Three-Phase Solutions of the Kadomtsev–Petviashvili Equation

By B. A. Dubrovin, Ron Flickinger, and Harvey Segur

The Kadomtsev–Petviashvili (KP) equation is known to admit explicit periodic and quasiperiodic solutions with N independent phases, for any integer N, based on a Riemann theta-function of N variables. For $N = 1$ and 2, these solutions have been used successfully in physical applications. This article addresses mathematical problems that arise in the computation of theta-functions of three variables and with the corresponding solutions of the KP equation. We identify a set of parameters and their corresponding ranges, such that *every* real-valued, smooth KP solution associated with a Riemann theta-function of three variables corresponds to exactly one choice of these parameters in the proper range. Our results are embodied in a program that computes these solutions efficiently and that is available to the reader. We also discuss some properties of three-phase solutions.

1. Introduction and main results

In their original paper, Korteweg and deVries [1] derived an equation equivalent to

$$u_t + (3u^2)_x + u_{xxx} = 0 \qquad \text{(KdV)}$$

Address for correspondence: Professor Harvey Segur, University of Colorado, Boulder, CO 80309-0526.

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to describe approximately the slow evolution of long water of moderate amplitude as they propagate under the influence of gravity in one direction in shallow water of uniform depth. We now know that the KdV equation describes approximately the evolution of long, one-dimensional waves in many physical settings, including long internal waves in a density-stratified ocean, ion-acoustic waves in a plasma, acoustic waves on a crystal lattice, and more [2].

If one relaxes the restriction that the waves be strictly one-dimensional, then one often derives instead a natural generalization of KdV that was first discovered by Kadomtsev and Petviashvili [3],

$$u_{xt} + (3u^2)_{xx} + u_{xxxx} + 3u_{yy} = 0. \tag{KP}$$

Depending on the physical problem, one can derive one of two KP equations, which differ in the sign of their u_{xxxx}-terms. The equation given above is sometimes called KP2. In particular, this KP equation describes approximately the slow evolution of gravity-induced waves of moderate amplitude on shallow water of uniform depth when the waves are nearly one-dimensional. (For example, see [4] for a derivation of the KP equation in this setting.)

The KP equation admits a large family of exact quasiperiodic solutions. Each such solution has N independent phases. Recent comparisons with experiments [5–7] show that the family of two-phase solutions of the KP equation describes waves in shallow water with surprising accuracy. This success suggests that more complicated KP solutions might provide accurate physical models of more complex wave phenomena.

The purpose of this article is to develop a larger family of KP solutions, in order to make these solutions available as physical models. Specifically, we address mathematical problems arising in the computation of Riemann theta-functions of three variables and in their application to three-phase solutions of the KP equation. To make these three-phase solutions as accessible as possible, our results are encoded in a computer program that we provide to the reader.

The organization of this article is as follows. In this section, we review briefly what is known about one- and two-phase solutions of KP, and then we state our main results on three-phase solutions. These results form the basis of a computer program that permits one to specify the parameters of a three-phase KP solution and then to view that solution as it evolves in time. In Section 2, we discuss some three-phase solutions obtained in this way, and we explore some wave phenomena described by three-phase solutions. We show that three-phase solutions differ from one- and two-phase solutions in the following important way: almost every one- or two-phase solution is time independent in some uniformly translating coordinate system; almost every three-phase solution is time dependent, in every coordinate system. Thus, nontrivial time dependence is an essential feature of a three-phase solution, and viewing its time evolution is necessary to understand its behavior. We provide two methods for this viewing: either watching a video of specific solutions or accessing the program and creating one's own animations.

To simplify the presentation of results, technical details and mathematical proofs are deferred to a series of appendices to the article. Appendix A gives detailed instructions to run the program, which allows the reader to supplement the solutions discussed in Section 2. Subsequent appendices provide proofs of the theorems presented below.

All of the solutions considered in this article have zero mean:

$$\int_{-\infty}^{\infty} u(x,y,t)\,dx = 0. \tag{1.1}$$

Among the simplest KP solutions are periodic traveling waves. These are plane-wave solutions of the form

$$u(x,y,t) = U(\phi), \qquad \phi = kx + ly + \omega t + \phi_0, \tag{1.2}$$

where $U(\phi)$ is a 2π-periodic function and ϕ_0 is an arbitrary phase constant. These solutions can be expressed in terms of a Jacobian elliptic function

$$U(\phi) = 2\left(\frac{kK}{\pi}\right)^2\left(\kappa^2 cn^2\left[\frac{K}{\pi}\phi;\kappa\right] - \beta\right), \tag{1.3a}$$

where $cn[z;\kappa]$ is an elliptic function with modulus κ $(0 \le \kappa \le 1)$, and

$$\beta = \frac{E}{K} - 1 + \kappa^2; \tag{1.3b}$$

the wavenumbers and frequency are related by a nonlinear dispersion relation,

$$\frac{\omega k + 3l^2}{k^4} = 4\frac{K^2}{\pi^2}\left[3\frac{E}{K} - 2 + \kappa^2\right]; \tag{1.3c}$$

$K(\kappa)$ and $E(\kappa)$ are the complete elliptic integrals of the first and second kinds, respectively [8]. Solutions of this form (with $l = 0$) were named *cnoidal waves* by Korteweg and deVries [1]. A cnoidal wave is completely specified by three parameters (e.g., k, l, κ), plus one arbitrary phase constant (ϕ_0). Figure 1 shows a cnoidal wave solution of KP, for one choice of the parameters.

If $l = 0$, then these results reduce to those for the KdV equation. A cnoidal wave solution with $l \neq 0$ can be transformed into one with $l = 0$ by using the invariance of KP with respect to a two-parameter group of transformations of the form

$$\begin{aligned} &x \to bx + ab^2 y - 3a^2 b^3 t, \qquad y \to b^2 y - 6ab^3 t, \\ &t \to b^3 t, \qquad u \to b^{-2} u, \end{aligned} \tag{1.4}$$

for arbitrary real numbers $\{a, b\}$ with $b \neq 0$. Clearly, every KdV solution is a y-independent solution of KP. Conversely, for a y-independent solution, the KP equation can be integrated once in x. If the solution also satisfies (1.1), then the constant of integration must vanish, so the solution must satisfy KdV.

Alternatively, we can represent these solutions in terms of Riemann theta-functions:

$$u(x,y,t) = 2\partial_x^2 \log \theta(kx + ly + \omega t + \phi_0 + \pi; \kappa), \tag{1.5a}$$

where

$$\theta(\psi; \kappa) = 1 + 2\sum_{m=1}^{\infty} q^{m^2} \cos m\psi, \tag{1.5b}$$

$$q(\kappa) = \exp\left(-\pi \frac{K'(\kappa)}{K(\kappa)}\right) \qquad \text{and} \qquad K'(\kappa) = K\left(\sqrt{1-\kappa^2}\right). \tag{1.5c}$$

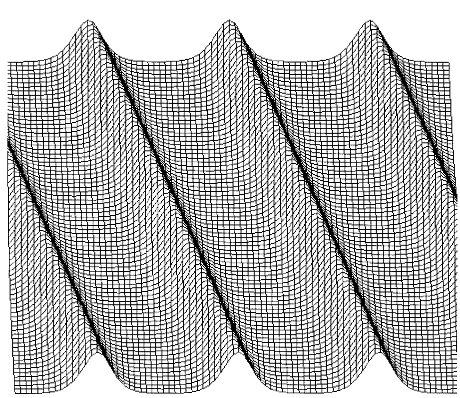

Figure 1. A cnoidal wave solution of the KP equation, with parameters: $\kappa^2 = 0.99$, $k = 1$, $l = 0.3$, $\omega = -1.2954$. Every cnoidal wave solution is one dimensional, and it is time independent in a uniformly translating coordinate system.

When $\omega k + 3l^2 - k^4 = 0$, then necessarily $\kappa = 0$, $K'/K = +\infty$, and $q = 0$, corresponding to the trivial solution $u \equiv 0$. (Observe that $\omega k + 3l^2 - k^4 = 0$ is just the dispersion relation for a plane-wave solution of the linearization of KP.) For a small negative $\varepsilon = \omega k + 3l^2 - k^4$, one obtains approximately a plane-wave solution,

$$u \sim A\cos(kx + ly + \omega t + \phi_0), \tag{1.6}$$

with the small amplitude $A = \sqrt{2|\varepsilon|/3}$.

More general *N-phase* solutions of KP have the form

$$u(x,y,t) = U(k_1 x + l_1 y + \omega_1 t + \phi_{01}, \dots, k_N x + l_N y + \omega_N t + \phi_{0N}), \tag{1.7}$$

for arbitrary constants $\{\phi_{01}, \dots, \phi_{0N}\}$, where the smooth function $U(\phi_1, \dots, \phi_N)$ is 2π-periodic in each variable separately (i.e., it is quasiperiodic). Construction of such multiphase solutions for any number of phases was first proposed by I. M. Krichever [9, 10] using algebro-geometric methods. A KP solution of the form (1.7) requires

$$U(\phi_1, \dots, \phi_N) = 2\partial_x^2 \log \theta(\phi_1, \dots, \phi_N | Z), \tag{1.8a}$$

where $\partial_x := \sum_{i=1}^N k_i(\partial/\partial\phi_i)$, and a theta-function of N variables is defined in terms of an N-fold Fourier series,

$$\theta(\phi_1, \dots, \phi_N | Z) = \sum_{m_1, \dots, m_N} c_{m_1, \dots, m_N} \exp\big(i(m_1\phi_1 + \cdots + m_N\phi_N)\big), \tag{1.8b}$$

the summation is over all choices of integers $\{m_1, \dots, m_N\}$, $i = \sqrt{-1}$, the Fourier coefficients have the form

$$c_{m_1, \dots, m_n} = \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} z_{ij} m_i m_j \right), \tag{1.8c}$$

and Z is an $N \times N$, symmetric, real, positive-definite matrix that we call the *period matrix* of the theta-function. The entries z_{ij} of the period matrix can be considered as free parameters of the theta-function. (To compare notation, we note that [4, 11] refer to a *Riemann matrix*, $B = -Z$.) Because Z is positive definite, the series in (1.8b) converges for arbitrary values of the phase variables $\{\phi_1, \dots, \phi_N\}$. However, we cannot in general distinguish the phases uniquely. Indeed, any linear transformation of the phases,

$$\phi_i' = \sum a_{ij} \phi_j \tag{1.9}$$

with integer coefficients a_{ij} and with $\det(a_{ij}) = 1$, gives another representation of the same solution with different wavenumbers and frequencies.

The period matrix Z, wave vectors $k = (k_1, \ldots, k_N)$, $l = (l_1, \ldots, l_N)$, and frequencies $\omega = (\omega_1, \ldots, \omega_N)$ are not all independent. An N-phase solution is specified by $3N$ independent parameters, plus N phase constants. The other parameters are related to these by a complicated system of "dispersion relations" that generalize (1.3c). We discuss these relations for three-phase solutions in Appendices G and H.

For the one-phase case, $N = 1$, we obtain just the traveling wave solution discussed above. For $N > 1$, if Z is exactly diagonal, then these formulae are degenerate and they do not give KP solutions. If Z is close to being diagonal with numerically large diagonal entries $\{z_{11}, z_{22}, \ldots, z_{NN}\}$, then the N-phase solution can be approximately represented as a sum of N small-amplitude waves:

$$u \sim -4 \sum_{j=1}^{N} \sqrt{\varepsilon_j}\, k_j^2 \cos \phi_j, \tag{1.10}$$

$$\begin{gathered} \varepsilon_j = \exp\{-z_{jj}\}, \qquad \phi_j = k_j x + l_j y + \omega_j t + \phi_{0j}, \\ \omega_j k_j + 3 l_j^2 \sim k_j^4, \qquad j = 1, 2, \ldots, N. \end{gathered} \tag{1.11}$$

The off-diagonal entries of the period matrix describe interactions between the phases. This result follows from the expansion of the theta-function:

$$\begin{aligned} \theta(\phi_1, \phi_2, \ldots, \phi_N) \sim\ & 1 + 2 \sum_{j=1}^{N} \sqrt{\varepsilon_j} \cos \phi_j \\ & + \sum_{i \neq j}^{N} \sqrt{\varepsilon_i \varepsilon_j} \left[e^{-z_{ij}} \cos\{\phi_i + \phi_j\} + e^{z_{ij}} \cos\{\phi_i - \phi_j\} \right]. \end{aligned} \tag{1.12}$$

When all $\varepsilon_j \to 0$, one obtains in the leading-order approximation:

$$z_{ij} \sim \log \left[\frac{k_i^2 k_j^2 (k_i + k_j)^2 + (k_i l_j - k_j l_i)^2}{k_i^2 k_j^2 (k_i - k_j)^2 + (k_i l_j - k_j l_i)^2} \right], \qquad i \neq j. \tag{1.13}$$

This interpretation fails when the period matrix, Z, is not almost diagonal.

Two-phase solutions of KP (i.e., $N = 2$) fall into two categories. If $l_1 k_2 - l_2 k_1 = 0$, then the solution is actually one dimensional, and a transformation of the form (1.4) with $\{a = l_1 / k_1 = l_2 / k_2, b = 1\}$ transforms it into a two-phase solution of KdV. Alternatively, if $l_1 k_2 - l_2 k_1 \neq 0$, then the solution is genuinely two dimensional, and it is spatially periodic in two independent directions in the $x-y$ plane. Two-phase solutions with $l_1 k_2 - l_2 k_1 \neq 0$ are time independent in a uniformly translating (or "Galilean") coordinate system,

$$u(x,y,t) = v(x + \xi t, y + \eta t), \tag{1.14a}$$

where

$$\xi = \frac{\omega_2 l_1 - \omega_1 l_2}{l_2 k_1 - l_1 k_2}, \qquad \eta = \frac{\omega_1 k_2 - \omega_2 k_1}{l_2 k_1 - l_1 k_2}. \tag{1.14b}$$

In other words, every two-phase solution of KP that is genuinely two dimensional has permanent form in an appropriately moving coordinate system. The spatial structure of this wave of permanent form can be found by solving one of three versions of the Boussinesq equation ($\sigma = \pm 1$, 0, or -1),

$$3\sigma v_{xx} = 3v_{yy} + v_{xxxx} + (3v^2)_{xx}, \tag{1.15}$$

because $v(x + \xi t, y + \eta t)$ satisfies one of these equations after a transformation of the form (1.4). We call a KP solution *stationary* if it is time independent in some Galilean coordinate system.

Figure 2 shows a two-phase solution of KP, for one choice of the parameters. The wave pattern in Figure 2 is spatially periodic, and the basic cell of the pattern is a hexagon: six steep wave crests form the edges of each hexagon and a broad wave trough fills each interior. The hexagon need not be regular, and that shown in Figure 2 is not a regular hexagon. However, the six crests surrounding a trough can be identified in pairs: opposite crests are parallel; they have equal amplitudes and equal lengths along the crests. The direction of propagation of the hexagon is not obvious from the figure itself, but it can be found from (1.14).

Every two-phase solution that is two-dimensional, like that in Figure 2, is spatially periodic in two directions, but it need not be periodic in either the x- or y-directions. A subset of the solutions that are periodic both in x and y were called *symmetric* solutions in [5]. A two-phase solution is specified by six $(= 3N)$ parameters, but a symmetric two-phase solution has only three independent parameters, because it requires $\{z_{11} = z_{22},\ k_1 = k_2,$ and $l_1 = -l_2\}$. Symmetric solutions propagate purely in the x-direction. An example is shown in Figure 3.

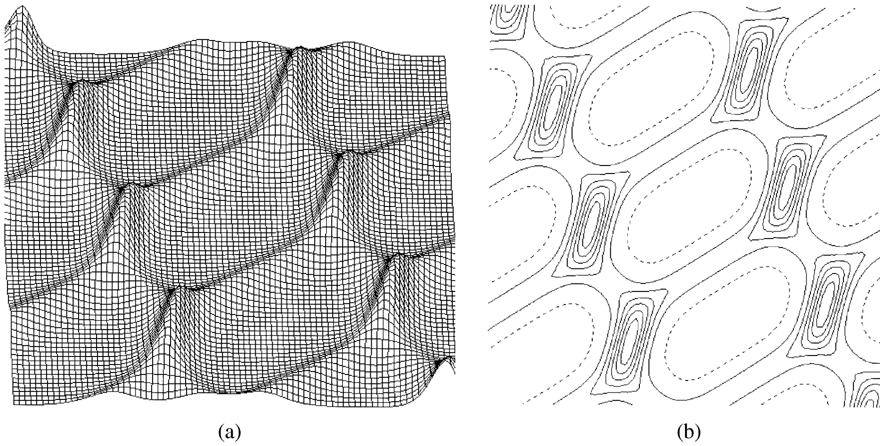

Figure 2. A two-phase KP solution, with parameters: $z_{11}=2$, $z_{12}=0.8$, $z_{22}=2.82$ (so $\alpha=2$, $\beta=2.5$, $\lambda=0.4$ in (1.16)), $k_1=0.6$, $k_2=0.8$, $l_1=0.2$, $l_2=-0.8059$, $\omega_1=-1.9065$, $\omega_2=-4.0238$. This solution is stationary, as are all two-phase solutions that are genuinely two-dimensional. (a) Perspective view of the solution. (b) Overhead view, with contour lines shown.

Two-phase solutions of KP were first computed in [4]. The results of these computations turned out to be in very good agreement with measurements from physical experiments on spatially periodic waves of permanent form in shallow water [5–7]. This good agreement suggests that *every* spatially periodic, two-phase solution of KP might well be of theta-function form, as

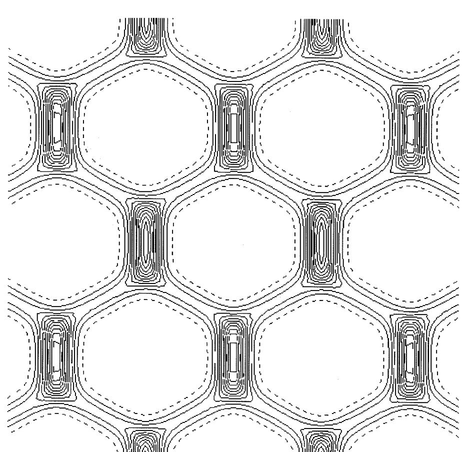

Figure 3. A symmetric two-phase solution, with parameters: $\alpha=2$, $\beta=1.68$, $\lambda=0.4$ (so $z_{11}=z_{22}=2$, $z_{12}=0.8$) $k_1=k_2=0.8$, $l_1=-l_2=0.6155175$, $\omega_1=\omega_2=-5.924798$. Every symmetric two-phase solution is periodic in x and in y, and it translates purely in the x-direction.

conjectured in [4]. The main goal of this article is computation of three-phase solutions of KP, along the lines of [4, 12–15]. As we explain below, three-phase solutions differ from two-phase solutions in two important respects.

(i) Every two-phase solution that is genuinely two dimensional is periodic in two spatial directions. A typical three-phase solution is not periodic in any direction; it is quasiperiodic in space. As a result, two-phase solutions can be obtained by solving the KP equation with appropriate periodic boundary conditions, but three-phase solutions generally cannot be obtained in this way.

(ii) Every two-phase solution that is genuinely two-dimensional is stationary (i.e., time independent in some Galilean coordinate system); almost every three-phase solution is time dependent, in every Galilean coordinate system. (See Theorem 4 for a precise statement of this assertion.) Thus, the three-phase solutions are among the simplest KP solutions that exhibit intrinsic time dependence.

We now summarize our main results about real-valued, three-phase solutions of the KP equation. A three-phase solution is determined by nine $(=3N)$ real parameters with dynamical significance, plus three $(=N)$ arbitrary phase constants $\{\phi_{01}, \phi_{02}, \phi_{03}\}$ [11]. Of the nine parameters, three are wavenumbers $\{k_1, k_2, l_1\}$, and six specify the period matrix:

$$Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{pmatrix} = \begin{pmatrix} \alpha & \alpha\lambda & \alpha\mu \\ \alpha\lambda & \alpha\lambda^2 + \beta & \alpha\lambda\mu + \beta\nu \\ \alpha\mu & \alpha\lambda\mu + \beta\nu & \alpha\mu^2 + \beta\nu^2 + \gamma \end{pmatrix}. \quad (1.16)$$

We use both parameterizations of the period matrix in what follows.

As discussed in Appendices B and C, a given period matrix can be transformed into another (equivalent) period matrix, so each theta-function has several equivalent representations, as does the corresponding KP solution. A *fundamental region* $\mathscr{D}$ is defined to be a closed set in the space of period matrices with two properties:

(i) Every period matrix is equivalent to some matrix in $\mathscr{D}$.

(ii) If two matrices in D are equivalent, then each belongs to the boundary of $\mathscr{D}$.

THEOREM 1. *A fundamental region of parameters of a real-valued theta-function of three variables is given by the following inequalities*:

$$\begin{aligned} &0 < z_{11} \le z_{22} \le z_{33}, \\ &0 \le 2z_{12} \le z_{11}, \qquad 0 \le 2z_{13} \le z_{11}, \\ &2|z_{23}| \le z_{22}, \qquad 2(z_{12} + z_{13} - z_{23}) \le z_{11} + z_{22}. \end{aligned} \quad (1.17)$$

In what follows, we refer to the set defined by (1.17) as *the fundamental region*. Its significance is that it contains a period matrix of every three-phase KP solution of interest. Theorem 1 is proved in Appendix D.

A second problem with theta-function representations of KP solutions is that some theta-functions give only trivial KP solutions. Specifically, a period matrix is said to be *decomposable* if it can be transformed into block-diagonal form,

$$Z = \begin{pmatrix} Z' & 0 \\ 0 & Z'' \end{pmatrix}, \tag{1.18}$$

and *indecomposable* otherwise. (See Appendix B for the set of allowable transformations.) A decomposable period matrix corresponds to a theta-function that factors into a product of two theta-functions with fewer variables and to a trivial KP solution.

THEOREM 2. *In the fundamental region, the only decomposable period matrices are in block-diagonal form. Therefore, if a period matrix lies in the fundamental region and if*

$$(\lambda\mu)^2 + (\lambda\nu)^2 + (\mu\nu)^2 > 0, \tag{1.19}$$

then the period matrix is indecomposable and the corresponding KP solution is nontrivial.

Theorem 2 is proved in Appendix E. (To compare terminology, we note that in [4] a period matrix for a two-phase solution of KP is said to be "in basic form" if it lies in the fundamental region and is indecomposable.)

Given an indecomposable period matrix, Z, and three wavenumbers, $\{k_1, k_2, l_1\}$, the next step is to compute the remaining wavenumbers and frequencies. Following [11], we show in Appendix H that if the period matrix is indecomposable, then k_3 satisfies an algebraic equation of degree 4; the coefficients in this equation depend on k_1 and k_2 and on the period matrix. After k_3 has been found, the wavenumbers (l_1, l_2, l_3) are determined up to the ambiguity

$$(l_1, l_2, l_3) \rightarrow \pm\left[(l_1, l_2, l_3) + a(k_1, k_2, k_3)\right], \tag{1.20}$$

where $\{a\}$ is an arbitrary parameter corresponding to that in (1.4). The ambiguity can be resolved by choosing a definite value of (say) l_1. After such a choice, the wavenumbers (l_1, l_2, l_3) are determined uniquely, up to a transformation

$$(l_1, l_2, l_3) \to -(l_1, l_2, l_3) + 2\frac{l_1}{k_1}(k_1, k_2, k_3). \tag{1.21}$$

The last step is the computation of the frequencies $(\omega_1, \omega_2, \omega_3)$. These are determined uniquely once the wavenumbers are fixed. The validity of these steps is assured by

THEOREM 3. *Let Z be a 3×3 indecomposable period matrix in the fundamental region. Then the quartic equation* (H.2) *has four one-parameter families of nonzero, real-valued solutions $k = (k_1, k_2, k_3)$, considered as curves on the real projective plane. For each such solution, the real-valued vectors $l = (l_1, l_2, l_3)$ and $\omega = (\omega_1, \omega_2, \omega_3)$ are determined uniquely, up to the ambiguity in* (1.20). *For each set of parameters obtained in this way,* (1.7) *and* (1.8) *provide a smooth, real-valued KP solution with three phases. This procedure generates all smooth, real-valued, three-phase KP solutions that can be expressed by* (1.7) *and* (1.8a) *in terms of Riemann theta-functions of three variables.*

Theorem 3 is part of a longer theorem (Proposition 2) that is stated in Appendix H and finally proved in Appendix I.

Once the free parameters of the solution have been chosen and the remaining parameters determined in this way, the KP solution is defined in terms of a multiple Fourier series according to (1.7)–(1.8). These series necessarily converge (because the period matrix is positive definite), but the convergence could be very slow. However, the equivalent representations of a theta-function allow us to write these series in more than one way. In Appendix F, we show that every period matrix in the fundamental region has a representation in which the multiple series converge quickly. It follows that theta-function representations of three-phase solutions of KP are always computationally efficient. These efficient representations are used in the computer program discussed in Appendix A.

Finally, we note that an alternative approach to the computation of multiphase KP solutions was proposed and implemented by Bobenko and Bordag [14]. In their approach, multiphase solutions are parameterized by configurations of circles in a plane, and the solutions themselves are computed in terms of certain Poincaré series. To our knowledge, the two approaches produce the same solutions. In fact, we used the numerical results in [14, 15] to validate our own computer program. A more refined comparison of the methods would be based on the computational efficiency of the two approaches, but such a comparison has not yet been undertaken.

2. Properties of three-phase solutions of the KP equation

2.1. Time dependence

THEOREM 4. *Let* $k=(k_1,k_2,k_3)$, $l=(l_1,l_2,l_3)$, $\omega=(\omega_1,\omega_2,\omega_3)$ *be the parameters of a real-valued KP solution with three phases.*

(i) *If*

$$k_1 l_2 - l_1 k_2 = k_1 l_3 - l_1 k_3 = k_2 l_3 - l_2 k_3 = 0, \tag{2.1}$$

then the solution is one dimensional. It can be transformed into a three-phase solution of KdV, using (1.4). *Every bounded three-phase solution of KdV is necessarily time dependent, in every Galilean coordinate system.*

(ii) *If any part of* (2.1) *is false, then the KP solution is genuinely two dimensional. The necessary and sufficient condition that such a solution be stationary* (*i.e.*, *time independent in some Galilean coordinate system*) *is that*

$$\det\begin{pmatrix} k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \\ \omega_1 & \omega_2 & \omega_3 \end{pmatrix} = 0. \tag{2.2}$$

(iii) *If* (2.2) *is false, then the three-phase solution is time dependent in every Galilean coordinate system. Moreover, then the KP solution is genuinely two dimensional and there is a Galilean coordinate system in which the solution is periodic in time.*

(iv) *Under transformations of the form* (1.4), *KP solutions that are stationary* (*or not*) *transform into other solutions that are stationary* (*or not*); *i.e.*, *stationarity is not affected by such transformations.*

It follows that a generic three-phase solution of KP is genuinely two dimensional, periodic in time in a uniformly translating coordinate system, and not stationary.

Proof of Theorem 4: (i) Given (2.1), using (1.4) with $\{a=-l_1/k_1=-l_2/k_2=-l_3/k_3,\ b=1\}$ transforms the solution into one that is y-independent. If it also satisfies (1.1), then it must satisfy KdV. But a KdV solution that is stationary satisfies a third-order ordinary differential equation. A bounded solution of this equation is an elliptic function, so it has one phase, not three. Therefore a three-phase solution of KdV cannot be stationary.

(ii), (iii) For definiteness, assume that $k_1 l_2 - k_2 l_1 \neq 0$. A three-phase solution of KP has the form

$$u(x,y,t) = U(k_1 x + l_1 y + \omega_1 t, k_2 x + l_2 y + \omega_2 t, k_3 x + l_3 y + \omega_3 t). \tag{2.3}$$

Under a Galilean change of coordinates,

$$x = x' + \xi t', \qquad y = y' + \eta t', \qquad t = t', \tag{2.4a}$$

with $\{\xi, \eta\}$ given by (1.14b), the solution becomes

$$u(x,y,t) = U(k_1 x' + l_1 y', k_2 x' + l_2 y', k_3 x' + l_3 y' + \Omega_3 t'), \tag{2.4b}$$

where

$$\Omega_3 = \det\begin{pmatrix} k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \\ \omega_1 & \omega_2 & \omega_3 \end{pmatrix} \Bigg/ \det\begin{pmatrix} k_1 & k_2 \\ l_1 & l_2 \end{pmatrix}. \tag{2.4c}$$

If (2.2) holds, then $\Omega_3 = 0$ and the solution is t'-independent. If (2.2) fails, then the solution is periodic in t', because it is constructed to be quasiperiodic in its three phases. (This observation is due to Martin Kruskal.) Any other Galilean change of coordinates would make one of the other two phases time dependent, so there is no Galilean coordinate system in which the solution is time independent.

(iv) Under a transformation of the form (1.4), wave vectors change according to

$$\begin{aligned} k_j &\to b^{-1} k_j, \qquad l_j \to b^{-2} l_j - ab^{-1} k_j, \\ \omega_j &\to b^{-3}\omega_j - 6ab^{-2} l_j - 3a^2 b^{-1} k_j. \end{aligned} \tag{2.5}$$

Substituting these into (2.2) shows that the determinant is changed by a factor of b^{-6} and is independent of a. Thus, a nonzero determinant remains nonzero under (1.4), and a nonstationary KP solution remains nonstationary. One can show that for a generic three-phase solution of KP, this determinant does not vanish (see Appendix I below). This completes the proof.

Figure 4 shows a three-phase solution of KP, for one choice of the parameters. The solution is time dependent, and the figure shows the solution at four different times. We now note some features of the solution.

(i) One can think of a three-phase solution of the KP equation as a nonlinear superposition of three independent single-phase waves, with the superposition specified by (1.7) and (1.8). In Figure 4a we have drawn three sets of parallel lines, corresponding roughly to the crests of three underlying plane waves. For this particular solution, it is evident that one of the three underlying plane waves is stronger than the other two.

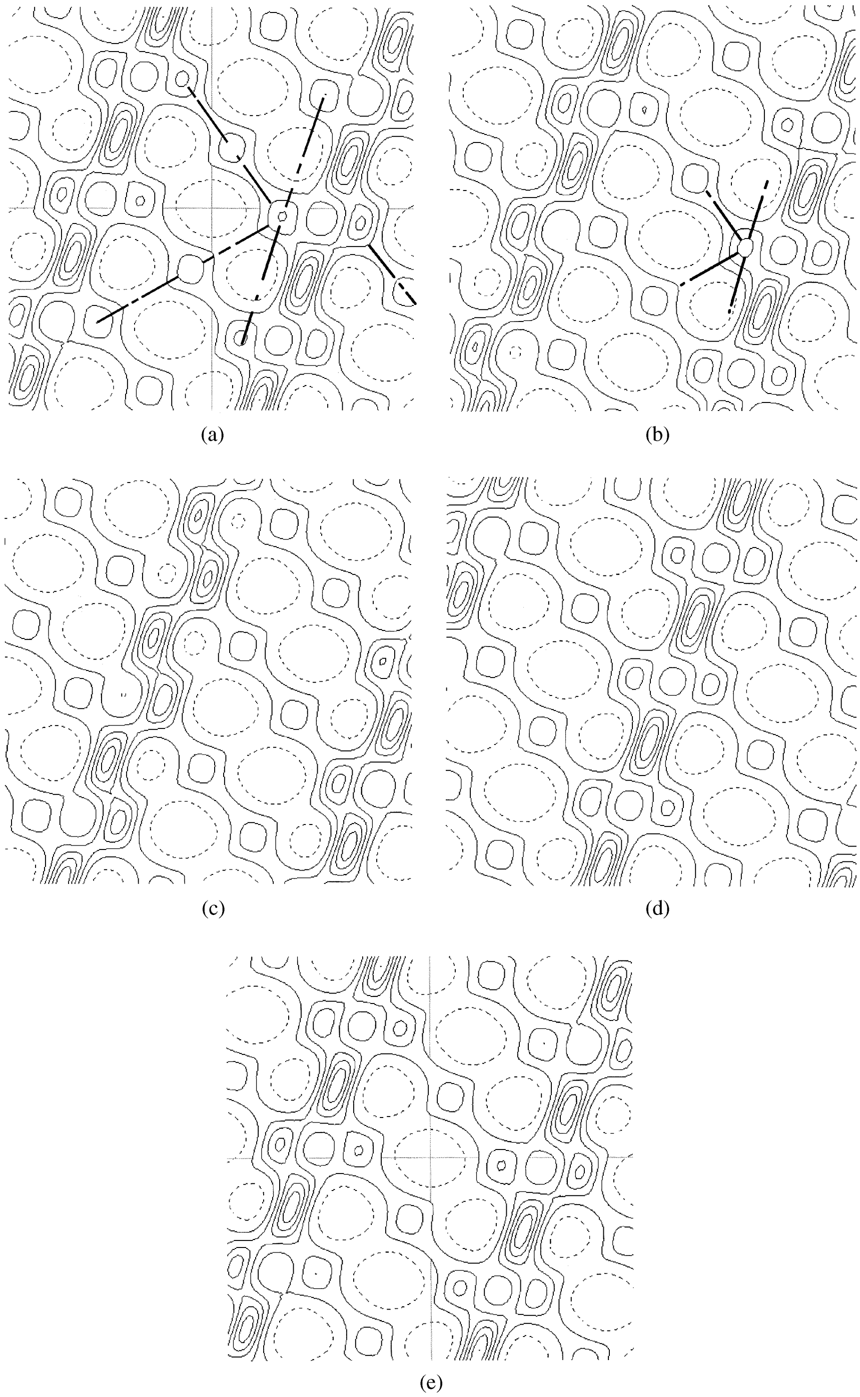

Figure 4. A three-phase KP solution, with parameters: $\alpha = 2$, $\beta = 4$, $\gamma = 4$, $\lambda = 0.5$, $\mu = 0.5$, $\nu = 0.1$, $k_1 = 0.5$, $k_2 = 1.0$, $k_3 = 1.2060$, $l_1 = -0.2$, $l_2 = -1.3974$, $l_3 = 0.6148$, $\omega_1 = -1.1427$, $\omega_2 = -6.2228$, $\omega_3 = -0.3940$, $+\sqrt{\ }$. This solution is periodic in a moving frame, with a period $T = 3.1908$. It is shown at four times: (a) $t = 0.0$, (b) $t = 0.5$, (c) $t = 1.5$, (d) $t = 3.1908$, (e) $t = 3.1908$, but translated according to (1.14).

(ii) The waves interact nonlinearly (according to KP), so each wave crest in Figure 4a undergoes a phase shift wherever it interacts with another wave crest. For this solution, the most obvious phase shifts are those experienced by the two weaker waves where they interact with the single strong wave.

(iii) At each time shown in Figure 4, one can find wave crests corresponding to those identified in Figure 4a. Thus, it is meaningful to discuss wave "crests" and "troughs" in this time-dependent solution. However, the amplitude of a wave is not uniform along a crest. Instead there are localized "peaks" where two (or three) underlying crests intersect. One such peak is identified in Figure 4a.

(iv) These peaks evolve in time as they propagate. For example, the peak identified in Figure 4a is also identified in Figure 4b, and one can see that it has grown larger in Figure 4b.

(v) The wave troughs also evolve in time as they propagate in space. Typical troughs are not hexagonal. A single wave trough (i.e., a shallow valley surrounded by mountains) can grow in size, or shrink, or disappear, or coalesce with a neighboring trough.

(vi) The entire solution is periodic (in time) in a moving coordinate system. The solution shown in Figure 4e is the same as that shown in Figure 4a, but Figure 4e is drawn one period later with x shifted by (2.25387) and y shifted by (-12.59596), in accord with (1.14b) and (2.4).

The solution shown in Figure 4 is time dependent, and one gains a better sense of its time evolution by watching it evolve in time, rather than by viewing a set of snapshots, as in Figure 4. Two methods to observe this evolution are available. First, a set of short videos, showing the time evolution of several three-phase solutions of KP, have been placed on the worldwide web at http://amath.colorado.edu/appm/other/kp/kp.html.

Second, Appendix A provides instructions to run the computer program (called `kp`) that we used to produce both the snapshots in Figure 4 and the videos mentioned above. The program is configured to run on any one of several UNIX platforms. To observe the time-dependent behavior of the particular solution in Figure 4, one needs to follow the instructions in Appendix A, using the parameters listed in the caption of Figure 4.

2.2. Nearly stationary solutions

As discussed above, a typical three-phase solution of KP is two dimensional, and it is not stationary. To our surprise, however, we found large families of nearly stationary solutions. These solutions are not strictly stationary (the determinant in (2.2) is not zero), but they appear to the eye to be stationary, and the determinant in (2.2) might differ from zero only in the second or third decimal place.

Figure 5 shows one such solution, at a particular time. As time changes, this solution appears simply to translate directly to the right (i.e., purely in the x-direction) with a constant speed. (The reader can observe this motion by viewing one of the videos mentioned above, or by running the `kp` program, using the parameter values listed in the caption to Figure 5.) In fact, the solution also evolves as it translates, but the nontranslational motion is very slow and very weak. Using (2.4c) for this solution, one obtains $\Omega_3 = 0.00487$, corresponding to a period of $T = 1290$ (instead of $T = 3.2$ for the solution in Figure 4).

Figure 6 shows another nearly stationary solution. As with that in Figure 5, this solution appears to translate purely in the x-direction. In this respect, these nearly stationary three-phase solutions are similar to symmetric two-phase waves, like that shown in Figure 3, but there is an important difference. Two-phase waves that are two-dimensional are spatially periodic, and the basic template of the periodic pattern is the hexagonal cell. If either of the wave patterns in Figures 5 and 6 is spatially periodic, then the basic template of the pattern must be much larger than the simple hexagonal cell. Both figures exhibit spatial patterns of hexagonal cells that vary in the x-direction.

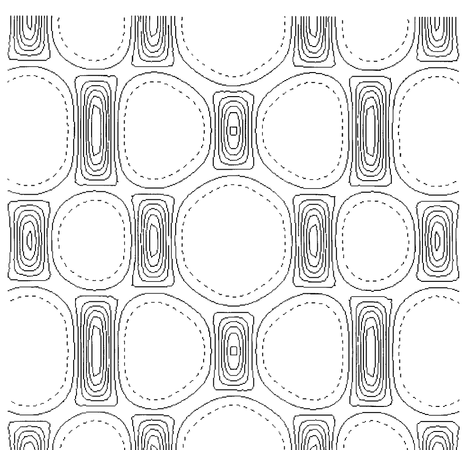

Figure 5. A three-phase KP solution that is nearly stationary, with parameters: $\alpha = 3$, $\beta = 63/25$, $\gamma = 81/35$, $\lambda = \mu = 0.4$, $\nu = 2/7$, $k_1 = k_2 = 1.0$, $k_3 = 0.574979$, $l_1 = -l_2 = 0.6412115$, $l_3 = 0.0$, $\omega_1 = \omega_2 = -3.260237$, $\omega_3 = -1.882164$, $+\sqrt{\ }$. The period matrix of this solution is completely symmetric: $z_{11} = z_{22} = z_{33} = 3$, $z_{ij} = 6/5$ for $i \neq j$. Consequently, there is another, identical solution with the same period matrix and with the phases renumbered: $k_1 = 0.574979$, $k_2 = k_3 = 1.0$, $l_1 = 0$, $l_2 = -l_3 = 0.6412115$.

All of the nearly stationary solutions that we found appear to translate purely in the x-direction. Their parameters fit into one of two categories:

(i) $z_{11} = z_{22} \le z_{33},\ z_{13} = z_{23},\ k_1 = k_2 = 1.0,\ l_1 = -l_2,\ l_3 = 0,\ \omega_1 = \omega_2$;
(ii) $z_{11} \le z_{22} = z_{33},\ z_{12} = z_{13},\ k_2 = k_3 = 1.0,\ l_1 = 0,\ l_2 = -l_3,\ \omega_2 = \omega_3$.

We had no trouble finding nearly stationary solutions within these categories. For each such solution, the wave pattern is periodic in y (with a spatial periodic of $2\pi/l_2$), but it is not periodic in x unless $\{k_1, k_2, k_3\}$ are rationally related. Even for those solutions that are periodic in x, the basic template of the pattern is not a simple hexagonal cell unless $k_1 = k_2 = k_3$. Every nearly stationary solution in one of these categories generates a two-parameter family of other nearly stationary solutions via (1.4). Whether nearly stationary solutions exist elsewhere in parameter space is unknown.

As stated above, two-dimensional two-phase solutions of KP have the identifying property that they are almost the only KP solutions that are stationary. However, the existence of large families of three-phase solutions that are nearly stationary means that the property of stationarity does not provide a practical means to identify two-phase solutions. A simple and effective method to identify the number of phases in a KP solution is unknown.

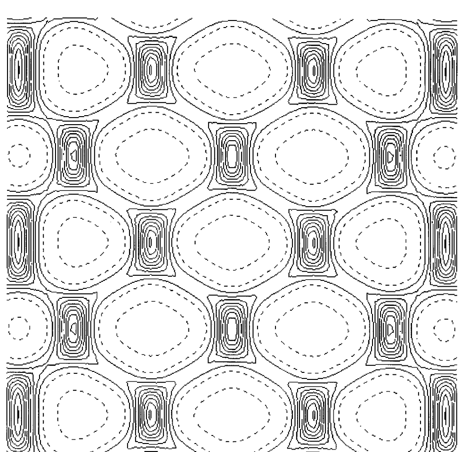

Figure 6. Another three-phase KP solution that is nearly stationary, with parameters: $\alpha = 3/2,\ \beta = 63/25,\ \gamma = 81/35,\ \lambda = \mu = 0.4,\ \nu = 2/7,\ k_1 = 0.3248453,\ k_2 = k_3 = 1.0,\ l_1 = 0.0,$ $l_2 = -l_3 = 0.818674,\ \omega_1 = -1.631909,\ \omega_2 = \omega_3 = -5.004516,\ -\sqrt{\ }$.

Appendix A. Running the kp program

1. Starting

When you enter the kp program, two windows open on the screen: a long "control panel" and a viewing window in which a picture of the solution appears. Separate these windows, and enlarge the viewing window if desired. (To do this, move the arrow to the lower right corner of the viewing window, then hold down the left button on the mouse while moving the corner to the desired location.)

2. Choosing parameters

Choose the parameters of the KP solution by moving the arrow into the appropriate input box of the control panel, and then changing the value in this box. Press "enter" after your changes.

(a) Choose values for $\{\alpha, \beta, \gamma, \lambda, \mu, \nu\}$. These determine the period matrix, Z, according to (1.16). Your choices must satisfy the inequalities in (1.17). For example: $\alpha = 3$, $\beta = 2.52$, $\gamma = 10$, $\lambda = \mu = 0.4$, $\nu = 4/7 = 0.285714286$, "enter".

Comment 1: The solution drawn in the viewing window changes each time you change a parameter of the solution, perhaps after a pause. To speed up the process, click on "Hold," then enter all changes, then click on "Hold" again.

(b) Choose nonzero values for k_1 and k_2. For example: $k_1 = 0.8$, $k_2 = 0.8$, "enter".

(c) The program now solves a quartic equation for k_3, so there are up to four real-valued choices for k_3. Go to the input box for k_3, hold down the left button, and up to four choices for k_3 will appear. Slide down to the desired choice and release the left button. For example: $k_3 = 0.459\ldots$.

Comment 2: Every real value of k_3 corresponds to a solution of the KP equation. However, very large values of $|k_3|$ correspond to solutions with short wavelengths and large amplitudes. In physical problems, the KP equation typically arises in the limit of small amplitudes and long wavelengths, so KP solutions with large values of $|k_3|$ are nonphysical and should be rejected. Moreover, the kp program itself does not draw solutions with very large values of $|k_3|$ accurately.

Comment 3: The kp program is designed to generate three-phase solutions of the KP equation, but it also generates two-phase solutions or single-phase cnoidal waves. This feature is discussed in more detail at the end of this appendix.

(d) Choose l_1. This corresponds to choosing the parameter "a" in (1.4). For example: $l_1 = 0.41045$, "enter".

(e) Choose "Sqrt (P_{ij})" by clicking on this button. This corresponds to the choice of a sign in equation (H.9). For example: $+$.

At this point you have selected the KP solution. All other choices affect how the solution is displayed in the viewing window, but not the solution itself.

3. Representing the solution

A theta-function of three variables has four representations, all of which involve nested infinite series, discussed in detail in Appendix F. The representations are:

1. A triple Fourier series;
2. A double Fourier series + a sum of "solitary" waves;
3. A single Fourier series + a double sum of solitary waves;
4. A triple sum of solitary waves.

Depending on the period matrix, the series in the different representations can have very different convergence rates, so it can be computationally efficient to use one representation instead of another. "Recommended Form = 3" means that for the period matrix chosen, representation 3 is the most efficient way to represent the solution in a neighborhood of $x = 0, y = 0, t = 0$, and the minimal number of terms in each series (m_1, m_2, m_3) is shown. The total number of terms in the nested series is: $\{2m_1 + 1\} * \{2m_2 + 1\} * \{2m_3 + 1\}$. These numbers are chosen so that θ and its derivatives are computed at $\{x = y = t = 0\}$ accurately to the level of ϵ in the control panel. (Independently, the theta-constants are computed to an accuracy of 10^{-20}). The kp program uses as many terms as it needs to achieve this accuracy. The tolerance (and hence the number of terms) for theta-constants cannot be seen or altered by the user.) For θ itself, you may choose any of the four representations (by clicking in "Displayed Form") and you may choose the number of terms in each series. This choice of representation affects only the display. It does not affect the (double precision) accuracy of the parameters in the control panel.

Comment 4: The recommended form always provides an accurate representation of the KP solution with relatively few terms in each series. For the same solution, the three representations that are not recommended might require *many* more terms to achieve the same accuracy.

Comment 5: Only representation 1 is uniformly valid in space time. Each of the other representations has a limited region of validity. You can see this in the example being developed in this appendix by using form 3 (which is recommended), then by clicking "Recommended," and then comparing the representation of the solution shown with that obtained after clicking on "222," which has more terms in its series. The "Recommended" picture is accurate near $x = 0, y = 0, t = 0$, but not far away from it. In this case, one should use "222" or higher. In fact, it is rarely wise to go below "222." For example: use form 3, 222.

4. Viewing the solution at a fixed time

(a) Everything is drawn on a mesh (that you often do not see). To increase the accuracy of the figure, increase the number shown in the "Mesh" input box. However, higher mesh resolution implies slower computation, so keep this number low while you are deciding what you want to see. For example: mesh = 35, "enter".

(b) The height of the solution surface can be displayed in different ways: by using contour lines, or a mesh, or shading, or any combination of the three. If you want to see contour lines drawn in, click on the "Contour (c)" button. (For the buttons at the bottom of the control panel, up = off, down = on.) The contours may look ragged. If you increase the "Mesh" (e.g., $35 \to 50$, "enter"), the contour lines smooth out.

(c) The input box marked "Contours" gives the spacing between contour lines. When the contouring option is on, zero is always the value of one contour line. If you want more contour lines, increase the resolution in this input box, then press "enter." As with the mesh, more resolution means less speed.

(d) Click on the "Mesh" button to see the mesh superimposed on the figure. The mesh becomes useful when the figure is viewed as a graphic projection. To do this, move the arrow from the control panel to the viewing window. Now the perspective can be changed in several ways.

(i) Hold down the left button on the mouse and move it north, south, east, or west. The picture rotates as you move the mouse (and "Mesh" becomes useful).

(ii) Hold the center button down and move the mouse again. The picture translates.

(iii) Hold the right bottom down, move the mouse, and rotate the picture about an axis through the origin, straight out of the screen.

Comment 6: As seen originally, the x-direction is to the right. Once you use (iii), then the x-axis is wherever you put it.

Comment 7: To see the coordinate axes in the figure, click on the "Axes" button.

5. Time dependence

All of the pictures shown so far have been at whatever time is shown in the input box marked "t" (probably $t = 0$). To view the same solution at a different time, enter the desired time in the "t"-input box, and press "enter."

Comment 8: For every representation except 1, one can choose "t" large enough that the figure drawn in the viewing window is quite inaccurate unless the number of terms in its series is increased. One can check for this possibility by increasing (by one) the number of terms in each series, and seeing whether the figure changes noticeably.

Alternatively, go to the "Δt"-input box, enter a value for Δt, and press "enter." The "proper" size of Δt depends on the values of $\{\omega_1, \omega_2, \omega_3\}$ in the solution in question. Now the solution can be updated in increments of Δt, in two different ways.

(i) To increase t by Δt, click on "step," then wait for the picture to change.

(ii) To watch a solution evolve in time (i.e., to observe a sequence of snapshots, Δt apart), click on "pause." To stop the animation, click again on "pause."

Comment 9: If the animation goes too slowly, increase the computational speed by decreasing mesh size, or by turning off "Shade," or by turning off "Contour," etc. ("Shade" is slowest, then "Contour"; "Mesh" is relatively fast.)

Comment 10: In the animation mode, commands are sometimes carried out slowly. *Do not* enter a command a second time.

6. Other options available in the kp program:

(a) If the input box marked "XY Range" shows "10," then the solution is computed in a square: $-10 \le x \le 10, -10 \le y \le 10$. To view the solution in a larger or smaller square, change the number in "XY Range," and press "enter."

(b) By changing the number in the "Scale" input box, one can magnify or shrink the figure drawn. (To change "Scale" interactively, hold down both the shift key and the middle button on the mouse, and move the mouse forward or backward.)

(c) By changing the number in the “Z Scale” input box, you can magnify or shrink the vertical scale (only) of the figure. (To change “Z Scale” interactively, hold down both the alt key and the middle button on the mouse, and move the mouse forward or backward.)

Comment 11: A transformation of the form (1.4) with $\{a = 0, b \neq 1\}$ also rescales the KP solution, without distorting the figure. In a sense, this is an alternative to “Z Scale.”

(d) Clicking on the “Rendering” button opens up another control panel, from which you can control colors in the figure, features of shading, and some other options shown in the control panel.

(e) The input boxes marked $\{\phi_{01}, \phi_{02}, \phi_{03}\}$ allow for changes in the three-phase constants in (1.7).

(f) “Reset View” returns the viewing screen to its original configuration, with the picture centered. The viewer is above the surface, looking directly down on it.

(g) Once you have found a suitable set of parameters, “Save Parameters” allows you to store the list of parameters in a file that you name. Use “Load Parameters” to retrieve the list from the file.

(h) To save the figure itself (instead of the parameters that generated the figure), use “Save Picture As.” The figure can be saved in any one of several formats.

(i) “Print Picture” pipes a Postscript version of the figure to the UNIX command specified in the input box. You may choose whether to invert the colors in the saved figure.

(j) “Make Movie” allows you to save a sequence of figures, with a temporal spacing of Δt, to create a movie. Two formats are available: GIF and FLI. A program to view the movie, called “xanim,” is available at http://www.portal.com/podlipec/home.html.

(k) “Dump Solution” writes the values of the solution on a grid to a file that you name.

(l) “Options” opens another control panel with several options that can be displayed in a window as the program runs, for debugging purposes. “M” is the coefficient matrix implicit in (H.1); “d4x(theta-hat)” is the matrix of fourth derivatives of the theta-constants, also in (H.1). “Quartic” gives the coefficients of the quartic equation defined by (H.2). “Pij” are defined by (H.8).

7. Constructing one- and two-phase solutions

The `kp` program can be used to generate approximate two-phase solutions of the KP equation or single-phase cnoidal waves. To obtain a two-phase solution, let $\gamma \to \infty$ with k_3 bounded. To obtain a cnoidal wave, let $\beta \to \infty$

and $\gamma \to \infty$ while satisfying the constraints in (1.17) with (α, k_2, k_3) all bounded.

In practice, for $0 < \alpha \le \alpha\lambda^2 + \beta \le 5$, it is usually sufficient to take $\gamma \ge 30$ (with a similar condition on β when generating single-phase waves). To decide whether γ (or β) is large enough, check whether the relevant parameters of the solution (such as ω_1) change when γ (or β) is increased further.

If γ is large enough then the approximate two-phase solution does not depend (to within numerical precision) on $\{\mu, \nu$, and the choice of $\sqrt{\ }\}$, all of which should affect primarily the third phase. However, the kp program cannot construct some two-phase solutions unless these "irrelevant" parameters lie in a particular range. Specifically, the program requires that k_3 be real valued, and even though the numerical value of k_3 does not affect the two-phase solution, the program stops if k_3 is not real. Consequently, if the program refuses to construct a one-phase or two-phase solution, it may be necessary to adjust the "irrelevant" parameters.

The KP solutions shown in Figures 1, 2, and 3 were obtained from kp in this way. The input values used for these solutions are the following:

(a) Figure 1: $\alpha = 2.677326$, $\beta = 25$, $\gamma = 30$, $\lambda = \mu = 0.5$, $\nu = 0.2$, $k_1 = k_2 = 1.0$, $k_3 = 1.09$, $l_1 = 0.3$, $-\sqrt{\ }$.

(b) Figure 2: $\alpha = 2$, $\beta = 2.5$, $\gamma = 30$, $\lambda = \mu = 0.4$, $\nu = 0.2$, $k_1 = 0.6$, $k_2 = 0.8$, $k_3 = 0.3535$, $l_1 = 0.2$, $+\sqrt{\ }$.

(c) Figure 3: $\alpha = 2$, $\beta = 1.68$, $\gamma = 30$, $\lambda = \mu = 0.4$, $\nu = 0.2$, $k_1 = k_2 = 0.8$, $k_3 = 3.59$, $l_1 = 0.6155175$, $+\sqrt{\ }$.

Appendix B. Basic properties of multidimensional theta-functions

This appendix contains basic information about theta-functions of several variables. More details can be found in [11, 16]. A general theta-function of N variables is defined in terms of an N-fold Fourier series

$$\theta(\phi_1, \ldots, \phi_N | Z) = \sum_{m_1, \ldots, m_N} c_{m_1, \ldots, m_N} \exp\big(i(m_1\phi_1 + \cdots + m_N\phi_N)\big), \quad \text{(B.1)}$$

where the summation is over all choices of integers $\{m_1, \ldots, m_N\}$, $i = \sqrt{-1}$, the Fourier coefficients have the form

$$c_{m_1, \ldots, m_n} = \exp\left(-\frac{1}{2}\sum_{i,j=1}^{N} z_{ij} m_i m_j\right), \quad \text{(B.2)}$$

and $Z=(z_{ij})_{1\le i,j\le N}$ is an $N\times N$ symmetric, complex-valued matrix with positive-definite real part:

$$Z = Z^T, \mathrm{Re}\{\mathbf{v}^T\cdot Z\cdot\mathbf{v}\} > 0, \text{ for all real-valued } N\text{-component vectors } \mathbf{v} \neq 0. \tag{B.3}$$

Z is the *period matrix*, and its entries z_{ij} can be considered to be the parameters of the theta-function. The space of all period matrices is called the *Siegel* (right) *half-plane*. The series in (B.1) defines an entire function of N complex variables $\{\phi_1,\ldots,\phi_N\}$, and it is 2π-periodic with respect to any of the ϕ_j.

There are natural identifications in the space of parameters of theta-functions. Two period matrices Z and Z' are called *equivalent* if they are related by a *Siegel modular transformation*

$$Z' = -2\pi i(AZ-2\pi iB)(CZ-2\pi iD)^{-1}. \tag{B.4}$$

Here the $2N\times 2N$ matrix, $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, must belong to the group $\mathrm{Sp}(N,\mathbf{Z})$. This means that all the entries of the $N\times N$ matrices A,B,C,D are integers and these matrices satisfy the condition

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\begin{pmatrix} A & B \\ C & D\end{pmatrix}^T = \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix} \tag{B.5a}$$

(I is the identity matrix) or equivalently

$$AB^T = BA^T, \qquad AD^T - BC^T = I, \qquad CD^T = DC^T. \tag{B.5b}$$

For equivalent period matrices the corresponding theta-functions coincide up to an appropriate change of arguments $(\phi_1,\ldots,\phi_N)=\phi$ and multiplication by an exponential of a quadratic form,

$$\theta(\phi_1',\ldots,\phi_n'|Z') = \exp\left[\frac{1}{2}\phi(CZ-2\pi iD)^{-1}C\phi^T - \frac{1}{2}\beta Z\beta^T - i\alpha\phi^T + is\right]$$

$$\cdot\sqrt{\det(CZ-2\pi iD)}\,\theta(\phi_1,\ldots,\phi_n|Z), \tag{B.6a}$$

where

$$\phi' = (\phi'_1,\dots,\phi'_n) = -2\pi i\phi(CZ-2\pi iD)^{-1} + 2\pi\alpha + i\beta Z' \quad \text{(B.6b)}$$

$$\alpha = (\alpha_1,\dots,\alpha_n), \qquad \beta = (\beta_1,\dots,\beta_n) \quad \text{(B.6c)}$$

$$\alpha_k = \frac{1}{2}\sum_{j=1}^{n} a_{kj}b_{kj} \quad \text{(B.6d)}$$

$$\beta_k = \frac{1}{2}\sum_{j=1}^{n} c_{kj}d_{kj} \quad \text{(B.6e)}$$

$$A = (a_{ij}), \qquad B = (b_{ij}), \qquad C = (c_{ij}), \qquad D = (d_{ij}). \quad \text{(B.6f)}$$

The constant s does not depend on Z or on ϕ (see [16]). We call these two *equivalent theta-functions*. Equivalent theta-functions give essentially the same solution of the KP equation.

We restrict our attention to the theta-functions with real period matrices Z. They are real valued for real arguments $\phi_1,\dots,\phi_n$. For brevity we call them *real theta-functions*. For real theta-functions we consider equivalencies of theta-functions with respect to the subgroup[1] of $\mathrm{Sp}(N,\mathbf{Z})$ of transformations of the form

$$Z \mapsto Z' = AZA^T, \qquad \begin{pmatrix} A & 0 \\ 0 & A^{T^{-1}} \end{pmatrix} \in \mathrm{Sp}(N,\mathbf{Z}). \quad \text{(B.7)}$$

This coincides with action of the group GL $(N,\mathbf{Z})$ of invertible $N \times N$ integer matrices (the determinant of these matrices equals ± 1). Following Minkowski [17], we call the two matrices Z', Z related by the transformation (B.7) *arithmetically equivalent*. The law (B.6a)–(B.6f) of transformation of the theta-functions becomes for (B.7) very simple:

$$\theta(\phi'_1,\dots,\phi'_N \mid Z') = \theta(\phi_1,\dots,\phi_N \mid Z), \quad \text{(B.8a)}$$

$$(\phi'_1,\dots,\phi'_N) = (\phi_1,\dots,\phi_N)A^T. \quad \text{(B.8b)}$$

As motivation for considering here only real theta-functions we refer to the theory of the KP2 equation[2] where only real theta-functions give real-valued

[1] The transformations (B.7) are not the only ones that preserve reality of the period matrix. For example, the inversion $Z' = Z^{-1}$ (one should put, in (B.4), $A = D = 0$ and $B = -C = I$) also preserves reality. We motivate our restriction of the class of general Siegel modular transformations (B.4) to the transformation (B.7) by the algebraic-geometrical theory of KP2: according to this theory (see below), Z should be the matrix of periods of holomorphic differentials on a Riemann surface of genus N, with a real structure (i.e., with an antiholomorphic involution). The corresponding basic a-cycles on the Riemann surface should coincide with ovals of this involution. Ambiguity in the choice of a-cycles precisely gives the ambiguity (B.7) in the period matrix.

[2] In the theory of the KP1 equation, other types of real theta-functions also occur [11, 18]. We do not consider them in this article.

smooth solutions that we construct in this article for $N=3$ (see also Appendix I below).

Appendix C. Need for a fundamental region

Theorem 1 identifies a fundamental region in the space of 3×3 period matrices. This appendix explains why such a region is needed: Without a fundamental region, it is virtually impossible to calculate theta-functions effectively. This difficulty might seem surprising, since the Fourier coefficients in the theta-series (1.8), (B.1) eventually decrease faster than exponentially. Moreover, in the case $N=1$ (elliptic theta-functions of Jacobi), the coefficients decrease monotonically, and it is easy to estimate how many terms are needed to achieve whatever accuracy is desired.

For $N\geq 2$, however, computation is a nontrivial problem, as we now demonstrate. Take the 2×2 matrix with entries

$$z_{11} = 111.207, \qquad z_{12} = 96.616, \qquad z_{22} = 83.943. \tag{C.1}$$

The first few Fourier coefficients are as follows:

$$\begin{aligned} &c_{0,0}=1,\ c_{0,1}=c_{0,-1}=10^{-114},\ c_{1,0}=c_{-1,0}=10^{-152},\\ &\quad c_{1,1}=c_{-1,-1}=10^{-530},\dots. \end{aligned} \tag{C.2}$$

Nevertheless, the theta-function is far from being identically 1:

$$\begin{aligned} \theta(\phi_1,\phi_2) \cong\ &1+0.41\cos(7\phi_1-8\phi_2)+0.11\cos(6\phi_1-7\phi_2)\\ &+0.11\cos(13\phi_1-15\phi_2) \end{aligned}$$

(the truncation error is less than 10^{-2}).

It is easy to understand how to find the relevant coefficients $c_{m_1\dots m_N}$ of the theta-series in (1.8c): the coefficient exceeds a given positive ϵ if and only if the integer vector $(m_1,\dots,m_N)$ is inside of the ellipsoid defined by the inequality

$$\sum_{i,j=1}^{N} z_{ij}m_im_j \leq 2\log 2\epsilon^{-1}. \tag{C.3}$$

In the above example, the ellipsoids (in fact, ellipses) are squeezed along one of the axes. So the points $(0,0)$, $\pm(7,-8)$, $\pm(6,-7)$, $\pm(13,-15)$ of the integer lattice give the main contributions in the theta-series. In general, a numerical code for computation of the theta-series with a given period matrix must contain an algorithm for finding integer solutions $m_1,\dots,m_N$ of the inequality (C.3) for a given period matrix (z_{ij}) and a given accuracy ϵ. The significance of the fundamental region identified in Theorem 1 is that for a period matrix in this region, the lowest terms in the series (i.e., the points closest to the origin) satisfy the inequality (C.3).

A second difficulty in the computation of theta-functions occurs because two different period matrices that are equivalent under (B.7) lead to the same solution of the KP equation. To eliminate this ambiguity, we need a *fundamental region* for the action (B.7) of the group $\mathrm{GL}(N,\mathbf{Z})$ in the space of period matrices. This is defined to be a closed set $\mathscr{D}$ in the space of period matrices that satisfies the following two requirements.

(i) Every period matrix is equivalent to some matrix belonging to $\mathscr{D}$.
(ii) If two matrices in $\mathscr{D}$ are equivalent then each belongs to the boundary of $\mathscr{D}$.

In other words, we need to describe a canonical form to which one can reduce any positive definite symmetric matrix Z by the transformations (B.7), where all the entries of the invertible matrix A are integers. (In [19], this canonical form was called "basic form".) The basis of the theory of reduction was created by Lagrange, Hermite, and Minkowski (see [20]). Here we give an explicit description of the fundamental region for the case $N=3$.

Example: For real theta-functions of two variables, the fundamental region of the parameters z_{11}, z_{12}, z_{22} was described in [4, 19]. This is specified by the inequalities

$$0 < z_{11} \le z_{22} \tag{C.4a}$$

$$0 \le 2z_{12} \le z_{11}. \tag{C.4b}$$

(A quadratic form $z_{11}p^2 + 2z_{12}pq + z_{22}q^2$ satisfying (C.4a)–(C.4b) is sometimes called *Lagrange reduced* [20].) On the boundary of the domain, at least one of the inequalities (C.4a)–(C.4b) becomes equality.

A fundamental region is not determined uniquely. The particular region (C.4a)–(C.4b) is a convenient one for calculation of theta-series. Indeed, for the example in (C.1), applying a transform (B.7) with

$$A = \begin{pmatrix} 7 & -8 \\ -6 & 7 \end{pmatrix},$$

one obtains the following parameters of the transformed theta-function:

$$z'_{11} = 0.503, \qquad z'_{12} = 0.250, \qquad z'_{22} = 0.915.$$

The main contributions in the corresponding theta-series come from the points closest to the origin:

$$\theta(\phi'_1, \phi'_2) \cong 1 + 0.41\cos\phi'_1 + 0.11\cos\phi'_2 + 0.11\cos(\phi'_1 - \phi'_2). \quad \text{(C.5)}$$

Similarly for $N = 3$, the problem is to find a "good" fundamental region on which the summation of the theta-series can be done easily, i.e., with the property that the first few terms of the theta-series dominate the rest of the series.

After this brief discussion of the problem, let us formulate our first main result.

THEOREM 1. *The fundamental region of parameters of real theta-functions of three variables is given by the following inequalities*:

$$0 < z_{11} \leq z_{22} \leq z_{33} \tag{C.6a}$$

$$0 < 2z_{12} \leq z_{11} \tag{C.6b}$$

$$0 < 2z_{13} \leq z_{11} \tag{C.6c}$$

$$2|z_{23}| \leq z_{22} \tag{C.6d}$$

$$2(z_{12} + z_{13} - z_{23}) \leq z_{11} + z_{22}. \tag{C.6e}$$

This theorem is proved in Appendix D.[3]

[3] A description of the fundamental region announced in [21, Theorem 2] seems to be wrong. Indeed the two positive-definite matrices

$$B = \begin{pmatrix} 1 & 0.1 & 1 \\ 0.1 & 1 & 0.1 \\ 1 & 0.1 & 2 \end{pmatrix}, \qquad B' = \begin{pmatrix} 1 & 0.1 & 0 \\ 0.1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

both belong to the fundamental region of [21], and moreover B belongs to its inner part. But they are equivalent: $B' = ABA^T$ for

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Appendix D. A fundamental region of parameters for real theta-functions of three variables

The theory of fundamental regions for the group SL(N,**Z**) of unimodular $N \times N$ matrices with integer entries acting as in (B.7) on the space of positive−definite symmetric matrices was created by Minkowski [17] (see also [20]). We need to formulate his result to prove our Theorem 1.

For a given symmetric $N \times N$ matrix $Z = z_{ij}$, let us denote by $Z(m_1,\dots,m_N)$ the positive−definite quadratic form

$$Z(m_1,\dots,m_N) = \sum_{i,j=1}^{N} z_{ij} m_i m_j. \tag{D.1}$$

We say that the quadratic form is *Minkowski reduced* (or, briefly, *M-reduced*) if the following infinite set of inequalities holds true:

$$z_{11} = Z(1,0,\dots,0) \le Z(m_1,\dots,m_N), \qquad M_1(m_1,\dots,m_N)$$

for all relatively prime $m_1,\dots,m_N$

$$z_{22} = Z(0,1,\dots,0) \le Z(m_1,\dots,m_N), \qquad M_2(m_1,\dots,m_N)$$

for all m_1 and relatively prime $m_2,\dots,m_N$

$$\vdots$$

$$z_{N-1,N-1} = Z(0,0,\dots,1,0) \le Z(m_1,\dots,m_N), \qquad M_{N-1}(m_1,\dots,m_N)$$

for all $m_1,\dots,m_{N-2}$, and relatively prime m_{N-1} and m_N

$$z_{NN} = Z(0,0,\dots,0,1) \le Z(m_1,\dots,m_N), \qquad M_N(m_1,\dots,m_N)$$

for all $m_1,\dots,m_{N-1}$, and $m_N = \pm 1$.

For every choice of $\{m_1,\dots,m_N\}$, each of the inequalities $M_k(m_1,\dots,m_N)$ is a linear inequality among the entries of the matrix z_{ij}.

THEOREM OF MINKOWSKI [20]. *The set of all M-reduced positive-definite, symmetric, $N \times N$ matrices is a fundamental region of the group SL(N,**Z**)* *acting as in* (B.7) *in the space of real, symmetric, positive-definite* $N \times N$ *matrices. In other words,*

(i) *every Z-matrix is equivalent under* (B.4) *to some matrix in this set, and*
(ii) *if two matrices in this set are equivalent under* (B.4), *then for each matrix, at least one inequality* $M_k(m_1, \ldots, m_N)$ *must be an equality.*

Moreover, the fundamental region is specified by a finite number of the inequalities $M_1(m_1, \ldots, m_N), \ldots, M_N(m_1, \ldots, m_N)$.

EXAMPLE: For $N = 2$ we have the following three basic Minkowski inequalities:

$$
\begin{aligned}
z_{11} &\le z_{22}, && M_1(0,1) \\
-z_{11} &\le 2z_{12}, && M_2(1,1) \\
2z_{12} &\le z_{11}, && M_2(1,-1).
\end{aligned}
$$

Other inequalities $M_1(m_1, m_2), M_2(m_1, m_2)$, follow from these three. This gives a fundamental region for the group $\mathrm{SL}(2, \mathbf{Z})$ acting in the space of symmetric positive-definite 2×2 matrices. Note that the condition $0 < z_{11}$ together with the above inequalities provides positive definiteness of the Z-matrix.

Note also that the group of transformations is slightly bigger than $\mathrm{SL}(2, \mathbf{Z})$: We are allowed, particularly, to change the sign of any of the basic vectors. Using this we can always make z_{12} nonnegative. This gives the fundamental region (C.4a)–(C.4b) of parameters of real theta-functions of two variables.

Using Minkowski's theorem it is easy to obtain an algorithm to reduce any 2×2 symmetric positive-definite matrix to the M-reduced form.

Let us come to the case of three variables. Proof of Theorem 1 is based on two lemmas.

LEMMA 1. *Let*

$$
Z(p,q,r) = z_{11}p^2 + 2z_{12}pq + z_{22}q^2 + 2z_{13}pr + 2z_{23}qr + z_{33}r^2 \quad \text{(D.2)}
$$

be a positive-definite quadratic form with

$$
0 \le z_{12}, \qquad 0 \le z_{13}. \quad \text{(D.3)}
$$

Then the infinite set of Minkowski inequalities,

$$z_{11} \le Z(p,q,r) \text{ for all relatively prime } p,q,r, \qquad M_1(p,q,r)$$

$$z_{22} \le Z(p,q,r) \text{ for all } p,q,r \text{ with relatively prime } q,r, \qquad M_2(p,q,r)$$

$$z_{33} \le Z(p,q,r) \text{ for all } p,q,r \text{ with } r = 1, \qquad M_3(p,q,r).$$

is satisfied if and only if the following seven inequalities are satisfied:

$$z_{11} \le z_{22}, \qquad M_1(0,1,0)$$

$$z_{22} \le z_{33}, \qquad M_2(0,0,1)$$

$$2z_{12} \le z_{11}, \qquad M_2(1,-1,0)$$

$$2z_{13} \le z_{11}, \qquad M_3(1,0,-1)$$

$$\pm 2z_{23} \le z_{22}, \qquad M_3(0,1,\mp 1)$$

$$2(z_{12}+z_{13}-z_{23}) \le z_{11}+z_{22}, \qquad M_3(-1,1,1).$$

Proof immediately follows from

LEMMA [Minkowski]. *The inequalities $M_1(1,p,q), M_2(p,1,q), M_3(p,q,1)$ for p,q equal ± 1 or 0 imply all other inequalities $M_l(p,q,r)$ for arbitrary integers p,q,r.*

For completeness we reproduce here the proof following [17].

It is sufficient to prove the inequalities $M_l(p,q,r)$ for nonnegative integers p,q,r only. Indeed, if some of p,q,r are not positive then we change the signs

$$(p,q,r) \mapsto (p',q',r') = (\pm p, \pm q, \pm r)$$

$$p' \ge 0, q' \ge 0, r' \ge 0$$

$$z_{ij} \mapsto z'_{ij} = \pm z_{ij}$$

in such a way that

$$Z(p,q,r) = Z'(p',q',r').$$

Note that $z'_{ll} = z_{ll}$. So the inequality $M_l(p,q,r)$ for the coefficients of the quadratic form (z_{ij}) coincides with the inequality $M_l(p',q',r')$ for the quadratic form (z'_{ij}).

Let us redenote

$$(p,q,r) = (m_1,m_2,m_3).$$

We need to prove the inequalities

$$z_{ll} \le Z(m_1,m_2,m_3), \qquad (M_l(m_1,m_2,m_3))$$

for any $l=1,2,3$ where $m_l, m_{l+1}, \dots$ are nonnegative and not all equal to zero. It is enough to consider only the inequality $M_3(m_1,m_2,m_3)$ where all m_1,m_2,m_3 are positive (otherwise the problem reduces to quadratic forms with two arguments).

Let

$$m := \min_{k=1,2,3} m_k > 0$$

and

$$j := \max_{m_k = m} k.$$

We introduce the vector (m'_1,m'_2,m'_3) putting

$$m'_k = \begin{cases} m_k - m & \text{if } k \neq j \\ m_j & \text{for } k = j. \end{cases}$$

We have

$$m'_k \ge 0 \text{ for any } k, \qquad m'_3 > 0.$$

Using the elementary identity

$$Z(m_1,m_2,m_3) - Z(m'_1,m'_2,m'_3)$$
$$= m^2\left[Z(1,1,1) - z_{jj}\right] + 2m \sum_{k \neq j}\left[(m_k - m)\sum_{l \neq j} z_{kl}\right]$$

we obtain

$$Z(m'_1,m'_2,m'_3) \le Z(m_1,m_2,m_3)$$

since

$$z_{jj} \le Z(1,1,1)$$

due to $M_j(1,1,1)$,

$$0 \le \sum_{l \ne j} z_{kl}$$

(follows for $N=3$ from $M_k(p,q,r)$, where one of p,q,r is 0 and two others are ± 1).

Thus the inequality $M_3(m_1, m_2, m_3)$ follows from the inequality $M_3'(m_1', m_2', m_3')$ where m_i' are still nonnegative and smaller than m_i, $i=1,2,3$. This gives the recursion procedure to prove the lemma.

Remark: The same arguments show that the lemma holds true also for $N=4$. Starting from $N=5$ the theory of Minkowski inequalities is more complicated.

LEMMA 2. *Inequalities* (C.6a)–(C.6e) *imply that the quadratic form in* (D.2) *is positive definite*.

Proof: According to a theorem of Sylvester [22, p. 306], the positive definiteness of the quadratic form follows from the following three inequalities:

$$z_{11} > 0, \qquad \det\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} > 0, \qquad \det\begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} > 0.$$

The first two are obvious from (C.6a) and (C.6b). For the 3×3 determinant we obtain from (C.6a)–(C.6d) the inequalities

$$\begin{aligned} &z_{11}z_{22}z_{33} - z_{11}z_{23}^2 + 2z_{12}z_{13}z_{23} - z_{12}^2 z_{33} - z_{13}^2 z_{22} \\ &\quad \ge z_{11}z_{22}z_{33} - \frac{z_{11}z_{22}^2}{4} - 2z_{12}z_{13}|z_{23}| - \frac{z_{11}^2 z_{33}}{4} - \frac{z_{11}^2 z_{22}}{4} \\ &\quad \ge \frac{1}{4}z_{11}z_{22}z_{33} - 2z_{12}z_{13}|z_{23}| \\ &\quad \ge \frac{1}{4}z_{11}z_{22}z_{33} - \frac{1}{4}z_{11}^2 z_{22} \ge 0. \end{aligned} \tag{D.4}$$

If any one of the inequalities among the z_{ij} in (C.6a)–(C.6d) is strict, then at least one of the inequalities in (D.4) is strict, and we have shown that the 3×3 determinant is positive. Otherwise, we have from (C.6a)–(C.6e) that

$$z_{11} = z_{22} = z_{33} = 2z_{12} = 2z_{13} = \pm 2z_{23} = a$$

for some positive constant a and for just one sign before $2z_{23}$. The 3×3 determinant equals

$$a^3 - \frac{1}{4}a^3 \pm \frac{1}{4}a^3 - \frac{1}{4}a^3 - \frac{1}{4}a^3.$$

This is positive if $z_{23} = \frac{1}{2}a$. For $z_{23} = -\frac{1}{2}a$, the determinant vanishes. But such a quadratic form does not satisfy the inequality (C.6e). Indeed, for

$$z_{11} = z_{22} = z_{33} = 2z_{12} = 2z_{13} = a, \qquad z_{23} = -\frac{1}{2}a,$$

this inequality (C.6e) reads

$$2a \geq 3a.$$

This contradicts the positivity of $a > 0$. Lemma 2 is proved.

Proof of Theorem 1: Let us prove first that any 3×3 positive-definite matrix Z can be reduced by the transformations (B.7) to a Z-matrix from the fundamental region (1.17). Indeed, due to Minkowski's theorem we can find an integer unimodular matrix A such that the matrix $Z' = AZA^T$ satisfies the inequalities of Lemma 1. Changing, if necessary, signs of the coordinates ϕ_2 and ϕ_3, we can always meet the requirements $z'_{12} \geq 0, z'_{13} \geq 0$. Such changes do not violate the inequalities of Lemma 1. As the result we obtain a matrix in the fundamental region (1.17).

Let us prove now that in the interior of the fundamental region (i.e., in the region of parameter space where all the inequalities in (1.17) are strict), no two matrices are equivalent with respect to (B.7). Let us assume that some two matrices $Z^{(1)}$ and $Z^{(2)}$ satisfying (1.17) are equivalent, $Z^{(2)} = AZ^{(1)}Z^T$, where A is an invertible matrix with integer entries. The case $\det A = 1$ is impossible due to Minkowski's theorem since (C.6a)–(C.6e) are part of the Minkowski fundamental region. If $\det A = -1$, then $\det(-A) = 1$, and $Z^{(2)} = (-A)Z^{(1)}(-A)^T$. Again, this contradicts Minkowski's theorem.

To complete the proof of Theorem 1 we note that, due to Lemma 2, the condition of positive definiteness of the Z-matrix is contained in the inequalities (1.17). The theorem is proved.

Appendix E. Indecomposable period matrices

As discussed in Appendix C, one practical difficulty in calculating theta-functions is solved by restricting one's attention to period matrices that lie in the fundamental region, defined in Theorem 1. A second difficulty remains: If a period matrix Z can be reduced to block-diagonal form,

$$Z' = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}, \tag{E.1}$$

by a Siegel modular transformation (B.4), then its theta-function factors into a product of two theta-functions with fewer variables, and the corresponding KP solution is trivial.[4]

The problem of decomposability is to specify explicitly the period matrices inside the fundamental region that correspond to decomposable theta functions. For theta-functions of two variables ($N = 2$), this problem was solved in [19], where it was shown that a period matrix in the fundamental region is decomposable if and only if it is diagonal.

We show next that for $N = 3$, a corresponding statement holds.

THEOREM 2. *In the fundamental region, the only decomposable period matrices are in block-diagonal form. Therefore, if a period matrix lies in the fundamental region and if*

$$(\lambda\mu)^2 + (\lambda\nu)^2 + (\mu\nu)^2 > 0, \tag{E.2}$$

then the period matrix is indecomposable and the corresponding KP solution is nontrivial.

Proof: In this proof (also in the summation formulae of the next appendix) we use Jacobi's decomposition of a symmetric positive-definite matrix [22, p. 41]. We recall that the Jacobi theorem provides a unique

[4] The main motivation for addressing the problem of decomposability again comes from the algebraic-geometrical theory of the KP equation: Only theta-functions associated with (connected) Riemann surfaces provide solutions of KP, and all of these theta-functions are indecomposable. An important starting point for the application of our results to nonlinear equations like KP is the theorem [20] (see also [11]) that for $N \leq 3$, every indecomposable theta-function is associated with a Riemann surface; i.e., there are no other constraints. For $N > 3$, this is not true: The Schottky problem arises [11, 23, 24]. For $N > 3$ an approach of [12–15] based on a representation of the parameters of the multiphase solutions of KP by Burnside-type series could be useful in solving the Schottky constraints for the period matrix. However, in this way one could face the problem of improving convergence of Burnside series. Thus, for $N > 3$, our three main problems are still important in the calculation of theta-functions, but other problems arise as well.

factorization of a symmetric, positive-definite matrix Z into the product

$$Z = SPS^T, \tag{E.3a}$$

where P is a diagonal matrix and S is a lower triangular matrix with ones on the diagonal. For $N=3$ this reads

$$Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{pmatrix} = \begin{pmatrix} \alpha & \lambda\alpha & \mu\alpha \\ \lambda\alpha & \beta + \lambda^2\alpha & \nu\beta + \lambda\mu\alpha \\ \mu\alpha & \nu\beta + \lambda\mu\alpha & \gamma + \nu^2\beta + \mu^2\alpha \end{pmatrix} \tag{E.3b}$$

$$P = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{E.3c}$$

$$S = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \mu & \nu & 1 \end{pmatrix}. \tag{E.3d}$$

For the parameters $\alpha, \beta, \gamma, \lambda, \mu, \nu$ we obtain

$$\alpha = z_{11}, \qquad \beta = \frac{z_{11}z_{22} - z_{12}^2}{z_{11}}, \qquad \gamma = \frac{\det Z}{z_{11}z_{22} - z_{12}^2}, \tag{E.4a}$$

$$\lambda = \frac{z_{12}}{z_{11}}, \qquad \mu = \frac{z_{13}}{z_{11}}, \qquad \nu = \frac{z_{11}z_{23} - z_{12}z_{13}}{z_{11}z_{22} - z_{12}^2}. \tag{E.4b}$$

LEMMA 3. *A matrix Z in the Jacobi representation* (E.3a)–(E.3d) *belongs to the fundamental region* (1.17) *if and only if the parameters* $\alpha, \beta, \gamma, \lambda, \mu, \nu$ *satisfy the following inequalities*:

$$0 < \alpha \tag{E.5a}$$

$$0 \le \lambda, \mu \le \frac{1}{2} \tag{E.5b}$$

$$(1 - \lambda^2)\,\alpha \le \beta \tag{E.5c}$$

$$\alpha(\lambda^2 - \mu^2) + \beta(1 - \nu^2) \le \gamma \tag{E.5d}$$

$$\nu \le \frac{1}{2} + \lambda(\lambda - 2\mu)\frac{\alpha}{2\beta} \tag{E.5e}$$

$$-\frac{1}{2} + \frac{\alpha}{2\beta}(\lambda + 2\mu - 1)(1 - \lambda) \le \nu \qquad \text{for } \lambda + \mu > \frac{1}{2} \tag{E.5f}$$

$$-\frac{1}{2} - \lambda(\lambda + 2\mu)\frac{\alpha}{2\beta} \le \nu \qquad \text{for } \lambda + \mu \le \frac{1}{2}. \tag{E.5g}$$

Proof can be obtained by direct substitution of (E.3b) into (1.17).

LEMMA 4. *The following inequalities hold for a matrix* (E.3b) *in the fundamental region* (1.17):

$$0 < \frac{\alpha}{\beta} \leq \frac{4}{3} \tag{E.6a}$$

$$|\nu| \leq \frac{2}{3} \tag{E.6b}$$

$$\frac{\gamma}{\alpha} \geq \frac{11}{36} \tag{E.6c}$$

$$\frac{\gamma}{\beta} \geq \frac{11}{36}. \tag{E.6d}$$

Proof: From (E.5c) we obtain

$$0 < \frac{\alpha}{\beta} \leq \frac{1}{1-\lambda^2}. \tag{E.7}$$

This, together with (E.5b) gives (E.6a).

Let us prove now that $\nu \leq \frac{2}{3}$. For $\lambda \geq 2\mu$ we obtain, from (E.5e) and (E.7),

$$\nu \leq \frac{1}{2} + \frac{\lambda^2 - 2\lambda\mu}{2(1-\lambda^2)} = \frac{1-2\lambda\mu}{2(1-\lambda^2)}.$$

The RHS of the inequality in the domain $0 \leq \lambda \leq \frac{1}{2}$, $0 \leq \mu \leq \frac{1}{2}$, $2\mu \leq \lambda$, has its maximum ($\frac{2}{3}$) at the point $\lambda = \frac{1}{2}$, $\mu = 0$. So in this domain, $\nu \leq \frac{2}{3}$. For $\lambda < 2\mu$, from (E.5e), we obtain $\nu \leq \frac{1}{2}$. The inequality $\nu \leq \frac{2}{3}$ is proved.

Let us prove that $-\frac{2}{3} \leq \nu$. In the domain $\lambda + \mu \leq \frac{1}{2}$, from (E.5g) and (E.7) we obtain

$$-\frac{1+2\lambda\mu}{2(1-\lambda^2)} = -\frac{1}{2} - \frac{\lambda(\lambda+2\mu)}{2(1-\lambda^2)} \leq \nu.$$

The LHS of the inequality in the domain $0 \leq \lambda$, $\mu \leq \frac{1}{2}$, $\lambda + \mu \leq \frac{1}{2}$ has its minimum ($-\frac{2}{3}$) at the point $\lambda = \frac{1}{2}$, $\mu = 0$. So in this domain, $-\frac{2}{3} \leq \nu$. If

$\lambda+\mu>\frac{1}{2}$, $\lambda+2\mu-1\leq 0$, then from (E.5f) we have $-\frac{1}{2}\leq\nu$. Finally, if $\lambda+\mu>\frac{1}{2}$ and $\lambda+2\mu-1>0$, then from (E.5f) and (E.7),

$$\nu \geq -\frac{1}{2}+\frac{(\lambda+2\mu-1)(1-\lambda)}{2(1-\lambda^2)} = \frac{\mu-1}{\lambda+1}.$$

The RHS has its minimum $(-\frac{2}{3})$ at the point $\lambda=\frac{1}{2}$, $\mu=0$ of the above domain. This completes the proof of (E.6b).

For the ratio γ/α, from (E.5c) and (E.5d) we now have

$$1-\mu^2-\nu^2(1-\lambda^2) = \lambda^2-\mu^2+(1-\lambda^2)(1-\nu^2) \leq \frac{\gamma}{\alpha}.$$

The LHS has its minimum $(\frac{11}{36})$ at the points $\lambda=0$, $\mu=\frac{1}{2}$, $\nu=\pm\frac{2}{3}$. This proves (E.6c).

We now prove (E.6d). From (E.5d) we obtain

$$\frac{\alpha}{\beta}(\lambda^2-\mu^2)+1-\nu^2 \leq \frac{\gamma}{\beta}.$$

For $\lambda\leq\mu$, this and (E.6b) give $\frac{5}{9}\leq\gamma/\beta$. Let $\lambda>\mu$. We obtain

$$\frac{1-\mu^2}{1-\lambda^2}-\nu^2 = \frac{\lambda^2-\mu^2}{1-\lambda^2}+1-\nu^2 \leq \frac{\gamma}{\beta}.$$

The LHS has its minimum $(\frac{11}{36})$ at the points $\lambda=0$, $\mu=\frac{1}{2}$, $\nu=\pm\frac{2}{3}$. Lemma 4 is proved.

Remark: The inequality (E.6b) can be attained, say, for the matrix

$$Z = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

Proof of Theorem 2: [Note: In this proof, we renormalize the period matrix $Z\rightarrow 2\pi Z$ for the sake of technical simplicity. This change does not affect the inequalities in (E.5a)–(E.5g) or (E.6a)–(E.6d).] For $N = 3$, there are three classes of block-diagonal symmetric matrices:

$$Z' = \begin{pmatrix} z'_{11} & 0 & 0 \\ 0 & z'_{22} & z'_{23} \\ 0 & z'_{23} & z'_{33} \end{pmatrix}, \qquad Z'' = \begin{pmatrix} z''_{11} & 0 & z''_{13} \\ 0 & z''_{22} & 0 \\ z''_{13} & 0 & z''_{33} \end{pmatrix},$$

$$Z''' = \begin{pmatrix} z'''_{11} & z'''_{12} & 0 \\ z'''_{12} & z'''_{22} & 0 \\ 0 & 0 & z'''_{33} \end{pmatrix}.$$

(Recall that the real parts of the matrices must be positive–definite.) In fact, these are equivalent with respect to Siegel modular transformations (B.4), (B.5a, b). Indeed, taking

$$A = D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = C = 0,$$

we obtain a transformation (B.4) of a matrix of the third type into one of the first type, and vice-versa. Similarly, the modular transformation with

$$A = D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = C = 0,$$

interchanges the first and the second classes of block-diagonal matrices.

So we need to prove the following statement. If a real symmetric matrix Z, in the fundamental region (1.17), is equivalent to a block-diagonal matrix of the first type,

$$iZ' \equiv i \begin{pmatrix} z'_{11} & 0 & 0 \\ 0 & z'_{22} & z'_{23} \\ 0 & z'_{23} & z'_{33} \end{pmatrix} = (iAZ + B)(iCZ + D)^{-1}, \tag{E.8}$$

where the 3×3 matrices A, B, C, D with integer values satisfy (B.5a)–(B.5b), then Z itself is a block-diagonal matrix.

Let us substitute the Jacobi representation (E.3a) into (E.8). We obtain

$$iZ' = (i\hat{A}P + \hat{B})(i\hat{C}P + \hat{D})^{-1}, \tag{E.9}$$

where

$$\begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & S^{T^{-1}} \end{pmatrix} \tag{E.10}$$

is again a real symplectic matrix (*not* integer!) as a product of two symplectic matrices. So the matrices

$$\hat{A} = AS, \qquad \hat{B} = BS^{T^{-1}}, \qquad \hat{C} = CS, \qquad \hat{D} = DS^{T^{-1}} \tag{E.11}$$

still satisfy (B.5a)–(B.5b). Note that the action (E.9) of the full symplectic group $\mathrm{Sp}(N, \mathbf{R})$ is also well defined on the Siegel half-plane (B.3).

Let us put

$$iZ' = U + iV, \tag{E.12}$$

where U, V are real symmetric matrices, and V is positive–definite. Rewriting (E.9) as

$$(U + iV)(i\hat{C}P + \hat{D}) = i\hat{A}P + \hat{B}, \tag{E.13}$$

and separating the real and imaginary parts, we obtain

$$\hat{A} = U\hat{C} + V\hat{D}P^{-1}, \tag{E.14}$$

$$\hat{B} = U\hat{D} - V\hat{C}P. \tag{E.15}$$

Substitute these into the equation $\hat{A}\hat{D}^T - \hat{B}\hat{C}^T = I$. Taking into account the equation $\hat{C}\hat{D}^T = \hat{D}\hat{C}^T$, we obtain

$$V\left(\hat{D}P^{-1}\hat{D}^T + \hat{C}P\hat{C}^T\right) = I. \tag{E.16}$$

We now write out explicitly certain entries of the matrix equations (E.14)–(E.16). Denote by $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ the entries of the matrices A, B, C, D (these are integers), and by $\hat{a}_{ij}, \hat{b}_{ij}, \hat{c}_i, \hat{d}_{ij}$ the entries of the matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ (real numbers). The relations (E.11) for the first rows of these matrices read as follows:

$$\hat{a}_{11} = a_{11} + \lambda a_{12} + \mu a_{13} \tag{E.17a}$$

$$\hat{a}_{12} = a_{12} + \nu a_{13} \tag{E.17b}$$

$$\hat{a}_{13} = a_{13} \tag{E.17c}$$

$$\hat{b}_{11} = b_{11} \tag{E.18a}$$

$$\hat{b}_{12} = -\lambda b_{11} + b_{12} \tag{E.18b}$$

$$\hat{b}_{13} = (-\mu + \lambda\nu) b_{11} - \nu b_{12} + b_{13} \tag{E.18c}$$

$$\hat{c}_{11} = c_{11} + \lambda e_{12} + \mu c_{13} \tag{E.19a}$$

$$\hat{c}_{12} = c_{12} + \nu c_{13} \tag{E.19b}$$

$$\hat{c}_{13} = c_{13} \tag{E.19c}$$

$$\hat{d}_{11} = d_{11} \tag{E.20a}$$

$$\hat{d}_{12} = -\lambda_{11} + d_{12} \tag{E.20b}$$

$$\hat{d}_{13} = (-\mu + \lambda\nu) d_{11} - \nu d_{12} + d_{13}. \tag{E.20c}$$

Also set

$$x = u_{11}, \qquad y = v_{11} > 0. \tag{E.21}$$

(These are the nonzero entries in the first rows of the matrices U and V.)
The $(1,1)$-entry of (E.16) is particularly important:

$$\alpha^{-1}\hat{d}_{11}^2 + \beta^{-1}\hat{d}_{12}^2 + \gamma^{-1}\hat{d}_{13}^2 + \alpha\hat{c}_{11}^2 + \beta\hat{c}_{12}^2 + \gamma\hat{c}_{13}^2 = y^{-1}. \tag{E.22}$$

Note that because $(y, \alpha, \beta, \gamma)$ are all finite and positive, (E.22) implies that the entries in the first rows of the matrices $\hat{C}, \hat{D}$ cannot all vanish simultaneously.

The first lines of (E.14) and (E.15) read

$$\hat{a}_{11} = \alpha^{-1} y\hat{d}_{11} + x\hat{c}_{11} \tag{E.23a}$$

$$\hat{a}_{12} = \beta^{-1} y\hat{d}_{12} + x\hat{c}_{12} \tag{E.23b}$$

$$\hat{a}_{13} = \gamma^{-1} y\hat{d}_{13} + x\hat{c}_{13} \tag{E.23c}$$

$$\hat{b}_{11} = x\hat{d}_{11} - \alpha y\hat{c}_{11} \tag{E.24a}$$

$$\hat{b}_{12} = x\hat{d}_{12} - \beta y\hat{c}_{12} \tag{E.24b}$$

$$\hat{b}_{13} = x\hat{d}_{13} - \gamma y\hat{c}_{13}. \tag{E.24c}$$

Let us assume that the matrix Z is not block diagonal, so that (E.2) holds. We show that this assumption is not consistent with the equations (E.15)–(E.22).

The first step is to derive the following equations:

$$a_{13} d_{11} = b_{11} c_{13}, \tag{E.25}$$

$$a_{12} d_{11} = b_{11} c_{12}. \tag{E.26}$$

To prove (E.25), multiply (E.23c) by d_{11} and (E.24a) by c_{13}, and subtract the results. Using (E.19c), (E.20a), and (E.22), one obtains

$$\begin{aligned} a_{13} d_{11} - b_{11} c_{13} &= y\left(\gamma^{-1}\hat{d}_{11}\hat{d}_{13} + \alpha\hat{c}_{11}\hat{c}_{13}\right) \\ &= \frac{\gamma^{-1}\hat{d}_{11}\hat{d}_{13} + \alpha\hat{c}_{11}\hat{c}_{13}}{\alpha^{-1}\hat{d}_{11}^2 + \gamma^{-1}\hat{d}_{13}^2 + \alpha\hat{c}_{11}^2 + \gamma\hat{c}_{13}^2 + \beta^{-1}\hat{d}_{12}^2 + \beta\hat{c}_{12}^2}. \end{aligned}$$

Set

$$r = \sqrt{\alpha^{-1}\hat{d}_{11}^2 + \gamma^{-1}\hat{d}_{13}^2}, \qquad \rho = \sqrt{\alpha\hat{c}_{11}^2 + \gamma\hat{c}_{13}^2}, \tag{E.27}$$

and

$$\frac{\hat{d}_{13}}{\sqrt{\gamma}} = r\cos\theta, \qquad \frac{\hat{d}_{11}}{\sqrt{\alpha}} = r\sin\theta,$$

$$\sqrt{\gamma}\,\hat{c}_{13} = \rho\cos\phi, \qquad \sqrt{\alpha}\,\hat{c}_{11} = \rho\sin\phi.$$

We obtain

$$a_{13}d_{11} - b_{11}c_3 = \frac{1/2\sqrt{(\alpha/\gamma)}\,(r^2 \sin 2\theta + \rho^2 \sin 2\phi)}{r^2 + \rho^2 + \beta^{-1}\hat{d}_{12}^2 + \beta\hat{c}_{12}^2}.$$

The RHS cannot be more than $\frac{1}{2}\sqrt{\alpha/\gamma}$. From (E.6c) we conclude that

$$|a_{13}d_{11} - b_{11}c_{13}| \le \frac{1}{2}\sqrt{\frac{\alpha}{\gamma}} \le \frac{3}{\sqrt{11}} < 1.$$

Since $a_{13}d_{11} - b_{11}c_{13}$ is an integer, we obtain (E.25).

Equation (E.26) can be derived in a similar way. Multiply (E.23b) by $d_{11} = \hat{d}_{11}$, multiply (E.24a) by $\hat{c}_{12}$ and subtract the results. Using (E.25) and (E.22) we obtain

$$\begin{aligned} a_{12}d_{11} - b_{11}c_{12} &= \frac{\beta^{-1}\hat{d}_{11}\hat{d}_{12} + \alpha\hat{c}_{11}\hat{c}_{12}}{\alpha^{-1}\hat{d}_{11}^2 + \beta^{-1}\hat{d}_{12}^2 + \alpha\hat{c}_{11}^2 + \beta\hat{c}_{12}^2 + \gamma^{-1}\hat{d}_{13}^2 + \gamma\hat{c}_{13}^2} \\ &= \frac{1/2\sqrt{(\alpha/\beta)}\,(r^2 \sin 2\theta + \rho^2 \sin 2\phi)}{r^2 + \rho^2 + \gamma^{-1}\hat{d}_{13}^2 + \gamma\hat{c}_{13}^2}, \end{aligned}$$

where we have used

$$r = \sqrt{\alpha^{-1}\hat{d}_{11}^2 + \beta^{-1}\hat{d}_{12}^2}, \qquad \rho = \sqrt{\alpha\hat{c}_{11}^2 + \beta\hat{c}_{12}^2},$$

$$\frac{\hat{d}_{11}}{\sqrt{\alpha}} = r\sin\theta, \qquad \frac{\hat{d}_{12}}{\sqrt{\beta}} = r\cos\theta,$$

$$\sqrt{\alpha}\,\hat{c}_{11} = \rho\cos\phi, \qquad \sqrt{\beta}\,\hat{c}_{12} = \rho\sin\phi.$$

Hence,

$$|a_{12}d_{11} - b_{11}c_{12}| \le \frac{1}{2}\sqrt{\frac{\alpha}{\beta}} \le \frac{1}{\sqrt{3}} < 1.$$

This proves (E.26).

Let us now prove that if

$$\hat{d}_{11}^2 + \hat{c}_{11}^2 \neq 0, \qquad \text{(E.28)}$$

then

$$a_{11}d_{11} = b_{11}c_{11} + 1, \tag{E.29}$$

and

$$\hat{c}_{12} = \hat{c}_{13} = \hat{d}_{12} = \hat{d}_{13} = 0. \tag{E.30}$$

To do this we obtain from Equations (E.23a), (E.24a) the equation

$$\hat{a}_{11}d_{11} - b_{11}\hat{c}_{11} = \frac{\alpha^{-1}\hat{d}_{11}^2 + \alpha\hat{c}_{11}^2}{\alpha^{-1}\hat{d}_{11}^2 + \alpha\hat{c}_{11}^2 + \beta^{-1}\hat{d}_{12}^2 + \beta\hat{c}_{12}^2 + \gamma^{-1}\hat{d}_{13}^2 + \gamma\hat{c}_{13}^2}. \tag{E.31}$$

From (E.25), (E.26), (E.17a), and (E.19a), it follows that

$$\hat{a}_{11}d_{11} - b_{11}\hat{c}_{11} = a_{11}d_{11} - b_{11}c_{11}.$$

So (E.31) and (E.28) imply

$$0 < a_{11}d_{11} - b_{11}c_{11} \leq 1.$$

Because $a_{11}d_{11} - b_{11}c_{11}$ is an integer, it must equal 1. This gives (E.29). Also, this implies that the RHS in (E.31) equals 1. This gives (E.30).

Let us assume now that $d_{11} \neq 0$. We show that this is impossible for a non-block-diagonal matrix Z. From $\hat{d}_{12} = 0$ (see (E.20b) and (E.30)) we obtain

$$\lambda = \frac{d_{12}}{d_{11}}. \tag{E.32}$$

So λ must be a rational number, $\lambda = p/q$. Since $0 \leq \lambda \leq \frac{1}{2}$, we have for $\lambda \neq 0$,

$$q \geq 2.$$

From (E.32) it follows that

$$d_{12} = mp, \qquad d_{11} = mq$$

for some integer m. From $\hat{b}_{12} = 0$ (see E.18b), (E.24b), and (E.30)) we obtain

$$b_{12} = \lambda b_{11}.$$

So the integers b_{11} and b_{12} have the form

$$b_{11} = nq, \qquad b_{12} = np$$

for some integer n. Equation (E.29) now reads

$$q(ma_{11} - nc_{11}) = 1.$$

This is impossible for $q \geq 2$. Therefore $\lambda = 0$. We also conclude that $d_{12} = b_{12} = 0$. From $\hat{d}_{13} = 0$ and from (E.20c) we obtain that $d_{13} = \mu d_{11}$, so

$$\mu = \frac{d_{13}}{d_{11}} = \frac{p}{q}$$

is a rational number. If $\mu = 0$ then we obtain a block-diagonal matrix. So $\mu \neq 0$, and $q \geq 2$ (since $0 \leq \mu \leq \frac{1}{2}$). From (E.30) and (E.23c) we see that $\hat{b}_{13} = 0$. Due to (E.18c) this, together with $b_{12} = 0$, $\lambda = 0$, gives

$$b_{13} = \mu b_{11}.$$

As above, we have

$$d_{13} = mp, \qquad d_{11} = mq, \qquad b_{13} = np, \qquad b_{11} = nq$$

for some integers m, n. Then (E.29) becomes

$$q(ma_{11} - nc_{11}) = 1,$$

which is impossible for $q > 2$. We have now proved that the assumption $d_{11} \neq 0$ contradicts Equations (E.22)–(E.24c) for a non-block-diagonal matrix Z.

So $d_{11} = 0$. It follows from (E.25) and (E.26) that

$$b_{11}c_{13} = 0, \qquad b_{11}c_{12} = 0. \tag{E.33}$$

Also, (E.20b)–(E.20c) read

$$\hat{d}_{12} = d_{12}, \qquad \hat{d}_{13} = d_{13} - \nu d_{12}.$$

We now prove that

$$a_{13}d_{12} = b_{12}c_{13}. \tag{E.34}$$

As above, taking a linear combination of (E.23c) and (E.24b) we obtain, using (E.33),

$$a_{13}d_{12} - b_{12}c_{13} \equiv \hat{a}_{13}d_{12} - \hat{b}_{12}c_{13} = \frac{\gamma^{-1}\,\hat{d}_{12}\hat{d}_{13} + \beta\hat{c}_{12}\hat{c}_{13}}{\beta^{-1}\hat{d}_{12}^2 + \gamma^{-1}\hat{d}_{13}^2 + \beta\hat{c}_{12}^2 + \gamma\hat{c}_{13}^2 + \alpha\hat{c}_{11}^2}$$

$$= \frac{1/2\sqrt{(\beta/\gamma)}\,(r^2\sin 2\theta + \rho^2\sin 2\phi)}{r^2 + \rho^2 + \alpha\hat{c}_{11}^2},$$

where

$$r = \sqrt{\beta^{-1}\hat{d}_{12}^2 + \gamma^{-1}\hat{d}_{13}^2}, \qquad \rho = \sqrt{\beta\hat{c}_{12}^2 + \gamma\hat{c}_{13}^2},$$

$$\frac{\hat{d}_{12}}{\sqrt{\beta}} = r\cos\theta, \qquad \frac{\hat{d}_{13}}{\sqrt{\gamma}} = r\sin\theta,$$

$$\sqrt{\beta}\,\hat{c}_{12} = \rho\cos\phi, \qquad \sqrt{\gamma}\,\hat{c}_{13} = \rho\sin\phi.$$

From this and from (E.6d) we obtain

$$|a_{13}d_{12} - b_{12}c_{13}| \le \frac{1}{2}\sqrt{\frac{\beta}{\gamma}} \le \frac{3}{\sqrt{11}} < 1.$$

This proves (E.34).

Next we prove that if

$$\hat{d}_{12}^2 + \hat{c}_{12}^2 \neq 0,$$

then

$$a_{12}d_{12} = b_{12}c_{12} + 1, \tag{E.35}$$

and

$$\hat{c}_{11} = \hat{c}_{13} = \hat{d}_{13} = 0. \tag{E.36}$$

The procedure is similar to the proof of (E.29) and (E.30). From (E.23b) and (E.24b), using (E.22), (E.33), and (E.34), we obtain

$$0 < a_{12}d_{12} - b_{12}c_{12} = \frac{\beta^{-1}\hat{d}_{12}^2 + \beta\hat{c}_{12}^2}{\beta^{-1}\hat{d}_{12}^2 + \beta\hat{c}_{12}^2 + \gamma^{-1}\hat{d}_{13}^2 + \gamma\hat{c}_{13}^2 + \alpha\hat{c}_{11}^2} \le 1.$$

This gives (E.35) and (E.36).

Let us show that $d_{12} = 0$. If not, then from $d_{11} = 0$, $\hat{d}_{13} = 0$, and (E.20c) we obtain

$$\nu = \frac{d_{13}}{d_{12}}.$$

So ν is a rational number, $\nu = p/q$. If $\nu \neq 0$ then $q \geq 2$ (since $|\nu| \leq \frac{2}{3}$) and $d_{13} = mp$, $d_{12} = mq$, for some integer m. From (E.36) and from (E.24c), using (E.18c) and $b_{11} = 0$ (because of (E.24a) and $\hat{d}_{11} = \hat{c}_{11} = 0$) we obtain

$$b_{13} = \nu b_{12}.$$

This gives

$$b_{13} = np, \qquad b_{12} = nq.$$

After substitution into (E.35), we obtain a contradiction. Hence $\nu = 0$, and $d_{13} = b_{13} = 0$. From (E.36) and from (E.23a), (E.23c), we obtain $\hat{a}_{13} = a_{13} = 0$, $\hat{a}_{11} = a_{11} + \lambda a_{12} = 0$. Together with $\hat{c}_{11} = \hat{c}_{13} = 0$, this gives

$$a_{11} + \lambda a_{12} = 0,$$

$$c_{11} + \lambda c_{12} = 0.$$

From this and from (E.35) we obtain, by the now familiar argument, that $\lambda = 0$, so Z is a block-diagonal matrix. This contradiction shows that $d_{12} = 0$.

From $d_{11} = d_{12} = 0$ and (E.34), we obtain also that

$$b_{12} c_{13} = 0. \tag{E.37}$$

Let us prove that for

$$d_{13}^2 + c_{13}^2 \neq 0$$

the equations

$$a_{13} d_{13} = b_{13} c_{13} + 1 \tag{E.38}$$

and

$$\hat{c}_{11} = \hat{c}_{12} = 0 \tag{E.39}$$

hold. From Equations (E.23c), (E.24c), using $b_{11} = 0$ and (E.37), we obtain

$$0 < a_{13}d_{13} - b_{13}c_{13} = a_{13}d_{13} - \hat{b}_{13}\hat{c}_{13} = \frac{\gamma^{-1}d_{13}^2 + \gamma c_{13}^2}{\gamma^{-1}d_{13}^2 + \gamma c_{13}^2 + \beta\hat{c}_{12}^2 + \alpha\hat{c}_{11}^2} \leq 1.$$

This gives (E.38), (E.39). From (E.39) and from $\hat{d}_{11} = 0$, $\hat{d}_{12} = 0$, we obtain, using (E.23a)–(E.23b) and (E.24a)–(E.24b),

$$\hat{b}_{11} = \hat{b}_{12} = 0, \qquad \hat{a}_{11} = \hat{a}_{12} = 0.$$

Let us show that $d_{13} = 0$. If not, then from (E.38) we see that both a_{13} and c_{13} cannot vanish simultaneously. From $\hat{a}_{12} = a_{12} + \nu a_{13} = 0$, $\hat{c}_{12} = c_{12} + \nu c_{13} = 0$ we obtain, as above, that ν is a rational number, $\nu = p/q$. If $\nu \neq 0$, then $q \geq 2$, $a_{12} = -mp$, $a_{13} = mq$, $c_{12} = -np$, $c_{13} = nq$. Equation (E.38) reads

$$q(md_{13} - nb_{13}) = 1$$

with $q \geq 2$, so necessarily $\nu = 0$. Then we have $a_{12} = 0$, $c_{12} = 0$, and from $\hat{a}_{11} = a_{11} + \mu a_{13} = 0$, $\hat{c}_{11} = c_{11} + \mu c_{13} = 0$ it follows that μ is a rational number, $\mu = p/q \neq 0$ (otherwise the Z-matrix is block diagonal), $q \geq 2$, so $a_{11} = -mp$, $a_{13} = mq$, $c_{11} = -np$, $c_{13} = np$, and (E.38) gives a contradiction. We have proved that for a non-block-diagonal matrix Z the equations (E.22)–(E.24c) imply $d_{11} = d_{12} = d_{13} = 0$.

Let us show that $c_{13} = 0$. Otherwise from (E.3) it follows that $b_{11} = 0$, and from (E.37) we obtain $b_{12} = 0$. Hence $\hat{b}_{12} = 0$. Since also $\hat{d}_{12} = 0$, from (E.24a)–(E.24b) we conclude that $\hat{c}_{11} = \hat{c}_{12} = 0$. From (E.38) we have also

$$b_{13}c_{13} = -1.$$

so

$$b_{13} = \pm 1, \qquad c_{13} = \mp 1.$$

Then $\hat{c}_{12} = c_{12} + \nu c_{13} = 0$ implies

$$|c_{12}| = |\nu| \leq \tfrac{2}{3}.$$

Hence $c_{12} = \nu = 0$. Then from $\hat{c}_{11} = c_{11} + \mu c_{13} = 0$ we obtain

$$|c_{11}| = |\mu| \leq \tfrac{1}{2}.$$

Hence $c_{11} = \mu = 0$. This is impossible for a non-block-diagonal matrix Z. Therefore, $c_{13} = 0$.

Let us prove that $\hat{c}_{12} = c_{12} = 0$. Otherwise from (E.35) we obtain

$$b_{12} c_{12} = -1,$$

or

$$b_{12} = \pm 1, \qquad c_{12} = \mp 1.$$

From (E.33) we have $b_{11} = 0$. Then from (E.24a) it follows that $c_{11} + \lambda c_{12} = \hat{c}_{11} = 0$. This gives

$$|c_{11}| = \lambda \leq \tfrac{1}{2}.$$

Hence $c_{11} = \lambda = 0$. Now from $\hat{d}_{13} = \hat{c}_{13} = 0$ and from (E.24c) we obtain $b_{13} - \nu b_{12} = \hat{b}_{13} = 0$. So

$$|b_{13}| = |\nu| \leq \tfrac{2}{3}.$$

Hence $b_{13} = \nu = 0$. Again we obtain a contradiction. This proves that $c_{12} = 0$.

The final step: to prove that $c_{11} = 0$. If $c_{11} \neq 0$, then from (E.29) we obtain

$$b_{11} c_{11} = -1,$$

$$b_{11} = \pm 1, \qquad c_{11} = \mp 1.$$

From $\hat{d}_{12} = \hat{c}_{12} = 0$ and (E.24b) we infer $b_{12} - \lambda b_{11} = \hat{b}_{12} = 0$, $|b_{12}| = \lambda \leq \frac{1}{2}$, so $b_{12} = \lambda = 0$. From (E.24c) and $\hat{c}_{13} = \hat{d}_{13} = 0$ we obtain in similar way that $-\mu b_{11} + b_{13} = \hat{b}_{13} = 0$, $|b_{13}| = \mu \leq \frac{1}{2}$. So $b_{13} = \mu = 0$. This contradiction shows that $c_{11} = 0$.

We have proved that for a non-block-diagonal matrix Z in the fundamental region (C.6a)–(C.6e) Equations (E.22)–(E.24c) imply $c_{11} = c_{12} = c_{13} = d_{11} = d_{12} = d_{13} = 0$. But this contradicts (E.22). Theorem 2 is proved.

Appendix F. Summation formulae for theta-functions of three variables

Let Z be a 3×3 real symmetric matrix in the fundamental region (1.17), and let

$$\theta = \theta(\phi_1, \phi_2, \phi_3 | Z)$$

be the corresponding theta-function. In this section we obtain efficient formulae for summation of the theta-series in different parts of the fundamental region. Note that even though the sum of the theta-series is a real-valued function (for real arguments ϕ_1, ϕ_2, ϕ_3), our formulae often are written in complex form.

We use the Jacobi representation (E.3a)–(E.3d) for the period matrix Z. Let us rewrite the theta-series in the form

$$\theta = \sum_m \exp\Big(-\frac{1}{2} mZm^T + im\phi^T\Big),$$

where the summation is taken over all three-component vectors of integers, $m = (m_1, m_2, m_3)$; here $\phi = (\phi_1, \phi_2, \phi_3)$. Using (E.3a), we obtain

$$\begin{aligned} -\frac{1}{2} mZm^T &+ im\phi^T \\ &= -\frac{1}{2} mSPS^T m^T + im\phi^T \\ &= -\frac{1}{2}\hat{m}P\hat{m}^T + i\hat{m}\phi^T \\ &= -\frac{1}{2}\big(\alpha\hat{m}_1^2 + \beta\hat{m}_2^2 + \gamma\hat{m}_3^2\big) + i\big(\hat{m}_1\hat{\phi}_1 + \hat{m}_2\hat{\phi}_2 + \hat{m}_3\hat{\phi}_3\big), \end{aligned} \tag{F.1a}$$

where we have put

$$\phi = \hat{\phi} S^T \tag{F.1b}$$

$$\hat{m} = mS. \tag{F.1c}$$

In the coordinate form these read

$$\big(\hat{m}_1, \hat{m}_2, \hat{m}_3\big) = (m_1 + m_2\lambda + m_3\mu, m_2 + \nu m_3, m_3) \tag{F.2a}$$

$$\big(\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3\big) = (\phi_1, -\lambda\phi_1 + \phi_2, (-\mu + \lambda\nu)\phi_1 - \nu\phi_2 + \phi_3). \tag{F.2b}$$

Using (F.1a)–(F.1c) and (F.2a)–(F.2b) we can rewrite the theta-series in the nested form

$$\begin{aligned}\theta = &\sum_{m_3} \exp\left[-\frac{1}{2}\gamma m_3^2 + i m_3 \hat{\phi}_3\right]\\ &\cdot \sum_{m_2} \exp\left[-\frac{1}{2}\beta(m_2+\nu m_3)^2 + i(m_2+\nu m_3)\hat{\phi}_2\right]\\ &\cdot \sum_{m_1} \exp\left[-\frac{1}{2}\alpha(m_1+m_2\lambda+m_3\mu)^2 + i(m_1+m_2\lambda+m_3\mu)\hat{\phi}_1\right].\end{aligned} \tag{F.3}$$

This formula gives an efficient way to compute θ if α, β, γ are all large.

If some of $\{\alpha, \beta, \gamma\}$ are small, then one or more of the sums in (F.3) converge slowly. We can improve convergence by applying an appropriate Siegel modular transform (B.4), (B.6a)–(B.6f). We need to do this only for three particular modular transforms; for these we derive the corresponding transformation laws of θ directly, using the Poisson sum formula:

$$\sum_{-\infty < m < \infty} e^{-m^2(\sigma/2)+imx} = \left(\frac{2\pi}{\sigma}\right)^{1/2} \sum_{-\infty < n < \infty} e^{-(2\pi^2/\sigma)(n-x/2\pi)^2} \tag{F.4}$$

This holds for $\operatorname{Re}\sigma > 0$ [25].

First we apply (F.4) to the m_1-sum in (F.3). After regrouping, this gives

$$\begin{aligned}\theta = &\left(\frac{2\pi}{\alpha}\right)^{1/2} \sum_{n_1} \exp\left[-\frac{2\pi^2}{\alpha}\left(n_1 - \frac{\hat{\phi}_1}{2\pi}\right)^2\right]\\ &\cdot \sum_{m_3} \exp\left[-\frac{\gamma}{2} m_3^2 + i m_3\left(\hat{\phi}_3 + 2\pi n_1(\mu - \lambda\nu)\right)\right]\\ &\cdot \sum_{m_2} \exp\left[-\frac{\beta}{2}(m_2+\nu m_3)^2 + i(m_2+\nu m_3)\left(\hat{\phi}_2 + 2\pi n_1 \lambda\right)\right].\end{aligned} \tag{F.5}$$

This is an efficient way to compute θ if α is small but β and γ is large.

Next, applying (F.4) to the m_2-sum in (F.5) gives

$$\begin{aligned}\theta = &\frac{2\pi}{\sqrt{\alpha\beta}} \sum_{n_1} \exp\left[-\frac{2\pi^2}{\alpha}\left(n_1 - \frac{\hat{\phi}_1}{2\pi}\right)^2\right] \sum_{n_2} \exp\left[-\frac{2\pi^2}{\beta}\left(n_2 - \lambda n_1 - \frac{\hat{\phi}_2}{2\pi}\right)^2\right]\\ &\cdot \sum_{m_3} \exp\left[-\gamma\frac{m_3^2}{2} + i m_3\left(\hat{\phi}_3 + 2\pi n_1(\mu - \lambda\nu) + 2\pi\nu n_2\right)\right].\end{aligned} \tag{F.6}$$

This works well if α, β are small and γ is large.

Finally, applying (F.4) to the m_3-sum in (F.6) yields the fourth representation of the theta-series:

$$\theta = \frac{(2\pi)^{3/2}}{\sqrt{\alpha\beta\gamma}} \sum_{n_1} e^{-(2\pi^2/\alpha)(n_1 - \hat{\phi}_1/2\pi)^2} \sum_{n_2} e^{-(2\pi^2/\beta)(n_2 - \lambda n_1 - \hat{\phi}_2/2\pi)^2}$$

$$\times \sum_{n_3} e^{-(2\pi^2/\gamma)(n_3 + n_1(\mu - \lambda\nu) + n_2\nu + \hat{\phi}_3/2\pi)^2}. \tag{F.7}$$

This choice is computationally efficient if α, β, γ are all small.

The four representations of θ, given in (F.3), (F.5), (F.6), (F.7), each converge quickly in a particular region of parameter space, as stated above. In fact, these four choices cover the entire range of parameters allowed. Combinations not covered above (e.g., β small with α, γ large) are excluded by (E.6).

When substituted appropriately into the KP equation, an indecomposable theta-function of three variables generates a three-phase, quasiperiodic solution of the KP equation, with each variable corresponding to one phase. Each of these solutions can be viewed as an exact nonlinear superposition of three cnoidal waves. These solutions are inherently unsteady, in every coordinate system obtained by a Galilean-type shift (1.14a).

The four limiting situations discussed above also can be interpreted in terms of the underlying cnoidal waves.

(i) If α, β, γ are all large, then each of the three cnoidal waves has small amplitude; i.e., they are nearly sinusoidal, and their interaction is weak.

(ii) If α is small with β, γ large, then one of the three cnoidal waves can be represented as a periodic train of widely separated, nearly solitary waves, while the other two cnoidal waves are of small amplitude and are nearly sinusoidal.

(iii) Parameters α and β small, with γ large, corresponds to two trains of nearly solitary waves, interacting with one nearly sinusoidal wave train.

(iv) If α, β, γ are all small, then the KP solution represents the nonlinear interaction of three trains of nearly solitary waves.

Appendix G. Dispersion relations for the multiphase solutions of KP

We study the dispersion relations for the theta-functional solutions of KP. By definition, these are the constraints imposed by the KP equation on the parameters $k_1, \ldots, k_N, l_1, \ldots, l_N, \omega_1, \ldots, \omega_N$, $Z = (z_{ij})$ of a theta-functional solution to KP of the form

$$u(x,y,t) = 2\partial_x^2 \log\theta(k_1x + l_1y + \omega_1 t + \phi_{01},\ldots,$$
$$k_Nx + l_Ny + \omega_N T + \phi_{0N}|Z). \qquad (G.1)$$

To formulate these dispersion relations explicitly we introduce theta-functions with characteristics corresponding to the doubled period matrix Z and doubled argument ϕ:

$$\hat{\theta}[m_1,\ldots,m_N](\phi|Z)$$
$$:= \sum_{k_1,\ldots,k_N} \exp\left[-\sum z_{ij}\left(k_i + \frac{m_i}{2}\right)\left(k_j + \frac{m_j}{2}\right) + 2i\sum\left(k_j + \frac{m_j}{2}\right)\phi_j\right] \qquad (G.2)$$

for any integers $m_1,\ldots,m_N$ taking values 0 or 1. The whole vector $[m] = [m_1,\ldots,m_N]$ is called the *characteristic* of the modified theta-function. The function $\hat{\theta}(\phi|Z)$ is again an even entire function of ϕ with certain conditions of periodicity and quasiperiodicity [16]. The values at $\phi = 0$ of the function (G.2) are called *theta-constants*. They depend only on Z and on the characteristic $[m]$. We suppress the arguments $\phi = 0$ and Z in the notations for theta-constants, keeping only their dependence on the characteristic. Values at $\phi = 0$ of the derivatives (necessarily of even order) of these functions we also call theta-constants and denote them as

$$\hat{\theta}_{ij}[m_1,\ldots,m_N] := \left[\partial^2\hat{\theta}[m_1,\ldots,m_N](\phi|Z)/\partial\phi_i\,\partial\phi_j\right]_{\phi=0}, \qquad (G.3)$$

$$\hat{\theta}_{ijkl}[m_1,\ldots,m_N] := \left[\partial^4\hat{\theta}[m_1,\ldots,m_N](\phi|Z)/\partial\phi_i\,\partial\phi_j\,\partial\phi_k\,\partial\phi_l\right]_{\phi=0}. \qquad (G.4)$$

For computation of the theta-constants for a matrix Z in the fundamental region, we use the techniques of Appendix F. We introduce also certain polynomials of $k = (k_1,\ldots,k_N)$, $l = (l_1,\ldots,l_N)$, $\omega = (\omega_1,\ldots,\omega_N)$ with the theta-constants as the coefficients

$$\partial_k^4\hat{\theta}[m] := \sum k_ik_jk_kk_l\hat{\theta}_{ijkl}[m] \qquad (G.5)$$

$$\partial_\omega\partial_k\hat{\theta}[m] := \sum k_i\omega_j\hat{\theta}_{ij}[m] \qquad (G.6)$$

$$\partial_l^2\hat{\theta}[m] := \sum l_il_j\hat{\theta}_{ij}[m]. \qquad (G.7)$$

Proposition 1. *The function* (1.7), (1.8) *satisfies the KP equation for any phase shift* ϕ_0 *iff the parameters* k, l, ω *and the period matrix* Z *satisfy the following system of* 2^N *equations*

$$\partial_k^4 \hat{\theta}[m] + \left(\partial_\omega \partial_k + 3\partial_l^2\right)\hat{\theta}[m] + d\hat{\theta}[m] = 0. \tag{G.8}$$

The characteristics $[m] = [m_1, \ldots, m_N]$, $m_i = 0$ *or* 1 *number the equations. An auxiliary unknown d is to be eliminated from Equations* (G.8).

Proof of this proposition can be obtained by direct substitution of the ansatz (1.7) to KP using the addition theorem for the theta-functions [11, 26]; cf. [27].

When solving the system (G.8), it is important to keep in mind its invariance with respect to transformation of the form

$$\begin{aligned} k &\mapsto bk \\ l &\mapsto b^2 l + ak \\ \omega &\mapsto b^3\omega - 6abl - 3\frac{a^2}{b}k \\ d &\mapsto b^4 d \\ Z &\mapsto Z. \end{aligned} \tag{G.9}$$

This is a manifestation of invariance of KP with respect to the transformations (1.4).

Example 1: For $N = 1$ eliminating d we obtain the dispersion relation in the form

$$\omega k + 3l^2 = k^4 g(z) \tag{G.10}$$

(the period matrix Z here is just a positive number z), where

$$g(z) = -\frac{\hat{\theta}^{IV}[0]\,\hat{\theta}[1] - \hat{\theta}^{IV}[1]\,\hat{\theta}[0]}{\hat{\theta}''[0]\,\hat{\theta}[1] - \hat{\theta}''[1]\,\hat{\theta}[0]}. \tag{G.11}$$

This is the dispersion relation of the traveling waves (1.3). Of course, when $z \to +\infty$ this equation goes to $\omega k + 3l^2 = k^4$.

Example 2: For $N = 2$ eliminating d from the system (G.8) one obtains three independent equations for the period matrix

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}. \tag{G.12}$$

These can be used to determine the entries $\{z_{11}, z_{12}, z_{22}\}$ of the period matrix for a given set of wavenumbers and frequencies. However, it is simpler computationally to start from a given period matrix Z and to solve the dispersion relations (G.8) for $\{k, l, \omega\}$, although not uniquely. In this way, a generic 2×2 period matrix Z generates a family of two-phase solutions of the KP equation. If the period matrix is real, then the KP solutions so obtained are real valued and smooth. For 2×2 period matrices the genericity condition, which assures the consistency of the dispersion relations, is that the period matrix Z cannot be transformed into a diagonal matrix with any transformation of the form (B.4) [11]. In other words, Z must be indecomposable; if Z lies in the fundamental domain, then it must be nondiagonal [19]. The first computations of two-phase solutions of the KP equation were effected in this way in [4].

As discussed in Section 2, two-phase solutions that are genuinely two dimensional are distinguished among all multiphase solutions because they are necessarily stationary; i.e., time-independent in some Galilean coordinate system. KP solutions that are genuinely two dimensional and have nontrivial time dependence must have at least three phases.

Analysis of the dispersion relations for three-phase solutions is more complicated. We sometimes need to use the algebraic-geometrical constructions of Appendix I to prove, say, that the dispersion relations are compatible with each other. Our proofs in Appendix H are more sketchy, but more details can be found in [11, 18, 23].

Appendix H. On computation of three-phase solutions: Solving the dispersion relations

We consider now the dispersion relations (G.8) for the case $N=3$. They have the explicit form

$$\begin{aligned}&\left(\omega_1 k_1 + 3l_1^2\right)\hat{\theta}_{11}[m] + \left(\omega_1 k_2 + \omega_2 k_1 + 6 l_1 l_2\right)\hat{\theta}_{12}[m] + \cdots \\ &\qquad + \left(\omega_3 k_3 + 3 l_3^2\right)\hat{\theta}_{33}[m] + d\hat{\theta}[m] + \partial_k^4 \hat{\theta}[m] = 0, \qquad \text{(H.1)}\end{aligned}$$

where the characteristic $[m]$ takes one of the eight values $[0,0,0], [1,0,0], \ldots, [0,1,1], [1,1,1]$. (Recall that the theta-constants are functions of the period matrix.) We describe first all complex solutions of the system. For a given matrix Z and a given vector k we can consider (H.1) as a linear system of eight equations with seven unknowns

$$\omega_1 k_1 + 3l_1^2, \omega_1 k_2 + \omega_2 k_1 + 6 l_1 l_2, \ldots, \omega_3 k_3 + 3 l_3^2, d.$$

This suggests that the vector k cannot be arbitrary: It must satisfy the compatibility condition

$$\det\left(\hat{\theta}_{11}[m], \hat{\theta}_{12}[m], \ldots, \hat{\theta}_{33}[m], \hat{\theta}[m], \partial_k^4 \hat{\theta}[m]\right) = 0. \tag{H.2}$$

(The characteristic $[m]$ numbers the eight rows of this square matrix.) This gives a quartic homogeneous equation for the vector $k = (k_1, k_2, k_3)$.

LEMMA 5. *For an arbitrary indecomposable period matrix Z the quartic equation* (H.2) *has a family of solutions $k = (k_1, k_2, k_3)$ depending on one complex parameter. The 8×7 matrix of theta-constants*

$$\left(\hat{\theta}_{11}[m], \hat{\theta}_{12}[m], \ldots, \hat{\theta}_{33}[m], \hat{\theta}[m]\right) \tag{H.3}$$

for an indecomposable matrix Z has rank 7.

The second statement of this lemma was proved in [11]. The main idea in the proof of the first statement comes from the construction of Appendix I. According to this algebraic-geometrical construction, the formulae (I.1) and (I.6)–(I.9) determine a solution of the system (H.1) for arbitrary Riemann surface R, arbitrary point $P_0 \in R$ and arbitrary complex local coordinate z on R near the point P_0. Another important point in the proof is the following algebraic-geometrical statement [28]: Any indecomposable 3×3 period matrix Z is a matrix of periods of holomorphic differentials on a Riemann surface of genus 3. The complex parameter determining the solutions k of the quartic equation (H.2) is the marked point P_0 of the corresponding Riemann surface. It is worth noting that for a generic period matrix the quartic equation (H.2) determines a realization of the corresponding Riemann surface on the complex projective plane consisting of all nonzero vectors $k = (k_1, k_2, k_3)$ considered up to multiplication by a nonzero complex number λ. For particular 3×3 matrices the quartic (H.2) becomes a perfect square

$$\det\left(\hat{\theta}_{11}[m], \hat{\theta}_{12}[m], \ldots, \hat{\theta}_{33}[m], \hat{\theta}[m], \partial_k^4 \hat{\theta}[m]\right) = \left[R(k_1, k_2, k_3)\right]^2 \tag{H.4}$$

of a homogeneous quadratic polynomial $R(k_1, k_2, k_3)$. Such period matrices form a 10-dimensional surface in the 12-dimensional space of all (complex) period matrices. Any such indecomposable matrix Z consists of periods of holomorphic differentials on a *hyperelliptic* Riemann surface of genus 3

$$w^2 = z^7 + a_1 z^6 + \cdots + a_7.$$

Because of this we call Z satisfying (H.4) a *hyperelliptic* period matrix.

For a given indecomposable Z, let $[m^1],\dots,[m^7]$ be characteristics such that the 7×7 matrix

$$\begin{pmatrix} \hat\theta_{11}[m^1] & \cdots & \hat\theta_{33}[m^1] & \hat\theta[m^1] \\ \cdots & \cdots & \cdots & \cdots \\ \hat\theta[m^7] & \cdots & \hat\theta_{33}[m^7] & \hat\theta[m^7] \end{pmatrix} \tag{H.5}$$

is not degenerate. We denote by (a_m^{ij}, a_m), $m \in \{m^1,\dots,m^7\}$ the entries of the inverse matrix (they are functions of Z)

$$\begin{pmatrix} a_{m^1}^{11} & \cdots & a_{m^7}^{11} \\ a_{m^1}^{12} & \cdots & a_{m^7}^{12} \\ \cdots & \cdots & \cdots \\ a_{m^1}^{33} & \cdots & a_{m^7}^{33} \\ a_{m^1} & \cdots & a_{m^7} \end{pmatrix}. \tag{H.6}$$

Polynomials $Q_{ij}(k)$ and $P_{ij}(k)$ of k of degrees 4 and 6, respectively, are defined by

$$Q_{ij}(k) := -\sum_{m\in\{m^1,\dots,m^7\}} a_m^{ij}\, \partial_k^4 \hat\theta[m], \tag{H.7}$$

$$P_{ij}(k) = \tfrac{1}{3}\left[k_i^2 Q_{jj}(k) - k_i k_j Q_{ij}(k) + k_j^2 Q_{ii}(k)\right]. \tag{H.8}$$

The system (H.1) can be rewritten in the form

$$k_i l_j - k_j l_i = \sqrt{P_{ij}(k)}, \qquad i < j \tag{H.9}$$

$$\omega_i = \frac{Q_{ii}(k) - 3l_i^2}{k_i} \tag{H.10}$$

for some choice of the signs of the square roots.

LEMMA 6. (i) *For any nonzero solution k of the quartic equation* (H.2), *the identity*

$$k_1\sqrt{P_{23}(k)} - k_2\sqrt{P_{13}(k)} + k_3\sqrt{P_{12}(k)} = 0 \tag{H.11}$$

is valid for some choice of signs of the radicals. For this choice of signs the vectors l and ω can be found from Equations (H.9), (H.10) *uniquely within the ambiguity* (1.20).

(ii) *If Z is a hyperelliptic period matrix then the quartic equation* (H.2) *is compatible with the equations*

$$P_{12}(k) = P_{13}(k) = P_{23}(k) = 0. \tag{H.12}$$

For any common solution of (H.2) *and* (H.12), $l=(0,0,0)$, $\omega_i = Q_{ii}(k)/k_i$ *satisfy* (H.1). *Then* (G.1) *gives a solution of the KdV equation.*

Proof of Lemma 6 can be found in [11].

This statement completes the construction of (complex-valued) three-phase solutions of KP for any indecomposable period matrix Z.

We now explain the statements necessary to obtain all smooth, real-valued KP solutions that can be expressed in terms of Riemann theta-functions of three variables.

PROPOSITION 2. (i) *Let Z be a real symmetric positive-definite indecomposable 3×3 matrix. Then the quartic equation* (H.2) *has four one-parameter families of nonzero real-valued solutions $k=(k_1,k_2,k_3)$ considered as curves on the real projective plane. For any such real solution k the polynomials $P_{ij}(k)$ take real positive values. So the real vectors l and ω can be found uniquely* (*within the ambiguity* (1.20)) *from Equations* (H.9), (H.10).

(ii) *If Z is a real hyperelliptic period matrix then the system* (H.2), (H.12) *has eight* (*up to rescaling*) *real nonzero solutions k. For any of them* (G.1) *reduces to a solution of KdV.*

(iii) *Two three-phase real-valued smooth solutions of KP constructed from real period matrices Z and Z' and two real vectors k and k' respectively satisfying* (H.2) *coincide for arbitrary real phase shift iff the matrices Z and Z' are arithmetically equivalent*

$$Z' = AZA^T \tag{H.13}$$

and the vectors are related by the equation

$$k' = kA^T. \tag{H.14}$$

Proof of Proposition 2 is given at the end of Appendix I.

Appendix I. Algebraic-geometrical construction of multiphase solutions of KP

We proceed here, following [9, 10], to the general theory of the theta-functional solutions to KP. We describe first more general complex meromorphic solutions of the form

$$u(x,y,t) = 2\partial_x^2 \log\theta(k_1x + l_1y + \omega_1t + \phi_{01},\dots, k_Nx + l_Ny + \omega_Nt + \phi_{0N}|Z) + c, \tag{I.1}$$

These will satisfy KP for those x, y, t when $\theta(kx + ly + \omega t + \phi_0) \neq 0$. It is convenient to parametrize these solutions not by the wavenumbers and frequencies but by a collection of algebraic-geometrical data that we describe now.

The main part of our collection of parameters is a compact connected Riemann surface R of genus $N > 0$. Topologically R looks like a sphere with N handles. However, we need to consider R as a one-dimensional complex variety. Then for a given genus $N > 0$ we obtain many biholomorphically inequivalent Riemann surfaces. They form a two-dimensional family for $N = 1$ and a $(6N - 6)$-dimensional family for $N > 1$. This is a part of our parameters, and this part determines the period matrix of the theta-functional multiphase solution.

We also need to fix a point P_0 of the Riemann surface R (this adds two more (real) parameters) and a complex local coordinate z near the point P_0 such that $z(P_0) = 0$. Change of the local coordinate

$$z \mapsto z' = f(z) \tag{I.2}$$

for a holomorphic function $f(z)$ satisfying $f(0) = 0$ does not affect the solution $u(x, y, t)$ to be constructed if

$$f(z) = z + O(z^4). \tag{I.3}$$

The equivalency class of the local coordinate z with respect to transformations (I.2), (I.3) adds six more parameters to our list. Krichever's construction gives thus a $(6N + 2)$-dimensional family of solutions of KP depending also on N arbitrary phase shifts. Vanishing of the (complex) mean value reduces the number of parameters to $6N$. We recall that, in general, these solutions are complex functions with poles; requiring reality and smoothness finally reduces Krichever's family of solutions to a $3N$-dimensional one (see below).

To represent Krichever's solutions by the theta-functional formulae we need to fix additional data on the Riemann surface R: a symplectic basis of cycles (i.e., classes of closed oriented loops in the homology group $H_1(R; \mathbf{Z})$) $a_1, \dots, a_N, b_1, \dots, b_N$. By definition, the intersection numbers of these cycles must have the canonical form

$$a_i \cdot a_j = b_i \cdot b_j = 0, \qquad a_i \cdot b_j = \delta_{ij} \tag{I.4}$$

(the Kronecker delta). Before writing the formula down we want to emphasize that the KP solution does not depend on the choice of the basis on R (there are many of them) but the theta-functional representation of the solution does depend on this choice of basis.

We proceed now to the formulae. A basis of cycles a_i, b_j uniquely determines a basis $\Omega_1, \dots, \Omega_n$ of holomorphic differentials on R normalized by the conditions

$$\oint_{a_k} \Omega_j = \delta_{kj}. \tag{I.5}$$

The period matrix $Z = (z_{kj})$ of the surface R corresponding to the symplectic basis a_i, b_j has the form

$$z_{kj} = \frac{1}{i} \oint_{b_k} \Omega_j, \qquad k, j = 1, \dots, N. \tag{I.6}$$

This matrix is symmetric and its real part is positive definite [29]. This is just the period matrix of the theta-functional solution we are constructing. Another choice of the basis on R gives an equivalent period matrix Z', i.e., related to Z by a transformation of the form (B.4). We recall that the solution of KP does not depend on the choice of basis.

We now give the formulae for the wavenumbers k_i and l_i and frequencies ω_i. This is the place where the marked point P_0 and the local complex coordinate z enter into the game.

Let us represent the basic holomorphic differentials in the form

$$\Omega_j = \varphi_j(z)\, dz, \qquad j = 1, \dots, N, \tag{I.7}$$

where the functions $\varphi_1(z), \dots, \varphi_N(z)$ are holomorphic near $z = 0$. The first coefficients of the expansions of these functions near $z = 0$ are just the wavenumbers and frequencies (up to some elementary factors)

$$\varphi_j(z) = i\left(k_j + l_j z - \tfrac{1}{4}\omega_j z^2 + O(z^3)\right), \qquad j = 1, \dots, N. \tag{I.8}$$

Observe that the vectors $\{k, l, \omega\}$ are linearly dependent iff there exists a meromorphic function on R with a single pole at P_0 of at most order 3 [28]. In particular, for an arbitrary Riemann surface of genus 3 and for a generic point P_0 (i.e., for a non-Weierstrass point), such a function does not exist [28]. Therefore the vectors $\{k, l, \omega\}$ for a generic three-phase solution of KP are linearly independent. (This justifies a claim made in the proof of Theorem 4.)

We explain now, following [18], how to specify smooth real solutions of KP among all the above theta-functional solutions. First, R must be a *real Riemann surface*. By definition the Riemann surface R is called real if it admits an antiholomorphic involution $\sigma\colon R \to R$, $\partial\sigma/\partial z = 0$ $\sigma^2 = \text{identity}$. Existence of such an involution specifies a $(3N-3)$-dimensional subfamily (one dimensional for $N=1$) among all the Riemann surfaces. The involution σ on R will be a part of the data determining a real smooth solution. We also need to impose some topological constraints onto the pair (R,σ). To do this we consider the set of fixed points of the involution σ on R. They form some number of ovals (closed smooth curves) that cannot be greater than $N+1$. Our topological restriction on the pair (R,σ) requires the number of real ovals of σ to be exactly equal to $N+1$. In this case the pair (R,σ) is called *real Riemann surface of maximal type*.

For example, for genus 1, any Riemann surface R can be represented by an algebraic equation of the form

$$w^2 = z^3 + az^2 + bz + c. \tag{I.9}$$

The complex numbers a,b,c are parameters of the Riemann surface, subject to one restriction: The roots z_1, z_2, z_3 of the right-hand side of (I.9) must be distinct. The Riemann surface R will be real if all the numbers a,b,c are real. The antiholomorphic involution then has the form

$$\sigma(z,w) = (\bar{z},\overline{w}), \tag{I.10}$$

where the bar indicates complex conjugation. The Riemann surface corresponding to (I.9) is of maximal type if all the roots z_1, z_2, z_3 are real. Ordering them as $z_1 < z_2 < z_3$ we obtain the fixed ovals of the involution σ as the closed contours laying on R over the segments $[z_1, z_2]$ and $[z_3, \infty]$. We bring to the attention of the reader that there is another real structure σ' on the same Riemann surface

$$\sigma'(z,w) = (\bar{z}, -\overline{w}). \tag{I.11}$$

Generically this is not equivalent to the real structure σ.

The last restriction providing reality and smoothness of the theta-functional solution requires the marked point P_0 to be fixed with respect to the involution σ

$$\sigma(P_0) = P_0 \tag{I.12}$$

(i.e., P_0 belongs to one of the ovals) and the complex local coordinate z must take real values on this oval. The latter can be reformulated in the form

$$z(\sigma(P)) = \overline{z(P)} \tag{I.13}$$

for any P close to P_0. The restriction leaves only four real parameters determining the choice of P_0 on R and of the class of equivalence of the local coordinate z. As shown in [18], all these conditions are necessary and sufficient for reality and smoothness of the solution. We obtain thus a $(3N+1)$-dimensional family of real smooth theta-functional solutions of KP (depending also on N arbitrary real phase shifts). Vanishing of the mean value of u reduces this to a $3N$-dimensional subfamily.

We can represent these real smooth solutions by real theta-functions, i.e., by theta-functions with all real arguments and with a real period matrix. To do this we adjust properly the symplectic basis a_i, b_j on R. Let us denote the ovals of σ by $\Gamma_1, \dots, \Gamma_{N+1}$,

$$\sigma\,|_{\Gamma_k} = \mathrm{id}, \qquad k = 1, \dots, N+1. \tag{I.14}$$

One can take any N of the ovals $\Gamma_{i_1}, \dots, \Gamma_{i_N}$ with an appropriate orientation to construct basic a-cycles

$$a_1 = \Gamma_{i_1}, \dots, a_N = \Gamma_{i_N}. \tag{I.15}$$

These cycles are invariant with respect to σ,

$$\sigma(a_j) = a_j, \qquad j = 1, \dots, N. \tag{I.16}$$

The a-cycles can be completed to a symplectic basis by appropriate b-cycles $b_1, \dots, b_N$ being anti-invariant with respect to σ

$$\sigma(b_j) = -b_j, \qquad j = 1, \dots, N. \tag{I.17}$$

Then the entries of the period matrix Z, the wavenumbers k_j, l_j, and the frequencies ω_j will be real numbers. The formula (G.1) in this case gives a real-valued smooth solution for arbitrary real-valued phase shifts $\phi_{01}, \dots, \phi_{0N}$.

More generally, one can take an integer linear combination of the cycles

$$a_1, \dots, a_N \mapsto a'_1, \dots, a'_N, \tag{I.18}$$

where

$$(a_1, \dots, a_N) = (a'_1, \dots, a'_N)A \tag{I.19}$$

for an $N \times N$ matrix A with integer entries and with determinant equal to ± 1. Particularly, another choice of N ovals $\Gamma'_{i_1}, \ldots, \Gamma'_{i_N}$ as the a'-cycles gives such a transformation since in the homologies of the Riemann surface the equation

$$\Gamma_1 + \cdots + \Gamma_{N+1} = 0$$

holds true. As above, we complete this basis to a symplectic basis on R by some σ-anti-invariant b'-cycles. We call any such a basis $a'_1, \ldots, a'_N, b'_1, \ldots, b'_N$ on a real Riemann surface R of the maximal type *compatible* with the real structure σ and R. For any such a compatible basis we still obtain a real symmetric positive-definite period matrix Z'. It is related to the period matrix Z by the transformation

$$Z' = AZA^T, \qquad A \in \mathrm{GL}(N, \mathbf{Z}). \tag{I.20}$$

As explained above, this is a particular class of Siegel modular transformations (recall that the two matrices Z and Z' are called arithmetically equivalent).

Summarizing the discussion of this section we formulate the following theorem, which was proved in [18].

PROPOSITION 3. *For any real Riemann surface (R, σ) of genus N of the maximal type with a marked point P_0 satisfying* (I.12) *and a complex coordinate z near P_0 satisfying* (I.13) *and for any symplectic basis on R compatible with the real structure σ*, (I.1) *provides a real-valued smooth solution of KP for arbitrary real phase shifts. The solution does not depend on the choice of the compatible basis of cycles. Any real-valued theta-functional solutions, smooth for arbitrary real phase shifts, can be obtained by this construction.*

Proof of Proposition 2 (from Appendix H): Let (R, σ) be any nonhyperelliptic Riemann surface of genus three of maximal type (i.e., with four real ovals of σ). We choose first an arbitrary symplectic basis (a'_j, b'_j) on R compatible with the real structure σ. Then the period matrix Z' computed by (I.6) is a real-valued, symmetric, positive-definite matrix. It is equivalent under (I.20, with $N = 3$) to a real-valued, symmetric matrix Z in the fundamental region, which is defined by (1.17). We introduce a new symplectic basis (a_j, b_j) on R:

$$(a_1, a_2, a_3) = (a'_1, a'_2, a'_3) A,$$

$$(b_1, b_2, b_3) = A^T (b'_1, b'_2, b'_3). \tag{I.21}$$

This basis is still compatible with the real structure σ, due to the reality of A. From (I.6), periods of holomorphic differentials on R with respect to the basis (a_j, b_j) coincide with the elements of Z.

Next we show that for this matrix, Z, the quartic equation in (H.2) has four one-parameter families of nonzero real-valued solutions, considering each as a smooth closed curve on the real projective plane. First, let P_0 be any fixed point of the involution σ. Then the numbers $\{k_j = k_j(P_0),\ l_j = l_j(P_0),\ \omega_j = \omega_j(P_0)\}$, defined by (I.8), are real for $j = 1, 2, 3$. They satisfy the system (H.2), in which the theta-functions are computed from the matrix Z, because u in (I.1) solves KP for arbitrary phase shifts, ϕ_{0j}. Now choose a local coordinate, z, near P_0 satisfying (I.13). Changes of z transform the vectors (k, l, ω) according to (G.9). We obtain in this way a well-defined map

$$P_0 \to \{k_1(P_0) : k_2(P_0) : k_3(P_0)\} \tag{I.22}$$

from any of the real ovals $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ to the real projective plane. Because R is not a hyperelliptic curve, this map is a smooth embedding. (This is also true for complex points $P_0 \in R$, considering (I.21) as a map to the complex projective plane.) Thus, the set of real-valued solutions of the quartic equation (H.2) contain at least four components, corresponding to the four real ovals on R. It is easy to see that (H.2) has no other real-valued solutions, using again that (I.22) is a smooth embedding. Then due to the uniqueness of solutions of (H.9), (H.10), for any real-valued choice of $\{k_1(P_0), k_2(P_0), k_3(P_0)\}$, we obtain real-valued solutions $\{k_j(P_0), l_j(P_0), \omega_j(P_0)\}$ for the system (H.1).

We remark that for a real-valued hyperelliptic surface R of maximal type (i.e., with eight real branch points), the map (I.22) is a degree 2 covering of R onto a smooth of rational curve on the real projective plane. Thus in the hyperelliptic case, the real solutions of the quartic equation (H.2) form four smooth curvilinear segments on the real projective plane $(k_1 : k_2 : k_3)$. The eight endpoints of these segments are the images of the branchpoints of R. If $(k_1 : k_2 : k_3)$ is one of these endpoints, then (G.1) provides a solution of KdV.

Now let us deform the period matrix, keeping it within the fundamental region. Denote by Z_s the deformed matrix, and by Z_0 the original matrix. By definition, Z_0 is the period matrix of the chosen Riemann matrix (R, σ), and Z_0 is indecomposable. We may assume that during the deformation, Z_s remains indecomposable because the set of decomposable matrices has codimension 2 in the full parameter space. As Z_s deforms continuously, the solution of the quartic equation (H.2) also deforms continuously in the complex projective plane. But the map (I.22) is a smooth embedding, so this solution corresponds to a continuously deforming, smooth Riemann surface,

R_s. This Riemann surface admits an antiholomorphic involution,

$$\sigma : k \to \bar{k},$$

because the coefficients of (H.2) are real numbers. As s changes, the topological type of the pair (R_s, σ) depends continuously on s, so it remains constant during the deformation. In other words, for any s the Riemann surface R_s with involution σ is of maximal type. So for any s the real solutions of the quartic equation form four components. Finally, the connectedness of the fundamental region completes the proof of the first two statements of Proposition 2 and the proof of Theorem 3.

To complete the description of all real-valued, smooth KP solutions associated with a theta-function of three variables, we use Proposition 3 (above). According to this result, any three-phase, real-valued, smooth solution of KP can be constructed from the formulae above, starting from a real Riemann surface (R, σ) of maximal type, a real point P_0, and a symplectic basis (a_j, b_j). Another choice of compatible symplectic basis (a'_j, b'_j) gives the same solution of KP, while the period matrices Z and Z' and the wave vectors k and k' are related by (H.13), (H.14). Indeed, a change to a compatible symplectic basis can be described by (I.21), after which the corresponding normalized holomorphic differentials are related by

$$(\Omega'_1, \ldots, \Omega'_N) = (\Omega_1, \ldots, \Omega_N) A^T,$$

the corresponding period matrices, Z, Z' by (H.13), and the wave vectors by (H.14). This completes the proof of Proposition 2.

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SISSA, TRIESTE, ITALY
EVOLVING VIDEO TECHNOLOGIES, ARVADA, CO
UNIVERSITY OF COLORADO, BOULDER

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