, l$ (it will be shown that $l = d$). Then the deformed Fröbenius algebra $\eta_{\alpha\beta}, c_{\alpha}{}^{\gamma}{}_{\beta}(t)$ can be considered as the perturbed chiral algebra of a model of topological conformal field theory. Here $q_{\alpha}$ are the charges of the primary fields and $d$ is the dimension of the model. In the conformal point $t = 0$ (if it belongs to the coupling space) eqs. (4.1) imply the tree-level superselection rules [10] for the primary correlators: $\langle \phi_{\alpha_1}\phi_{\alpha_2}\dots \rangle \neq 0$ only for $\alpha_1 + \alpha_2 + \dots = d$. In this case the WDVV equations can be reduced to a system of ODEs. This ODE system will be investigated in this section.

Only decomposable case will be considered here. For a self-similar solution of the WDVV equation the rotation coefficients satisfy the similarity condition

$$\gamma_{ij}(ku) = k^{-1}\gamma_{ij}(u). \tag{4.2}$$

So the similarity reduction of the gauge equivalent system (2.24) has the following "standardized" form:

$$\partial_k\gamma_{ij} = \gamma_{ik}\gamma_{ki} \quad \text{for distinct } i, j, k = 1, \dots, N,$$

$$\partial\gamma_{ij} = 0, \quad \partial = \sum_{i=1}^{N}\partial_i, \quad \gamma_{ji} = \gamma_{ij},$$

$$\sum_{k=1}^{N} u^k\partial_k\gamma_{ij} = -\gamma_{ij}. \tag{4.3}$$

For $N = 2$ the system (4.3) is linear and can be solved easily (see example 5.1 in the sect. 5).

For the first non-trivial case $N = 3$ the ODE (4.3) can be written in the form

$$\Gamma'_{23} = \Gamma_{21}\Gamma_{13}, \qquad (z\Gamma_{13})' = -\Gamma_{12}\Gamma_{23}, \qquad [(z-1)\Gamma_{12}]' = \Gamma_{13}\Gamma_{23}. \quad (4.4a)$$

Here

$$z = \frac{u^1 - u^3}{u^2 - u^3}, \qquad \gamma_{ij}(u) = \frac{1}{u^2 - u^3}\Gamma_{ij}(z) = \gamma_{ji}(u). \qquad (4.4b)$$

The system (4.4a) can be reduced [30] to a system of the second order equivalent to a particular case of the Painlevé-VI equation using the first integral

$$\Gamma_{23}^2 + (z\Gamma_{13})^2 + [(z-1)\Gamma_{12}]^2 = \text{const.} \qquad (4.4c)$$

For any $N \geqslant 3$ one obtains a non-linear system of ODE of order $N(N-1)/2$ (multicomponent generalization of the Painlevé-VI). The isomonodromic deformations method can be used for solving the system. I will show here how one can calculate the scaling dimensions $q_\alpha$, $d$, $l$ in the framework of the isomonodromic deformations theory. It will be shown that in the self-similar case one can express the deformation $(c_\alpha{}^\beta{}_\gamma(t), \eta_{\alpha\beta})$ via $\gamma_{ij}(u)$ by algebraic operations and quadratures.

Let us start with a commutation representation of (4.3).

*Proposition 4.1.* The system (4.3) is equivalent to the equations of compatibility of the linear problem (2.26) with the system

$$z\partial_z\psi = (zU - [U, \Gamma])\psi. \qquad (4.5)$$

Here

$$\psi = (\psi_1, \ldots, \psi_N)^{\mathrm{T}}, \qquad (4.6a)$$

$$U = \text{diag}(u^1, \ldots, u^N), \qquad (4.6b)$$

$$\Gamma = (\gamma_{ij}(u)). \qquad (4.6c)$$

*Proof.* The system (2.26) under the condition (4.2) is invariant with respect to the transformations

$$u \mapsto cu, \qquad z \mapsto c^{-1}z, \qquad \psi \mapsto \psi. \qquad (4.7)$$

Hence it commutes with the operator

$$\sum_{i=1}^{N} u^i\partial_i - z\partial_z. \qquad (4.8)$$

Action of this operator on the solutions of (2.26) can be written in the form (4.5). The proposition is proved.

The system (4.5) is a linear system of ODEs with rational coefficients in $z$ depending on the parameters $u^1, \ldots, u^N$. The standard corollary of the proposition (see, e.g. ref. [31]) that the dependence of the coefficients on the parameters $u^1, \ldots, u^N$ is an isomonodromy deformation of the equation (4.5). So the monodromy matrices of (4.5) parametrize the general solution of (4.3). Any solution of (4.3) locally is an analytic function of the variables $u^1, \ldots, u^N$. Solutions of (4.3) with different reality conditions of the form $\gamma_{ij}^* = \pm \gamma_{ij}$ were constructed in ref. [26]. See appendix B for general solutions of (4.3).

If a solution $\gamma_{ij}(u)$ of (4.3) is defined in a neighbourhood of the diagonal hyperplanes $u^i = u^j$ then for $u^i - u^j \to 0$ one has

$$\gamma_{ij}(u) = \frac{\mu_{ij}}{u^i - u^j} + O(1) \tag{4.9a}$$

for some constants $\mu_{ij} = -\mu_{ji}$,

$$\gamma_{kl}(u) = O(1) \quad \text{for} \quad (k, l) \neq (i, j). \tag{4.9b}$$

It would be interesting to investigate dependence of the parameters $\mu_{ij}$ on the monodromy matrix of eq. (4.5). I hope to do this in the next paper.

Another corollary of proposition 4.1 is very important here

*Proposition 4.2.* Eigenvalues of the matrix $[U, \Gamma]$ are integrals of the system (4.3). If a vector-function $\psi = (\psi_1, \ldots, \psi_N)^T$ obeys the system (2.28) and the similarity condition

$$\psi(cu) = c^P \psi(u), \tag{4.10}$$

then $\psi$ is an eigenvector of the matrix $[U, \Gamma]$ with the eigenvalue $-p$.

*Proof.* From (2.26), (4.5) for $z = 0$ one obtains

$$\partial_k [U, \Gamma] = [[E_k, \Gamma], [U, \Gamma]] \quad k = 1, \ldots, N \tag{4.11a}$$

where the matrix $E_k$ has the form

$$(E_k)_{ij} = \delta_{ik} \delta_{kj}. \tag{4.11b}$$

Hence the eigenvalues of $[U, \Gamma]$ do not depend on $u^k$. From the Euler identity

$$\sum u^k \partial_k \psi = p \psi$$

for the homogeneous function $\psi$ satisfying (2.26), for $z = 0$ one obtains

$$[U, \Gamma]\psi = -p\psi.$$

The proposition is proved.   $\square$

Note that the eigenvalues of $[U, \Gamma]$ are the local monodromy indices of (4.5) near the singular point $z = 0$.

The following statement is a rigidity theorem for *real* decomposable Fröbenius algebras with positive invariant scalar product.

*Corollary.*   Any self-similar potential deformation of real decomposable Fröbenius algebra with positive invariant scalar product is a trivial one: $c_\alpha{}^\gamma{}_\beta(t) = $ const..

*Proof.*   For positive metric $\mathrm{d}s^2$ the canonical coordinates $u^1, \dots, u^N$ are real (for real $t$). The diagonal entries $g_{ii}(u)$ of the metric in these coordinates also are real and positive. Hence the rotation coefficients $\gamma_{ij}(u) = \gamma_{ji}(u)$ are real, and $[U, \Gamma]$ is a real skew-symmetric matrix. All non-zero eigenvalues of the matrix are imaginary. So the matrix $[U, \Gamma]$ should equal zero identically. This means that the deformation is a trivial one.   $\square$

*Theorem 4.1.*   For any self-similar potential deformation of a decomposable Fröbenius algebra with the indices $q_1 = 0$, $q_2, \dots, q_N$, $d$, $l$ (see (4.1)) the corresponding matrix $[U, \Gamma]$ is a diagonalizable one with the eigenvalues

$$\mu_\alpha = -\tfrac{1}{2}d + q_\alpha \quad \alpha = 1, \dots, N - 1, \qquad \mu_N = \tfrac{1}{2}d. \tag{4.12a}$$

The corresponding eigenvectors $\psi^\alpha$ have the form

$$\psi_i^\alpha = g_{ii}^{-1/2}\partial_i t^\alpha \quad \alpha = 1, \dots, N - 1, \qquad \psi_i^N = g_{ii}^{1/2}, \tag{4.12b}$$

and obey the normalization conditions

$$\eta^{\alpha\beta} = \sum_{i=1}^{N} \psi_i^\alpha \psi_i^\beta, \qquad \left(\eta^{\alpha\beta}\right) = \left(\eta_{\alpha\beta}\right)^{-1}. \tag{4.12c}$$

Conversely, let $\Gamma = (\gamma_{ij})$ be any solution of the system (4.3) such that the matrix $[U, \Gamma]$ is a diagonalizable one. Let the eigenvalues $\mu_1, \dots, \mu_N$ of the matrix are ordered in such a way that

$$\mu_{N-\alpha+1} = -\mu_\alpha. \tag{4.13}$$

Then a self-similar potential deformation $(c_\alpha{}^\gamma{}_\beta(t), \eta_{\alpha\beta})$ of a decomposable Fröbenius algebra with the canonical diagonal coordinates $u^1, \dots, u^N$ exists with the indices $d$, $l$, $q_1, \dots, q_N$ of the form

$$d = 2\mu_N, \qquad l = d, \qquad q_\alpha = \tfrac{1}{2}d + \mu_\alpha. \tag{4.14}$$

If $\overline{\psi}^{\alpha}$ is any basis of eigenvectors of the matrix $[U, \Gamma]$,

$$[U, \Gamma]\overline{\psi}^{\alpha} = \mu_{\alpha}\overline{\psi}^{\alpha}, \qquad (4.15)$$

then the metric $g_{ii}$ and the flat coordinates $t^{\alpha}$ can be determined by the formulae (4.12b) where the vectors $\psi^{\alpha}$ have the form

$$\psi^{\alpha} = T_{\beta}^{\alpha}\overline{\psi}^{\beta} \qquad (4.16)$$

for some matrix $T_{\beta}^{\alpha} = T_{\beta}^{\alpha}(u)$. If the eigenvalues $\mu_{\alpha}$ of the matrix $[U, \Gamma]$ are simple then the matrix $T_{\beta}^{\alpha}(u)$ is a diagonal one, and its diagonal entries can be found by quadratures. The invariant scalar product $\eta_{\alpha\beta}$ has the form (4.12c), and the structure constants $c_{\alpha}{}^{\gamma}{}_{\beta}(t)$ have the form

$$c^{\alpha\beta\gamma} = \eta^{\alpha\lambda}\eta^{\beta\mu}c_{\lambda}{}^{\gamma}{}_{\mu} = \sum_{i=1}^{N} \frac{\psi_i^{\alpha}\psi_i^{\beta}\psi_i^{\gamma}}{\psi_i^{N}}. \qquad (4.17)$$

*Proof.* For a self-similar deformation with the indices $d$, $l$, $q_1, \ldots, q_N$ the corresponding solutions $\psi^{\alpha}$ of the system (2.28) have the form (4.12b). They are homogeneous vector-functions of the weights $p_{\alpha} = -q_{\alpha} + \frac{1}{2}d$. So the first part of the theorem follows from proposition 4.2.

To prove the second part we use the equation (4.11a). The equation means that the operators

$$\partial_k - [E_k, \Gamma] \quad k = 1, \ldots, N, \qquad (4.18)$$

commute with the matrix $[U, \Gamma]$. Also they commute pairwise because of (2.26). So they can be diagonalized simultaneously due to diagonalizability of the matrix $[U, \Gamma]$. The diagonalization procedure has the form (4.16). For the case of non-degeneracy of the spectrum $[U, \Gamma]$ one has for any $\alpha$

$$\left(\partial_k - [E_k, \Gamma]\right)\overline{\psi}^{\alpha} = f_k^a\overline{\psi}^{\alpha} \quad k = 1, \ldots, N, \qquad (4.19)$$

for some functions $f_k^{\alpha}(u)$. The diagonal matrix $T_{\beta}^{\alpha} = T_{\alpha}^{\alpha} \cdot \delta_{\beta}^{\alpha}$ can be found by quadratures from the equations

$$\partial_k \log T_{\alpha}^{\alpha} = f_k^{\alpha} \quad k = 1, \ldots, N. \qquad (4.20)$$

The theorem is proved. $\square$

*Remark 4.1.* If $\mu$ is a degenerate eigenvalue of the diagonalizable matrix $[U, \Gamma]$ of the multiplicity $l$, and $\overline{\psi}^{(1)}, \ldots, \overline{\psi}^{(l)}$ are corresponding linearly-independent eigenvectors then instead of (4.19) one will have

$$\left(\partial_k - [E_k, \Gamma]\right)\overline{\psi}^{(a)} = \sum_{b=1}^{l} f_{kb}^{(a)}\overline{\psi}^{(b)}, \qquad (4.21)$$

for some matrices $f_{kb}^{(a)}$. These matrices commute pairwise for different $k$. If they can be diagonalized simultaneously then the corresponding basis $\psi^{(1)}, \ldots, \psi^{(l)}$ of solutions of (2.15b, c) also can be found by quadratures.

*Remark 4.2.* The functions $f_k^\alpha$ can be found explicitly. E.g. if an eigenvector $\bar{\psi}^\alpha$ of $[U, \Gamma]$ is normalized in such a way that

$$\sum_{i=1}^{N} \bar{\psi}_i^\alpha = 1 \tag{4.22a}$$

then

$$f_k^\alpha = \sum_{i=1}^{N} \gamma_{ik}\left(\bar{\psi}_i^\alpha - \bar{\psi}_k^\alpha\right) \quad k = 1, \ldots, N. \tag{4.22b}$$

The solutions $\psi_\alpha = (\psi_{i\alpha}(u, z))$ of the linear system (4.5), can be chosen in such a way that

$$\psi_{i\alpha}(cu, c^{-1}z) = c^{\bar{\mu}_\alpha}\psi_{i\alpha}(u, z),$$

$$\bar{\mu}_\alpha = q_\alpha - \tfrac{1}{2}d. \tag{4.23}$$

They satisfy the equation

$$z\partial_z\psi_\alpha = (zU - [U, \Gamma])\psi_\alpha - \bar{\mu}_\alpha\psi_\alpha. \tag{4.24}$$

For the coefficients

$$\psi_\alpha(u, z) = \sum_{p=0}^{\infty} \psi_{\alpha,p}(u)z^p, \qquad \psi_{\alpha,p}(u) = \left(\psi_{i(\alpha,p)}(u)\right) \tag{4.25}$$

one has

$$[U, \Gamma]\psi_{\alpha,0} = -\bar{\mu}_\alpha\psi_{\alpha,0}, \tag{4.26}$$

so

$$\psi_{i(\alpha,0)}(u) = \eta_{\alpha\beta}\psi_i^\beta(u), \tag{4.27a}$$

$$([U, \Gamma] + p + \bar{\mu}_\alpha)\psi_{\alpha,p}(u) = U\psi_{\alpha,p-1} \quad p \geqslant 1. \tag{4.27b}$$

In the non-resonant case where

$$\bar{\mu}_\alpha - \bar{\mu}_\beta + p \neq 0 \quad p = 1, 2 \ldots, \tag{4.28a}$$

or, equivalently

$$q_\alpha - q_\beta + p \neq 0 \quad p = 1, 2, \ldots, \quad \beta = 1, \ldots, N, \tag{4.28b}$$

this gives a recurrence procedure for finding the functions $\psi_{i\alpha}(u, z)$. The formula (A.13) then provides an expression for the functions $x_\alpha(z)$ (and, particularly, for the flat coordinates $t_\alpha$), and the formula (A.12) gives the expression for the function $V_{\alpha\beta}(z, w)$.

The densities $h_{\alpha,p}(t)$ and the coefficients $V_{(\alpha,p),(\beta,q)}(t)$ with respect to the scaling transformations $t^\alpha \mapsto c^{1-q_\alpha}t^\alpha$ transform in the following way:

$$h_{\alpha,p} \mapsto c^{1+q_\alpha+p-d}h_{\alpha,p}, \tag{4.29a}$$

$$V_{(\alpha,p),(\beta,q)} \mapsto c^{p+q+q_\alpha+q_\beta+1-d}V_{(\alpha,p),(\beta,q)}. \tag{4.29b}$$

So the hierarchy (3.47) is invariant with respect to the transformations

$$t^\alpha \mapsto c^{1-q_\alpha}t^\alpha$$

$$T^{\alpha,p} \mapsto c^{1-q_\alpha-p}T^{\alpha,p}. \tag{4.30}$$

The $\tau$-function (3.74) with respect to these transformations has the weight $3 - d$. This gives the identity

$$\sum (p + q_\alpha - 1)T^{\alpha,p}\partial_{T^{\alpha,p}} \log \tau_{1,1} = (d - 3) \log \tau_{1,1}. \tag{4.31}$$

Note that the term with the number $(\alpha, p) = (1, 1)$ cancels in the l.h.s. of (4.31). That means that the $\tau_0$-function (3.60) satisfies the same equation (4.31). I recall that the $\tau_0$-funtion is a homogeneous function of the degree 2 (see (3.63)). This can be used (as in ref. [19]) for construction of self-similar solutions of the hierarchy (3.47) with other similarity indices. Let us fix $(\alpha, p)$. The $\tau$-function $\tau_{\alpha,p}(T)$ is defined (formally *) by the shift

$$\tau_{\alpha,p}(T) = \tau_0(T^{1,0}, \ldots, T^{\alpha,p} + 1, \ldots) = \tau_{1,1}(T^{1,0}, \ldots, T^{1,1} - 1, \ldots, T^{\alpha,p} + 1, \ldots).$$

$$\tag{4.32a}$$

The corresponding solution $t = t(T)$ of (3.47) is determined by the variational problem

$$\nabla\left(\Phi_T(t) - h_{\alpha,p}\right) = 0, \tag{4.32b}$$

---

* The shift (4.32a) might not be well defined due to the gradient catastrophe for the system of hydrodynamic type (3.47) (see ref. [20] for discussion of the role of self-similar solutions of the Whitham hierarchy in investigation of dispersionless shock-waves).

cf. (3.57), (3.73)). This is the self-similar solution of (3.47) w.r.t. the transformations (cf. ref. [10])

$$t^\beta \mapsto c^{1-q_\beta} t^\beta \tag{4.33a}$$

$$T^{\beta,r} \mapsto c^{q_\alpha - q_\beta + p - r} T^{\beta,r}. \tag{4.33b}$$

The shifted $\tau$-function satisfies the equation

$$\sum_{\beta,r} (q_\beta - q_\alpha - p + r) T^{\beta,r} \partial_{T^{\beta,r}} \log \tau_{\alpha,p} = \left[ d - 1 + 2(p + q_\alpha) \right] \log \tau_{\alpha,p}. \tag{4.33c}$$

Therefore the self-similar solutions of the hierarchy of hydrodynamic type (3.47) are in one-to-one correspondence with the multi-critical topological models (see ref. [5]). In the multi-critical point (4.33) one has in the notation of ref. [10]

$$\gamma_{\text{string}} = (d - 1)/(p + q_\alpha). \tag{4.33d}$$

We end this section with the discussion of bi-hamiltonian structure of the hierarchy (3.47) for self-similar potential deformations of decomposable Fröbenius algebras. In this case another flat diagonal Egoroff metric is determined on the coupling space M. Let

$$\mathrm{d}\tilde{s}^2 = \sum_{i=1}^{N} \frac{g_{ii}(u)}{u^i} (\mathrm{d}u^i)^2. \tag{4.34}$$

*Lemma 4.1.* For a flat Egoroff metric $\mathrm{d}s^2 = \sum_{i=1}^{N} g_{ii}(u)(\mathrm{d}u^i)^2$ with scaling-invariant rotation coefficients $\gamma_{ij}(cu) = c^{-1}\gamma_{ij}(u)$ the metric

$$\mathrm{d}s_{a,b}^2 = \sum_{i=1}^{N} \frac{g_{ii}(u)}{a + bu^i} (\mathrm{d}u^i)^2 \tag{4.35}$$

also is a flat one.

*Proof.* The rotation coefficients of the metric (4.35) with respect to the coordinates

$$\bar{u}^i = \log(a + bu^i) \tag{4.36}$$

equal

$$\tilde{\gamma}_{ij} = \frac{1}{b} \exp\left( \frac{\bar{u}^i + \bar{u}^j}{2} \right) \gamma_{ij}\left( \frac{\mathrm{e}^{\bar{u}^1} - a}{b}, \ldots, \frac{\mathrm{e}^{\bar{u}^N} - a}{b} \right). \tag{4.37}$$

Since $\gamma_{ij}(u) = \gamma_{ji}(u)$ obey the same system (2.24) the coefficients $\tilde{\gamma}_{ij}(\bar{u})$ obey the same system with respect to the $\bar{u}$-variables.  □

*Proposition 4.3.* The Poisson brackets $\{\ ,\ \}_{\mathrm{d}s^2}$ and $\{\ ,\ \}_{\mathrm{d}\tilde{s}^2}$ are compatible (i.e. any linear combination of them again is a Poisson bracket). Functionals from the canonical lagrangian subspace $\mathscr{H}$ (3.46) commute with respect to $\{\ ,\ \}_{\mathrm{d}\tilde{s}^2}$. For any homogeneous function $h(u) \in \mathscr{H}$, $h(cu) = c^q h(u)$, the following identity holds:

$$\left\{ \cdot \ , \int h \, \mathrm{d}x \right\}_{\mathrm{d}\tilde{s}^2} = \left( \frac{d-1}{2} + q \right) \left\{ \cdot \ , \int \partial^{-1} h \, \mathrm{d}x \right\}_{\mathrm{d}s^2}. \tag{4.38}$$

*Proof.* Let $g_{ij}$ any $\tilde{g}_{ij}$ be any two flat metrics with the corresponding Levi-Civita connections $\Gamma^i_{jk}$ and $\tilde{\Gamma}^i_{jk}$. They determine two Poisson brackets $\{\ ,\ \}_{\mathrm{d}\tilde{s}^2}$ by formula (3.1).

*Lemma 4.2.* The Poisson brackets $\{\ ,\ \}_{\mathrm{d}s^2}$ and $\{\ ,\ \}_{\mathrm{d}\tilde{s}^2}$ are compatible iff the metric $ag^{ij} + b\tilde{g}^{ij}$ is a flat one for any $a$, $b$ and the tensor

$$T^j_{st} = \tilde{\Gamma}^j_{st} - \Gamma^j_{st} \tag{4.39}$$

satisfies the condition

$$g^{it}\tilde{g}^{ks}T^j_{st} = g^{kt}\tilde{g}^{is}T^j_{st}. \tag{4.40}$$

*Proof.* As it was proved in ref. [23], formula (3.1) determines a Poisson bracket (for non-degenerate matrix $g^{ij}$) iff the coefficients $g^{ij}$ are the contravariant components of a flat metric and $\Gamma^i_{jk}$ are the Christoffel symbols of the corresponding Levi-Civita connections. For the linear combination $a\{\ ,\ \}_{\mathrm{d}s^2} + b\{\ ,\ \}_{\mathrm{d}\tilde{s}^2}$ the coefficient before $\delta'(x - y)$ equals $ag^{ij} + b\tilde{g}^{ij}$. So this metric should be a flat one. Eq. (4.40) is equivalent to the symmetry of the corresponding connection. $\square$

In our case one has

$$ag^{ij} + b\tilde{g}^{ij} = g^{ii}(a + bu^i)\delta^{ij}. \tag{4.41}$$

Flatness of this metric was proved in lemma 4.1.
The tensor (4.39) has the form

$$T^j_{st} = -\frac{\delta_{st}\delta_{sj}}{2u^j}. \tag{4.42}$$

Eq. (4.40) for this tensor can be verified straightforward.
To prove the commutativity of functionals from $\mathscr{H}$ it is sufficient to verify (4.38) since the homogeneous functions $h_{(\alpha, p)}$ span $\mathscr{H}$. We recall that for a flat Egoroff metric

$$\mathrm{d}\tilde{s}^2 = \sum_{i=1} \tilde{g}_{ii}(\tilde{u})(\mathrm{d}\tilde{u}^i)^2$$

the functional $H = \int h \, dx$ belongs to the canonical lagrangian family iff the vector-function

$$\tilde{\psi}_i^h = \tilde{g}_{ii}^{-1/2} \tilde{\partial}_i h,$$

$\tilde{\partial}_i = \partial/\partial \tilde{u}^i$, satisfies the system

$$\tilde{\partial}_j \tilde{\psi}_i^h = \tilde{\gamma}_{ij} \tilde{\psi}_j^h \quad i \neq j. \tag{4.43}$$

Then

$$\left\{ \tilde{u}^i(x), \int h(\tilde{u}) \, dy \right\}_{d\tilde{s}^2} = \tilde{g}_{ii}^{-1/2} \tilde{\partial} \tilde{\psi}_i^h, \tag{4.44a}$$

$$\tilde{\partial} = \sum_{i=1}^{N} \tilde{\partial}_i. \tag{4.44b}$$

In our case in the coordinates $\tilde{u}^i = \log u^i$ the components of the Egoroff metric (4.34) are

$$\tilde{g}_{ii} = u^i g_{ii}, \tag{4.45a}$$

and the rotation coefficients are

$$\tilde{\gamma}_{ij} = \sqrt{u^i u^j} \, \gamma_{ij}. \tag{4.45b}$$

So

$$\tilde{\psi}_i^h = \sqrt{u^i} \, \psi_i^h, \tag{4.45c}$$

where $\psi_i^h = g_{ii}^{-1/2} \partial_i h$. So the function $\tilde{\psi}_i^h$ satisfies the system (4.43) iff the function $\psi_i^h$ satisfies the system

$$\partial_j \psi_i^h = \gamma_{ij} \psi_j^h.$$

That means that the canonical lagrangian subspaces $\tilde{\mathcal{H}}$ and $\mathcal{H}$ coincide.

The formula (4.38) follows from (4.44a) since

$$\tilde{\partial} = \sum_{i=1}^{N} = u^i \partial_i.$$

The proposition is proved.   □

   *Corollary.*   In the non-resonant case

$$\frac{d+1}{2} - q_\alpha + p \neq 0 \quad \text{for any } p = 0, 1, \dots, \tag{4.46}$$

all the equations of the hamiltonian hierarchy (3.47) are also hamiltonian systems with respect to the second Poisson structure $\{\ ,\ \}_{d\tilde{s}^2}$.

As follows from (4.38) the recursion operator for the compatible pair of Poisson structures coincides (up to multiplication by a constant matrix) with the operator $\partial^{-1}$.

For the example 5.3 of sect. 5 (below) the non-resonance conditions (4.46) are valid. But for the example 5.4 there are $g$ "resonant" systems in the hierarchy (3.47). The functionals $\int t^\alpha dX$, $\alpha = n, \ldots, n + g - 1$, belong to the annihilator of both the Poisson brackets $\{\ ,\ \}_{ds^2}$ and $\{\ ,\ \}_{d\tilde{s}^2}$.

*Remark 4.3.* The difference tensor $T^j_{s\,t}$ (4.39), (4.42) determines a new multiplication of vector fields (see sect. 2) by the formula

$$\tilde{\partial}_i \times \tilde{\partial}_j = \delta_{ij}\tilde{\partial}_i, \tag{4.47}$$

$\tilde{\partial}_i = \partial/\partial\tilde{u}^i$, $\tilde{u}^i = \log u^i$ (we omit the coefficient $-1/2$). The metrics $ds^2_{a,b}$ (4.35) are invariant for this multiplication for any $a$, $b$. So

$$\tilde{c}_\alpha{}^\gamma{}_\beta = \sum_{i=1}^N \frac{1}{u^i}\frac{\partial u^i}{\partial t^\alpha}\frac{\partial u^i}{\partial t^\beta}\frac{\partial t^\gamma}{\partial u^i}$$

is a potential deformation. But the normalization condition (2.8b) does not hold, since the unity $\tilde{\partial} = \sum_i \tilde{\partial}_i$ is not covariant constant.

## 5. Main examples

Any solutions $\gamma_{ij}(u)$ of the system (2.24) determines a $N$-parametric family of solutions of the WDVV equations. The system (2.24) can be solved using the standard machinery of the "inverse spectral transform" (see, e.g. [28,29] for localized solutions, [29] for algebraic geometry solutions, [26] for self-similar solutions). There are also solutions in elementary functions.

*Example 5.1.* $N = 2$. The system (2.24) is linear. The general self-similar solution has the form

$$\gamma_{12}(u) = \gamma_{21}(u) = \frac{i\mu}{u^1 - u^2}, \tag{5.1}$$

for some real $\mu$. The basis $\psi_{i\alpha}(u)$ of solutions of the system (2.28) for $z = 0$ has the form

$$\psi_1 = \frac{a}{\sqrt{2}}\begin{pmatrix} r^\mu \\ ir^\mu \end{pmatrix}, \qquad \psi_2 = \frac{a^{-1}}{\sqrt{2}}\begin{pmatrix} r^{-\mu} \\ -ir^{-\mu} \end{pmatrix}, \qquad r = u^1 - u^2, \tag{5.2}$$

for any $a \neq 0$. Let $\mu \neq -\frac{1}{2}$. Then the flat coordinates have the form

$$t^1 = \frac{u^1 + u^2}{2}, \qquad t^2 = \frac{a^2 r^{2\mu+1}}{2(2\mu+1)}. \tag{5.3}$$

The metric $ds^2$ has the form

$$ds^2 = \tfrac{1}{2}a^2\left[(u^1-u^2)^{2\mu}(du^1)^2 - (u^1-u^2)^{2\mu}(du^2)^2\right] = 2\,dt^1\,dt^2. \tag{5.4}$$

Here

$$q_1 = 0, \qquad q_2 = d = -2\mu. \tag{5.5}$$

For $\mu \neq \pm 1/2,\ -3/2$ the primary free-energy has the form

$$F = \tfrac{1}{2}(t^1)^2 + a^{-4}\frac{(1+2\mu)^3}{2(1-2\mu)(2\mu+3)}\left[2a^{-2}(2\mu+1)\right]^{-4\mu/(2\mu+1)}(t^2)^{(2\mu+3)/(1+2\mu)}. \tag{5.6}$$

For $\mu = -1/2$ $(d = 1)$ the flat coordinates are

$$t^1 = \frac{u^1 + u^2}{2}, \qquad t^2 = \frac{a^2}{2}\log(u^1 - u^2). \tag{5.7}$$

The potential $F(t)$ is not a homogeneous function

$$F(t) = \tfrac{1}{2}(t^1)^2 t^2 + 2^{-6}a^2\,\exp 4a^{-2}t^2 \tag{5.8}$$

(the free energy of the $CP^1$-model [5]). For $\mu = 1/2,\ -3/2$ the formula (5.3) still is valid. And logarithms should be involved into formula (5.6). The deformed chiral algebra of two elements $e_1 = e,\ e_2$ has the form

$$e_1 e_1 = e_1,\ e_1 e_2 = e_2, \qquad e_2 e_2 = a^{-4}\left[2a^{-2}(1-d)t^2\right]^{-2d/(1-d)}e_1, \quad d \neq 1, \tag{5.9}$$

and

$$e_2 e_2 = a^{-4}\left(\exp 4a^{-2}t^2\right)e_1, \quad d = 1. \tag{5.10}$$

In fact, all the formulae were obvious a priori. Only their dependence on the "inverse data" is not obvious. And this dependence is important for calculation of the $\tau$-function.

The solutions $\psi_{i\alpha}(u, z)$ of the linear system (2.26) have the form

$$
\begin{pmatrix} \psi_{11} \\ \psi_{21} \end{pmatrix} = \frac{a}{\sqrt{2}} \left( \frac{2}{z} \right)^{\mu} \sqrt{rz} \begin{pmatrix} I_{\mu-1/2}(rz) + I_{\mu+1/2}(rz) \\ i\left[ I_{\mu-1/2}(rz) - I_{\mu+1/2}(rz) \right] \end{pmatrix} e^{z(u^1+u^2)}, \qquad (5.11)
$$

$$
\begin{pmatrix} \psi_{12} \\ \psi_{22} \end{pmatrix} = \frac{a^{-1}}{\sqrt{2}} \left( \frac{z}{2} \right)^{\mu} \sqrt{rz} \begin{pmatrix} I_{-\mu-1/2}(rz) + I_{-\mu+1/2}(rz) \\ i\left[ I_{-\mu+1/2}(rz) - I_{-\mu-1/2}(rz) \right] \end{pmatrix} e^{z(u^1+u^2)} \qquad (5.12)
$$

Here $r = u^1 - u^2$; the modified Bessel function $I_\nu(x)$ (see ref. [35]) is determined as the solution of the equation

$$
x^2 I_\nu'' + x I_\nu' - (x^2 + \nu^2) I_\nu = 0, \qquad (5.13a)
$$

of the form

$$
I_\nu(x) = \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m! \Gamma(\nu + m + 1)}. \qquad (5.13b)
$$

The function $V_{\alpha\beta}(z, w)$ can be calculated using the standard expansions for products of Bessel functions.

*Example 5.2.* Let us consider 4-dimensional Fröbenius algebra with a basis $P = e$, $Q$, $R$, $S$ and with the multiplication given in the table

| × | P | Q | R | S |
|---|---|---|---|---|
| P | P | Q | R | S |
| Q | Q | $f'''(t_Q)R$ | S | 0 |
| R | R | S | 0 | 0 |
| S | S | 0 | 0 | 0 |

(5.14)

Here $f$ is an arbitrary function. This deformation is indecomposable. The symmetric scalar product has the form $\eta_{PS} = \eta_{RQ} = 1$, otherwise zero. Let $t_P$, $t_Q$, $t_R$, $t_S$ be the corresponding flat coordinates on the coupling space. The flat coordinates $x_\alpha(t, z)$ of the perturbed connection (2.14) can be found explicitly:

$$
x_P = \left[ t_S + z t_Q t_R + z^2 \left( t_Q f'(t_Q) - 2f(t_Q) \right) \right] e^{z t_P},
$$

$$
x_Q = \left[ t_R + z f'(t_Q) \right] e^{z t_P},
$$

$$
x_R = t_Q e^{z t_P},
$$

$$
x_S = z^{-1} [ e^{z t_P} - 1 ]. \qquad (5.15)
$$

The generating function for the double correlators $V_{\alpha\beta}(t; z, w) = V_{\beta\alpha}(t; w, z)$ have the form

$$V_{PP}(t; z, w) = \left[ t_S + (z + w)t_Q t_R + (z^2 - zw + w^2)(t_Q f'(t_Q) - 2f(t_Q)) \right.$$

$$\left. + zw(t_Q f''(t_Q) - f'(t_Q)) \right] e^{(z+w)t_P},$$

$$V_{PQ}(t; z, w) = \left[ t_R + zt_Q f''(t_Q) + (w - z)f'(t_Q) \right] e^{(z+w)t_P},$$

$$V_{PR}(t; z, w) = t_Q \, e^{(z+w)t_P},$$

$$V_{PS}(t; z, w) = (z + w)^{-1}[e^{(z+w)t_P} - 1],$$

$$V_{QQ}(t; z, w) = f''(t_Q) \, e^{(z+w)t_P},$$

$$V_{QR}(t; z, w) = (z + w)^{-1}[e^{(z+w)t_P} - 1], \tag{5.16}$$

other components of the matrix $V_{\alpha\beta}$ vanish. This completes the solution of the model. Particularly, for the case

$$f'''(t) = bt + e^{ct} \tag{5.17}$$

($b$, $c$ are some constants) one obtains the correlators of the topological sigma model with a Calabi–Yau target space being considered in ref. [5] (in the case the Calabi–Yau manifold has the smallest possible Hodge numbers $b_{0,0} = b_{1,1} = b_{2,2} = b_{3,3} = 1$).

*Example 5.3.* (see ref. [10]). The coupling space for the $A_{n-1}$-topological minimal model is a set of all polynomials of given degree $n$ of the form

$$M = \left\{ \lambda(p) = p^n + a_{n-2}p^{n-2} + \ldots + a_0 \mid a_0, \ldots, a_{n-2} \in \mathbb{C} \right\}. \tag{5.18}$$

Here $N = n - 1$. For any polynomial $\lambda \in M$ (it is called a Landau–Ginzburg potential) the Fröbenius algebra $A = A_\lambda$ is the algebra of truncated polynomials

$$A_\lambda = \frac{\mathbb{C}[p]}{(\lambda'(p) = 0)}, \tag{5.19}$$

with the scalar product

$$\langle f(p), g(p) \rangle = -\frac{1}{n} \operatorname*{res}_{p=\infty} \frac{f(p)g(p)}{\lambda'(p)}. \tag{5.20}$$

It is easy to see that (4.19), (4.20) is a Fröbenius algebra for any $\lambda(p)$. This is a potential deformation as it was proved in ref. [10].

The affine structure on M is introduced in the following way. Let $\phi_\alpha = \phi_\alpha(p; \lambda)$, $\alpha = 1, \ldots, n - 1$, be the orthogonal basis of $A_\lambda$

$$\langle \phi_\alpha, \phi_\beta \rangle = \eta_{\alpha\beta} \equiv \delta_{\alpha+\beta,n}, \quad \deg \phi_\alpha = \alpha - 1. \tag{5.21}$$

For the polynomials $\phi_\alpha$ in ref. [10] was obtained the formula

$$\phi_\alpha(p; \lambda) = \frac{n}{\alpha} \frac{\mathrm{d}}{\mathrm{d}p} \left[ \lambda^{\alpha/n}(p) \right]_+. \tag{5.22}$$

Here $[\ ]_+$ means the polynomial (in $p$) part of the series $\lambda^{\alpha/n}(p)$. The dependence of $\lambda(p)$ (i.e. of its coefficients) on the flat coordinates $t^\alpha$ are determined by the equations

$$\frac{\partial \lambda(p)}{\partial t^\alpha} = -\phi_\alpha(p; \lambda) \quad \alpha = 1, \ldots, n - 1. \tag{5.23}$$

It was observed in ref. [13] that eqs. (4.23) and their solution in ref. [10] have a natural interpretation in the theory of the dispersionless Gelfand–Dikii (GD) hierarchy. Also a notion of $\tau$-function of this dispersionless hierarchy was proposed in ref. [13] to obtain a formula for the primary partition function of the model. I recall that the GD (or generalized KdV) hierarchy has the form

$$\partial_{\tau^q} L = \left[ L, (L^{q/n})_+ \right], \tag{5.24a}$$

where

$$L = \partial^n + a_{n-2}(x)\partial^{n-2} + \ldots + a_0(x), \tag{5.24b}$$

is an ordinary differential operator, $\partial = \partial/\partial x$, and $(\ )_+$ means the differential part of the pseudo-differential operator $L^{q/n}$. The dispersionless approximation can be obtained from (5.24) by the substitution

$$a_i = a_i(X, T^1, \ldots), \quad X = \epsilon x, \quad T^q = \epsilon \tau^q, \quad \epsilon \to 0, \tag{5.25}$$

and taking the leading term in $\epsilon$. The FFM representation of the dispersionless GD hierarchy has the form

$$\partial_{T^q} \mathrm{d}p \mid_{\lambda = \mathrm{const.}} = \partial_X \mathrm{d}\Phi_q \mid_{\lambda = \mathrm{const.}}, \tag{5.26a}$$

$$\Phi_q = \left[ \lambda^{q/n}(p) \right]_+. \tag{5.26b}$$

It can be rewritten via wronskians [13]

$$\partial_{T^q}\lambda(p) = \partial_p\lambda(\partial_X\Phi_q)_{p=\text{const.}} - \partial_p\Phi_q(\partial_X\lambda)_{p=\text{const.}} \tag{5.27}$$

The standard calculation of ref. [18] gives that the diagonal coordinates (Riemann invariants) for the hierarchy (5.26) are the critical values $u^1,\ldots,u^{n-1}$ of $\lambda(p)$

$$u^i = \lambda(p_i) \quad i = 1,\ldots,N = n-1, \tag{5.28a}$$

where

$$\lambda'(p_i) = 0. \tag{5.28b}$$

The characteristic speeds are

$$v_{q,i}(u) = \left.\frac{d\Phi_q}{dp}\right|_{p=p_i}, \tag{5.29a}$$

$$\partial_{T^q}u^i = v_{q,i}(u)\partial_X u^i \quad i = 1,\ldots,N. \tag{5.29b}$$

It can be shown that the hierarchy (5.29) coincides with (3.47) for the model (up to normalization of the variables $T^q$). The algebra $A_\lambda$ is decomposable if the polynomial $\lambda'(p)$ has simple roots. So $u^1,\ldots,u^N$ are the canonical diagonal coordinates. The metric $ds^2$ in these coordinates equals

$$ds^2 = \sum_{i=1}^{N} \frac{(du^i)^2}{\lambda''(p_i)}. \tag{5.30}$$

Let us prove now that the functions

$$h_{\alpha,q} = -\frac{1}{(\alpha/n)_q} \operatorname*{res}_{p=\infty} \lambda^{(\alpha/n)+q}\,dp \quad \alpha = 1,\ldots,n-1, \quad q = 0,1,\ldots, \tag{5.31}$$

$$(a)_q \equiv a(a+1)\ldots(a+q-1) \tag{5.32}$$

on the space (5.18) are the basic conservation laws (3.43) of the hierarchy (5.29). Indeed, from (5.26) it is obvious that (5.31) are densities of conservation laws. One needs to verify only the recursion relations

$$\partial h_{\alpha,q} = h_{\alpha,q-1}, \quad q \geqslant 1, \tag{5.33a}$$

$$\partial h_{\alpha,0} = \text{const.}, \tag{5.33b}$$

$$\partial = \sum_{i=1}^{N}\partial_i, \quad \partial_i = \partial/\partial u^i. \tag{5.33c}$$

To do it let us observe that the translation along $\lambda$ of the form

$$\lambda \mapsto \lambda + \epsilon, \quad p \mapsto p, \quad a_i \mapsto a_i, \quad i \neq 0, \quad a_0 \mapsto a_0 + \epsilon, \tag{5.34}$$

is equivalent to the translation along $u^1 + \ldots + u^N$. So

$$\partial h_{\alpha,q} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} h_{\alpha,q}(u^1 + \epsilon, \ldots, u^N + \epsilon)_{\epsilon=0}$$

$$= \frac{1}{(\alpha/n)_q} \operatorname*{res}_{p=\infty} \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left( (\lambda + \epsilon)^{(\alpha/n)+q} \, \mathrm{d}p(\lambda + \epsilon) \right)_{\epsilon=0}$$

$$= \frac{1}{(\alpha/n)_{q-1}} \operatorname*{res}_{p=\infty} \lambda^{(\alpha/n)+q-1} \, \mathrm{d}p. \tag{5.35}$$

This gives (5.33a) for $q \geqslant 1$ and

$$\partial h_{\alpha,0} = \operatorname*{res}_{p=\infty} \lambda^{(\alpha/n)-1} \mathrm{d}p = -\delta_{\alpha,n-1}. \tag{5.36}$$

The formulae (5.33) are proved. Particularly one obtains the flat coordinates on M

$$t^\alpha = -n \operatorname*{res}_{p=\infty} \frac{\lambda^{(n-\alpha)/n}}{n-\alpha} \mathrm{d}p \quad \alpha = 1, \ldots, N = n - 1. \tag{5.37}$$

The generating functions $x_\alpha(t, z) = \Sigma h_{\alpha,q}(t) z^q$ can be written in the form

$$x_\alpha(t, z) = -\frac{n}{\alpha} \operatorname*{res}_{p=\infty} {}_1F_1\left(1; 1 + \frac{\alpha}{n}; z\lambda\right) \mathrm{d}p \quad \alpha = 1, \ldots, N. \tag{5.38}$$

Here ${}_1F_1(a; c; z)$ is the Kummer (or confluent hypergeometric) function [35]

$${}_1F_1(a; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \frac{z^m}{m!}. \tag{5.39}$$

The scaling dimensions of the variables $t^1, \ldots, t^N$ equal 1, $(n-1)/n, \ldots, 2/n$ respectively, and $d = (n-2)/n$. So the eigenvalues of the matrix $[U, \Gamma]$ (see sect. 4) are

$$\mu_\alpha = -\tfrac{1}{2} + \frac{\alpha}{n} \quad \alpha = 1, \ldots, n - 1. \tag{5.40}$$

I recall that these are the local monodromy indices of the system (4.5) near the point $z = 0$. It would be interesting to calculate all the monodromy data of the solutions (5.38) in $z$-plane. Note that the non-resonance conditions (4.28) are valid

in this case. So all the functions $h_{\alpha,q}$ can be expressed in algebraic way via the coordinates $t^\alpha$ and $u^i$ and the rotation coefficients.

The generating function of two-point correlators have the form

$$V_{\alpha\beta}(t; z, w) = \eta^{\mu\nu}(z + w)^{-1}\left\{\left[\operatorname*{res}_{p=\infty} \lambda^{(\alpha/n)-1}{}_1F_1\left(1; \frac{\alpha}{n}; z\lambda\right)\phi_\mu(p)\, \mathrm{d}p\right]\right.$$

$$\left.\times\left[\operatorname*{res}_{p=\infty} \lambda^{(\beta/n)-1}{}_1F_1\left(1; \frac{\beta}{n}; w\lambda\right)\phi_\nu(p)\, \mathrm{d}p\right] - \eta_{\alpha\beta}\right\}. \quad (5.41)$$

*Example 5.4.* A non-zero genus generalization of the LG machinery of the previous example was constructed in refs. [15,16]. An appropriate moduli space of algebraic curves is taken as the coupling space. More precisely, let $M_{g,n}$ be the moduli space of dimension $N = 2g + n - 1$ of smooth algebraic curves C of genus $g$ with a marked point $Q_\infty \in C$ and with a marked meromorphic function $\lambda$ on C of degree $n$ with a pole only in $Q_\infty$. If $P_1,\ldots,P_N$ are the branch points of C,

$$\mathrm{d}\lambda\,|_{P_i} = 0 \qquad\qquad (5.42)$$

then local coordinates on $M_{g,n}$ can be constructed as

$$u^i = \lambda(P_i) \quad i = 1,\ldots,N = 2g + n - 1. \qquad (5.43)$$

The one-dimensional affine group $\lambda \mapsto \alpha\lambda + \beta$ acts on $M_{g,n}$ as

$$u^i \mapsto \alpha u^i + \beta \quad i = 1,\ldots,N. \qquad (5.44)$$

For $g = 0$ the space $M_{0,n}$ coincides with the space (5.18) of polynomials of degree $n$.

For $g > 0$ the coupling space M of the model is the covering of $M_{g,n}$ being obtained by fixation of a symplectic basis $a_1,\ldots,a_g, b_1,\ldots,b_g \in H_1(C, \mathbb{Z})$ and of a local parameter $k^{-1}$ in a neighbourhood of $Q_\infty$ such that

$$k^n = \lambda, \quad \lambda \to \infty. \qquad (5.45)$$

Sometimes I will denote $k$ as $\lambda^{1/n}$ for simplicity of notations.

Let us fix an abelian differential $\mathrm{d}p$ on C such that

$$\mathrm{d}p = \mathrm{d}(k + O(1)), \quad \lambda \to \infty, \qquad (5.46a)$$

$$\oint_{a_s} \mathrm{d}p = 0 \quad s = 1,\ldots,g. \qquad (5.46b)$$

The flat Egoroff $\partial$-invariant metric on M in the local coordinates (5.43) has the form [15]

$$ds^2 = \sum_{i=1}^{N} g_{ii}(u)(du^i)^2, \tag{5.47a}$$

$$g_{ii}(u) = -\frac{1}{n} \operatorname*{res}_{P_i} \frac{(dp)^2}{d\lambda} \tag{5.47b}$$

(for $n = 2$ this formula was obtained in ref. [26]). The flat coordinates $t^1, \ldots, t^N$ for $ds^2$ have the form [15,16]

$$t^\alpha = -n \operatorname*{res}_{Q_\alpha} \frac{\lambda^{(n-\alpha)/n}}{n-\alpha} dp \quad \alpha = 1, \ldots, n-1, \tag{5.48a}$$

$$t^{n-1+\alpha} = \frac{1}{2\pi i} \oint_{a_\alpha} p \, d\lambda \qquad \alpha = 1, \ldots, g, \tag{5.48b}$$

$$t^{g+n-1+\alpha} = \oint_{b_\alpha} dp \qquad \alpha = 1, \ldots, g. \tag{5.48c}$$

The metric $ds^2$ in the coordinates (5.48) has the form

$$\langle dt^\alpha, dt^\beta \rangle = \delta^{\alpha+\beta} \quad \text{for} \quad 1 \leqslant \alpha, \beta \leqslant n-1, \tag{5.49a}$$

$$\langle dt^{n-1+\alpha}, dt^{g+n-1+\beta} \rangle = \delta^{\alpha\beta} \quad \text{for} \quad 1 \leqslant \alpha, \beta \leqslant g, \tag{5.49b}$$

otherwise zero.

The amazing point is in *global* definition on M of the flat coordinates $t^1, \ldots, t^N$. So M is an unramified covering over some domain in $\mathbb{C}^N$.

The coordinates $t^1, \ldots, t^N$ have a simple interpretation in the hamiltonian theory of GD hierarchy [20]. Let us consider the first hamiltonian structure of the hierarchy [36]. The annihilator of it is generated by $(n-1)$ local functionals of $L$. The values of these functionals on the family M of $g$-gap solutions equal $t^1, \ldots, t^{n-1}$. Furthermore, $t^n, \ldots, t^{g+n-1}$ are the action variables on M and $t^{g+n}, \ldots, t^N$ are the components of the wave number vector. The same interpretation of $t^1, \ldots, t^{n-1}$ is true for the example 5.3 above (the case $M = M_{0,n}$). The action variables and wave numbers for the "0-gap" solutions (constants) are not defined.

Let us construct the hierarchy (3.47). Let

$$p = \int_{P_0}^{P} dp \tag{5.50}$$

be an abelian integral (a multivalued function) on C. Here the base point $P_0$ is chosen in such a way that $\lambda(P_0) \equiv 0$ (in some domain on M).

The multivalued differential $p \, d\lambda$ on C has the form

$$p \, d\lambda = k \, d\lambda - \left( \sum_{\alpha=1}^{n-1} t^{\alpha} k^{\alpha-1} + O(k^{-1}) \right) dk, \quad k = \lambda^{1/n} \to \infty, \quad (5.51a)$$

$$\oint_{a_s} p \, d\lambda = 2\pi i t^{n+s-1} \quad s = 1, \ldots, g, \tag{5.51b}$$

$$\Delta_{a_s}(p \, d\lambda) = 0, \qquad \Delta_{b_s}(p \, d\lambda) = t^{g+n-1+s} \, d\lambda \quad s = 1, \ldots, g. \tag{5.51c}$$

Here $\Delta_{a_s}$, $\Delta_{b_s}$ are the increments along $a$- and $b$-cycles

$$\Delta_{a_s}(f(P)) = f(P + a_s) - f(P), \qquad \Delta_{b_s}(f(P)) = f(P + b_s) - f(P) \tag{5.52}$$

for any function (or differential) on C.

The primary differentials (may be, multivalued) $\phi_\alpha$ on C are defined by the formula

$$\phi_\alpha = \partial_\alpha(p \, d\lambda)_{\lambda = \text{const.}} = -\partial_\alpha(\lambda \, dp)_{p = \text{const.}} \quad \alpha = 1, \ldots, N, \tag{5.53}$$

$\partial_\alpha = \partial/\partial t^\alpha$. So the function $\lambda = \lambda(p)$, $p = \int dp$, is the LG potential of the model. More explicitly,

$$\phi_\alpha \equiv dp^{(\alpha)} = \left( -k^{\alpha-1} + O(k^{-2}) \right) dk, \quad k = \lambda^{1/n} \to \infty, \tag{5.54a}$$

$$\oint_{a_s} dp^{(\alpha)} = 0 \quad s = 1, \ldots, g \tag{5.54b}$$

(the normalized abelian differential of the second kind, $\phi_1 = dp^{(1)} = -dp$);

$$\phi_{n-1+\alpha} \equiv \omega_\alpha \quad \alpha = 1, \ldots, g, \tag{5.55a}$$

are the normalized holomorphic differentials on C,

$$\oint_{a_s} \omega_\alpha = 2\pi i \delta_{\alpha s}, \tag{5.55b}$$

$$\phi_{g+n-1+\alpha} \equiv \sigma_\alpha \quad \alpha = 1, \ldots, g, \tag{5.56a}$$

is a holomorphic (modulo $d\lambda$) everywhere on C multivalued differential with the increments

$$\Delta_{a_\beta}\sigma_\alpha = 0, \qquad \Delta_{b_\beta}\sigma_\alpha = \delta_{\alpha\beta} \, d\lambda. \tag{5.56b}$$

The primary part (3.20) of the hierarchy (3.47) here has the FFM form

$$\partial_{T^{\alpha,0}} \, \mathrm{d}p = -\partial_X \phi_\alpha \quad \alpha = 1, \ldots, N. \tag{5.57}$$

Equivalently, in the diagonal variables $u^1, \ldots, u^N$

$$\partial_{T^{\alpha,0}} u^i = - \left. \frac{\phi_\alpha}{\mathrm{d}p} \right|_{P_i} \partial_X u^i. \tag{5.58}$$

The equations (5.57) for $\alpha = 1, \ldots, n - 1$ (together with the corresponding part of the hierarchy (3.47)) can be obtained from the GD hierarchy (5.24) by the averaging procedure [17–20,23] over the family M of $g$-gap solutions of GD. This part for $g \geqslant 0$ was used in ref. [14] to construct solutions of multicut loop equations [21]. But for $g > 0$ without the extension (5.57) for $\alpha = n, \ldots, N$ it is impossible to construct closed primary operator algebra on the base of the averaged GD hierarchy.

For the particular solution (3.77) of (5.57) where $T^{\alpha,0} \equiv T^\alpha$ and $\partial_X u^i \equiv 1$, $i = 1, \ldots, N$, one obtains

$$\partial_{T^{\alpha,0}} u^i \equiv \partial_\alpha u^i = - \left. \frac{\phi_\alpha}{\mathrm{d}p} \right|_{P_i} \quad i = 1, \ldots, N. \tag{5.59}$$

This coincides with (5.53).

The formulae (5.58), (2.34) immediately give the residue representation of the primary correlators

$$\eta_{\alpha\beta} = \sum_{i=1}^N \operatorname*{res}_{P_i} \frac{\phi_\alpha \phi_\beta}{\mathrm{d}\lambda}, \tag{5.60a}$$

$$c_{\alpha\beta\gamma}(t) = - \sum_{i=1}^N \operatorname*{res}_{P_i} \frac{\phi_\alpha \phi_\beta \phi_\gamma}{\mathrm{d}\lambda \, \mathrm{d}p}. \tag{5.60b}$$

The primary operator algebra (5.60) can be represented via relations among some quadratic differentials

$$\phi_\alpha \phi_\beta = c_\alpha{}^\gamma{}_\beta \phi_\alpha \, \mathrm{d}p \ (\text{modulo } \mathrm{d}\lambda\text{-divisible differentials}). \tag{5.61}$$

Let us call this model as $M_{g,n}$-model.

The charges $q_\alpha$ of the primary fields $\phi_\alpha$ equal

$$q_\alpha = \frac{\alpha - 1}{n} \quad \alpha = 1, \ldots, n - 1, \tag{5.62a}$$

$$q_{n-1+\alpha} = -\frac{1}{n} \qquad \alpha = 1, \ldots, g, \tag{5.62b}$$

$$q_{g+n-1+\alpha} = \frac{n-1}{n}, \qquad \alpha = 1, \ldots, g \tag{5.62c}$$

The dimension of the $M_{g,n}$-model equals

$$d = \frac{n-2}{n}. \tag{5.63}$$

There are resonances of the form (4.28) (for $p = 1$). So the recurrence procedure (4.27) for the vectors $\psi_{i(\alpha,1)}$ is not well-defined.

Note that one can choose any of the primary differentials $\phi_\alpha$ to construct another $\partial$-invariant Egoroff metric

$$ds_\alpha^2 = \sum g_{ii_\alpha}(du^i)^2, \tag{5.64a}$$

$$g_{ii_\alpha} = \mathop{\mathrm{res}}_{P_i} \frac{(\phi_\alpha)^2}{d\lambda}, \tag{5.64b}$$

with the same rotation coefficients. This possibility is a reflection of arbitrariness in the choice of the solution $\psi_{i1}$ of the system (2.26). For the metric (5.64) one obtains another global affine coordinate system on M. The formulae (5.62), (5.63) for the scaling dimensions also will change.

Let us construct now the generating functions $x_\alpha(t, z)$ and $V_{\alpha\beta}(t; z, w)$. For the generating function $x_\alpha(t, z)$ of the complete family of the conservation laws (3.43) of (5.57) the following formulae are valid:

$$x_\alpha(t, z) = -\frac{n}{\alpha} \mathop{\mathrm{res}}_{Q_\infty} \lambda^{\alpha/n} {}_1F_1\left(1; 1 + \frac{\alpha}{n}; z\lambda\right) dp \qquad \alpha = 1, \ldots, n-1, \tag{5.65a}$$

$$x_{\alpha'}(t, z) = \oint_{b_\alpha} e^{z\lambda} dp \qquad \alpha' = n - 1 + \alpha, \; 1 \leqslant \alpha \leqslant g, \tag{5.65b}$$

$$x_{\alpha''}(t, z) = \frac{1}{2\pi i} \oint_{a_\alpha} p \, e^{\lambda z} d\lambda \qquad \alpha'' = g + n - 1 + \alpha, \; 1 \leqslant \alpha \leqslant g. \tag{5.65c}$$

The proof is similar to the previous example.

The generating function $V_{\alpha\beta}(t; z, w)$ for double correlators in the $M_{g,n}$-model coupled to gravity can be calculated now via the formula (3.62), where

$$\partial_\mu x_\alpha(t, z) = \mathop{\mathrm{res}}_{Q_\infty} \lambda^{(\alpha-n)/n} {}_1F_1\left[1; \frac{\alpha}{n}; z\lambda\right] \phi_\mu(\lambda) \qquad 1 \leqslant \alpha \leqslant n-1, \tag{5.66a}$$

$$\partial_\mu x_{\alpha'}(t, z) = \eta_{\mu\alpha'} - z \oint_{b_\alpha} e^{z\lambda} \phi_\mu(\lambda) \quad \alpha' = n - 1 + \alpha, \ 1 \leqslant \alpha \leqslant g, \quad (5.66b)$$

$$\partial_\mu x_{\alpha''}(t, z) = \frac{1}{2\pi i} \oint_{a_\alpha} e^{z\lambda} \phi_\mu(\lambda) \quad \alpha'' = g + n - 1 + \alpha, \ 1 \leqslant \alpha \leqslant g. \quad (5.66c)$$

(In ref. [15,16] formulae of another type for the coefficients $V_{(\alpha,p),(\beta,q)}$ were obtained.) Particularly, one obtains [15,16]

$$V_{(\alpha',0),(\beta',0)} \equiv \langle \phi_{\alpha'} \phi_{\beta'} \rangle = -\tau_{\alpha\beta} \quad \alpha' = n - 1 + \alpha, \ \beta' = n - 1 + \beta, \ 1 \leqslant \alpha, \ \beta \leqslant g,$$

$$(5.67a)$$

where

$$\tau_{\alpha\beta} = \oint_{b_\beta} \omega_\alpha \qquad (5.67b)$$

is the period matrix of holomorphic differentials on the algebraic curve C. Thus the problem of specification of the solution of the WDVV being described in this example (say, from the point of view of the isomonodromy deformation theory for (4.5)) seems to be very important for solving the Schottky problem of specification of the period matrices of holomorphic differentials on Riemann surfaces [37].

In ref. [15] also more general models of this type were considered where M is a covering over the moduli space $M_{g;n_1,\ldots,n_m}$ of algebraic curves of given genus $g$ with $m$ marked points and with marked meromorphic function $\lambda$ with poles only in these marked points of given orders $n_1,\ldots,n_m$. I will not consider this example here.

In ref. [15,16] the $M_{g,n}$-model was called as minimal model of non-zero genus. To avoid an abusement I stress again that $M_{g,n}$ gives a tree-level primary correlators for a model of topological field theory coinciding with the $A_{n-1}$-minimal model for $g = 0$.

It seems plausible, nevertheless, that $M_{g,n}$-models for $g > 0$ can be obtained from minimal models as a result of phase transitions. I'll outline here this points for the simplest example of $n = 2$. Let us consider the $k = 3$ model of "pure gravity" [2]. Here one has only one primary field $\phi_1 = \mathscr{P}$. The dilaton operators for this multicritical point coincides with $\sigma_3(\phi_1)$. Let us consider the dependence of the primary correlator $t = \langle \mathscr{P}\mathscr{P} \rangle$ on the couplings $T^{1,0} = X$ and $T^{1,1} = T$, other couplings vanish. Let the couplings $X, T$ be real. The dependence is specified by the string equation

$$X + tT = \tfrac{1}{3} t^3. \qquad (5.68)$$

The function $t = t(X, T)$ is smooth for $T < 0$. For $T = 0$ it has the form $t = (3X)^{1/3}$. So triple correlators have a singularity in this point. After formal extension of $t(X,$

$T$) for positive $T$ one would obtain a three-valued function $t = (t_1(X) < t_2(X) < t_3(X))$ (for fixed $T > 0$) in some domain $X_- < X < X_+$.

Instead of this multi-valued correlators I propose to use the prescription of the dispersive hydrodynamics [20,28,32] having been elaborated in description of the dispersive analogue of shock-waves. The idea is to consider the $M_{1,2}$-model to describe behaviour of the correlator inside some interval $X'_-(T) < X < X'_+(T)$, $X'_+(T) = A_+ T^{3/2}$ for some constants $A_+$. Matching with solutions of (5.68) in the edge points of the interval (and also the position of the edge points) is specified by the assumptions of $C^1$-smoothness for $T \neq 0$ of the correlator $\langle \mathscr{P}\mathscr{P} \rangle$. This is provided by an appropriate squeezing of the elliptic curve (being the point of $M_{1,2}$). In the left-edge $X'_-$ the $a$-cycle of the curve should be pinched, in the right-edge $X'_+$ the $b$-cycle should be pinched.

For other multicritical points of minimal models coupled to gravity other $M_{g,n}$ models (with any $g$) can be obtained as a result of phase transitions.

The models $M_{g;n_1,\ldots,n_m}$ could be obtained by "fusion" of a number of $M_{g,n}$-models. I hope to describe this picture of phase transitions of $M_{g,n}$-models in the next publication.

## Appendix A. Inverse problem formulae for string correlators

Here I am going to express the $\tau$-function in the decomposable case via solutions of the linear problem (2.26). It is sufficient to obtain an expression for the generating function (3.62).

Let us fix a basis $\psi_i^\alpha(u)$, $\alpha = 1,\ldots,N$ in the space of solutions of the linear system (2.28). Here $u = (u^1,\ldots,u^N)$ are the canonical coordinates (2.18). I recall that the basis $\psi_i^\alpha$ relates to the flat coordinates $t^\alpha$ and the scalar product $\eta_{\alpha\beta}$ by the formulae

$$\eta^{\alpha\beta} = \sum_{i=1}^N \psi_i^\alpha(u)\psi_i^\alpha(u), \tag{A.1}$$

$$g_{ii}^{1/2} = \eta_{1\alpha}\psi_i^\alpha, \tag{A.2}$$

$$\partial_i t^\alpha = g_{ii}^{1/2}\psi_i^\alpha. \tag{A.3}$$

Let

$$\psi_{i\alpha}(u) = \eta_{\alpha\beta}\psi_i^\beta(u), \qquad (\eta_{\alpha\beta}) = (\eta^{\alpha\beta})^{-1}, \tag{A.4}$$

and

$$\psi_{i\alpha}(u, z) = \sum_{p=0}^{\infty} \psi_{i(\alpha,p)}(u) z^p \tag{A.5}$$

be a solution of the system (2.26) being specified by the normalization

$$\psi_{i\alpha}(u, 0) = \psi_{i\alpha}(u). \tag{A.6}$$

It can be normalized also by the equations

$$\sum_{i=1}^{N} \psi_{i\alpha}(u, z)\psi_{i\beta}(u, -z) = \eta_{\alpha\beta}. \tag{A.7}$$

It relates to the functions $x_\alpha(t, z)$ by the formula

$$\psi_{i\alpha}(u, z) = g_{ii}^{-1/2}(u)\partial_i x_\alpha(t, z) \tag{A.8}$$

(here $t = t(u)$). For the coefficients $\psi_{i(\alpha,p)}$ this implies

$$\psi_{i,(\alpha,p)} = g_{ii}^{-1/2}(u)\partial_i h_{\alpha,p}. \tag{A.9}$$

Thus the gradient of $x_\alpha(t, z)$ in the diagonal coordinates $u^1, \ldots, u^N$ has the form

$$\nabla x_\alpha(t, z) = \sum_{i=1}^{N} \frac{\psi_{i\alpha}(u, z)}{\psi_{i(1,0)}(u)}\partial_i. \tag{A.10}$$

So the hierarchy (3.47) in the coordinates $u^1, \ldots, u^N$ has the form

$$\partial_{T^{\alpha,p}} u^i = \frac{\psi_{i(\alpha,p)}(u)}{\psi_{i(1,0)}(u)}\partial_X u^i \quad i = 1, \ldots, N. \tag{A.11}$$

For the generating formula (A.8) gives

$$V_{\alpha\beta}(u; z, w) = (z + w)^{-1}\left[\sum_{i=1}^{N} \psi_{i\alpha}(u, z)\psi_{i\beta}(u, w) - \eta_{\alpha\beta}\right]. \tag{A.12}$$

Particularly,

$$x_\alpha(u, z) = z^{-1}\left[\sum g_{ii}^{1/2}(u)\psi_{i\alpha}(u, z) - \eta_{\alpha 1}\right]. \tag{A.13}$$

Completeness of the conservation laws $h_{\alpha,p}$ follows from the following statement.

*Proposition A.1.*   Let $h_{\alpha,p}(u)$, $\alpha = 1, \ldots, N$, $p = 0, 1 \ldots$ be a family of densities of conservation laws of a hamiltonian system of hydrodynamic type. Let us assume that the system have a diagonal form with different characteristic speeds in the coordinates $u^1, \ldots, u^N$ and the Poisson brackets for the system have the form (3.10) where the metric $ds^2$ in the coordinates $u^1, \ldots, u^N$ is a diagonal Egoroff $\partial$-invariant metric. If the densities $h_{\alpha,p}(u)$ satisfies the recursion relations

$$\partial h_{\alpha,p+1} = h_{\alpha,p} \quad p \geqslant 0, \tag{A.14a}$$

where

$$\partial = \sum_{i=1}^{N} \partial_i, \tag{A.14b}$$

and the densities $h_{\alpha,0} = t_\alpha$ span the annihilator of the Poisson bracket, then linear combinations of the functionals

$$\int h_{\alpha,p} \, dX \tag{A.15}$$

form a dense subset in the space of all conservation laws of the system.
   This can be proved using ref. [25].

## Appendix B. Inverse spectral transform for the similarity reduction of the WDVV

Here I outline the solution of the system (4.3) via reduction to a matrix Riemann problem.
   Let $R_{ij}$, $1 \leqslant i, j \leqslant N$, be rays in the complex $z$-plane of the form

$$\text{Re } z(u^i - u^j) = 0, \quad \text{Re } z \, e^{i\epsilon}(u^i - u^j) < 0 \quad \text{for} \quad \epsilon > 0, \quad z \in R_{ij}. \tag{B.1}$$

Let R be a line via the origin in the $z$-plane not containing any of the rays (B.1). It divides the $z$-plane into two half-planes $\pi_+$ and $\pi_-$ (with respect to the standard orientation of the $z$-plane). Let $R_+$ and $R_-$ be respectively the positive and the negative part of R with respect to the above orientation.
   The datum of the inverse problem for solving (4.3) is a Stokes matrix $S$. It is a $N \times N$ matrix $S = (s_{ij})$ with the properties

$$s_{ii} = 1 \quad i = 1, \ldots, N, \qquad s_{ij} = 0 \quad \text{if} \quad R_{ij} \subset \pi_-. \tag{B.2}$$

There are $N(N-1)/2$ independent parameters in $S$.

The formulation of the Riemann problem is as follows. Let $\Psi_+(u, z)$, $\Psi_-(u, z)$ be two $N \times N$-matrix-valued functions analytic in $z$ in the half-planes $\pi_+$ and $\pi_-$ respectively satisfying the following boundary conditions:

$$\Psi_+ \mid_{R_+} = \Psi_- \mid_{R_+} S, \qquad \Psi_+ \mid_{R_-} = \Psi_- \mid_{R_-} S^T \tag{B.3a}$$

($S^T$ means the transposed matrix) with the asymptotics on $z \to \infty$ of the form

$$\Psi_\pm(u, z) \, e^{-zU} = \mathbb{1} + O(z^{-1}), \quad z \to \infty. \tag{B.3b}$$

($\mathbb{1}$ is the unity). The solution of this Riemann problem can be reduced to solution of linear integral equations in the standard way (see, e.g. ref. [31]).

*Proposition B.1.* If $\Psi_\pm(u, z)$ is a solution of the Rieman problem (B.3) then the matrix

$$\Gamma(u) = \left( \gamma_{ij}(u) \right) = \lim_{z \to \infty} z \left[ \Psi_\pm(u, z) \, e^{-zU} - \mathbb{1} \right] \tag{B.4}$$

satisfies (4.3).

The proof is standard for the isomonodromy deformation theory.

Note that the eigenvalues of the matrix $[U, \Gamma]$ can be calculated as

$$\text{eigen } [U, \Gamma] = \frac{1}{2\pi i} \text{eigen } \log SS^{T-1}. \tag{B.5}$$

This is the analogue of the cyclic relations [31].

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