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INTEGRABLE FUNCTIONAL EQUATIONS AND ALGEBRAIC GEOMETRY

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Introduction. The main goal of this paper is to show that certain functional equations can be solved using an appropriate version of the inverse spectral method. (We refer the reader to the books [1]–[5] where the basic ideas of the application of the inverse spectral method to integrable systems of differential equations are explained.) Here we shall demonstrate our method by considering in detail the functional equation

$$\frac{q(x, y)q(y, z)}{q(x, z)} = r(x, y) - r(z, y) + p(x, z). \tag{0.1}$$

Note that, in the case

$$p(x, y) = p(x - y), \qquad q(x, y) = q(x - y), \qquad r(x, y) = r(x - y), \tag{0.2}$$

where $r(z)$ is an odd function, this equation reduces to the more simple functional equation

$$\frac{q(x)q(y)}{q(x + y)} = r(x) + r(y) + p(x + y), \tag{0.3}$$

introduced by Calogero and Bruschi in connection to integrable many-body problems [6].

The usual way to solve functional equations of this type is to derive a differential equation for the involved functions assuming appropriate smoothness. This was done for equation (0.3) in [7]. (The differentiated form of the functional equation (0.3) was also used by I. Krichever [14] in his theory of action-angle variables for Calogero-Moser systems with an elliptic potential.) Here we shall solve the functional equation (0.1) without assuming smoothness or even continuity for the functions p, q, r. We only assume that these functions are Lebesgue measurable.

The equation (0.1) has a rich symmetry group. Indeed, one can transform the functions as follows:

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$$q(x, y) \to \frac{v(x)}{v(y)} q(x, y), \qquad r \to r, \qquad p \to p; \tag{0.4}$$

$$q \to q, \qquad r(x, y) \to r(x, y) + f(y) + g(x), \qquad p(x, y) \to p(x, y) - g(x) + g(y); \tag{0.5}$$

$$p, q, r \to kp, kq, kr. \tag{0.6}$$

Here $v(x)$, $f(x)$, $q(x)$ are arbitrary functions and k is constant. Also, one can change the arguments by an arbitrary function $z(x)$:

$$p(x, y), \qquad q(x, y), \qquad r(x, y) \to p(z(x), z(y)), \qquad q(z(x), z(y)), \qquad r(z(x), z(y)). \tag{0.7}$$

Our main result is that, modulo the ambiguity (0.4)–(0.7), a generic solution of the functional equation has the form (0.2) with

$$p(z) = \zeta(z_0) - \zeta(z + z_0), \qquad q(z) = \frac{\sigma(z - z_0)}{\sigma(z_0)\sigma(z)}, \qquad r(z) = \zeta(z), \tag{0.8}$$

(cf. [7]) where σ and ζ are the Weierstrass elliptic σ- and ζ-functions (see [9]) and z_0 is a complex number.

Our method of solution of the functional equation (0.1) involves first relating (0.1) to the algebraic equation

$$\frac{q_{ij} q_{jk}}{q_{ik}} = r_{ij} - r_{kj} + p_{ik}. \tag{0.9}$$

We then construct a commutation representation ("Lax pair") for equation (0.9). Because of the existence of such a representation, we can call (0.9) an *integrable algebraic equation* and equation (0.1) an *integrable functional equation.* We shall show that the problem of solving the functional equation (0.1) can be reduced to the problem of classifying certain commutative algebras of λ-matrices (λ plays the role of spectral parameter). Using ideas of the algebraic-geometric integration method (see surveys [10]–[12]) we obtain such a classification.

We note that equation (0.9) is not the first algebraic equation to be solved by the inverse spectral method. In fact, Krichever [8] used algebraic-geometric techniques to classify two-component solutions of the Yang-Baxter equation. However, the nature of integrability of equations (0.1) and (0.9) is substantially different from that of the Yang-Baxter equation.

Remark. Within the ambiguity (0.4), the function

$$q(x, y) = \frac{\sigma(x - y - z_0)}{\sigma(x_0)\sigma(x - y)}$$

solving the functional equation (0.1) coincides with the Baker-Akhiezer function

$$\psi(x, y; z_0) = \exp(z_0\zeta(x) - z_0\zeta(y))q(x, y).$$

We recall that $\psi(x, y; z_0)$ as a function of x is a single-valued function on the elliptic curve

$$x \in \mathbb{C}/\{2m\omega + 2n\omega'\},$$

and it has an essential singularity in $x = 0$

$$\psi(x, y; z_0) \simeq \exp\frac{z_0}{x} \qquad \text{for } x \to 0.$$

A generalization of the Baker-Akhiezer function (essentially, a generalization of the exponential function on the Riemann sphere) plays a very important role in the machinery of the algebraic-geometric method of integration of nonlinear equations [11]. We have found that Baker-Akhiezer functions on Riemann surfaces of genus g are related to the integrable functional equation[1]

$$\frac{q(x, y)q(y, z)}{q(x, z)} = r(x, y) - r(z, y) + \sum_{k=1}^{g} s_k(y)p_k(x, z). \tag{0.10}$$

(See Appendix for the precise formulations.)

It would be interesting to derive all the standard facts of the theory of Baker-Akhiezer functions from the functional equation (0.10). We are also tempted to consider solutions of the integro-functional equation

$$\frac{q(x, y)q(y, z)}{q(x, z)} = r(x, y) - r(z, y) + \int_M s(y, t)p(x, z, t)\, dt \tag{0.11}$$

(integral over a space M with a measure dt) as Baker-Akhiezer functions of infinite genus. We will consider these problems in subsequent publications.

1. Commutation representation for the functional equation. Let X be a set of at least 5 elements. Let $p(x, y)$, $q(x, y)$, $r(x, y)$ be complex-valued functions on $X \times X\backslash$(diagonal) satisfying the functional equation

$$q(x, y)q(y, z) = q(x, z)[r(x, y) - r(z, y)] + p(x, z), \tag{1.1}$$

$$x, y, z \text{ distinct}.$$

[1] In [15] it was shown that the Baker-Akhiezer function on a Riemann surface of any genus g as a function of z_0 (i.e., of a point of the Jacobian of the Riemann surface) satisfies certain functional equations. These coincide with ours only for $g = 1$.

Let $x_1, \ldots, x_n$ be arbitrary distinct points of the set X, $n \geqslant 4$. Let us fix n numbers $\gamma_1, \ldots, \gamma_n$ satisfying

$$\gamma_1 + \cdots + \gamma_n = 0, \tag{1.2a}$$

$$|\gamma_1|^2 + \cdots + |\gamma_n|^2 \neq 0, \tag{1.2b}$$

and the following condition: there exist n pairwise distinct numbers $a_1, \ldots, a_n$ such that

$$a_1 + \cdots + a_n = 0 \qquad \text{and } \gamma_1 a_1 + \cdots + \gamma_n a_n = 0. \tag{1.2c}$$

We introduce $n \times n$ off-diagonal matrices

$$p_{ij} := p(x_i, x_j), \qquad q_{ij} := q(x_i, x_j)\gamma_j, \qquad r_{ij} := r(x_i, x_j)\gamma_j. \tag{1.5}$$

In these formulae $i \neq j$. We put zeros on the diagonals of the matrices.

LEMMA 1.1. *If $p(x, y)$, $q(x, y)$, $r(x, y)$ satisfy the functional equation (1.1), then the matrices (1.3) satisfy the following system of algebraic equations:*

$$q_{ij}q_{jk} = q_{ik}[r_{ij} - r_{kj}] + \gamma_j p_{ik}, \tag{1.4}$$

$$i, j, k \text{ distinct}.$$

Proof. Obvious.

We construct now a "Lax pair" (more precisely, a *commutation representation*) for the system of algebraic equations (1.4). This system will be represented as the condition of commutativity of a family of $n \times n$ matrices depending on an additional spectral parameter λ. We will call these matrices *λ-matrices*. The idea to consider equations of commutativity of a pair of λ-matrices as "integrable algebraic equations" was proposed by one of the authors (P. S.) in [13]. Here we develop this idea further by considering multidimensional commutative algebras of λ-matrices.

Let us fix n numbers $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfying (1.2). Let $\mathscr{L}^\gamma$ be a commutative algebra of $n \times n$ λ-matrices L generated by the subspace $\mathscr{L}_1^\gamma$ of the form

$$\mathscr{L}_1^\gamma = \{L = \lambda A + U \mid A = \operatorname{diag}(a_1, \ldots, a_n),\ U = (u_{ij})\} \tag{1.5a}$$

where $a_1, \ldots, a_n$ are *arbitrary* complex numbers satisfying (1.2c). We assume the algebra $\mathscr{L}^\gamma$ to be *irreducible*; i.e., there exists no nontrivial subspace $V \subset \mathbb{C}^n$ which is invariant with respect to $\mathscr{L}^\gamma$. This is guaranteed by the inequalities

$$\sum_{s \neq i} (|u_{is}|^2 + |u_{si}|^2) \neq 0 \tag{1.5b}$$

for any $i = 1, \ldots, n$ and for some matrix $U = (u_{ij})$ such that $L = \lambda A + U \in \mathscr{L}_1^\gamma$.

LEMMA 1.2. *There exist off-diagonal matrices $Q = (q_{ij})$ and $R = (r_{ij})$ such that all the elements of $\mathcal{L}_1^\gamma$ can be represented in the form*

$$L \equiv L_A := \lambda A + [A, Q] + D_A + cI \tag{1.6}$$

where D_A is a diagonal matrix with entries given by the vector $[R, A]$ $(1, 1, \ldots, 1)^T$, i.e.,

$$D_A = \operatorname{diag}([R, A](1, 1, \ldots, 1)^T), \tag{1.7}$$

c is an arbitrary constant, and I is the unity matrix.

Proof. Let $\mathcal{A}$ be the space of matrices $A = \operatorname{diag}(a_1, \ldots, a_n)$ satisfying (1.2c). The map

$$\mathcal{L}_1^\gamma/\{cI\} \to \mathcal{A}, \qquad \lambda A + U \to A \tag{1.8}$$

is an isomorphism of linear spaces. Indeed, let $A \in \mathcal{A}$ be a matrix with pairwise distinct diagonal elements. Let $L = \lambda A + U$ be an element of $\mathcal{L}_1^\gamma$. We define an off-diagonal matrix Q by the equality

$$U = [A, Q] + D, \qquad D = \operatorname{diag}(d_1, \ldots, d_n). \tag{1.9}$$

For any $L' = \lambda A' + U' \in \mathcal{L}$, the commutativity of L and L' implies $[A, U'] = [A', U]$, or using (1.9)

$$U' = [A', Q] + D', \qquad D' = \operatorname{diag}(d_1', \ldots, d_n').$$

Let $L_i = \lambda A + [A, Q] + D_i$, $i = 1, 2$ be two elements of $\mathcal{L}_1^\gamma$. Here D_1, D_2 are some diagonal matrices. Then $D_1 - D_2 \in \mathcal{L}^\gamma$. So $D_1 - D_2 = cI$ because of the irreducibility condition (1.5b). So the linear map (1.8) is an isomorphism. The inverse map is just given by (1.6). (The formula (1.7) gives the general form of a diagonal matrix $D = D_A$ linearly depending on the traceless matrix A.) Lemma 1.2 is proved.

The matrices Q, R can be considered as parameters of commutative algebras of λ-matrices of the above form. The matrix Q is determined uniquely by the commutative algebra, while the ambiguity in determining R is of the form

$$r_{ij} \sim r_{ij} + p_i \gamma_j \tag{1.10}$$

for arbitrary p_i.

In the coordinate form, the formulae of Lemma 1.2 mean that for any matrix A satisfying (1.2) there exists a unique matrix $L_A \in \mathcal{L}$ of the form

$$(L_A)_{ij} = \left[\lambda a_i + \sum_s r_{is}(a_s - a_i)\right]\delta_{ij} + (a_i - a_j)q_{ij}, \qquad i, j = 1, \ldots, n. \tag{1.11}$$

We obtain (together with λI) an $(n-1)$-dimensional family of commuting $n \times n$ λ-matrices.

PROPOSITION 1.1. *The equations of commutativity of the λ-matrices (1.11), satisfying (1.2c), are equivalent to the algebraic equations (1.3).*

Proof. Let $A = \operatorname{diag}(a_1, \ldots, a_n)$, $B = \operatorname{diag}(b_1, \ldots, b_n)$ be two diagonal matrices satisfying (1.2c). The commutativity of L_A and L_B implies

$$b_i \sum (a_j - a_l) c_{ilj} - b_j \sum (a_i - a_l) c_{ilj} + (a_i - a_j) \sum b_l c_{ilj} = 0, \tag{1.12}$$

where

$$c_{ilj} = q_{il} q_{lj} + (r_{jl} - r_{il}) q_{ij}, \tag{1.13}$$

and the sums are over l with $l \neq i, j$. Since equation (1.12) is an identity for any B satisfying (1.2c), this guarantees the existence of numbers p_{ij} such that

$$c_{ilj} = \gamma_l p_{ij}, \tag{1.14}$$

$$i, j, l \text{ are distinct}.$$

Proposition 1.1 is proved.

Remark. More generally, one can consider commutative algebras of λ-matrices with the generators of the form (1.5) where the diagonal leading terms $A = \operatorname{diag}(a_1, \ldots, a_n)$ satisfy $g + 1$ independent linear constraints

$$a_1 + \cdots + a_n = 0, \tag{1.15a}$$

$$\sum_{j=1}^{n} \gamma_j^l a_j = 0, \qquad l = 1, \ldots, g, \tag{1.15b}$$

where the numbers γ_j^l satisfy the conditions

$$\sum_{j=1}^{n} \gamma_j^l = 0, \qquad l = 1, \ldots, g. \tag{1.16}$$

Then under certain regularity conditions for the linear space (1.15), the equations of commutativity of the $(n-g)$-dimensional family of λ-matrices (1.11) read

$$q_{ij} q_{jk} = q_{ik} [r_{ij} - r_{kj}] + \sum_{l=1}^{g} \gamma_j^l p_{ik}^l, \tag{1.17}$$

$$i, j, k \text{ distinct},$$

for some matrices $(p_{ik}^1), \ldots, (p_{ik}^g)$. We leave as an exercise to the reader to derive the equations (1.19) for arbitrary g from the commutation representation. For $g = 1$ this gives (1.4). The case $g = 0$ is even simpler; but the corresponding algebraic equations (1.17) for $d = 0$ can be solved in an elementary way:

$$q_{ij} = \frac{v_i}{v_j} \frac{\lambda_j}{z_i - z_j}, \qquad r_{ij} = \frac{\lambda_j}{z_i - z_j} + f_j, \tag{1.18}$$

where λ_j, v_j, f_j, z_j are arbitrary complex numbers.

2. Spectral curve of the integrable algebraic system. The idea of the spectral curve corresponding to the integrable algebraic system (1.4) (or, equivalently, to the commutative algebra of λ-matrices (1.11)) is very simple. We consider the family of common eigenvectors of the commuting operators L_A,

$$L_A \psi = \mu_A \psi .$$

Here μ_A is some algebraic function of λ (depending linearly on A) defined on an algebraic curve consisting of all common eigenvectors considering them up to a factor. This is the spectral curve we need. In our case the spectral curve is an elliptic curve; we will give a very explicit description of it. Investigating the analytic properties of the eigenvectors ψ as functions on the spectral curve is the core of the algebraic-geometric integration method [10]–[12]. In our case this investigation provides us with explicit formulae for solutions of (1.4) in terms of elliptic functions.

Let us assume that

$$\gamma_1 + \gamma_2 + \gamma_3 \neq 0. \tag{2.1}$$

Choose two vectors (b_1, b_2, b_3), (c_1, c_2, c_3) with nonzero components satisfying

$$\gamma_1 b_1 + \gamma_2 b_2 + \gamma_3 b_3 = 0 \tag{2.2a}$$

and

$$\gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_3 = 0, \tag{2.2b}$$

such that the points (b_1, c_1), (b_2, c_2), (b_3, c_3) are not on a line. We assume also that all the matrix elements q_{ij} for $i, j = 1, 2, 3$, $i \neq j$ do not vanish.

Let

$$B = \operatorname{diag}(b_1, b_2, b_3, 0, \ldots, 0), \qquad C = \operatorname{diag}(c_1, c_2, c_3, 0, \ldots, 0) \tag{2.3}$$

be two diagonal matrices of order n. Let us consider the corresponding matrices L_B, L_C of the form (1.11) with the substitution B and C instead of A. The matrices

B, C are not traceless. Note that adding a scalar matrix to the matrix A in (1.11) shifts the corresponding λ-matrix L_A by $\alpha\lambda I$. So the matrices L_B, L_C still commute,

$$L_B, L_C \in \mathscr{L}^{\gamma}. \tag{2.4}$$

We first prove the existence of a common eigenvector ψ of the commuting λ-matrices L_B, L_C. The eigenvector ψ satisfies

$$L_B\psi = u\psi, \tag{2.5a}$$

$$L_C\psi = v\psi, \tag{2.5b}$$

where u, v, the corresponding eigenvalues are certain functions of λ,

$$u = u(\lambda), \qquad v = v(\lambda). \tag{2.6}$$

As the first step we will eliminate λ from the system (2.6) and prove that the eigenvalues u, v are related by a cubic equation. This equation will give us an explicit realization of the spectral curve.

Multiplying equation (2.5a) by C, equation (2.5b) by B, and subtracting the results, we obtain

$$M\phi = 0 \tag{2.7}$$

where $\phi = (\psi_1, \psi_2, \psi_3)^T$ and $M = M(u, v)$ is a 3×3 matrix of the form

$$M(u, v) = \begin{bmatrix} b_1 v - c_1 u + d_1 & \beta_{12} q_{12} & \beta_{13} q_{13} \\ \beta_{21} q_{21} & b_2 v - c_2 u + d_2 & \beta_{23} q_{23} \\ \beta_{31} q_{31} & \beta_{32} q_{32} & b_3 v - c_3 u + d_3 \end{bmatrix}. \tag{2.8}$$

Here

$$\beta_{ij} = b_i c_j - b_j c_i = \rho \gamma_k, \tag{2.9}$$

for some $\rho \neq 0$. Here i, j, k is an even substitution of 1, 2, 3, and

$$d_i = -\sum_{l=1}^{3} r_{il} \beta_{il}. \tag{2.10}$$

LEMMA 2.1. *If a nonzero common eigenvector ψ of the λ-matrices L_B, L_C exists, then*

$$\Delta(u, v) := \det M(u, v) = 0. \tag{2.11}$$

Proof. If $\Delta(u, v) \neq 0$, then $\psi_1 = \psi_2 = \psi_3 = 0$. If such nonzero common eigenvectors ψ of L_B, L_C exist, then they form an invariant subspace of all the commutative algebra $\mathcal{L}^\gamma$. This contradicts irreducibility of $\mathcal{L}^\gamma$. Lemma 2.1 is proved.

The algebraic curve (2.11) is a plane cubic curve. Generically this is an elliptic curve; for certain values of the parameters it can degenerate to a rational one. Let us assume this curve to be generic (i.e., an elliptic one). We will consider it as a 3-sheet covering of the complex u-plane. There are 3 infinite points on the curve,

$$u, v \to \infty, \qquad \frac{v}{u} \to \frac{c_i}{b_i}, \qquad i = 1, 2, 3. \tag{2.12a}$$

We will denote them by ∞_1, ∞_2, ∞_3 respectively. Near these points the ratio v/u has the expansion

$$\frac{v}{u} = \frac{c_i}{b_i} - \frac{d_i}{b_i} u^{-1} - [\beta_{ji} q_{ji} q_{ij} + \beta_{ki} q_{ki} q_{ik}] u^{-2} + O(u^{-3}) \tag{2.12b}$$

where i, j, k is a permutation of 1, 2, 3. Equation (2.12b) follows easily from (2.11).

Let us study the analytic properties of the eigenvector $\phi = (\psi_1, \psi_2, \psi_3)^T$ of the matrix $M(u, v)$ on the curve (2.11).

PROPOSITION 2.1. *The components ψ_2, ψ_3 of the eigenvector ϕ of $M(u, v)$ normalized by*

$$\psi_1 = 1 \tag{2.13}$$

are rational functions on the curve (2.11). They vanish at the point ∞_1; ψ_2 and ψ_3 have a pole at ∞_2 and ∞_3 respectively of the form

$$\psi_2 = -\frac{u}{b_2 q_{12}} + O(1) \text{ near } \infty_2, \tag{2.14a}$$

$$\psi_3 = -\frac{u}{b_3 q_{13}} + O(1) \text{ near } \infty_3. \tag{2.14b}$$

Also

$$\psi_3(\infty_2) = \frac{q_{32}}{q_{12}}, \tag{2.15a}$$

$$\psi_2(\infty_3) = \frac{q_{23}}{q_{13}}. \tag{2.15b}$$

For $|u| < \infty$, the functions ψ_2, ψ_3 have a pole at the point (u_0, v_0),

$$u_0 = b_3 \frac{\beta_{12}}{\beta_{13}} \frac{q_{12} q_{23}}{q_{13}} + b_2 \frac{\beta_{13}}{\beta_{12}} \frac{q_{13} q_{32}}{q_{12}} + \frac{b_2 d_3 - b_3 d_2}{\beta_{23}}, \tag{2.16a}$$

$$v_0 = c_3 \frac{\beta_{12}}{\beta_{13}} \frac{q_{12} q_{23}}{q_{13}} + c_2 \frac{\beta_{13}}{\beta_{12}} \frac{q_{13} q_{32}}{q_{12}} + \frac{c_2 d_3 - c_3 d_2}{\beta_{23}}. \tag{2.16b}$$

Proof. Let $\Delta_{ij}(u, v)$ be the cofactor of the (i, j)th element in the determinant of the matrix $M(u, v)$. Then the eigenvector can be represented as

$$\psi_2 = \frac{\Delta_{i2}(u, v)}{\Delta_{i1}(u, v)}, \qquad \psi_3 = \frac{\Delta_{i3}(u, v)}{\Delta_{i1}(u, v)}, \tag{2.17}$$

for any $i = 1, 2, 3$. Hence ψ_2, ψ_3 are rational functions on the curve (2.11). The behavior of the eigenvector at infinity follows immediately from the expansion (2.13). Let us look for finite poles of ψ_2, ψ_3. As it follows from (2.17), they could be only at the point (u, v) where

$$\Delta_{11}(u, v) = \Delta_{21}(u, v) = \Delta_{31}(u, v) = 0.$$

Let us write explicitly a part of this system:

$$\Delta_{21}(u, v) = \beta_{13}\beta_{32}q_{13}q_{32} - \beta_{12}q_{12}(b_3 v - c_3 u + d_3) = 0, \tag{2.18a}$$

$$\Delta_{31}(u, v) = \beta_{12}\beta_{23}q_{12}q_{23} - \beta_{13}q_{13}(b_2 v - c_2 u + d_2) = 0. \tag{2.18b}$$

Solving the linear system (2.18), we find the unique solution $u = u_0$, $v = v_0$ of the form (2.16).

Let us prove that the point (u_0, v_0) belongs to the curve (2.11). (This implies also that $\Delta_{11}(u_0, v_0) = 0$.) Indeed, otherwise the functions ψ_2, ψ_3 would have no poles for $|u| < \infty$. But any of them has precisely one pole at infinity. It is impossible for a rational function on an elliptic curve to have one pole. Proposition 2.1 is proved.

Let us consider now the other components of the common eigenvector (2.5).

PROPOSITION 2.2. *The components $\psi_4, \ldots, \psi_n$ of the common eigenvector ψ of the matrices L_B, L_C normalized by (2.13) are rational functions on the curve (2.11). They vanish at the point ∞_1. In the point (u_0, v_0) all of them have poles; each component $\psi_j, j \geqslant 4$, also has a pole at the point*

$$\infty_j := (u_j, v_j), j = 4, \ldots, n, \tag{2.19a}$$

$$u_j = \sum_{l=1}^{3} r_{jl} b_l, \qquad v_j = \sum_{l=1}^{3} r_{jl} c_l, \tag{2.19b}$$

of the form

$$\psi_j = \frac{\tilde{v}_j}{u - u_j} + O(1), \qquad j = 4, \dots, n, \tag{2.20}$$

where

$$\tilde{v}_j = \Big\{ q_{j1} b_1 \Big[\sum_{l=1}^{3} (r_{jl} - r_{2l}) \beta_{2l} \sum_{l=1}^{3} (r_{jl} - r_{3l}) \beta_{3l} - \beta_{23}\beta_{32} q_{23} q_{32} \Big]$$
$$+ q_{j2} b_2 \Big[\beta_{23}\beta_{31} q_{23} q_{31} - \beta_{21} q_{21} \sum_{l=1}^{3} (r_{jl} - r_{3l}) \beta_{3l} \Big]$$
$$+ q_{j3} b_3 \Big[\beta_{32}\beta_{21} q_{32} q_{21} - \beta_{31} q_{31} \sum_{l=1}^{3} (r_{jl} - r_{2l}) \beta_{2l} \Big] \Big\} \Big/$$
$$\Big[\sum_{l=1}^{3} (r_{jl} - r_{2l}) \beta_{2l} \sum_{l=1}^{3} (r_{jl} - r_{3l}) \beta_{3l} - \beta_{23}\beta_{32} q_{23} q_{32} \Big]. \tag{2.21}$$

Proof. For $j \geqslant 4$, the jth equation in the system (2.5) reads

$$\psi_j = \frac{\sum_{l=1}^{3} (b_j - b_l) q_{jl} \psi_l}{u - u_j}, \tag{2.22a}$$

$$\psi_j = \frac{\sum_{l=1}^{3} (c_j - c_l) q_{jl} \psi_l}{v - v_j}, \tag{2.22b}$$

where u_j, v_j have the form (2.19). So all the ψ_j are rational functions on the same curve (2.11). The statements about the pole at (u_0, v_0) and the behavior at infinity are obvious from the analytic properties of ψ_2 and ψ_3. The function ψ_j also *can* have a pole only at the point (u_j, v_j) of the form (2.19) (zero of the denominators in (2.22)). Let us prove that the point (u_j, v_j) belongs to the curve (2.11) for any $j = 4, \dots, n$. Indeed, otherwise the function ψ_j would have only one pole at (u_0, v_0). So it should be a constant. Hence $\psi_j = 0$ since $\psi_j(\infty_1) = 0$. This contradicts to irreducibility of the commutative algebra $\mathscr{L}^{\gamma}$.

To prove (2.20)–(2.21) we use (2.22) and the formulae

$$\psi_2 = \frac{\Delta_{12}}{\Delta_{11}}, \qquad \psi_3 = \frac{\Delta_{13}}{\Delta_{11}}.$$

Proposition 2.2 is proved.

It is time now to remember the spectral parameter λ. We will show that λ is also a rational function on the curve (2.11).

PROPOSITION 2.3. *The spectral parameter λ of the matrices L_B, L_C is a rational function on the curve (2.11) having poles only at the points $\infty_1, \ldots, \infty_n$ of the form*

$$\lambda = \frac{u}{b_1} + \tilde{\lambda}_0 + O(u^{-1}) \qquad \text{near } \infty_1;$$

$$\tilde{\lambda}_0 = -\frac{\sum_{l=1}^{3} r_{1l}(b_l - b_1)}{b_1}, \tag{2.23a}$$

$$\lambda = \frac{u}{b_2} + O(1) \qquad \text{near } \infty_2, \tag{2.23b}$$

$$\lambda = \frac{u}{b_3} + O(1) \qquad \text{near } \infty_3, \tag{2.23c}$$

$$\lambda \sim \frac{q_{1j}\tilde{v}_j}{u - u_j} \qquad \text{near } \infty_j, j = 4, \ldots, n, \tag{2.23d}$$

where $\tilde{v}_j$ are given by (2.21).

Proof. Taking the first equation of the system (2.5), we obtain

$$\lambda b_1 = u - \sum_{l=1}^{3} r_{1l}(b_l - b_1) - \sum_{l=1}^{3} (b_1 - b_l)q_{1l}\psi_l + b_1 \sum_{j=4}^{n} q_{1j}\frac{\sum_{l=1}^{3} b_l q_{jl}\psi_l}{u - u_j}, \tag{2.24a}$$

$$\lambda c_1 = v - \sum_{l=1}^{3} r_{1l}(c_l - c_1) - \sum_{l=1}^{3} (c_1 - c_l)q_{1l}\psi_l + c_1 \sum_{j=4}^{n} q_{1j}\frac{\sum_{l=1}^{3} b_l q_{jl}\psi_l}{v - v_i}. \tag{2.24b}$$

So λ is a rational function on the curve (2.11) having poles at the points $\infty_1, \ldots, \infty_n$ and, possibly, at the point (u_0, v_0). To prove that λ has no pole at (u_0, v_0) we consider the second equation in (2.5a):

$$\left(\lambda b_2 + \sum_{l=1}^{3} r_{2l}(b_l - b_2)\right)\psi_2 = u\psi_2 - \sum_{l=1}^{3} (b_2 - b_l)q_{2l}\psi_l + b_2 \sum_{j=4}^{n} q_{2j}\frac{\sum_{l=1}^{3} b_l q_{jl}\psi_l}{u - u_j},$$

and we proceed similarly for (2.5b). If λ has a pole at (u_0, v_0), then the left-hand side of this equation has a double pole at this point. But the right-hand side has at most a simple pole at this point. This contradiction completes the proof of Proposition 2.3.

PROPOSITION 2.4. *The common eigenvector ψ of L_B, L_C is an eigenvector of any operator L_A (1.13) of the commutative algebra $\mathscr{L}^\gamma$*

$$L_A\psi = \mu_A\psi. \tag{2.25}$$

The function μ_A is a rational one on the curve (2.11) having poles only at the points $\infty_1, \ldots, \infty_n$ of the form

$$\mu_A = \frac{a_1}{b_1} u + \tilde{\mu}_0 + O(u^{-1}) \qquad \text{near } \infty_1;\ \tilde{\mu}_0 = \sum_s r_{1s}(a_s - a_1), \tag{2.26a}$$

$$\mu_A = \frac{a_2}{b_2} u + O(1) \qquad \text{near } \infty_2, \tag{2.26b}$$

$$\mu_A = \frac{a_3}{b_3} u + O(1) \qquad \text{near } \infty_3, \tag{2.26c}$$

$$\mu_A \sim \frac{a_j q_{1j} \tilde{v}_j}{u - u_j} \qquad \text{near } \infty_j, j = 4, \ldots, n. \tag{2.26d}$$

Proof. Since ψ is the unique common eigenvector of the λ-matrices L_B and L_C, it should be also an eigenvector for all the operators of the commutative algebra. To describe the poles of the eigenvalue μ_A we have to consider the first equation of the system (2.25). To prove the cancellation of the pole at (u_0, v_0) it is enough to consider the second equation of (2.25). We omit here the calculations since they are similar to the previous proposition.

We conclude this section by obtaining explicitly the conditions for $q(x_i, x_j)$, $r(x_i, x_j)$, $i, j = 1, 2, 3$, which guarantee that the spectral curve (2.8) is nonsingular.

We first observe that the spectral curve does not depend on the choice of the solutions (b_1, b_2, b_3) and (c_1, c_2, c_3) of (2.2). Indeed, changing these vectors simply implies a linear transformation on the (u, v)-plane. Such a transformation preserves the genus of the spectral curve.

So we will use the following particular choice of the vectors:

$$(b_1, b_2, b_3) = (-\gamma_3, 0, \gamma_1) \tag{2.27a}$$

$$(c_1, c_2, c_3) = (0, -\gamma_3, \gamma_2). \tag{2.27b}$$

Renormalizing

$$u = \gamma_2\gamma_3 x, \qquad v = \gamma_2\gamma_3 y, \tag{2.28}$$

we obtain the following equation of the spectral curve:

$$\det \begin{bmatrix} -y + R_{13} - R_{12} & Q_{12} & -Q_{13} \\ -Q_{21} & x + R_{21} - R_{23} & Q_{23} \\ Q_{31} & -Q_{32} & y - x + R_{32} - R_{31} \end{bmatrix} = 0. \tag{2.29}$$

Here we introduce the notation

$$Q_{ij} = q(x_i, x_j), \qquad R_{ij} = r(x_i, x_j), \qquad i, j = 1, 2, 3. \tag{2.30}$$

The equation (2.29) can be written in the form

$$\alpha(x)y^2 + \beta(x)y + \gamma(x) = 0, \tag{2.31}$$

where

$$\alpha(x) = x + R_{21} - R_{23}, \tag{2.32a}$$

$$\beta(x) = x^2 + x(R_{31} - R_{32} + R_{13} - R_{12} + R_{21} - R_{23}) + Q_{12}Q_{21} - Q_{23}Q_{32}, \tag{2.32b}$$

$$\begin{aligned}\gamma(x) = {} & x^2(R_{12} - R_{13}) + x(Q_{13}Q_{31} - Q_{12}Q_{21}) \\ & + (R_{12} - R_{23})(R_{13} - R_{12})(R_{32} - R_{31}) + Q_{12}Q_{23}Q_{31} \\ & - Q_{13}Q_{32}Q_{21} + Q_{13}Q_{31}(R_{21} - R_{23}) + Q_{12}Q_{21}(R_{32} - R_{31}) \\ & + Q_{23}Q_{32}(R_{13} - R_{12}).\end{aligned} \tag{2.32c}$$

The ramification points over the x-plane of the Riemann surface (2.31) are the zeros of the discriminant

$$D(x) = \beta^2(x) + 4\alpha(x)\gamma(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4, \tag{2.33}$$

where

$$a_1 = 2R_{12} - 2R_{13} + 2R_{21} - 2R_{23} + 2R_{31} - 2R_{32}, \tag{2.34a}$$

$$\begin{aligned}a_2 = {} & -2Q_{12}Q_{21} + 4Q_{13}Q_{31} - 2Q_{23}Q_{32} + R_{12}^2 - 2R_{12}R_{13} + R_{13}^2 \\ & + 2R_{12}R_{21} - 2R_{13}R_{21} + R_{21}^2 - 2R_{12}R_{23} + 2R_{13}R_{23} - 2R_{21}R_{23} \\ & + R_{23}^2 - 2R_{12}R_{31} + 2R_{13}R_{31} + 2R_{21}R_{31} - 2R_{23}R_{31} + R_{31}^2 \\ & + 2R_{12}R_{32} - 2R_{13}R_{32} - 2R_{21}R_{32} + 2R_{23}R_{32} - 2R_{31}R_{32} + R_{32}^2,\end{aligned} \tag{2.34b}$$

$$\begin{aligned} a_3 = {} & 4Q_{12}Q_{23}Q_{31} - 4Q_{13}Q_{21}Q_{32} - 2Q_{12}Q_{21}R_{12} - 2Q_{23}Q_{32}R_{12} \\ & + 2Q_{12}Q_{21}R_{13} + 2Q_{23}Q_{32}R_{13} - 2Q_{12}Q_{21}R_{21} + 8Q_{13}Q_{31}R_{21} \\ & - 2Q_{23}Q_{32}R_{21} + 2Q_{12}Q_{21}R_{23} - 8Q_{13}Q_{31}R_{23} + 2Q_{23}Q_{32}R_{23} \\ & - 2Q_{12}Q_{21}R_{31} - 2Q_{23}Q_{32}R_{31} + 4R_{12}^2R_{21} - 4R_{12}R_{13}R_{31} \\ & - 4R_{12}R_{23}R_{31} + 4R_{13}R_{23}R_{31} + 2Q_{12}Q_{21}R_{32} + 2Q_{23}Q_{32}R_{32} \\ & - 4R_{12}^2R_{32} + 4R_{12}R_{13}R_{32} + 4R_{12}R_{23}R_{32} - 4R_{13}R_{23}R_{32}, \end{aligned} \tag{2.34c}$$

$$\begin{aligned} a_4 = {} & Q_{12}^2Q_{21}^2 - 2Q_{12}Q_{21}Q_{23}Q_{32} + Q_{23}^2Q_{32}^2 + 4Q_{12}Q_{23}Q_{31}R_{21} \\ & - 4Q_{13}Q_{21}Q_{32}R_{21} - 4Q_{23}Q_{32}R_{12}R_{21} + 4Q_{23}Q_{32}R_{13}R_{21} \\ & + 4Q_{13}Q_{31}R_{21}^2 - 4Q_{12}Q_{23}Q_{31}R_{23} + 4Q_{13}Q_{21}Q_{32}R_{23} \\ & + 4Q_{23}Q_{32}R_{12}R_{23} - 4Q_{23}Q_{32}R_{13}R_{23} - 8Q_{13}Q_{31}R_{21}R_{23} \\ & + 4Q_{13}Q_{31}R_{23}^2 - 4Q_{12}Q_{21}R_{21}R_{31} + 4R_{12}^2R_{21}R_{31} \\ & - 4R_{12}R_{13}R_{21}R_{31} + 4Q_{12}Q_{21}R_{23}R_{31} - 4R_{12}^2R_{23}R_{31} \\ & + 4R_{12}R_{13}R_{23}R_{31} - 4R_{12}R_{21}R_{23}R_{31} + 4R_{13}R_{21}R_{23}R_{31} \\ & + 4R_{12}R_{23}^2R_{31} - 4R_{13}R_{23}^2R_{31} + 4Q_{12}Q_{21}R_{21}R_{32} - 4R_{12}^2R_{21}R_{32} \\ & + 4R_{12}R_{13}R_{31}R_{32} - 4Q_{12}Q_{21}R_{23}R_{32} + 4R_{12}^2R_{23}R_{32} \\ & - 4R_{12}R_{13}R_{23}R_{32} + 4R_{12}R_{21}R_{23}R_{32} - 4R_{13}R_{21}R_{23}R_{32} \\ & - 4R_{12}R_{23}^2R_{32} + 4R_{13}R_{23}^2R_{32}. \end{aligned} \tag{2.34d}$$

Equation (2.31) is an elliptic curve unless the polynomial (2.33) has multiple zeros. This happens if and only if the discriminant

$$\begin{aligned} J(x_1, x_2, x_3) = {} & a_1^2a_2^2a_3^2 - 4a_2^3a_3^2 - 4a_1^3a_3^3 + 18a_1a_2a_3^3 - 27a_3^4 - 4a_1^2a_2^3a_4 \\ & + 16a_2^4a_4 + 18a_1^3a_2a_3a_4 - 80a_1a_2^2a_3a_4 - 6a_1^2a_3^2a_4 \\ & + 144a_2a_3^2a_4 - 27a_1^4a_4^2 + 144a_1^2a_2a_4^2 - 128a_2^2a_4^2 \\ & - 192a_1a_3a_4^2 + 256a_4^3 \end{aligned} \tag{2.35}$$

of the polynomial (2.33), whose coefficients are defined by (2.34), vanishes. We do not give the explicit form of $J(x_1, x_2, x_3)$ in terms of $q(x_i, x_j)$, $r(x_i, x_j)$, since the relevant formula is longer than the entire present paper.

We conclude that the condition

$$J(x_1, x_2, x_3) \neq 0$$

guarantees that (2.8) is a nonsingular elliptic curve.

3. Uniformization of the spectral curve. Elliptic parametrization of commutative algebras of λ-matrices. Propositions 2.1–2.4 justify calling the curve (2.11) the *spectral curve* of the commutative algebra $\mathcal{L}^{\gamma}$. The formulae of Section 2 can be interpreted as the solution of the direct spectral problem that assign to the algebra $\mathcal{L}^{\gamma}$ a set of spectral data: the spectral curve with marked points $\infty_1, \ldots, \infty_n$, the pole (u_0, v_0), and the parameters $\tilde{v}_2, \ldots, \tilde{v}_n$, where

$$\tilde{v}_2 = -\frac{1}{b_2 q_{12}}, \qquad \tilde{v}_3 = -\frac{1}{b_3 q_{13}}, \tag{3.1}$$

and the numbers $\tilde{v}_4, \ldots, \tilde{v}_n$ are given by the formula (2.21). In this section we solve the inverse problem: we reconstruct the commutative algebra for a given set of spectral data.

Let us fix a basis a, b of cycles on the spectral curve (appropriately oriented). We will use an elliptic uniformization of the spectral curve (2.11). For this we fix a holomorphic differential on the curve (2.11),

$$\Omega = \frac{du}{\Delta_v(u, v)}, \tag{3.2}$$

where

$$\begin{aligned}\Delta_v(u, v) \equiv \frac{\partial \Delta(u, v)}{\partial v} &= b_1(b_2 v - c_2 u + d_2)(b_3 v - c_3 u + d_3) \\ &+ b_2(b_1 v - c_1 u + d_1)(b_3 v - c_3 u + d_3) \\ &+ b_3(b_1 v - c_1 u + d_1)(b_2 v - c_2 u + d_2) + b_1 \beta_{32}^2 q_{23} q_{32} \\ &+ b_2 \beta_{13}^2 q_{13} q_{31} + b_3 \beta_{12}^2 q_{12} q_{21}.\end{aligned} \tag{3.3}$$

We denote by 2ω, $2\omega'$ the basic periods of the holomorphic differential, i.e.,

$$2\omega = \oint_a \Omega, \qquad 2\omega' = \oint_b \Omega. \tag{3.4}$$

Their periods satisfy the inequality

$$\operatorname{Im} \frac{\omega'}{\omega} > 0. \tag{3.5}$$

We dissect the spectral curve along the cycles a, b and introduce the complex coordinate

$$z = \int_{\infty_1}^{(u,v)} \Omega. \tag{3.6}$$

(Paths of all the integrals should not intersect the canonical cuts a and b.) The coordinate z is defined up to an integer linear combination of the periods 2ω, $2\omega'$. So the uniformization map (3.6) establishes an isomorphism of the spectral curve with the torus

$$\mathbb{C}/\{2m\omega + 2n\omega' | m, n \in \mathbb{Z}\}. \tag{3.7}$$

Rational functions on the spectral curve become elliptic functions on the torus (3.7). Let us obtain explicit formulae for the rational functions constructed in Section 2. Put

$$z_1 = 0, \qquad z_j = \int_{\infty_1}^{\infty_j} \Omega, \qquad j = 2, \ldots, n, \tag{3.8}$$

$$z_0 = \int_{\infty_1}^{(u_0, v_0)} \Omega. \tag{3.9}$$

PROPOSITION 3.1. *The following formulae hold:*

(i)

$$\lambda = \sum_{i=1}^{n} \lambda_i \zeta(z - z_i) + \lambda_0, \tag{3.10a}$$

where

$$\lambda_1 = -\frac{1}{\beta_{21}\beta_{31}}, \qquad \lambda_2 = \frac{1}{\beta_{12}\beta_{32}}, \qquad \lambda_3 = -\frac{1}{\beta_{13}\beta_{23}}; \qquad \lambda_j = \frac{q_{1j}\tilde{v}_j}{\Delta_j}, \qquad j \geqslant 4, \tag{3.10b}$$

$$\lambda_0 = \tilde{\lambda}_0 + \sum_{i=2}^{n} \lambda_i \zeta(z_i) - \frac{1}{2} \frac{b_1(b_1 d_2 - b_2 d_1)}{\beta_{21}}, \tag{3.10c}$$

and $\tilde{\lambda}_0$, $\tilde{v}_j$, Δ_j *are defined in (2.23a), by (2.21), and*

$$\begin{aligned}\Delta_j = {} & b_1 \sum_{l=1}^{3} (r_{jl} - r_{2l})\beta_{2l} \sum_{l=1}^{3} (r_{jl} - r_{3l})\beta_{32} \\ & + b_2 \sum_{l=1}^{3} (r_{jl} - r_{1l})\beta_{1l} \sum_{l=1}^{3} (r_{jl} - r_{3l})\beta_{32} \\ & + b_3 \sum_{l=1}^{3} (r_{jl} - r_{1l})\beta_{1l} \sum_{l=1}^{3} (r_{jl} - r_{2l})\beta_{2l} + b_1 \beta_{32}^2 q_{23} q_{32} \\ & + b_2 \beta_{13}^2 q_{13} q_{31} + b_3 \beta_{12}^2 q_{12} q_{21},\end{aligned} \tag{3.10d}$$

respectively.

(ii)

$$\mu_A = \sum_{i=1}^{n} a_i \lambda_i \zeta(z - z_i) + \mu_0, \tag{3.11a}$$

where

$$\mu_0 = \tilde{\mu}_0 + \sum_{i=1}^{n} a_i \lambda_i \zeta(z_i) - \frac{a_1}{2} \frac{b_1(b_1 d_2 - b_2 d_1)}{\beta_{21}}, \tag{3.11b}$$

and $\tilde{\mu}_0$ *is defined in (2.26a).*

(iii)

$$\psi_i = v_i \frac{\sigma(z)\sigma(z - z_i - z_0)}{\sigma(z - z_i)\sigma(z - z_0)}, \qquad i \geqslant 2, \tag{3.12a}$$

where

$$v_2 = \frac{q_{23}}{q_{13}} \frac{\sigma(z_3 - z_2)\sigma(z_3 - z_0)}{\sigma(z_3)\sigma(z_3 - z_2 - z_0)}; \qquad v_j = \frac{q_{j2}}{q_{12}} \frac{\sigma(z_2 - z_j)\sigma(z_2 - z_0)}{\sigma(z_2)\sigma(z_2 - z_j - z_0)}, \qquad j \geqslant 3. \tag{3.12b}$$

Proof. The formulae (3.10a), (3.11a), (3.12a) give the most general form of elliptic functions with the analytic properties described in Section 2. To calculate the coefficients λ_i, one should take into account that near ∞_1

$$\Delta_v(u, v) = \frac{\beta_{21}\beta_{31}}{b_1} u^2 + \frac{\beta_{31}}{b_1}(b_1 d_2 - b_2 d_1)u + O(1).$$

(This follows from (2.12b) and (3.3).) So near $z = 0$,

$$z = -\frac{b_1}{\beta_{21}\beta_{31}}u^{-1} + \frac{1}{2}\frac{b_1(b_1d_2 - b_2d_1)}{\beta_{21}^2\beta_{21}}u^{-2} + O(1), \tag{3.13a}$$

or equivalently,

$$\frac{1}{z} = -\frac{\beta_{21}\beta_{31}}{b_1}u - \frac{1}{2}\frac{\beta_{31}(b_1d_2 - b_2d_1)}{b_1} + O(u^{-1}). \tag{3.13b}$$

Similarly, near ∞_2

$$z - z_2 = -\frac{b_2}{\beta_{12}\beta_{32}}u^{-1} + O(u^{-2}), \tag{3.13c}$$

while near ∞_3

$$z - z_3 = -\frac{b_3}{\beta_{13}\beta_{23}}u^{-1} + O(u^{-2}). \tag{3.13d}$$

This (and the formulae (2.23a, b, c)) give $\lambda_1, \lambda_2, \lambda_3$. To obtain λ_j for $j \geqslant 4$, we use the obvious formula

$$\frac{dz}{du} = \frac{1}{\Delta_v(u, v)}.$$

Putting

$$\Delta_j = \Delta_v(u_j, v_j) \tag{3.14}$$

and using (2.23d), one obtains the formula for λ_j. The formula (3.10c) follows from (3.13b) and (2.23a). For μ_A the calculations are similar. The formulas for v_j follow by comparing the values of the functions ψ_i at the infinite points. Proposition 3.1 is proved.

COROLLARY. *The coefficients $\lambda_1, \ldots, \lambda_n$ in (3.10a) are proportional to $\gamma_1, \ldots, \gamma_n$:*

$$\lambda_i = \kappa\gamma_i, \qquad i = 1, \ldots, n, \tag{3.15a}$$

where

$$\kappa = -\frac{1}{\rho^2\gamma_1\gamma_2\gamma_3}, \tag{3.15b}$$

and the coefficient ρ is the same as in (2.9).

Proof. The sum of residues of the elliptic functions λ, μ should be equal to zero. This gives

$$\lambda_1 + \cdots + \lambda_n = 0$$

and also

$$a_1\lambda_1 + \cdots + a_n\lambda_n = 0$$

for arbitrary $a_1, \ldots, a_n$ satisfying (1.2c). So the proportionality (3.15a) holds true. The coefficient κ can be found from (3.10b). The corollary is proved.

PROPOSITION 3.3. *The following formulae hold for the matrices p_{ij}, q_{ij}, r_{ij} for $i, j > 1$:*

$$p_{ij} = \kappa^2[\zeta(z_0) - \zeta(z_i - z_j + z_0) - g_i + g_j] \tag{3.16a}$$

$$q_{ij} = \kappa \frac{v_i}{v_j} \gamma_j \frac{\sigma(z_j - z_i - z_0)}{\sigma(z_j - z_i)\sigma(z_0)} \tag{3.16b}$$

$$r_{ij} = \kappa\gamma_j\zeta(z_i - z_j) + f_i\gamma_j + g_j \tag{3.16c}$$

for some f_i, g_j.

Proof. We consider the ith equation in the system $L_A\psi = \mu_A\psi$. Letting, in this equation, $z \to z_j$ for $j \neq i$, we obtain (3.16b). Consider now the same equation near $z = z_i$. We obtain

$$\sum_{j=1}^{n} r_{ij}(a_i - a_j) = a_i\lambda_0 - \mu_0 + \sum (a_i - a_j)\lambda_j\zeta(z_i - z_j)$$

$$= \sum_{j=1}^{n} (a_i - a_j)[\lambda_j\zeta(z_i - z_j) + \lambda_j\zeta(z_j)] - \sum_j r_{1j}(a_j - a_1)$$

$$+ \frac{1}{2}(a_1 - a_i)b_1 \frac{b_1 d_2 - b_2 d_1}{\beta_{21}}.$$

(We used the formulae (2.26b), (3.10c), and (3.11b).) Both sides of these equations are linear functions of the vector $(a_1, \ldots, a_n)$ satisfying (1.2c). From this we obtain (3.16c). Substituting (3.16b, c) into the equation (1.4), we obtain (3.16a). Proposition 3.3 is proved.

Remark. Using the same ideas, one can prove that the algebraic integrable system (1.17) can be solved in terms of theta-functions of Riemann surfaces of genus g for $n \geqslant g$ (see Appendix).

4. From algebraic to functional equations and proof of the main theorem

THEOREM. *Let X be a set of at least 5 elements. Let $p(x, y)$, $q(x, y)$, $r(x, y)$ be complex-valued functions on $X \times X\backslash$(diagonal) satisfying the functional equation (1.1) and the genericity assumption*

$$J \equiv J(x_1, x_2, x_3) \neq 0, \tag{4.1}$$

for some distinct $x_1, x_2, x_3 \in X$, where J is defined by equation (2.35). Then these functions can be represented in the form

$$q(x, y) = k\frac{v(x)\,\sigma(z(y) - z(x) - z_0)}{v(y)\,\sigma(z_0)\sigma(z(y) - z(x))}, \tag{4.2a}$$

$$r(x, y) = k[\zeta(z(x) - z(y)) + f(y) + g(x)], \tag{4.2b}$$

$$p(x, y) = kq(x, y)[\zeta(z_0) - \zeta(z(x) - z(y) + z_0) - g(x) + g(y)]. \tag{4.2c}$$

Here $\sigma = \sigma(z|\omega, \omega')$, $\zeta = \zeta(z|\omega, \omega')$ are the Weierstrass elliptic functions with the same periods ω and ω', $v(x)$, $z(x)$, $f(x)$, $g(x)$ are arbitrary complex-valued functions on X, z_0 is a complex number such that $z_0 \neq 0$ (modulo periods of σ and ζ), and k is an arbitrary complex constant. Conversely, the formulae (4.2) determine a solution of the functional equation (1.1) for arbitrary ω, ω', z_0, k, and arbitrary functions $v(x)$, $z(x)$, $f(x)$, $g(x)$.

Proof. Let us take a solution $p(x, y)$, $q(x, y)$, $r(x, y)$ of the functional equation (1.1). We put now $n = 5$ and choose three points x_1, x_2, x_3 such that $q(x_i, x_j) \neq 0$ for $i = 1, 2, 3$, $i \neq j$, and such that the genericity assumption (4.1) as well as the irreducibility assumption (1.5b) hold. The spectral curve (2.11) then is an elliptic curve (J in (4.1) is the discriminant of (2.11)). We put also $x_4 = x$, $x_5 = y$. Taking arbitrary constants $\gamma_1, \ldots, \gamma_5$ with $\gamma_1\gamma_2\gamma_3 \neq 0$ (as in the beginning of Section 1 also satisfying (2.1)), we obtain from the Proposition 3.3 that, particularly,

$$p_{45} = \kappa^2[\zeta(z_0) - \zeta(z_4 - z_5 + z_0)],$$

$$q_{45} = \kappa\frac{v_4}{v_5}\gamma_5\frac{\sigma(z_5 - z_4 - z_0)}{\sigma(z_0)\sigma(z_5 - z_4)},$$

$$r_{45} = \kappa\gamma_5\zeta(z_4 - z_5),$$

up to adding inessential terms f_4, g_5. From the formulae of Sections 2 and 3 we infer that:

1. The parameters of the ζ- and σ-functions do not depend on $x_4 = x$, $x_5 = y$.
2. z_0 does not depend on x, y (see equations (2.16) and (3.9)).

3. z_4 and z_5 depend only on x, y respectively, and this dependence is given by the same function

$$z_4 = z(x), \qquad z_5 = z(y).$$

(The function $z(x)$ can be expressed explicitly using (2.19) and (3.8) via the spectral curve and via the functions $r(x, x_l)$, $l = 1, 2, 3$.)

4. The coefficients v_4, v_5 are functions depending only on x, y respectively, and this dependence is given by the same functions

$$v_4 = v(x), \qquad v_5 = v(y).$$

(The function $v(x)$ can be expressed explicitly using (3.12b) via the spectral curve and via the functions $q(x, x_2)$ and $z(x)$.) Recalling that $q_{45} = q(x, y)\gamma_5$, $r_{45} = r(x, y)\gamma_5$, we complete the proof of the theorem.

Remark. If $J = 0$, the general solution of (0.1) can be obtained from (4.2) by degeneration of the elliptic curve. We will not consider here all these possible degenerations.

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Appendix

Functional equation for Baker-Akhiezer functions of arbitrary genus g. Here we will describe the general solution of the functional equation (0.10). We will use notations of [16] of the theory of theta-functions on Riemann surfaces.

THEOREM. *Let X be a set of at least $3g + 2$ elements. Let $p(x, y)$, $q(x, y)$, $r(x, y)$ be complex-valued functions on $X \times X \backslash$(diagonal). Let $s_1(x), \ldots, s_g(x)$ be linearly independent complex-valued functions on X. Assume that the above functions satisfy the functional equation (0.10). Then, for a generic solution, the function $q(x, y)$ has the form*

$$q(x, y) = k \frac{v(x)}{v(y)} \frac{\theta(\int_{P(y)}^{P(x)} w - z_0)}{\theta(z_0)\varepsilon(P(x), P(y))}. \tag{A.1}$$

Here $\theta = \theta(z)$ is the Riemann theta-function of a Riemann surface Γ of genus g with a marked symplectic basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$. The vector $w = (w_1, \ldots, w_g)$ denotes the normalized holomorphic differentials on Γ,

$$\oint_{a_k} w_j = 2\pi i \delta_{jk}.$$

The map

$$P: X \to \Gamma, \qquad x \mapsto P(x),$$

is an arbitrary map, $\varepsilon(P(x), P(y))$ is expressed via the prime-form of Γ with respect to arbitrary local parameters $z(x)$ on Γ (such that $z(x)|_{P(x)} = 0$)

$$E(P(x), P(y)) = \varepsilon(P(x), P(y))(dz(x))^{-1/2}(dz(y))^{-1/2}.$$

The vector z_0 is an arbitrary g-component complex vector with $\theta(z_0) \neq 0$, $v(x)$ is an arbitrary function, and k is an arbitrary constant. Conversely, for any Riemann surface of genus g, the formula (A.1) (together with (A.2)–(A.4) below) gives a solution of (0.10).

The functions $r(x, y)$, $p_k(x, y)$, $s_k(x)$ can be found from the functional equation (0.10) using formulae of [16]:

$$r(x, y) = -k\frac{d \log E(P(y), P(x))}{dz(y)} \tag{A.2}$$

$$s_k(x) = \left.\frac{w_k(P)}{dz(x)}\right|_{P=P(x)} \tag{A.3}$$

$$p_k(x, y) = -\frac{k}{2\pi i}\oint_{a_k} \frac{\theta(\int_{P(y)}^{P} w - z_0)\theta(\int_{P(x)}^{P} w + z_0)E(P(x), P(y))}{\theta(z_0)\theta(\int_{P(y)}^{P(x)} w - z_0)E(P, P(x))E(P, P(y))}. \tag{A.4}$$

The genericity assumption of the theorem means that the corresponding spectral curve is nonsingular. The coefficients of the equation of the spectral curve can be expressed via values of $q(x_i, x_j)$, $r(x_i, x_j)$, $s(x_i)$ in arbitrary $2g + 1$ fixed points $x_1, \ldots, x_{2_g+1}$. All solutions of the functional equation can be obtained from the above formulae by degenerating the Riemann surface Γ.

The proof of the theorem will be published elsewhere. We hope that the reader familiar with Riemann surfaces and their theta-functions can reconstruct the proof after reading our paper (the only new point in the proof is an application of Clifford and Riemann-Roch theorems [17] to calculate the genus of the spectral curve).

In order to explain the relation of (A.1) to Baker-Akhiezer functions (more precisely, to Baker-Akhiezer 1/2-bidifferentials in $P(x)$, $P(y)$), let us fix a generic point $P_\infty \in \Gamma$. There exists a unique differential Ω^{z_0} of the second kind on Γ with a pole of order $\leqslant g + 1$ in the point P_∞ and with zero a-periods, such that the vector of b-periods coincides with z_0:

$$\oint_b \Omega^{z_0} = z_0.$$

We put

$$\psi(P, Q; z_0) = \exp\left[\int_P^Q \Omega^{z_0}\right] \frac{\theta(\int_Q^P w - z_0)}{\theta(z_0)E(P, Q)}. \tag{A.5}$$

This is a single-valued 1/2-bidifferential in $P(Q) \in \Gamma \times \Gamma$ with an essential singularity in $P = P_\infty$ and $Q = P_\infty$ and with residue 1 on the diagonal $P = Q$. It is clear that this can be obtained from (A.1) for $P = P(x)$, $Q = P(y)$ by an appropriate choice of $v(x)$.

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