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INTEGRABLE SYSTEMS AND CLASSIFICATION OF 2-DIMENSIONAL TOPOLOGICAL FIELD THEORIES.

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Abstract.

In this paper we consider from the point of view of differential geometry and of the theory of integrable systems the so-called WDVV equations as defining relations of 2-dimensional topological field theory. A complete classification of massive topological conformal field theories (TCFT) is obtained in terms of monodromy data of an auxillary linear operator with rational coefficients. Procedure of coupling of a TCFT to topological gravity is described (at tree level) via certain integrable bihamiltonian hierarchies of hydrodynamic type and their τ-functions. A possible role of bihamiltonian formalism in calculation of high genus corrections is discussed. As a biproduct of this discussion new examples of infinite dimensional Virasoro-type Lie algebras and their nonlinear analogues are constructed. As an algebro-geometrical applications it is shown that WDVV is just the universal system of integrable differential equations (high order analogue of the Painlevé-VI) specifying periods of Abelian differentials on Riemann surfaces as functions on moduli of these surfaces.

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This paper is an extended version of the talk given at the Workshop "Integrable Systems" (Luminy, July 1991). Also more recent results [39] are included. I dedicate it to the memory of J.-L.Verdier.

Introduction.

A quantum field theory (QFT) on a D-dimensional manifold M consists of:

1). a family of local fields $\phi_\alpha(x)$, $x \in M$ (functions or sections of a fiber bundle over M). A metric $g_{ij}(x)$ on M usualy is one of the fields (the gravity field).

2). A Lagrangian $L = L(\phi, \phi_x, ...)$. Classical field theory is determined by the Euler – Lagrange equations

$$\frac{\delta S}{\delta \phi_\alpha(x)} = 0, \quad S[\phi] = \int L(\phi, \phi_x, ...).$$

3). Procedure of quantization usualy is based on construction of an appropriate path integration measure $[d\phi]$. The partition function is a result of the path integration over the space of all fields $\phi(x)$

$$Z_M = \int [d\phi] e^{-S[\phi]}.$$

Correlation functions (non normalized) are defined by a similar path integral

$$< \phi_\alpha(x)\phi_\beta(y)\ldots >_M = \int [d\phi]\phi_\alpha(x)\phi_\beta(y)\ldots e^{-S[\phi]}.$$

Since the path integration measure is almost never well-defined (and also taking in account that different Lagrangians could give equivalent QFT) an old idea of QFT is to construct a self-consistent QFT by solving a system of differential equations for correlation functions. These equations were scrutinized in 2D conformal field theories where D=2 and Lagrangians are invariant with respect to conformal transformations

$$\delta g_{ij}(x) = \epsilon g_{ij}(x), \quad \delta S = 0.$$

This theory is still far from being completed since complexity (and, probably, nonintegrability) of the differential equations determining correlators.

Here I will consider another class of solvable QFT: topological field theories. These theories admit *topological invariance*: they are invariant with respect to arbitrary change of the metric $g_{ij}(x)$ on M

$$\delta g_{ij}(x) = \text{arbitrary}, \quad \delta S = 0.$$

On the quantum level that means that the partition function Z_M depends only on topology of M. All the correlation functions also are topological creatures: they depend only on the labels of operators and on topology of M but not on positions of operators

$$< \phi_\alpha(x)\phi_\beta(y)\ldots >_M \equiv < \phi_\alpha\phi_\beta\cdots >_M .$$

The simplest example is 2D gravity with the Hilbert – Einstein action

$$S = \int R\sqrt{g} d^2x = \text{Euler characteristic of } M.$$

There are two ways of quantization of this functional. The first one is based on an appropriate discrete version of the model ($M \to$ polihedron). This way leads to considering matrix integrals of the form [55]

$$Z_N(t) = \int_{X^*=X} \exp\{-\mathrm{tr}(X^2 + t_1 X^4 + t_2 X^6 + \ldots) dX$$

where the integral should be taken over the space of all $N \times N$ Hermitian matrices X. Here t_1, t_2 ... are called coupling constants. A solution of 2D gravity [1] is based on the observation that after an appropriate limiting procedure $N \to \infty$ (and a renormalization of t) the limiting partition function coincides with τ-function of the KdV-hierarchy.

Another approach to 2D gravity is based on an appropriate supersymmetric extension of the Hilbert – Einstein Lagrangian [2]. This reduces the path integral over the space of all metrics $g_{ij}(x)$ on a surface M of the given genus g to an integral over the finite-dimensional space of conformal classes of these metrics, i.e. over the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. Correlation functions of the model are expressed via intersection numbers of some cycles on the moduli space [2-4, 46, 49]

$$\phi_\alpha \leftrightarrow c_\alpha \in H_*(\mathcal{M}_g), \quad \alpha \in \mathbf{N}$$

$$< \phi_\alpha \phi_\beta \ldots >_g = \#(c_\alpha \cap c_\beta \cap \ldots)$$

(here the supscript g means correlators on a surface of genus g). This approach is often called *cohomological field theory*.

It was conjectured by Witten that the both approaches to 2D quantum gravity should give the same results. This conjecture was proved by Kontsevich [42-43]. He showed that the generating function

$$F(t) = \sum_g \sum_{\alpha,\beta\ldots} \frac{t_1^\alpha}{\alpha!} \frac{t_2^\beta}{\beta!} \ldots < \phi_\alpha \phi_\beta \ldots >_g$$

(the free energy of 2D gravity) is logarythm of τ-function of a solution of the KdV hierarchy (this was the original form of the Witten's conjecture). The τ-function is specified by the string equation (see eq. (3.16b) below).

Other examples of 2D TFT constructed in [2-6, 8-9, 46-49, 57] proved that these could have important mathematical applications, probably being the best tool for treating sophisticated topological objects. In these examples correlators can be expressed via intersection numbers on moduli spaces (or their coverings [48]) of holomorphyc maps of Riemann surfaces to a complex (or even almost complex) variety (*topological sigma-models* [2]) or via intersection form of a singularity in the catastrophe theory (*topological Landau – Ginsburg models* [9, 7]; see also [10]). (We do not discuss here interesting relations between these models.) This gives rise to the following

Problems. What could be an intrinsic origin of integrability in 2D TFT? How one can classify 2D TFT? Is it possible to find an analogue of the KdV hierarchy for calculating the partition function of a given TFT model?

In this paper an approach to these problems is proposed being based on differential geometry and on the theory of classical integrable systems of KdV type. Main ingredient of my approach is Hamiltonian formalism of integrable hierarchies of KdV type (see, e.g., [21, 25, 29]) and, especially, Hamiltonian analysis of semi-classical limits of these systems [18-21].

Let me start with considering *matter sector* of a 2D topological field theory. That means that the set of local fields $\phi_1(x), \ldots, \phi_n(x)$ (the so-called *primary fields* of the model) does not contain the metric. (Afterwards one should integrate over the space of metrics. This should give rise to a procedure of *coupling to topological gravity* that will be described below.) Then the correlators of the fields $\phi_1(x)$, ..., $\phi_n(x)$ obey very simple algebraic axioms (a consequence [51] of the general Atiyah's axioms [50] of a topological field theory).

Let

$$\eta_{\alpha\beta} =< \phi_\alpha \phi_\beta >_0$$

(0 means genus zero correlator),

$$c_{\alpha\beta\gamma} =< \phi_\alpha \phi_\beta \phi_\gamma >_0$$

Then

1) These tensors are symmetric and $\det(\eta_{\alpha\beta}) \neq 0$. I will use the tensor $\eta_{\alpha\beta}$ and the inverse $(\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}$ for lowering and raising indices.

2) $c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} c_{\alpha\beta\epsilon}$ is a tensor of structure constants of a commutative associative algebra A with a unity. That means that for a basis e_1, ..., e_n in A the multiplication law has the form

$$e_\alpha e_\beta = c^\gamma_{\alpha\beta} e_\gamma.$$

(We will normalise a basis in such a way that e_1 = the unity of A. So $c^\beta_{1\alpha} = \delta^\beta_\alpha$.)

3) Let $H = \eta^{\alpha\beta} e_\alpha e_\beta \in A$. Then for correlators of genus g the following formula holds

$$< \phi_\alpha \ldots \phi_\gamma >_g =< e_\alpha \ldots e_\gamma, H^g > .$$

On this way

Topologicaly invariant Lagrangian $\rightarrow$ correlators of local physical fields

we lose too much relevant information. To capture more information on a topological Lagrangian we will consider a topological field theory together with its deformations preserving topological invariance

$$L \rightarrow L + \sum t^\alpha L^{(pert)}_\alpha$$

(t^α are coupling constants). Here we use ideas and results of [8, 51]. In these papers it was proposed a general construction of a class of 2D TFT by twisting of N=2 superconformal field theories. So-called *topological conformal field theories* (TCFT) are obtained by this procedure. For any TCFT with n local observables (primary operators) it was constructed a *canonical* n-parameter deformation preserving topological invariance. All the correlators of the primary fields ϕ_1, ..., ϕ_n in the perturbed TCFT now depend on coupling parameters t_1, ..., t_n. This dependence is not arbitrary but obeys the following eguations:

1). $\eta_{\alpha\beta} \equiv \text{const in } t$

2). $c^{\alpha}_{1\beta} \equiv \delta^{\alpha}_{\beta}$

3). $c_{\alpha\beta\gamma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$ for some function $F(t)$ (primary free energy).

Equations of associativity give a system of nonlinear PDE for $F(t)$

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\sigma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\gamma \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\beta \partial t^\sigma} \tag{0.1}$$

with the constraint

$$\frac{\partial^3 F(t)}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}. \tag{0.2}$$

These equations were called in [39] *Witten – Dijkgraaf – E. Verlinde – H. Verlinde* (WDVV) equations. In fact in TCFT one should assume invariance of a solution with respect to scaling transformations of the form

$$t^\alpha \mapsto c^{1-q_\alpha} t^\alpha$$

$$\eta_{\alpha\beta} \mapsto c^{q_\alpha + q_\beta - d} \eta_{\alpha\beta}$$

$$c_{\alpha\beta} \mapsto c^{q_\alpha + q_\beta - q_\gamma} c^{\gamma}_{\alpha\beta}$$

for some numbers q_α (*charges* of the fields ϕ_α) and d (*dimension* of the model).

My program now is:

1. To classify 2D TFT as solutions of WDVV equations, and

2. For any solution of WDVV (I recall that this describes the matter sector of a TFT model) to construct (i.e., to calculate partition function and correlators) a complete TFT model (coupling of the given matter sector to topological gravity).

The problem 1 was investigated in [39]. A Lax pair for the WDVV equations was constructed. For the so-called *massive* TCFT models where the Fröbenius algebra for almost all t has no nilpotents it was shown that solutions of the WDVV equations form a $\frac{n(n-1)}{2}+1$-dimensional family (where n is the number of primaries). It turns out that the WDVV equations for massive case are equivalent to equations of isomonodromy deformations of an ordinary linear differential operator with rational coefficients. These isomonodromy deformation equations coincide with the Painlevé-VI equation (for $n = 3$) and with high order analogues of the Painlevé-VI for $n > 3$. Monodromy data (i.e. Stokes matrices) of the operator with rational coefficients serve as parameters of massive TCFT-models.

Concerning the second problem my conjecture is that the set of solutions of WDVV parametrizes a big class of hierarchies of 1+1-integrable systems. All the wellknown hierarchies are in this class but they are only isolated points in it.

The basic idea of construction of these integrable hierarchies comes from the standard in quantum field theory Feynmann diagram expansion machinery. In 2D QFT it becomes a representation of partition function and correlators as a sum of contributions from surfaces of different genera g (*genus expansion*). The idea is that the genus expansion of the partition function should coincide with the small dispersion expansion (see below) of the τ-function of some integrable hierarchy.

A first step on this way has been done in [39]: for any solution of WDVV a hierarchy of integrable Hamiltonian equations of hydrodynamic type was constructed such that τ-function of a particular solution of this hierarchy coincides with the genus zero approximation of the correspondent TFT model coupled to gravity (see Section 3 below). From the point of view of WDVV equations the hierarchy determines a family of symmetries of the equation (0.1) (see below, Proposition 3.1). To go further one should solve a non-standard "inverse problem": to reconstruct integrable hierarchy from the zero-dispersion limit of it. Some examples of such a reconstruction are discussed in Sections 3, 4. Probably, bihamiltonian formalism could be useful to complete solution of this problem.

Examples of solutions of WDVV and of corresponding integrable hierarchies are described in Section 4. Almost all the known examples are obtained as a result of analysis of the semiclassical (particularly, dispersionless) limiting procedure in integrable hierarchies of KdV type. More precisely, let

$$\partial_{t_k} y^a = f_k^a(y, \partial_x y, \partial_x^2 y, \ldots),\ a = 1, \ldots, l,\ k = 0, 1, \ldots$$

be a commutative hierarchy of Hamiltonian integrable systems of the KdV type. "Hierarchy" means that the systems are ordered, say, by action of a recursion operator. Number of recursions determine a level of a system in the hierarchy. Systems of the level zero form a primary part of the hierarchy (these correspond to the primary operators in TFT); others can be obtained from the primaries by recursions. The hierarchy posesses a rich family of finite-dimensional invariant manifolds. Some of them can be found in a straightforward way; one needs to apply sophisticated algebraic geometry methods [28] to construct more wide class of invariant manifolds. Any of these manifolds after an extension to complex domain turns out to be fibered over some base M (a complex manifold of some dimension n) with m-dimensional tori as the fibers (common invariant tori of the hierarchy). For $m = 0$ M is nothing but the family of common stationary points of the hierarchy. For $m > 0$ M is a moduli space of Riemann surfaces of some genus g with certain additional structures: marked points, marked meromorphic function etc. These are the families of finite-gap ("g-gap") solutions of the hierarchy. The main observation is that any such M determines a solution of WDVV equation (M is a Fröbenius manifold in the terminology of Section 1 below, or the "small phase space" of a TFT theory in the terminology of [3-4, 46]). For $m = 0$ and the set M of stationary points of the Gelfand – Dickey hierarchy this essentialy follows from [8, 11]; for general case (including arbitrary genera g) a construction of solution of WDVV was given in [13-14] (see also recent preprint [44]).

To give an idea how an integrable Hamiltonian hierarchy of the above form induces tensors $c_{\alpha\beta}^{\gamma}$, $\eta_{\alpha\beta}$ on a finite dimensional invariant manifold M I need to introduce the notion of semiclassical limit of a hierarchy near a family M of invariant tori (sometimes it is called also a *dispersionless limit* or *Whitham averaging* of the hierarchy; see details in [15-21]). In the simplest case of the family of stationary solutions the semiclassical limit is defined as follows: one should substitute in the equations of the hierarchy

$$x \mapsto \epsilon x = X,\ t_k \mapsto \epsilon t_k = T_k$$

and tend ϵ to zero. For more general M (family of invariant tori) one should add averaging over the tori. As a result one obtains a new integrable Hamiltonian hierarchy where the dependent variables are coordinates v^1, ..., v^n on M and the independent variables are the slow variables X and T_0, T_1, ... This new hierarchy always has a form of a quasilinear system of PDE of the first order

$$\partial_{T_k} v^p = c_k{}^p_q(v)\partial_X v^q, \ k = 0, 1, \dots$$

for some matrices of coefficients $c_k{}^p_q(v)$. One can keep in mind the simplest example of a semiclassical limit (just the dispersionless limit) of the KdV hierarchy. Here M is the one-dimensional family of constant solutions of the KdV hierarchy. For example, rescaling the KdV one obtains

$$u_T = uu_X + \epsilon^2 u_{XXX}$$

(KdV with small dispersion). After $\epsilon \to 0$ one obtains

$$u_T = uu_X.$$

The semiclassical limit of all the KdV hierarchy has the form

$$\partial_{T_k} u = \frac{u^k}{k!}\partial_X u, \ k = 0, 1, \dots.$$

A semiclassical limit of spatialy discretized hierarchies (like Toda system) is obtained by a similar way. It still is a system of quasilinear PDE of the first order.

Let us come back to determination of tensors $\eta_{\alpha\beta}$, $c_{\alpha\beta}^\gamma$ on M. To introduce $\eta_{\alpha\beta}$ we need to use the Hamiltonian structure of the original hierarchy. A semiclassical limit (or "averaging") of this Hamiltonian structure in the sense of general construction of S.P.Novikov and the author induces a Hamiltonian structure of the semiclassical hierarchy: a Poisson bracket of the form

$$\{v^p(X), v^q(Y)\}_{\text{semiclassical}} = g^{ps}(v(X))[\delta_s^q \partial_X \delta(X - Y) - \Gamma_{sr}^q(v) v_X^r \delta(X - Y)]$$

where $g^{pq}(v)$ are contravariant components of a metric on M and $\Gamma_{pr}^q(v)$ are the Christoffel symbols of the Levi-Civitá connection for $g^{pq}(v)$ (the so-called *Poisson brackets of hydrodynamic type*). (Strictly speaking the metric and the connection are defined on a real part of M that parametrizes smooth solutions of the original hierarchy with some reality constraints. But the formulae for the metric and the connection admit an extension onto all M.) From the general theory of Poisson brackets of hydrodynamic type [18-21] one concludes that the metric $g^{pq}(v)$ on M should have zero curvature. So local flat coordinates t^1, ..., t^n on M exist such that the metric in this coordinates is constant

$$\frac{\partial t^\alpha}{\partial v^p}\frac{\partial t^\beta}{\partial v^q} g^{pq}(v) = \eta_{\alpha\beta} = \text{const.}$$

The Poisson bracket $\{\ ,\ \}_{\text{semiclassical}}$ in these coordinates has the form

$$\{t^\alpha(X), t^\beta(Y)\}_{\text{semiclassical}} = \eta^{\alpha\beta}\delta'(X - Y).$$

The tensor $(\eta_{\alpha\beta}) = (\eta^{\alpha\beta})^{-1}$ together with the flat coordinates t^α is the first part of a structure we want to construct. (The flat coordinates t^1, ..., t^n can be expressed via Casimirs of the original Poisson bracket and action variables and wave numbers along the invariant tori - see details in [18-21].)

To define a tensor $c_{\alpha\beta}^\gamma(t)$ on M (or, equivalently, the "primary free energy" $F(t)$) we need to use a semiclassical limit of the τ-function of the original hierarchy [11, 53-54, 61]

$$\log \tau_{\text{semiclassical}}(T_0, T_1, \ldots) = \lim_{\epsilon \to 0} \epsilon^{-2} \log \tau(\epsilon t_0, \epsilon t_1, \ldots).$$

Then

$$F = \log \tau_{\text{semiclassical}}$$

where $\tau_{\text{semiclassical}}$ should be considered as a function only on n of the slow variables of the same level. (To satisfy the normalization (0.2) one should choose properly these n slow variables and normalize values of others. This specifies uniquely the semiclassical τ-function.) The semiclassical τ-function as the function of all slow variables coincides with the tree-level partition function of the matter sector $\eta_{\alpha\beta}$, $c_{\alpha\beta}^\gamma$ coupled to topological gravity.

Summarizing, we can say that a structure of Fröbenius manifold (i.e., a solution of WDVV) on an invariant manifold M of an integrable Hamiltonian hierarchy is induced by a semiclassical limit of the Poisson bracket of the hierarchy and of the τ-function of the hierarchy. So the above conjecture can be reformulated as follows: WDVV equations just specify the semiclassical limits of τ-functions of Hamiltonian integrable hierarchies.

I do not consider in this paper one more type of integrable systems being involved in 2D TFT: the so-called equations of *topological-antitopological fusion* proposed in [40]. These equations describe the ground state metric on a given 2D TFT model. See [45] about the theory of integrability of these equations. An interesting relation of these equations to the theory of harmonic maps also was found in [45].

1. Geometry of Fröbenius manifolds.

I recall that A is called a Fröbenius algebra (over **R** or **C**) if it is a commutative associative algebra with a unity and with a nondegenerate invariant inner product

$$< ab, c > = < a, bc > . \tag{1.1}$$

If e is the unity of A then the invariant inner product on A can be written in the form

$$< a, b > = \omega_e(ab) \tag{1.2a}$$

where

$$\omega_e(a) = < e, a > . \tag{1.2b}$$

Moreover, for any linear functional $\omega \in A^*$ the inner product

$$< a, b >_\omega = \omega(ab) \tag{1.3}$$

is invariant. It is nondegenerate for generic ω. Any invariant inner product on a finite-dimensional Fröbenius algebra A (only finite-dimensional algebras will be considered) can be represented in the form (1.3).

If $e_i,\ i = 1, \ldots, n$ is a basis in A then the structure of Fröbenius algebra is specified by the coefficients η_{ij}, c_{ij}^k where

$$< e_i, e_j >= \eta_{ij} \tag{1.4a}$$

$$e_i e_j = c_{ij}^k e_k \tag{1.4b}$$

(summation over repeated indices will be assumed). The matrix η_{ij} and the structure constants c_{ij}^k satisfy the following conditions:

$$\eta_{ji} = \eta_{ij}, \quad \det(\eta_{ij}) \neq 0 \tag{1.5a}$$

$$c_{ij}^s c_{sk}^l = c_{is}^l c_{jk}^s \tag{1.5b}$$

(associativity),

$$c_{ijk} = \eta_{is} c_{jk}^s = c_{jik} = c_{ikj} \tag{1.5c}$$

(commutativity and invariance of the inner product). If $e = (e^i)$ is the unity of A then

$$e^s c_{sj}^i = \delta_j^i \tag{1.5d}$$

(the Kronecker delta).

1-dimensional Fröbenius algebras are parametrized by 1 number (length of the unity). Any semisimple n-dimensional Fröbenius algebra is isomorphic to the direct sum of n one-dimensional Fröbenius algebras

$$f_i f_j = \delta_{ij} f_i, \quad < f_i, f_j >= \eta_{ii} \delta_{ij}. \tag{1.6}$$

Moreover, any Fröbenius algebra without nilpotents is a semisimple one.

Let us consider a particular class of deformations of Fröbenius algebras.

Definition 1.1. A manifold M is called *quasi-Fröbenius* if it is equipped with three tensors $c = (c_{ij}^k(x))$, $\eta = (\eta_{ij}(x))$, $e = (e^i(x))$ satisfying (1.5) for any $x \in M$.

In other words these three tensors provide a structure of Fröbenius algebra in the space of smooth vector fields $Vect(M)$ over the ring $\mathcal{F}(M)$ of smooth functions on M:

$$[v \cdot w]^k(x) = c_{ij}^k(x) v^i(x) w^j(x), \tag{1.7a}$$

$$< v, w > (x) = \eta_{ij}(x) v^i(x) w^j(x) \tag{1.7b}$$

for any $v,\ w \in Vect(M)$.

Complex quasi-Fröbenius manifolds are defined in a similar way but the tensors c, η, e should be holomorphic. They provide a structure of Fröbenius algebra in the space of holomorphic vector fields over the ring of holomorphic functions.

Informaly speaking, n-dimensional quasi-Fröbenius manifolds are n-parameter deformations of n-dimensional Fröbenius algebras. For any $x \in M$ the tangent space T_xM is a

Fröbenius algebra with the structure constants $c_{ij}^k(x)$, invariant inner product $\eta_{ij}(x)$, and unity $e^i(x)$.

As it was explained above, in physical applications there are additional restrictions for quasi-Fröbenius manifolds.

Definition 1.2. A quasi-Fröbenius M is called *Fröbenius manifold* if the invariant metric

$$ds^2 = \eta_{ij}(x)dx^i dx^j \tag{1.8a}$$

is flat, the unity vector field e is covariantly constant

$$\nabla e = 0 \tag{1.8b}$$

(here ∇ is the Levi-Civitá connection for ds^2) and the tensor

$$\nabla_z < u \cdot v, w > \tag{1.8c}$$

is symmetric in the vectors u, ..., z.

Localy Fröbenius manifolds are in 1-1 correspondence with solutions of WDVV equations (i.e., with 2D TFTs). Indeed, for the flat metric (1.8a) localy flat coordinates t^α exist such that the metric is constant in these coordinates, $ds^2 = \eta_{\alpha\beta}dt^\alpha dt^\beta$, $\eta_{\alpha\beta} = \text{const}$. The covariantly constant vector field e in the flat coordinates has constant components; using a linear change of the coordinates one can obtain $e^\alpha = \delta_1^\alpha$. The tensor $c_{\alpha\beta\gamma}(t)$ in these coordinates satisfies the condition

$$\partial_\delta c_{\alpha\beta\gamma} = \partial_\gamma c_{\alpha\beta\delta}. \tag{1.9a}$$

This means that $c_{\alpha\beta\gamma}(t)$ can be represented in the form

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha\partial_\beta\partial_\gamma F(t) \tag{1.9b}$$

for some function $F(t)$ satisfying the WDVV equations.

The first step in solving WDVV is to obtain a "Lax pair" for these equations. The most convenient way is to represent them as the compatibility conditions of an overdetermined linear system depending on a spectral parameter λ.

Proposition 1.1. *A quasi-Fröbenius manifold is Fröbenius iff the unity e is covariantly constant and the pencil of connections*

$$\tilde\nabla_u(\lambda)v = \nabla_u v + \lambda u \cdot v, \quad u,\ v \in Vect(M) \tag{1.10}$$

is flat identicaly in λ.

Flatness of the pencil of connections (1.10) is equivalent to flatness of the metric η and to the equation

$$\nabla_u(v \cdot w) - \nabla_v(u \cdot w) + u \cdot \nabla_v w - v \cdot \nabla_u w = [u, v] \cdot w \tag{1.11}$$

for any three vector fields u, v, w. Here $[u, v]$ means the commutator of the vector fields. This equation is equivalent to the symmetry of the tensor (1.8c).

Corollary. *WDVV is an integrable system.*

Indeed, WDVV is equivalent to compatibility of the following linear system

$$\tilde{\nabla}_\alpha(\lambda)\xi = 0, \quad \alpha = 1, ..., n, \tag{1.12a}$$

(here ξ is a covector field), or, equivalently, in the flat coordinates t^α

$$\partial_\alpha \xi_\beta = \lambda c_{\alpha\beta}^\gamma(t)\xi_\gamma. \tag{1.12b}$$

Compatibility of the system (1.12) (identicaly in the spectral parameter λ) together with the symmetry of the tensor $c_{\alpha\beta\gamma} = \eta_{\alpha\epsilon}c_{\beta\gamma}^\epsilon$ is equivalent to WDVV.

It turns out that symmetries of Fröbenius manifolds play an important role in geometrical foundation of TFT. We start with the notion of *algebraic symmetry* of a Fröbenius manifold.

Definition 1.3. A diffeomorphism $f : M \to M$ of a Fröbenius manifold is called algebraic symmetry if it preserves the multiplication law of vector fields:

$$f_*(u \cdot v) = f_*(u) \cdot f_*(v) \tag{1.13}$$

(here f_* is the induced linear map $f_* : T_xM \to T_{f(x)}M$).

Proposition 1.2. *Algebraic symmetries of a Fröbenius manifold form a finite-dimensional Lie group $G(M)$.*

The generators of action of $G(M)$ on M (i.e. the representation of the Lie algebra of $G(M)$ in the Lie algebra of vector fields on M) are the vector fields w such that

$$[w, u \cdot v] = [w, u] \cdot v + [w, v] \cdot u \tag{1.14}$$

for any vector fields u, v.

Note that the group $G(M)$ always is nontrivial: it contains the one-parameter subgroup of shifts along the coordinate t^1. The generator of this subgroup coincides with the unity vector field e.

The group $G(M)$ can be calculated for the important class of *massive* Fröbenius manifolds.

Definition 1.4. A Fröbenius manifold is called massive if the algebra on T_xM is semisimple for any $x \in M$.

In physical language massive Fröbenius manifolds are coupling spaces of massive TFT models.

Main lemma. *The connect component of the identity in the group $G(M)$ of algebraic symmetries of a n-dimensional massive Fröbenius manifold is a n-dimensional commutative Lie group that acts localy transitively on M.*

This is a reformulation of the main lemma of [39].

Action of the group of algebraic symmetries provides a new affine structure on a massive Fröbenius manifold. The structure tensor c_{ij}^k is constant in this affine structure.

From the main lemma the following statement follows.

Theorem 1.1. [39] *On a massive Fröbenius manifold local coordinates u^1,...u^n exist such that the multiplication law of vector fields in these coordinates has the form*

$$\partial_i \cdot \partial_j = \delta_{ij}\partial_i, \tag{1.15}$$

where $\partial_i = \partial/\partial u^i$. The invariant metric η in these coordinates has a diagonal form

$$\eta_{\alpha\beta}dt^\alpha dt^\beta = \sum_{i=1}^n \eta_{ii}(u)(du^i)^2 \tag{1.16}$$

satisfying the equations

$$d(\sum_{i=1}^n \eta_{ii}(u)du^i) = 0, \tag{1.17a}$$

$$\sum_{k=1}^n \partial_k \eta_{ii} = 0. \tag{1.17b}$$

Conversely, for a flat diagonal metric with the properties (1.17) and $t^\alpha = t^\alpha(u)$, $\alpha = 1, ..., n$ being the flat coordinates for the metric the formulae

$$\eta_{\alpha\beta} = \sum_{i=1}^n \eta_{ii}(u)\frac{\partial u^i}{\partial t^\alpha}\frac{\partial u^i}{\partial t^\beta}, \tag{1.18a}$$

$$c_{\alpha\beta}^\gamma(t) = \sum_{i=1}^n \frac{\partial u^i}{\partial t^\alpha}\frac{\partial u^i}{\partial t^\beta}\frac{\partial t^\gamma}{\partial u^i}, \tag{1.18b}$$

$$e^\alpha = \sum_{i=1}^n \frac{\partial t^\alpha}{\partial u^i} \tag{1.18c}$$

determine (localy) a massive Fröbenius manifold.

The above coordinates u^1, ... , u^n on a massive Fröbenius manifold are determined uniquely up to permutations and shifts. They are called *canonical coordinates* on the massive Fröbenius manifold M. The tensor c of structure constants in these coordinates has the following canonical constant form

$$c_{ij}^k = \delta_{ij}\delta_j^k. \tag{1.19}$$

The canonical coordinates u^i can be found by solving an overdetermined system of differential equations

$$\frac{\partial t^\alpha}{\partial u^i}\frac{\partial t^\beta}{\partial u^j}c_{\alpha\beta}^\gamma = \delta_{ij}\frac{\partial t^\gamma}{\partial u^i}.$$

For massive conformal invariant Fröbenius manifolds (see the next section) they can be found in a pure algebraic way (the Proposition 2.4 below).

To complete local classification of massive TFT one has to classify flat diagonal metrics with the properties (1.17). This class of metrics was studied by Darboux [33] and Egoroff. Following Darboux, I will call them *Egoroff metrics*. Vanishing of the curvature of these metrics can be written in the form of the following system of PDE (*Darboux – Egoroff system*) for the *rotation coefficients*

$$\gamma_{ij}(u) = \frac{\partial_j \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}}, \ i \neq j \tag{1.20}$$

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad i, \ j, \ k \text{ are distinct}, \tag{1.21a}$$

$$\sum_{k=1}^{n} \partial_k \gamma_{ij} = 0, \ i \neq j \tag{1.21b}$$

$$\gamma_{ji} = \gamma_{ij}. \tag{1.21c}$$

It is interesting that the same equations (for even n) arise in the calculation [34] of multipoint correlators in impenetrable Bose-gas, see Appendix to [45].

Integrability of the Darboux – Egoroff system was observed in [23]. It essentialy coincides with the "pure imaginary reduction" of the n-wave system (see [26, 25]). This can be represented as the compatibility conditions of the following linear system (depending on a spectral parameter λ)

$$\partial_j \psi_i = \gamma_{ij}\psi_j, \quad i \neq j \tag{1.22a}$$

$$\sum_{k=1}^{n} \partial_k \psi_i = \lambda \psi_i. \tag{1.22b}$$

To complete local classification of massive Fröbenius manifolds one first should apply an appropriate version of inverse spectral transform (IST) to solve the Darboux – Egoroff system (1.21). Below I will give an example of IST for the important case of self-similar solutions of (1.21) (so called topological conformal field theories [8, 51]). To find the metric (1.16) and flat coordinates $t^\alpha = t^\alpha(u)$ for a given solution $\gamma_{ij}(u)$ one has to fix a basis $\psi_{i\alpha}(u)$, $\alpha = 1, ...n$ in the space of solutions of the system (1.22) for $\lambda = 0$

$$\partial_j \psi_{i\alpha} = \gamma_{ij}\psi_{j\alpha}, \quad i \neq j, \tag{1.23a}$$

$$\sum_k \partial_k \psi_{i\alpha} = 0, \tag{1.23b}$$

$\alpha = 1, ..., n$. Then we put

$$\eta_{ii}(u) = \psi_{i1}^2(u), \tag{1.24a}$$

$$\eta_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha}(u)\psi_{i\beta}(u), \tag{1.24b}$$

$$\frac{\partial t_\alpha}{\partial u^i} = \psi_{i1}(u)\psi_{i\alpha}(u), \tag{1.24c}$$

$$c_{\alpha\beta\gamma}(t(u)) = \sum_{i=1}^{n} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}. \tag{1.24d}$$

These formulae complete local classification of complex massive Fröbenius manifolds. They are parametrized (localy) by n arbitrary functions of 1 variable (the parametrization of solutions of the Darboux – Egoroff system) and also by n complex parameters because of the ambiguity in the choice of solutions ψ_{i1} in the formulae (1.24).

To classify real Fröbenius manifolds one should apply IST to various real forms of the Darboux – Egoroff system. We will not do it here (see [27] for discussion of real forms of the system (1.21) in algebraic-geometry IST).

Global topology of massive Fröbenius manifolds is rather poor. We say that a n-dimensional manifold M admits S_n-structure if the structure group of the tangent bundle TM can be reduced to the symmetric group S_n. An atlas of coordinates charts on M is *compatible* with the given S_n-structure if differentials of the transition functions are the correspondent elements of S_n (in the standard n-dimensional representation). Globaly a S_n-manifold M with a compatible atlas is determined by an affine representation of $\pi_1(M) \to S_n \to Aff_n$, i.e. the transition functions have the form

$$u^i \mapsto u^{\sigma(i)} + a^i_\sigma, \tag{1.25a}$$

$$a^i_{\sigma'\sigma} = a^{\sigma'(i)}_\sigma + a^i_{\sigma'}, \tag{1.25b}$$

for $\sigma,\ \sigma' \in S_n$. As an example of S_n-manifold one can have in mind the space of all polynomials $M = \{P(u) = u^n + a_1 u^{n-1} + ... + a_n |\ a_1, ..., a_n \in \mathbf{C}\}$ without multiple roots. Compatible coordinates are the roots of $P(u)$. The transition functions (1.25) are given by the standard n-dimensional representation of the braid group $\pi_1(M) = B_n$.

The Darboux – Egoroff system is well-defined on any S_n-manifold M with a marked compatible atlas. To obtain a massive Fröbenius structure on M one should find a solution $\gamma_{ij}(u)$ being covariant with respect to transformations of the form (1.25). This "boundary value problem" seems to be more complicated.

In all the examples (below) of massive TFT the coupling space M (massive Fröbenius manifold) can be extended by adding certain locus M_{sing} (at least of real codimension 2). The structure of Fröbenius manifold can be extended on $\hat{M} = M \cup M_{sing}$ but the algebra structure on the tangent spaces $T_x\hat{M}$ for $x \in M_{sing}$ has nilpotents. The flat metric $\eta_{\alpha\beta}$ is extended on $\hat{M}$ without degeneration. So $\hat{M}$ is still a localy Euclidean manifold.

Remark. The notion of Fröbenius manifolfd admits algebraic formalization in terms of the ring of functions on a manifold. More precisely, let R be a commutative associative algebra with a unity over a field k of characteristics $\neq 2$. We are interesting in structures of Fröbenius algebra over R in the R-module of k-derivations $Der(R)$ (i.e. $u(\kappa) = 0$ for $\kappa \in k, u \in Der(R)$) satisfying

$$\tilde{\nabla}_u(\lambda)\tilde{\nabla}_v(\lambda) - \tilde{\nabla}_v(\lambda)\tilde{\nabla}_u(\lambda) = \tilde{\nabla}_{[u,v]}(\lambda) \text{ identicaly in } \lambda \tag{1.26a}$$

$$\text{for } \tilde{\nabla}_u(\lambda)v = \nabla_u v + \lambda u \cdot v, \tag{1.26b}$$

$$\nabla_u e = 0 \text{ for all } u \in Der(R) \tag{1.26c}$$

where e is the unity of the Fröbenius algebra $Der(R)$. Non-degenerateness of the symmetric inner product

$$< \, , \, >: Der(R) \times Der(R) \to R$$

means that it provides an isomorphism $\mathrm{Hom}_R(Der(R), R) \to Der(R)$. I recall that the covariant derivative is a derivation $\nabla_u v \in Der(R)$ defined for any $u, \ v \in Der(R)$ being determined from the equation

$$< \nabla_u v, w >=$$

$$\frac{1}{2}[u < v, w > +v < w, u > -w < u, v > + < [u, v], w > + < [w, u], v > + < [w, v], u >] \tag{1.27}$$

for any $w \in Der(R)$ (here $[\, , \,]$ denotes the commutator of derivations). Note that the notion of infinitesimal algebraic symmetry also can be algebraicaly formalized in a similar way. It would be interesting to find a pure algebraic version of the theorem 1.1 . This could give an algebraic approach to the problem of classification of Fröbenius manifolds.

We consider in conclusion of this section a closure of the class of massive Fröbenius manifolds as the set of all Fröbenius manifolds with n-dimensional commutative group of algebraic symmetries. Let A be a fixed n-dimensional Fröbenius algebra with structure constants c_{ij}^k and an invariant inner nondegenerate inner product $\epsilon = (\epsilon_{ij})$. Let us introduce matrices

$$C_i = (c_{ij}^k). \tag{1.28}$$

An analogue of the Darboux – Egoroff system (1.21) for an operator-valued function

$$\gamma(u) : A \to A, \ \gamma = (\gamma_i^j(u)), \ u = (u^1, \ldots, u^n) \tag{1.29a}$$

(an analogue of the rotation coefficients) where the operator γ is symmetric with respect to ϵ,

$$\epsilon\gamma = \gamma^{\mathrm{T}}\epsilon \tag{1.29b}$$

has the form

$$[C_i, \partial_j\gamma] - [C_j, \partial_i\gamma] + [[C_i, \gamma], [C_j, \gamma]] = 0, \ i, j = 1, \ldots, n, \tag{1.30}$$

$\partial_i = \partial/\partial u^i$. This is an integrable system with the Lax representation

$$\partial_i\Psi = \Psi(\lambda C_i + [C_i, \gamma]), \ i = 1, \ldots, n. \tag{1.31}$$

It is convenient to consider $\Psi = (\psi_1(u), \ldots, \psi_n(u))$ as a function with values in the dual space A^*. Note that A^* also is a Fröbenius algebra with the structure constants $c_k^{ij} = c_{ks}^i \epsilon^{sj}$ and the invariant inner product $< \, , \, >_*$ determined by $(\epsilon^{ij}) = (\epsilon_{ij})^{-1}$.

Let $\Psi_\alpha(u)$, $\alpha = 1$, ... ,n be a basis of solutions of (1.31) for $\lambda = 0$

$$\partial\Psi_\alpha = \Psi_\alpha[C_i, \gamma], \ \alpha = 1, ..., n \tag{1.32a}$$

such that the vector $\Psi_1(u)$ is invertible in A^*. We put

$$\eta_{\alpha\beta} =< \Psi_\alpha(u), \Psi_\beta(u) >_* \tag{1.32b}$$

$$\mathrm{grad}_u t_\alpha = \Psi_\alpha(u) \cdot \Psi_1(u) \tag{1.32c}$$

$$c_{\alpha\beta\gamma}(t(u)) = \frac{\Psi_\alpha(u) \cdot \Psi_\beta(u) \cdot \Psi_\gamma(u)}{\Psi_1(u)}. \tag{1.32d}$$

Theorem 1.2. *Formulae (1.32) for arbitrary Fröbenius algebra A localy parametrize all Fröbenius manifolds with n-dimensional commutative group of algebraic symmetries.*

Considering u as a vector in A and $\Psi_1^2 = \Psi_1 \cdot \Psi_1$ as a linear function on A one obtains the following analogue of Egoroff metrics (on A)

$$ds^2 = \Psi_1^2(du \cdot du). \tag{1.33}$$

2. Conformal invariant Fröbenius manifolds and isomonodromy deformations.

Definition 2.1. A diffeomorphism $f : M \to M$ is called *conformal symmetry* if

$$f_*(u \cdot v) = \mu_f^c f_*(u) \cdot f_*(v) \tag{2.1a}$$

$$< f_*(u), f_*(v) >= \mu_f^\eta < u, v > \tag{2.1b}$$

$$f_*(e) = \mu_f^e e \tag{2.1c}$$

for some functions μ_f^c, μ_f^η, μ_f^e. A Fröbenius manifold M is called *conformal invariant* if it admits a one-parameter group of conformal symmetries $f^{(\tau)}$ such that the tensors $f_*^{(\tau)}(c)$, $f_*^{(\tau)}(\eta)$, $f_*^{(\tau)}(e)$ determine on M a Fröbenius structure for any τ.

Let v be the generator of the one-parameter group of conformal symmetries on a conformal invariant Fröbenius manifolds.

Proposition 2.1. *On a massive conformal invariant Fröbenius manifold an action of the one-parameter group of conformal symmetries is generated by the field*

$$v = \sum_{i=1}^n u^i \partial_i \tag{2.2}$$

(modulo obvious transformations $v \mapsto av + be$ for constant a and b). It acts on the tensors c, η, e by the following formulae

$$\mathcal{L}_v c = c \tag{2.3a}$$

$$\mathcal{L}_v e = -e \tag{2.3b}$$

$$\mathcal{L}_v \eta = (2 - d)\eta \tag{2.3c}$$

where d is a constant.

Here $\mathcal{L}_v$ means the Lie derivative along the vector field v.

Corollary. *For a massive conformal invariant Fröbenius manifold the rotation coefficients $\gamma_{ij}(u)$ satisfy the similarity condition*

$$\gamma_{ij}(cu) = c^{-1}\gamma_{ij}(u) \tag{2.4a}$$

or, equivalently

$$\sum_{k=1}^{n} u^k \partial_k \gamma_{ij}(u) = -\gamma_{ij}(u). \tag{2.4b}$$

For $n = 2$ the similarity reduction (2.4) of the Darboux – Egoroff system can be solved immediately:

$$\gamma_{12} = \gamma_{21} = \frac{id}{2} \frac{1}{u^1 - u^2}. \tag{2.5}$$

For the first nontrivial case $n = 3$ the system (1.21), (2.4) reads

$$\Gamma_1' = \Gamma_2 \Gamma_3 \tag{2.6a}$$

$$(z\Gamma_2)' = -\Gamma_1 \Gamma_3 \tag{2.6b}$$

$$((z-1)\Gamma_3)' = \Gamma_1 \Gamma_2 \tag{2.6c}$$

where

$$\gamma_{ij}(u) = \frac{1}{u^2 - u^3} \Gamma_k(z), \ i, \ j, \ k \text{ are distinct} \tag{2.7a}$$

$$z = \frac{u^1 - u^3}{u^2 - u^3}. \tag{2.7b}$$

It has an obvious first integral

$$\Gamma_1^2 + (z\Gamma_2)^2 + ((z-1)\Gamma_3)^2 = \text{const}. \tag{2.8}$$

Using this integral one can reduce [30] the system (2.6) to a particular case of the Painlevé-VI equation.

For $n > 3$ the system (1.21), (2.4) can be considered as a high-order analogue of the Painlevé-VI. To find solutions of this system one can use an appropriate version of IST: the so-called method of isomonodromy deformations [31]. This gives parametrization of solutions of the system (1.21), (2.4) by monodromy data of the following system of linear ODE with rational coefficients:

$$\lambda \frac{d\psi}{d\lambda} = (\lambda U - [U, \gamma])\psi. \tag{2.9a}$$

Here

$$U = \text{diag}(u^1, ..., u^n), \tag{2.9b}$$

$$\gamma = (\gamma_{ij}(u)). \tag{2.9c}$$

Solutions of this linear ODE have some monodromy properties, i.e. they are multivalued functions in the complex λ-plane.

Proposition 2.2. *The monodromy transformations of solutions of the system (2.9) do not depend on the parameters u iff the matrix $\gamma_{ij}(u)$ is a solution of the system (1.21), (2.4).*

The linear system (2.9) has two singular points: a regular singularity at $\lambda = 0$ and an irregular one at $\lambda = \infty$. Monodromy transformations of solutions of the system near $\lambda = 0$ have the form

$$\psi \mapsto \exp(-2\pi i[U, \gamma])\psi. \tag{2.10}$$

So the eigenvalues of the matrix $[U, \gamma]$ are first integrals of the system (1.21), (2.4). These generalise the first integral (2.8). Monodromy at infinity is determined by a $n \times n$ *Stokes matrix* S (see [31] for details). The diagonal terms of S equal 1; $n(n-1)/2$ of other entries of the matrix S vanish. Other matrix elements of S can be used as local parameters of massive conformal-invariant Fröbenius manifolds (just $n(n-1)/2$ arbitrary complex parameters; one should add one more parameter: a norming constant of a solution $\psi_{i1}(u)$ in (1.24) being an eigenvector of the matrix $[U, \gamma]$). The monodromy at $\lambda = 0$ can be expressed via S using cyclic relations (see [39]). If the Stokes matrix S is sufficiently close to the unity matrix then the inverse problem of the monodromy theory (i.e., to determine the coefficients of the linear operator (2.9) from the given monodromy data) always is solvable. The solution can be obtained by solving linear integral equations [39].

Let us assume that the monodromy of the operator (2.9) in the origin is semisimple. That means that the matrix $[U, \gamma]$ has pairwise different eigenvalues μ_1, ... μ_n. Let us order them in such a way that

$$\mu_\alpha + \mu_{n-\alpha+1} = 0. \tag{2.11}$$

Proposition 2.3. *Flat coordinates on a massive conformal invariant Fröbenius manifold with semisimple monodromy of (2.9) at $\lambda = 0$ can be chosen in such a way that the generator v of conformal symmetries has the form*

$$v = \sum (1 - q_\alpha) t^\alpha \partial_\alpha \tag{2.12}$$

for

$$q_\alpha = \mu_1 - \mu_\alpha \tag{2.13a}$$

where μ_α are the eigenvalues of the matrix $[U, \gamma]$ ordered as in (2.11).

In other words, the tensors c, η, e should be conformal covariant with respect to the following transformations

$$t^\alpha \mapsto c^{1-q_\alpha} t^\alpha \tag{2.14a}$$

$$c^\gamma_{\alpha\beta} \mapsto c^{q_\alpha + q_\beta - q_\gamma} c^\gamma_{\alpha\beta} \tag{2.14b}$$

$$\eta_{\alpha\beta} \mapsto c^{q_\alpha + q_\beta - d} \eta_{\alpha\beta} \tag{2.14c}$$

where

$$d = q_n = 2\mu_1 \tag{2.13b}$$

is the same as in (2.3c),

$$e \mapsto c^{-1} e. \tag{2.14d}$$

The equation (2.14c) means that $\eta_{\alpha\beta} \neq 0$ only for $q_\alpha + q_\beta = d$.

The numbers q_α are called *charges* of the TCFT model, d is called *dimension* of the model. For topological sigma-models it coincides with complex dimension of the target-space. Scaling laws (2.14) were obtained in [8] using the assumption that the TCFT model is obtained by twisting of a N=2 supersymmetric model of QFT. These imply superselection rules for tree-level correlators in the conformal point $t = 0$ (the stationary point of the field v (2.3)). In our approach the scaling laws follow from simple symmetry assumption on the Fröbenius manifold.

Summarizing we obtain

Theorem 2.1. *All massive conformal invariant Fröbenius manifolds are parametrized by monodromy data of the linear operator*

$$\Lambda = \lambda\partial_\lambda - \lambda U + M(u) \tag{2.15a}$$

$$U = \mathrm{diag}(u^1, \ldots, u^n). \tag{2.15b}$$

$$M^{\mathrm{T}} = -M. \tag{2.15c}$$

Manifolds with semisimple monodromy at the origin $\lambda = 0$ form a $[\frac{n(n-1)}{2} + 1]$-parameter family. The free energy $F(t)$ of such a Fröbenius manifold can be expressed via quadratures of a high-order analogue of the Painlevé-VI transcendents, i.e. solutions of the equations of isomonodromy deformations of (2.15).

For nonresonant conformal invariant Fröbenius manifolds (see (3.19) below) with a semisimple monodromy at the origin the structure functions $c_{\alpha\beta}^{\gamma}(t)$ can be expressed algebraicaly (i.e. without quadratures) via the above high-order analogue of the Painlevé-VI transcendents. Also one has

Proposition 2.4. *The canonical coordinates $u^1, \ldots, u^n$ on a massive conformal invariant Fröbenius manifold coincide with eigenvalues of the matrix*

$$\tilde U = (\tilde U_\beta^\gamma(t)) = ((1 + q_\beta - q_\gamma)F_\beta^\gamma(t)) \tag{2.16a}$$

$$F_\beta^\gamma(t) = \eta^{\gamma\epsilon}\partial_\beta\partial_\epsilon F(t). \tag{2.16b}$$

It would be interesting to understand a physical sense of the operator $\tilde U$ for TCFT models.

Remark. We saw that monodromy is an important invariant of a massive conformal invariant Fröbenius manifold. It can be defined also for arbitrary conformal invariant Fröbenius manifold by considering the linear operator

$$\tilde\Lambda = \lambda\partial_\lambda - \lambda\tilde U + \tilde M, \tag{2,17a}$$

where

$$\tilde M = (\tilde M_\beta^\gamma) = (q_\beta \delta_\beta^\gamma), \tag{2.17b}$$

the matrix $\tilde U$ has the form (2.16). WDVV equations determine isomonodromy deformations of $\tilde\Lambda$.

Monodromy properties of eigenfunctions of $\tilde\Lambda$ near irregular singularity $\lambda = \infty$ (i.e. Stokes matrices) strongly depend on algebraic structure of the multiplication on TM. These Stokes matrices are constrained by cyclic relations since monodromy near the origin $\lambda = 0$ is fixed by the given charges q_α. An advantage of the isomonodromy problem

(2.15) for massive Fröbenius manifolds is in universality (independence on the charges; the charges can be expressed via an arbitrary Stokes matrix of (2.15)). Note that a basis of common eigenfunctions of $\tilde\Lambda$

$$\tilde\Lambda\xi = \kappa\xi \tag{2.18a}$$

and (1.12) has the form

$$\xi_\beta(t,\lambda) = \partial_\beta h_\alpha(t,\lambda),\ \kappa = d - q_\alpha,\ \text{for any } \alpha = 1,\dots,n \tag{2.18b}$$

where the solutions $h_\alpha(t,\lambda)$ of (3.5) are normalized by (3.6).

3. Coupling to gravity. Systems of hydrodynamic type: their Hamiltonian formalism, solutions, and τ-functions.

Let us fix a Fröbenius manifold (i.e. a solution of the WDVV equations. Considering this as the primary free energy of the matter sector of a 2D TFT model, let us try to calculate the tree-level (i.e., the zero-genus) approximation of the complete model obtained by coupling of the matter sector to topological gravity. The idea to use hierarchies of Hamiltonian systems of hydrodynamic type for such a calculation was proposed by E.Witten [46] for the case of topological sigma-models. An advantage of my approach is in effective construction of these hierarchies for any solution of WDVV. The tree-level free energy of the model will be identified with τ-function of a particular solution of the hierarchy. For a TCFT-model (i.e. for a conformal invariant Fröbenius manifold) the hierarchy carries a bihamiltonian structure under a non-resonance assumption for charges and dimension of the model (this bihamiltonian structure was constructed in [39] for the case of massive perturbations of a TCFT model; here I generalize it for an arbitrary TCFT model). This gives an answer to a question of [46] (see p.283). As it was mentioned in the Introduction, the bihamiltonian structure could be useful for calculation higher genus corrections .

So let $c_{\alpha\beta}^\gamma(t)$, $\eta_{\alpha\beta}$ be a solution of WDVV, $t = (t^1,\dots,t^n)$. I will construct a hierarchy of the first order PDE systems linear in derivatives (*systems of hydrodynamic type*) for functions $t^\alpha(T)$, T is an infinite vector

$$T = (T^{\alpha,p}),\ \ \alpha = 1,\ \dots,\ n,\ \ p = 0,\ 1,\ \dots;\ T^{1,0} = X,$$

$$\partial_{T^{\alpha,p}} t^\beta = c_{(\alpha,p)}{}_\gamma^\beta(t)\partial_X t^\gamma \tag{3.1a}$$

for some matrices of coefficients $c_{(\alpha,p)}{}_\gamma^\beta(t)$. The marked variable $X = T^{1,0}$ usualy is called *cosmological constant*.

I will consider the equations (3.1) as dynamical systems (for any (α,p)) on the space of functions $t = t(X)$ with values in the Fröbenius manifold M.

A. Construction of the systems. I define a Poisson bracket on the space of functions $t = t(X)$ (i.e. on the loop space $\mathcal{L}(M)$) by the formula

$$\{t^\alpha(X), t^\beta(Y)\} = \eta^{\alpha\beta}\delta'(X-Y). \tag{3.2}$$

All the systems (3.1a) have hamiltonian form

$$\partial_{T^{\alpha,p}} t^\beta = \{t^\beta(X), H_{\alpha,p}\} \tag{3.1b}$$

with the Hamiltonians of the form

$$H_{\alpha,p} = \int h_{\alpha,p+1}(t(X))dX. \tag{3.3}$$

The generating functions of densities of the Hamiltonians

$$h_\alpha(t,\lambda) = \sum_{p=0}^{\infty} h_{\alpha,p}(t)\lambda^p, \ \ \alpha = 1, \dots, n \tag{3.4}$$

coincide with the flat coordinates of the perturbed connection $\tilde{\nabla}(\lambda)$ (see (1.10)). That means that they are determined by the system (cf. (1.12))

$$\partial_\beta \partial_\gamma h_\alpha(t,\lambda) = \lambda c_{\beta\gamma}^{\epsilon}(t)\partial_\epsilon h_\alpha(t,\lambda). \tag{3.5}$$

This gives simple recurrence relations for the densities $h_{\alpha,p}$. Solutions of (3.5) can be normalized in such a way that

$$h_\alpha(t,0) = t_\alpha = \eta_{\alpha\beta}t^\beta, \tag{3.6a}$$

$$< \nabla h_\alpha(t,\lambda), \nabla h_\beta(t,-\lambda) >= \eta_{\alpha\beta}. \tag{3.6b}$$

Here ∇ is the gradient (in t). It can be shown that the Hamiltonians (3.3) are in involution. So all the systems of the hierarchy (3.1) commute pairwise.

B. Specification of a solution $t = t(T)$. The hierarchy (3.1) admits an obvious scaling group

$$T^{\alpha,p} \mapsto cT^{\alpha,p}, \ \ \ t \mapsto t. \tag{3.7}$$

Let us take the nonconstant invariant solution for the symmetry

$$(\partial_{T^{1,1}} - \sum T^{\alpha,p}\partial_{T^{\alpha,p}})t(T) = 0 \tag{3.8}$$

(I identify $T^{1,0}$ and X. So the variable X is supressed in the formulae.) This solution can be found without quadratures from a fixed point equation for the gradient map

$$t = \nabla\Phi_T(t), \tag{3.9}$$

$$\Phi_T(t) = \sum_{\alpha,p} T^{\alpha,p}h_{\alpha,p}(t). \tag{3.10}$$

It can be proved existence and uniqueness of such a fixed point for sufficiently small $T^{\alpha,p}$ for $p > 0$ (more precisely, in the domain: $T^{\alpha,0}$ are arbitrary, $T^{1,1} = o(1)$, $T^{\alpha,p} = o(T^{1,1})$ for $p > 0$).

C. τ-function. Let us define coefficients $V_{(\alpha,p),(\beta,q)}(t)$ from the expansion

$$(\lambda+\mu)^{-1}(< \nabla h_\alpha(t,\lambda), \nabla h_\beta(t,\mu) > -\eta_{\alpha\beta}) = \sum_{p,q=0}^{\infty} V_{(\alpha,p),(\beta,q)}(t)\lambda^p\mu^q \equiv V_{\alpha\beta}(t,\lambda,\mu). \tag{3.11}$$

The infinite matrix of coefficients $V_{(\alpha,p),(\beta,q)}(t)$ has a simple meaning: it is the energy-momentum tensor of the commutative Hamiltonian hierarchy (3.1). That means that a matrix entry $V_{(\alpha,p),(\beta,q)}(t)$ is the density of flux of the Hamiltonian $H_{\alpha,p}$ along the flow $T^{\beta,q}$:

$$\partial_{T^{\beta,q}} h_{\alpha,p+1}(t) = \partial_X V_{(\alpha,p),(\beta,q)}(t). \tag{3.12}$$

Then

$$\tau(T) = \frac{1}{2} \sum V_{(\alpha,p),(\beta,q)}(t(T)) T^{\alpha,p} T^{\beta,q} + \sum V_{(\alpha,p),(1,1)}(t(T)) T^{\alpha,p} + \frac{1}{2} V_{(1,1),(1,1)}(t(T)) \tag{3.13}$$

Remark. More general family of solutions of (3.1) has the form

$$\nabla[\Phi_T(t) - \Phi_{T_0}(t)] = 0 \tag{3.14}$$

for arbitrary constant vector $T_0 = T_0^{\alpha,p}$. For massive Fröbenius manifolds these form a dense subset in the space of all solutions of (3.1) (see [22, 23, 39]). Formally they can be obtained from the solution (3.9) by a shift of the arguments $T^{\alpha,p}$. τ-function of the solution (3.14) can be formaly obtained from (3.13) by the same shift. For the example of topological gravity [3, 46] such a shift is just the operation that relates the tree-level free energies of the topological phase of 2D gravity and of the matrix model. It should be taken in account that the operation of such a time shift in systems of hydrodynamic type is a subtle one: it can pass through a point of gradient catastrophe where derivatives become infinite. The correspondent solution of the KdV hierarchy has no gradient catastrophes but oscillating zones arise (see [32] for details).

Theorem 3.1. *Let*

$$\mathcal{F}(T) = \log \tau(T), \tag{3.15a}$$

$$< \phi_{\alpha,p} \phi_{\beta,q} \ldots >_0 = \partial_{T^{\alpha,p}} \partial_{T^{\beta,q}} \ldots \mathcal{F}(T). \tag{3.15b}$$

Then the following relations hold

$$\mathcal{F}(T)|_{T^{\alpha,p}=0 \text{ for } p>0,\ T^{\alpha,0}=t^\alpha} = F(t) \tag{3.16a}$$

$$\partial_X \mathcal{F}(T) = \sum T^{\alpha,p} \partial_{T^{\alpha,p-1}} \mathcal{F}(T) + \frac{1}{2} \eta_{\alpha\beta} T^{\alpha,0} T^{\beta,0} \tag{3.16b}$$

$$< \phi_{\alpha,p} \phi_{\beta,q} \phi_{\gamma,r} >_0 = < \phi_{\alpha,p-1} \phi_{\lambda,0} >_0 \eta^{\lambda\mu} < \phi_{\mu,0} \phi_{\beta,q} \phi_{\gamma,r} >_0 . \tag{3.16c}$$

Let me establish now a 1-1 correspondence between the statements of the theorem and the standard terminology of QFT. In a complete model of 2D TFT (i.e. a matter sector coupled to topological gravity) there are infinite number of operators. They usualy are denoted by $\phi_{\alpha,p}$ or $\sigma_p(\phi_\alpha)$. The operators $\phi_{\alpha,0}$ can be identified with the primary operators ϕ_α; the operators $\phi_{\alpha,p}$ for $p > 0$ are called *gravitational descendants* of ϕ_α. Respectively one has infinite number of coupling constants $T^{\alpha,p}$. The formula (3.15a) expresses the tree-level (i.e. genus zero) partition function of the model of 2D TFT via logarythm of the τ-function (3.13). Equation (3.15b) is the standard relation between the correlators (of genus zero) in the model and the free energy. Equation (3.16a) manifests that before coupling to gravity the partition function (3.15a) coincides with the primary partition function of the given matter sector. Equation (3.16b) is the string equation for the free energy [3, 4, 8, 46]. And equations (3.16c) coincide with the genus zero recursion relations for correlators of a TFT [4, 46].

Particularly, from (3.15) one obtains

$$< \phi_{\alpha,p}\phi_{\beta,q} >_0 = V_{(\alpha,p),(\beta,q)}(t(T)), \tag{3.17a}$$

$$< \phi_{\alpha,p}\phi_{1,0} >_0 = h_{\alpha,p}(t(T)), \tag{3.17b}$$

$$< \phi_{\alpha,p}\phi_{\beta,q}\phi_{\gamma,r} >_0 = < \nabla h_{\alpha,p} \cdot \nabla h_{\beta,q} \cdot \nabla h_{\gamma,r}, [e - \sum T^{\alpha,p} \nabla h_{\alpha,p-1}]^{-1} > . \tag{3.17c}$$

The second factor of the inner product in the r.h.s. of (3.17c) is an invertible element (in the Fröbenius algebra of vector fields on M) for sufficiently small $T^{\alpha,p}$, $p > 0$. From the last formula one obtains

Proposition 3.1. *The coefficients*

$$c_{p,\alpha\beta}^{\gamma}(T) = \eta^{\gamma\mu}\partial_{T^{\alpha,p}}\partial_{T^{\beta,p}}\partial_{T^{\mu,p}} \log \tau(T) \tag{3.18}$$

for any p and any T are structure constants of a commutative associative algebra with the invariant inner product $\eta_{\alpha\beta}$.

As a rule such an algebra has no unity.

In fact the Proposition holds also for a τ-function of an arbitrary solution of the form (3.14).

We see that the hierarchy (3.1) determines a family of Bäcklund transforms of the WDVV equation (0.1)

$$F(t) \mapsto \tilde{F}(\tilde{t}),$$

$$\tilde{F} = \log \tau, \ \tilde{t}^{\alpha} = T^{\alpha,p}$$

for a fixed p and for arbitrary τ-function of (3.1). So it is natural to consider equations of the hierarchy as Lie – Bäcklund symmetries of WDVV.

Up to now I even did not use the scaling invariance (2.14). It turns out that this gives rise to a bihamiltonian structure of the hierarchy (3.1).

Let us consider a conformal invariant Fröbenius manifold, i.e. a TCFT model with charges q_α and dimension d. We say that a pair α, p is *resonant* if

$$\frac{d+1}{2} - q_\alpha + p = 0. \tag{3.19}$$

Here p is a nonnegative integer. The TCFT model is *nonresonant* if all pairs α, p are nonresonant. For example, models satisfying the inequalities

$$0 = q_1 \leq q_2 \leq \ldots \leq q_n = d < 1 \tag{3.20}$$

all are nonresonant.

Theorem 3.2. *1) For a conformal invariant Fröbenius manifold with charges q_α and dimension d the formula*

$$\{t^\alpha(X), t^\beta(Y)\}_1 = [(\frac{d+1}{2} - q_\alpha)F^{\alpha\beta}(t(X)) + (\frac{d+1}{2} - q_\beta)F^{\alpha\beta}(t(Y))]\delta'(X-Y) \quad (3.21)$$

$$F^{\alpha\beta}(t) = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\partial_{\alpha'}\partial_{\beta'}F(t)$$

determines a Poisson bracket compatible with the Poisson bracket (3.2). 2) For a nonresonant TCFT model all the equations of the hierarchy (3.1) are Hamiltonian equations also with respect to the Poisson bracket (3.21).

The nonresonancy condition is essential: equations (3.1) with resonant numbers (α, p) do not admit another Poisson structure.

Remark. According to the theory [18-21] of Poisson brackets of hydrodynamic type any such a bracket is determined by a flat Riemannian (or pseudo-Riemannian) metric $g_{\alpha\beta}(t)$ on the target space M (more precisely, one needs a metric $g^{\alpha\beta}(t)$ on the cotangent bundle to M). In our case the target space is the Fröbenius manifold M. The first Poisson structure (3.2) is determined by the metric being specified by the double-point correlators $\eta_{\alpha\beta}$. The second flat metric for the Poisson bracket (3.21) on a conformal invariant Fröbenius manifold M has the following geometrical interpretation. Let ω_1 and ω_2 be two 2-forms on M. We can multiply them $\omega_1,\ \omega_2 \mapsto \omega_1 \cdot \omega_2$ using the multiplication of tangent vectors and the isomorphism η between tangent and cotangent spaces. Then the new inner product $< \ , \ >_1$ is defined by the formula

$$< \omega_1, \omega_2 >_1 = \mathrm{i}_v(\omega_1 \cdot \omega_2). \quad (3.22)$$

Here i_v is the operator of contraction with the vector field v (the generator of conformal symmetries (2.3)). The metric (3.22) can be degenerate. The theorem states that, nevertheless, the Jacobi identity for the Poisson bracket (3.21) holds.

Main examples of solutions of WDVV and of corresponding hierarchies will be given in the next section. Here I will consider the simplest class of examples where $c_{\alpha\beta}^\gamma$ does not depend on t. They form structure constants of a Fröbenius algebra A with an invariant inner product $< \ , \ >$ ($\eta_{\alpha\beta}$ in a basis $e_1 = 1, \ldots, e_n$). Let

$$\mathbf{t} = t^\alpha e_\alpha \in A. \quad (3.23)$$

The linear system (3.5) can be solved easily:

$$h_\alpha(t, \lambda) = \lambda^{-1} < e_\alpha, e^{\lambda \mathbf{t}} - 1 > .$$

This gives the following form of the hierarchy (3.1)

$$\partial_{T^{\alpha,p}}\mathbf{t} = \frac{1}{p!} e_\alpha \mathbf{t}^p \partial_X \mathbf{t}. \quad (3.24)$$

The solution (3.9) is specified as the fixed point

$$G(\mathbf{t}) = \mathbf{t}, \quad (3.25a)$$

$$G(\mathbf{t}) = \sum_{p=0}^{\infty} \frac{\mathbf{T}_p}{p!} \mathbf{t}^p. \tag{3.25b}$$

Here I introduce A-valued coupling constants

$$\mathbf{T}_p = T^{\alpha,p} e_\alpha \in A, \; p = 0, \; 1, \ldots. \tag{3.26}$$

The solution of (3.25) has the well-known form

$$\mathbf{t} = G(G(G(\ldots))) \tag{3.27}$$

(infinite number of iterations). The τ-function of the solution (3.27) has the form

$$\log \tau = \frac{1}{6} < 1, \mathbf{t}^3 > - \sum_p \frac{< \mathbf{T}_p, \mathbf{t}^{p+2} >}{(p+2)p!} + \frac{1}{2} \sum_{p,q} \frac{< \mathbf{T}_p \mathbf{T}_q, \mathbf{t}^{p+q+1} >}{(p+q+1)p!q!}. \tag{3.28}$$

For the tree-level correlation functions of a TFT-model with constant primary correlators one immediately obtains

$$< \phi_{\alpha,p} \phi_{\beta,q} >_0 = \frac{< e_\alpha e_\beta, \mathbf{t}^{p+q+1} >}{(p+q+1)p!q!}, \tag{3.29a}$$

$$< \phi_{\alpha,p} \phi_{\beta,q} \phi_{\gamma,r} >_0 = \frac{1}{p!q!r!} < e_\alpha e_\beta e_\gamma, \frac{\mathbf{t}^{p+q+r}}{1 - \sum_{s \geq 1} \frac{\mathbf{T}_s \mathbf{t}^{s-1}}{(s-1)!}} > . \tag{3.29b}$$

For $n = 1$ the formulae (3.29) give the tree-level correlators of the topological gravity (see [3, 46]). For $n = 24$ one obtains the tree-level correlators of the topological sigma-model with a K3-surface as the target space. Here the algebra $A = H^*(K3)$ is a graded one: it has a basis P, Q_1, ..., Q_{22}, R of degrees 0, 1 (all the Q's) and 2 resp. The multiplication has the form

$$P \text{ is the unity}, \; Q_i Q_j = \eta_{ij} R, \; Q_i R = R^2 = 0 \tag{3.30}$$

for a nondegenerate symmetric matrix η_{ij}. The scalar product (the intersection number) has the form

$$\eta_{PR} = 1, \; \eta_{Q_i Q_j} = \eta_{ij}.$$

Let us consider now the second hamiltonian structure (3.21). I start with the most elementary case $n = 1$ (the pure gravity). Let me redenote the coupling constant

$$u = t^1.$$

The Poisson bracket (3.21) for this case reads

$$\{u(X), u(Y)\}_1 = \frac{1}{2}(u(X) + u(Y))\delta'(X - Y). \tag{3.31}$$

This is nothing but the Lie – Poisson bracket on the dual space to the Lie algebra of one-dimensional vector fields.

For arbitrary graded Fröbenius algebra A the Poisson bracket (3.21) also is linear in the coordinates t^α

$$\{t^\alpha(X), t^\beta(Y)\}_1 = [(\frac{d+1}{2} - q_\alpha)c_\gamma^{\alpha\beta} t^\gamma(X) + (\frac{d+1}{2} - q_\beta)c_\gamma^{\alpha\beta} t^\gamma(Y)]\delta'(X-Y). \quad (3.32)$$

It determines therefore a structure of an infinite dimensional Lie algebra on the loop space $\mathcal{L}(A^*)$ where A^* is the dual space to the graded Fröbenius algebra A. Theory of linear Poisson brackets of hydrodynamic type and of corresponding infinite dimensional Lie algebras was constructed in [34] (see also [18]). But the class of examples (3.32) is a new one. Note that the case $A = H^*(K3)$ is a nonresonant one.

Let us come back to the general (i.e. nonlinear) case of a TCFT model. I will assume that the charges and the dimension are ordered in such a way that

$$0 = q_1 < q_2 \leq \ldots \leq q_{n-1} < q_n = d. \quad (3.33)$$

Then from (3.21) one obtains

$$\{t^n(X), t^n(Y)\}_1 = \frac{1-d}{2}(t^n(X) + t^n(Y))\delta'(X-Y). \quad (3.34)$$

Since

$$\{t^\alpha(X), t^n(Y)\}_1 = [(\frac{d+1}{2} - q_\alpha)t^\alpha(X) + \frac{1-d}{2} t^\alpha(Y)]\delta'(X-Y), \quad (3.35)$$

the functional

$$P = \frac{2}{1-d}\int t^n(X)dX \quad (3.36)$$

generates spatial translations. We see that for $d \neq 1$ the Poisson bracket (3.21) can be considered as a nonlinear extension of the Lie algebra of one-dimensional vector fields. An interesting question is to find an analogue of the Gelfand – Fuchs cocycle for this bracket. I found such a cocycle for a more particular class of TCFT models. We say that a TCFT-model is *graded* if for any t the Fröbenius algebra $c_{\alpha\beta}^\gamma(t)$, $\eta_{\alpha\beta}$ is graded.

Theorem 3.3. *For a graded TCFT-model the formula*

$$\{t^\alpha(X), t^\beta(Y)\}\hat{}_1 = \{t^\alpha(X), t^\beta(Y)\}_1 + \epsilon^2 \eta^{1\alpha}\eta^{1\beta}\delta'''(X-Y) \quad (3.37)$$

determines a Poisson bracket compatible with (3.2) and (3.21) for arbitrary ϵ^2 (the central charge). For a generic graded TCFT model this is the only one deformation of the Poisson bracket (3.21) proportional to $\delta'''(X-Y)$.

For $n = 1$ (3.37) determines nothing but the Lie – Poisson bracket on the dual space to the Virasoro algebra

$$\{u(X), u(Y)\}\hat{}_1 = \frac{1}{2}[u(X) + u(Y)]\delta'(X-Y) + \epsilon^2\delta'''(X-Y) \quad (3.38)$$

(the second Poisson structure of the KdV hierarchy). For $n > 1$ and constant primary correlators (i.e. for a constant graded Fröbenius algebra A) the Poisson bracket (3.37) can be considered as a vector-valued extension (for $d \neq 1$) of the Virasoro.

Graded TCFT models occur as the topological sigma-models with a Calabi – Yau manifold of (complex) dimension d as the target space [2, 46, 57]. They are nonresonant for even d. Particularly, for $d = 2$ one obtains the K3-models where the primary correlators are constant. For $d > 2$ these are not constant because of instanton corrections [46, 47, 57]. As it was explained in [57], finding of these primary correlators for the Calabi – Yau models (and, therefore, graded solutions of WDVV) could be a crucial point in solving the problem of mirror symmetry.

The compatible pair of the Poisson brackets (3.2) and (3.37) generates an integrable hierarchy of PDE for a non-resonant graded TCFT using the standard machinery of the bihamiltonian formalism [52]

$$\partial_{T^{\alpha,p}} t^\beta = \{t^\beta(X), \hat{H}_{\alpha,p}\} = \{t^\beta(X), \hat{H}_{\alpha,p-1}\hat{\}}_1. \tag{3.39}$$

Here the Hamiltonians have the form

$$\hat{H}_{\alpha,p} = \int \hat{h}_{\alpha,p+1} dX, \tag{3.40a}$$

$$\hat{h}_{\alpha,p+1} = [\frac{d+1}{2} - q_\alpha + p]^{-1} h_{\alpha,p+1}(t) + \epsilon^2 \Delta \hat{h}_{\alpha,p+1}(t, \partial_X t, \ldots, \partial_X^p t; \epsilon^2) \tag{3.40b}$$

where $\Delta \hat{h}_{\alpha,p+1}$ are some polynomials determined by (3.39). They are graded-homogeneous of degree 2 where $\deg \partial_X^k t = k$, $\deg \epsilon = -1$. The small dispersion parameter ϵ also plays the role of the string coupling constant. It is clear that the hierarchy (3.1) is the zero-dispersion limit of this hierarchy. For $n = 1$ using the pair (3.2) and (3.38) one immediately obtains the KdV hierarchy. Note that this describes the topological gravity. It would be interesting to investigate relation of the hierarchies determined by the pair (3.2) and (3.37) to a nonperturbative (i.e., for all genera) description of the Calabi – Yau models (especially, of the K3 models) coupled to gravity. For a model with constant correlators (for a graded Fröbenius algebra A) the first nontrivial equations of the hierarchy are

$$\partial_{T^{\alpha,1}} \mathbf{t} = e_\alpha \mathbf{t} \mathbf{t}_X + \frac{2\epsilon^2}{3-d} e_\alpha e_n \mathbf{t}_{XXX}. \tag{3.41}$$

For non-graded TCFT models it could be of interest to find nonlinear analogues of the cocycle (3.37). These should be differential geometric Poisson brackets of the third order [58, 18] of the form

$$\{t^\alpha(X), t^\beta(Y)\hat{\}}_1 = \{t^\alpha(X), t^\beta(Y)\}_1 +$$

$$\epsilon^2 \{g^{\alpha\beta}(t(X))\delta'''(X-Y) + b_\gamma^{\alpha\beta}(t(X)) t_X^\gamma \delta''(X-Y) +$$

$$[f_\gamma^{\alpha\beta}(t(X)) t_{XX}^\gamma + h_{\gamma\delta}^{\alpha\beta}(t(X)) t_X^\gamma t_X^\delta] \delta'(X-Y) +$$

$$[p^{\alpha\beta}_{\gamma}(t)t^{\gamma}_{XXX} + q^{\alpha\beta}_{\gamma\delta}(t)t^{\gamma}_{XX}t^{\delta}_{X} + r^{\alpha\beta}_{\gamma\delta\lambda}(t)t^{\gamma}_{X}t^{\delta}_{X}t^{\lambda}_{X}]\delta(X-Y)\}. \tag{3.42}$$

I recall (see [58, 18]) that the form (3.42) of the Poisson bracket should be invariant with respect to nonlinear changes of coordinates in the manifold M. This implies that the leading term $g^{\alpha\beta}(t)$ transforms like a metric (may be, degenerate) on the cotangent bundle T_*M, $b^{\alpha\beta}_{\gamma}(t)$ are contravariant components of a connection on M etc. The Poisson bracket (3.42) is assumed to be compatible with (3.2). Then the compatible pair (3.2), (3.42) of the Poisson brackets generates an integrable hierarchy of the same structure (3.39), (3.40).

4. Examples.

I start with the most elementary examples of solutions of WDVV for $n = 2$. Only massive solutions are of interest here (a 2-dimensional nilpotent Fröbenius algebra has no nontrivial deformations). The Darboux – Egoroff equations in this case are linear. I consider only TCFT case (the similarity reduction of WDVV). Let us redenote the coupling constants

$$t^1 = u,\ t^2 = \rho. \tag{4.1}$$

For $d \neq 1$ the primary free energy F has the form

$$F = \frac{1}{2}\rho u^2 + \frac{g}{a(a+2)}\rho^{a+2}, \tag{4.2}$$

$$a = \frac{1+d}{1-d} \tag{4.3}$$

g is an arbitrary constant. The second term in the formula for the free energy should be understood as

$$\frac{g}{a(a+2)}\rho^{a+2} = \int\int\int g(a+1)\rho^{a-1}.$$

The linear system (3.5) can be solved via Bessel functions [39]. Let me give an example of equations of the hierarchy (3.1) (the $T = T^{1,1}$-flow)

$$u_T + uu_X + g\rho^a\rho_X = 0 \tag{4.4a}$$

$$\rho_T + (\rho u)_X = 0. \tag{4.4b}$$

These are the equations of isentropic motion of one-dimensional fluid with the dependence of the pressure on the density of the form $p = \frac{g}{a+2}\rho^{a+2}$. The Poisson structure (3.2) for these equations was proposed in [37]. For $a = 0$ (equivalently $d = -1$) the system coincides with the equations of waves on shallow water (the dispersionless limit [59] of the nonlinear Schrödinger equation (NLS)).

For $d = 1$ the primary free energy has the form

$$F = \frac{1}{2}\rho u^2 + ge^{\rho}. \tag{4.5}$$

This coincides with the free energy of the topological sigma-model with CP^1 as the target space. Note that this can be obtained from the same solution of the Darboux – Egoroff system as the semiclassical limit of the NLS (the case $d = -1$ above) for different choices of the eigenfunction ψ_{1i} (in the notations of (1.24)). The corresponding $T = T^{2,0}$-system of the hierarchy (3.1) reads

$$u_T = g(e^\rho)_X$$

$$\rho_T = u_X.$$

Eliminating u one obtains the long wave limit

$$\rho_{TT} = g(e^\rho)_{XX} \tag{4.6}$$

of the Toda system

$$\rho_{n\,tt} = e^{\rho_{n+1}} - 2e^{\rho_n} + e^{\rho_{n-1}}. \tag{4.7}$$

(The 2-dimensional version of (4.6) was obtained in the formalism of Whitham-type equations in [44].) It would be interesting to prove that the nonperturbative free energy of the CP^1-model coincides with the τ-function of the Toda hierarchy.

Example 2. Topological minimal models. I consider here the A_n-series models only. The Fröbenius manifold M here is the set of all polinomials (*Landau – Ginsburg superpotentials*) of the form

$$M = \{w(p) = p^{n+1} + a_1 p^{n-1} + \ldots + a_n |\ a_1, \ldots, a_n \in \mathbf{C}\}. \tag{4.8}$$

For any $w \in M$ the Fröbenius algebra $A = A_w$ is the algebra of truncated polynomials

$$A_w = \mathbf{C}[p]/(w'(p) = 0) \tag{4.9}$$

(the prime means derivative with respect to p) with the invariant inner product

$$< f, g >= \mathrm{res}_{p=\infty} \frac{f(p)g(p)}{w'(p)}. \tag{4.10}$$

The algebra A_w is semisimple if the polynomial $w'(p)$ has simple roots. The canonical coordinates (1.15) $u^1, \ldots, u^n$ are the critical values of the polynomial $w(p)$

$$u^i = w(p_i),\ \text{where } w'(p_i) = 0,\ i = 1, \ldots, n. \tag{4.11}$$

Let us take the following diagonal metric on M

$$\sum_{i=1}^n \eta_{ii}(u)(du^i)^2, \quad \eta_{ii}(u) = [w''(p_i)]^{-1}. \tag{4.12}$$

It can be proved that this is a flat Egoroff metric on M. The correspondent flat coordinates on M have the form

$$t^\alpha = -\frac{n+1}{n-\alpha+1} \mathrm{res}_{p=\infty} w^{\frac{n-\alpha+1}{n+1}}(p)dp,\ \alpha = 1, \ldots, n. \tag{4.13}$$

The metric (4.12) in these coordinates has the constant form

$$\sum_{i=1}^{n} \eta_{ii}(u)(du^i)^2 = \eta_{\alpha\beta} dt^\alpha dt^\beta, \quad \eta_{\alpha\beta} = \delta_{n+1,\alpha+\beta}. \tag{4.14}$$

The ortonormal basis in A_w with respect to this metric consists of the polynomials $\phi_1(p)$,... , $\phi_n(p)$ of degrees 0, 1, ... , $n-1$ resp. where

$$\phi_\alpha(p) = \frac{d}{dp}[w^{\frac{\alpha}{n+1}}]_+, \quad \alpha = 1, \dots, n, \tag{4.15}$$

Here $[\ ]_+$ means the polynomial part of the power series in p. This is a TCFT model with the charges and dimension

$$q_\alpha = \frac{\alpha - 1}{n+1}, \ d = q_n = \frac{n-1}{n+1}. \tag{4.16}$$

In fact one obtains a n-parameter family of TFT models with the same canonical coordinates u^i of the form (4.11) where

$$\eta_{ii}(u) \mapsto \eta_{ii}(u, c) = [w''(p_i)]^{-1} [\sum c_\alpha \phi_\alpha(p_i)]^2, \tag{4.17a}$$

$$t^\alpha \mapsto t^\alpha(c) = -\frac{n+1}{n-\alpha+1} \mathrm{res}_{p=\infty} w^{\frac{n-\alpha+1}{n+1}}(p)[\sum c_\gamma \phi_\gamma(p)]dp \tag{4.17b}$$

depending on arbitrary parameters c_1, ..., c_n. This reflects the ambiguity in the choice of the solution ψ_{i1} in the formulae (1.24). These models are conformal invariant if only one of the coefficients c_γ is nonzero.

The corresponding hierarchy of the systems of hydrodynamic type (3.1) coincides with the dispersionless limit of the Gelfand – Dickey hierarchy for the scalar Lax operator of order $n+1$. This essentialy follows from [8, 11]. I recall that the Gelfand – Dickey hierarchy for an operator

$$L = \partial^{n+1} + a_1(x)\partial^{n-1} + \dots + a_n(x)$$

$$\partial = d/dx$$

has the form

$$\partial_{t^{\alpha,p}} L = c_{\alpha,p}[L, [L^{\frac{\alpha}{n+1}+p}]_+], \ \alpha = 1, \dots, n, \ p = 0, 1, \dots \tag{4.18}$$

for some constants $c_{\alpha,p}$. Here $[\ ]_+$ denotes differential part of the pseudodifferential operator. The dispersionless limit of the hierarchy is defined as follows: one should substitute

$$x \mapsto \epsilon x = X, \ t^{\alpha,p} \mapsto \epsilon t^{\alpha,p} = T^{\alpha,p} \tag{4.19}$$

and tend ϵ to zero. The dispersionless limit of τ-function of the hierarchy is defined [11, 53-54, 61] as

$$\log \tau_{\text{dispersionless}}(T) = \lim_{\epsilon \to 0} \epsilon^{-2} \log \tau(\epsilon t). \tag{4.20}$$

Modified minimal model (4.17) is related to the same Gelfand – Dickey hierarchy with the following modification of the L-operator

$$L \mapsto \tilde{L} = \sum c_\gamma [L^{\frac{\gamma}{n+1}}]_+. \tag{4.21}$$

The linear equation (3.5) for the minimal model can be solved in the form [39]

$$h_\alpha(t;\lambda) = -\frac{n+1}{\alpha} \mathrm{res}_{p=\infty} w^{\frac{\alpha}{n+1}} {}_1F_1(1; 1+\frac{\alpha}{n+1}; \lambda w(p)) dp. \tag{4.22}$$

Here ${}_1F_1(a;c;z)$ is the Kummer (or confluent hypergeometric) function [35]

$$ {}_1F_1(a;c;z) = \sum_{m=o}^{\infty} \frac{(a)_m}{(c)_m} \frac{z^m}{m!}, \tag{4.23a}$$

$$(a)_m = a(a+1)\dots(a+m-1). \tag{4.23b}$$

The generating function (3.11) has the form

$$V_{\alpha\beta}(t;\lambda,\mu) = (\lambda+\mu)^{-1}[\eta^{\mu\nu}(\mathrm{res}_{p=\infty} w^{\frac{\alpha}{n+1}-1} {}_1F_1(1; \frac{\alpha}{n+1}; \lambda w(p))\phi_\mu(p)dp)\times$$

$$(\mathrm{res}_{p=\infty} w^{\frac{\beta}{n+1}-1} {}_1F_1(1; \frac{\beta}{n+1}; \mu w(p))\phi_\nu(p)dp) - \eta_{\alpha\beta}]. \tag{4.24}$$

From this one obtains formulae for the τ-function.

Example 3. $M_{g;n_0,\dots,n_m}$-models [13, 14]. Let $M = M_{g;n_0,\dots,n_m}$ be a moduli space of dimension

$$n = 2g + n_0 + \dots + n_m + 2m \tag{4.25}$$

of sets

$$(C; \infty_0, \dots, \infty_m; w; k_0, \dots, k_m; a_1, \dots, a_g, b_1, \dots, b_g) \in M_{g;n_0,\dots,n_m} \tag{4.26}$$

where C is a Riemann surface with marked points ∞_0, ..., ∞_m, and a marked meromorphic function

$$w : C \to CP^1, \quad w^{-1}(\infty) = \infty_0 \cup \dots, \cup \infty_m \tag{4.27}$$

having a degree $n_i + 1$ near the point ∞_i, and a marked symplectic basis $a_1, \dots, a_g$, $b_1, \dots, b_g \in H_1(C, \mathbf{Z})$, and marked branches of roots of w near ∞_0, ..., ∞_m of the orders $n_0 + 1$, ..., $n_m + 1$ resp.,

$$k_i^{n_i+1}(P) = w(P), \ P \text{ near } \infty_i. \tag{4.28}$$

(This is a connect manifold as it follows from [56].) We need the critical values of w

$$u^j = w(P_j), \ dw|_{P_j} = 0, \ j = 1, \dots, n \tag{4.29}$$

(i.e. the ramification points of the Riemann surface (4.27)) to be local coordinates in open domains in M where

$$u^i \neq u^j \text{ for } i \neq j \tag{4.30}$$

(due to the Riemann existence theorem). Another assumption is that the one-dimensional affine group acts on M as

$$(C;\infty_0,\dots,\infty_m;w;\dots)\mapsto(C;\infty_0,\dots,\infty_m;aw+b;\dots) \tag{4.31a}$$

$$u^i\mapsto au^i+b,\ i=1,\dots,n. \tag{4.31b}$$

Let dp be the normalized Abelian differential of the second kind on C with a double pole at ∞_0

$$dp=dk_0+\text{ regular terms} \tag{4.32a}$$

$$\oint_{a_i}dp=0,\ i=1,\dots,g. \tag{4.32b}$$

Using u^i as the canonical coordinates (1.15) I define a flat Egoroff metric on M by the formula

$$ds^2=\sum_{i=1}^n\eta_{ii}(u)(du^i)^2, \tag{4.33a}$$

$$\eta_{ii}(u)=\text{res}_{P_i}\frac{(dp)^2}{dw}. \tag{4.33b}$$

It can be extended globaly on M. The corresponding flat coordinates are

$$t^{i;\alpha}=-\frac{n_i+1}{n_i-\alpha+1}\text{res}_{\infty_i}k_i^{n_i-\alpha+1}dp,\ i=0,\dots,m,\ \alpha=1,\dots,n_i; \tag{4.34a}$$

$$p^i=\text{v.p.}\int_{\infty_0}^{\infty_i}dp=\lim_{Q\to\infty_0}\Big(\int_Q^{\infty_i}dp+k_0(Q)\Big),\ i=1,\dots,m; \tag{4.34b}$$

$$q^i=-\text{res}_{\infty_i}wdp,\ i=1,\dots,m; \tag{4.34c}$$

$$r^i=\oint_{b_i}dp,\ s^i=-\frac{1}{2\pi i}\oint_{a_i}wdp,\ i=1,\dots,g. \tag{4.34d}$$

The metric (4.33) in the coordinates has the following form

$$\eta_{t^{i;\alpha}t^{i;\beta}}=\frac{1}{n_i+1}\delta_{ij}\delta_{\alpha+\beta,n_i+1} \tag{4.35a}$$

$$\eta_{p^iq^j}=\delta_{ij} \tag{4.35b}$$

$$\eta_{r^is^j}=\delta_{ij}, \tag{4.35c}$$

other components of the η vanish. The unity vector field is a unit vector along the coordinate $t^{0;1}$.

Proposition 4.1. *The flat metric (4.35) is well-defined globaly on M and the flat coordinates (4.34) are globaly independent analytic functions on M.*

As a consequence we obtain that the moduli space M is an unramified covering over a domain in $\mathbf{C}^n$ (see [13, 14]).

Let us introduce primary differentials on C (or on a universal covering $\tilde{C}$ of $C \setminus \infty_0 \cup \ldots \cup \infty_m$) of the form

$$\phi_{t^A} = \partial_{t^A}(p dw)_{w=\text{const}} \tag{4.36}$$

where

$$p(P) = \int_{Q_0}^{P} dp, \tag{4.37a}$$

$$Q_0 \in C,\ w(Q_0) = 0, \tag{4.37b}$$

t^A is one of the flat coordinates (4.34). Note that the definition (4.36) of the primary differentials can be rewritten as

$$\phi_{t^A} = -\partial_{t^A}(w dp)_{p=\text{const}} \tag{4.38}$$

where the multivalued coordinate p on C is defined in (4.37). So $w(p)$ plays the role of the Landau – Ginsburg superpotential for the $M_{g;n_0,\ldots,n_m}$-models. More explicitly, $\phi_{t^{i;\alpha}}$ is a normalized Abelian differential of the second kind with a pole in ∞_i,

$$\phi_{t^{i;\alpha}} = -\frac{1}{\alpha} dk_i^\alpha + \text{ regular terms} \quad \text{near } \infty_i,$$

$$\oint_{a_j} \phi_{t^{i;\alpha}} = 0; \tag{4.39a}$$

ϕ_{p^i} is a normalized Abelian differential of the second kind on C with a pole only at ∞_i with the principal part of the form

$$\phi_{p^i} = dw + \text{ regular terms} \quad \text{near } \infty_i,$$

$$\oint_{a_j} \phi_{p^i} = 0; \tag{4.39b}$$

ϕ_{q^i} is a normalized Abelian differential of the third kind with simple poles at ∞_0 and ∞_i with residues -1 and +1 resp.;
ϕ_{r^i} is a normalized multivalued differential on C with increments along the cycles b_i of the form

$$\phi_{r^i}(P + b_j) - \phi_{r^i}(P) = -\delta_{ij} dw,$$

$$\oint_{a_j} \phi_{r^i} = 0; \tag{4.39c}$$

ϕ_{s^i} are the basic holomorphic differentials* on C normalized by the condition

$$\oint_{a_j} \phi_{s^i} = 2\pi i \delta_{ij}. \tag{4.39d}$$

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* 1-form pdw was used by Novikov and Veselov in their theory of algebro-geometric Poisson brackets [60]. The coordinates s^i are the algebro-geometric action variables of [60]. In [60] it also was an important point that derivatives $\partial_{s^i}(pdw)$ are the normalized holomorphic differentials.

The inner product (4.33) in terms of the primary differentials ϕ_{t^A} reads

$$\eta_{AB} = \sum_{i=1}^{n} \mathrm{res}_{P_i} \frac{\phi_{t^A}\phi_{t^B}}{dw}. \tag{4.40}$$

The structure functions $c_{ABC}(t)$ can be calculated as

$$c_{ABC}(t) = \sum_{i=1}^{n} \mathrm{res}_{P_i} \frac{\phi_{t^A}\phi_{t^B}\phi_{t^C}}{dw dp}. \tag{4.41}$$

Extension of the Fröbenius structure on all the moduli space M is given by the condition that the differential

$$\frac{\phi_{t^A}\phi_{t^B} - c_{AB}^{C}\phi_{t^C}dp}{dw} \tag{4.42}$$

is holomorphic for $|w| < \infty$. The Fröbenius algebra on T_tM will be nilpotent for Riemann surfaces $w : C \to CP^1$ with more than double branch points. This is a conformal invariant Fröbenius manifold with the dimension

$$d = \frac{n_0 - 1}{n_0 + 1} \tag{4.43a}$$

and charges

$$q_{t^{i;\alpha}} = \frac{\alpha}{n_i + 1} - \frac{1}{n_0 + 1} \tag{4.43b}$$

$$q_{r^i} = q_{p^i} = \frac{n_0}{n_0 + 1} \tag{4.43c}$$

$$q_{s^i} = q_{q^i} = -\frac{1}{n_0 + 1}. \tag{4.43d}$$

For the particular case $g = m = 0$ we obtain the Fröbenius manifolds of the minimal models (the previous example). For $g = 0$, $m > 0$ we obtain models with rational functions as superpotentials.

The generating functions $h_{t^A}(t;\lambda)$ (3.4) have the form

$$h_{t^{i;\alpha}}(t;\lambda) = -\frac{n_i + 1}{\alpha} \mathrm{res}_{p=\infty_i} k_i^{\alpha}\, {}_1F_1(1; 1 + \frac{\alpha}{n_i + 1}; \lambda w(p)) dp. \tag{4.44a}$$

$$h_{p^i} = \mathrm{v.p.} \int_{\infty_0}^{\infty_i} e^{\lambda w} dp \tag{4.44b}$$

$$h_{q^i} = \mathrm{res}_{\infty_i} \frac{e^{\lambda w} - 1}{\lambda} dp \tag{4.44c}$$

$$h_{r^i} = \oint_{b_i} e^{\lambda w} dp \tag{4.44d}$$

$$h_{s^i} = \frac{1}{2\pi i}\oint_{a_i} p e^{\lambda w} dw. \tag{4.44e}$$

Remark. Integrals of the form (4.44) seem to be interesting functions on the moduli space of the form $M_{g;n_0,\dots,n_m}$. A simplest example of such an integral for a family of elliptic curves reads

$$\int_0^{\omega} e^{\lambda\wp(z)} dz \tag{4.45}$$

where $\wp(z)$ is the Weierstrass function with periods 2ω, $2\omega'$. For real negative λ a degeneration of the elliptic curve ($\omega \to \infty$) reduces (4.45) to the standard probability integral $\int_0^\infty e^{\lambda x^2} dx$. So the integral (4.45) is an analogue of the probability integral as a function on λ and on moduli of the elliptic curve. I recall that dependence on these parameters is specified by the equations (3.5), (2.17).

Gradients of this functions on the moduli space M have the form

$$\partial_{t^A} h_{t^{i;\alpha}} = \text{res}_{\infty_i} k_i^{\alpha - n_i - 1}\, {}_1F_1(1; \frac{\alpha}{n_i+1}; \lambda w(p))\phi_{t^A}, \tag{4.46a}$$

$$\partial_{t^A} h_{p^i} = \eta_{t^A p^i} - \lambda \text{v.p.} \int_{\infty_0}^{\infty_i} e^{\lambda w}\phi_{t^A} \tag{4.46b}$$

$$\partial_{t^A} h_{q^i} = \text{res}_{\infty_i} e^{\lambda w}\phi_{t^A} \tag{4.46c}$$

$$\partial_{t^A} h_{r^i} = \eta_{t^A r^i} - \lambda \oint_{b_i} e^{\lambda w}\phi_{t^A} \tag{4.46d}$$

$$\partial_{t^A} h_{s^i} = \frac{1}{2\pi i}\oint_{a_i} e^{\lambda w}\phi_{t^A}. \tag{4.46e}$$

The generating function $V_{\alpha\beta}(t;\lambda,\mu)$ of coefficients of the τ-function (3.13) can be calculated via inner products (w.r.t. the matrix (4.35)) of (4.46). Particularly, for a part of the Hessian of the primary free energy $F(t)$ (a function on M) one obtains [13, 14]

$$\frac{\partial^2 F}{\partial s^i \partial s^j} = -\tau_{ij} = -\oint_{b_j} \phi_{s^i}. \tag{4.47}$$

This is nothing but the matrix of periods of holomorphic differentials on M. Other second derivatives of F also turn out to be certain periods of some Abelian differentials on C.

Conclusion. *WDVV is a universal system of integrable differential equations for periods of Abelian differentials on Riemann surfaces.*

I recall that this system is a high-order analogue of the Painlevé-VI equation (i.e. equations of isomonodromy deformations of (2.15)). To specify the solution of WDVV one needs to find the monodromy matrix of the linear operator (2.15) for the eigenfunctions of the form (4.44). I will do it in a forthcoming publication.

We obtain the following picture of "Painlevé uniformisation" of the moduli spaces $M_{g;n_0,\dots,n_m}$: (1) a global system of analytic coordinates on $M_{g;n_0,\dots,n_m}$; (2) periods of

Abelian differentials on curves $C \in M_{g;n_0,\dots,n_m}$ are certain high-order Painlevé transcendents as functions of these coordinates.

Remark. For any Hamiltonian $H_{A,p}$ of the form (3.3), (4.44) one can construct a differential $\Omega_{A,p}$ on C or on the covering $\tilde{C}$ with singulariries only at the marked infinite points such that

$$\frac{\partial}{\partial u^i} h_{t^A,p} = \mathrm{res}_{P_i} \frac{\Omega_{A,p} dp}{dw}, \ i = 1, \dots, n. \tag{4.48}$$

See[13] for an explicit form of these differentials (for $m = 0$ also see [14]). Using these differentials the hierarchy (3.1) can be written in the Flaschka – Forest –McLaughlin form [16]

$$\partial_{T^{A,p}} dp = \partial_X \Omega_{A,p} \tag{4.49}$$

(derivatives of the differentials are to be calculated with w =const.).

The matrix $V_{(A,p),(B,q)(t)}$ determines a pairing of these differentials with values in functions on the moduli space

$$(\Omega_{A,p}, \Omega_{B,q}) = V_{(A,p),(B,q)}(t) \tag{4.50}$$

Particularly, the primary free energy F as a function on M can be written in the form [13, 14]

$$F = -\frac{1}{2}(pdw, pdw). \tag{4.51}$$

Note that the differential pdw can be written in the form

$$pdw = \sum \frac{n_i + 1}{n_i + 2} \Omega^{(n_i+2)}_{\infty_i} + \sum t^A \phi_{t^A} \tag{4.52}$$

where $\Omega^{(n_i+2)}_{\infty_i}$ is the Abelian differential of the second kind with a pole at ∞_i of the form

$$\Omega^{(n_i+2)}_{\infty_i} = dk_i^{n_i+2} + \text{ regular terms } \quad \text{near } \infty_i. \tag{4.53}$$

For the pairing (4.50) one can obtain from [44] the following formula

$$(f_1 dw, f_2 dw) = \frac{1}{2} \int \int_C (\bar{\partial} f_1 \partial f_2 + \partial f_1 \bar{\partial} f_2) \tag{4.54}$$

where the differentials ∂ and $\bar{\partial}$ along the Riemann surface should be understood in the distribution sense. The meromorphic differentials $f_1 dw$ and $f_2 dw$ on the covering $\tilde{C}$ should be considered as piecewise meromorphic differentials on C with jumps on some cuts.

The corresponding hierarchy (3.1) is obtained by averaging along invariant tori of a family of g-gap solutions of a KdV-type hierarchy related to a matrix operator L of the matrix order $m + 1$ and of orders n_0, ..., n_m in $\partial/\partial x$. The example $m = 0$ (the averaged Gelfand – Dickey hierarchy) was considered in more details in [14]. The Poisson bracket (3.2) is a result of semiclassical limit (or averaging) [18-21] of the first haniltonian structure of the Gelfand – Dickey hierarchy; averaging of the second hamiltonian structure (the classical W-algebra) gives the Poisson structure (3.21). Therefore, (3.21) can be considered as semiclassical limit of classical W-algebras. The corresponding flat metric (3.22) on the moduli space $M_{g;n_0,\dots,n_m}$ is well-defined on a subset of Riemann surfaces having $w = 0$ a non-ramifying point.

Also for $g + m > 0$ one needs to extend the KdV-type hierarchy to obtain (3.1) (see [13-14]). To explain the nature of such an extension let us consider the simplest example of $m = 0$, $n_0 = 1$. The moduli space M consists of hyperelliptic curves of genus g with marked homology basis

$$y^2 = \prod_{i=1}^{2g+1} (w - w_i). \tag{4.55}$$

This parametrizes the family of g-gap solutions of the KdV. The L operator has the well-known form

$$L = -\partial_x^2 + u. \tag{4.56}$$

In real smooth periodic case $u(x + T) = u(x)$ the quasimomentum $p(w)$ is defined by the formula

$$\psi(x + T, w) = e^{ip(w)T} \psi(x, w) \tag{4.57}$$

for a solution $\psi(x, w)$ of the equation

$$L\psi = w\psi \tag{4.58}$$

(the Bloch – Floquet eigenfunction). The differential dp can be extended onto the family of all (i.e. quasiperiodic complex meromorphic) g-gap operators (4.55) as a normalized Abelian differential of the second kind with a double pole at the infinity $w = \infty$. (So the above superpotential (4.38) has the sense of the Bloch dispersion law, i.e. the dependence of the energy w on the quasimomentum p.) The Hamiltonians of the KdV hierarchy can be obtained as coefficients of expansion of dp near the infinity. To obtain a complete family of conservation laws of the averaged hierarchy (3.1) one needs to extend the family of the KdV integrals by adding nonlocal functionals of u of the form

$$\oint_{a_i} w^k dp, \quad \oint_{b_i} w^{k-1} dp, \; k = 1, 2, \dots. \tag{4.59}$$

As in (4.17) one can deform the above Fröbenius structure on the moduli space $M = M_{g;n_0,\dots,n_m}$ by changing the differential dp,

$$dp \mapsto \tilde{dp} = \sum c_A \phi_{t^A} \tag{4.60}$$

for arbitrary constant coefficients. (The deformed Fröbenius structure genericaly is well-defined only on a subset of M.) Particularly, if $\tilde{dp}$ is a differential of the third kind on C then the "dimension" d of this model always equals 1. The corresponding hierarchy (3.1) is obtained by averaging a Toda-type system.

Here I consider the simplest example of such a deformation. Let us consider the 3-dimensional family M of elliptic curves

$$y^2 = 4(w-c)^2 - g_2(w-c) - g_3 = 4(w-c-e_1)(w-c-e_2)(w-c-e_3) \qquad (4.61)$$

with ordered roots e_1, e_2, e_3. It is convenient to use the Weierstrass uniformization of (4.63)

$$w = \wp(z) + c \qquad (4.62a)$$

$$y = \wp'(z) \qquad (4.62b)$$

(I will use the standard notations [35] of the theory of elliptic functions). Let us use the holomorphic differential

$$dp = \frac{\pi i dz}{\omega} \qquad (4.63)$$

to construct a Fröbenius structure on M (here $\wp(\omega) = e_1$). The corresponding Landau – Ginsburg superpotential is the Weierstrass function (4.62a) where one should substitute $z = \omega p/\pi i$. The flat coordinates t^1, t^2, t^3 for the superpotential read

$$t^1 = -c + \frac{\eta}{\omega} \qquad (4.64a)$$

$$t^2 = -1/\omega \qquad (4.64b)$$

$$t^3 = 2\pi i\tau \ \text{ where } \tau = \omega'/\omega, \qquad (4.64c)$$

$\wp(\omega') = e_3$, $\eta = -\int_0^\omega \wp(z)dz$. The charges of this manifold are $q_0 = 0$, $q_1 = \frac{1}{2}$, $q_2 = d = 1$.

Remark. The above models with $m = 0$, $g > 0$ can be obtained [39] in a semiclassical description of correlators of multimatrix models (at the tree-level approximation for small couplings they correspond to various self-similar solutions of the hierarchy (3.1)) as functions of the couplings after passing through a point of gradient catastrophe. The idea of such a description is originated in the theory of a dispersive analogue of shock waves [32]; see also [18].

More general algebraic-geometrical examples of solutions of WDVV were constructed in [44]. In these examples M is a moduli space of Riemann surfaces of genus g with a marked normalized Abelian differential of the second kind dw with poles at marked points and with fixed b-periods

$$\oint_{b_i} = B_i, \ i = 1, \ldots, g.$$

For $B_i = 0$ one obtains the above Fröbenius structures on $M_{g;n_0,\ldots,n_m}$. Unfortunately, for $B \neq 0$ the Fröbenius structures of [44] does not admit a conformal invariance.

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