Dahmen W (1997) Wavelets and multiscale methods for operator equations. *Acta Numerica* 6: 55–228.
Daubechies I (1988) Orthonormal bases of compactly supported wavelets. *Communications in Pure and Applied Mathematics* 41: 909.
Daubechies I (1992) *Ten Lectures on Wavelets*. Philadelphia, PA: SIAM.
Donoho and Johnstone (1994) Ideal spatial adaptation via wavelet shrinkage. *Biometrika* 81: 425.
Grossmann A and Morlet J (1984) Decomposition of Hardy functions into square integrable wavelets of constant shape. *SIAM Journal of Mathematical Analysis*. 15(4): 723–736.
Lemarié P-G and Meyer Yves (1986) Ondolettes et bases hilbertiennes. *Revista Matematica Iberoamericano* 2: 1.
Mallat S (1989) Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$. *Transactions of the American Mathematical Society* 315: 69.
Mallat S (1998) *A Wavelet Tour of Signal Processing*. San Diego, CA: Academic Press.

WDVV Equations and Frobenius Manifolds

B Dubrovin, SISSA-ISAS, Trieste, Italy

Main Definition

WDVV equations of associativity (after E Witten, R Dijkgraaf, E Verlinde, and H Verlinde) is tantamount to the following problem: find a function $F(v)$ of n variables $v=(v^1,v^2,\ldots,v^n)$ satisfying the conditions [1], [3], and [4] given below. First,

$$\frac{\partial^3 F(v)}{\partial v^1 \partial v^\alpha \partial v^\beta} \equiv \eta_{\alpha\beta} \qquad [1]$$

must be a constant symmetric nondegenerate matrix. Denote $(\eta^{\alpha\beta})=(\eta_{\alpha\beta})^{-1}$ the inverse matrix and introduce the functions

$$c^\gamma_{\alpha\beta}(v) = \eta^{\gamma\epsilon}\frac{\partial^3 F(v)}{\partial v^\epsilon \partial v^\alpha \partial v^\beta}, \quad \alpha,\beta,\gamma = 1,\ldots,n \qquad [2]$$

The main condition says that, for arbitrary $v^1,\ldots,v^n$ these functions must be structure constants of an associative algebra, that is, introducing a v-dependent multiplication law in the n-dimensional space by

$$a\cdot b := \left(c^1_{\alpha\beta}(v)a^\alpha b^\beta,\ldots,c^n_{\alpha\beta}(v)a^\alpha b^\beta\right)$$

one obtains an n-parameter family of n-dimensional associative algebras (these algebras will automatically be also commutative). Spelling out this condition one obtains an overdetermined system of nonlinear PDEs for the function $F(v)$ often also called WDVV associativity equations

$$\frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F(v)}{\partial v^\mu \partial v^\gamma \partial v^\delta} = \frac{\partial^3 F(v)}{\partial v^\delta \partial v^\beta \partial v^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F(v)}{\partial v^\mu \partial v^\gamma \partial v^\alpha} \qquad [3]$$

for arbitrary $1\le \alpha,\beta,\gamma,\delta\le n$. (Summation over repeated indices will always be assumed.) The last one is the so-called quasihomogeneity condition

$$EF = (3-d)F + \tfrac{1}{2}A_{\alpha\beta}v^\alpha v^\beta + B_\alpha v^\alpha + C \qquad [4]$$

where

$$E = \left(a^\alpha_\beta v^\beta + b^\alpha\right)\frac{\partial}{\partial v^\alpha}$$

for some constants a^α_β, b^α satisfying

$$a^\alpha_1 = \delta^\alpha_1, \qquad b^1 = 0$$

$A_{\alpha\beta}, B_\alpha, C, d$ are some constants. E is called Euler vector field and d is the charge of the Frobenius manifold.

For $n=1$ one has $F(v)=(1/6)v^3$. For $n=2$ one can choose

$$F(u,v) = \tfrac{1}{2}uv^2 + f(u)$$

only the quasihomogeneity [4] makes a constraint for $f(v)$. The first nontrivial case is for $n=3$. The solution to WDVV is expressed in terms of a function $f=f(x,y)$ in one of the two forms (in the examples all indices are written as lower):

$$\begin{aligned} d\neq 0:\quad & F = \tfrac{1}{2}v_1^2 v_3 + \tfrac{1}{2}v_1 v_2^2 + f(v_2,v_3) \\ & f_{xxy}^2 = f_{yyy} + f_{xxx}f_{xyy} \\ d = 0:\quad & F = \tfrac{1}{6}v_1^3 + v_1 v_2 v_3 + f(v_2,v_3) \\ & f_{xxx}f_{yyy} - f_{xxy}f_{xyy} = 1 \end{aligned} \qquad [5]$$

The function $f(x,y)$ satisfies additional constraint imposed by [4]. Because of this the above PDEs [5] can be reduced (Dubrovin 1992, 1996) to a particular case of the Painlevé-VI equation (*see* Painlevé Equations).

The problem [1], [3], [4] is invariant with respect to linear changes of coordinates preserving the direction of the vector $\partial/\partial v^1$:

$$v^\alpha \mapsto \tilde v^\alpha = P^\alpha_\beta v^\beta + Q^\alpha,\ \det(P^\alpha_\beta)\neq 0, \quad P^\alpha_1 = \delta^\alpha_1$$

It is also allowed to add to $F(v)$ a polynomial of the degree at most 2. To consider more general non-linear changes of coordinates one has to give a coordinate-free form of the above equations [1], [3], [4]. This gives rise to the notion of Frobenius manifold introduced in Dubrovin (1992).

Recall that a Frobenius algebra is a pair $(A, <,>)$, where A is a commutative associative algebra with a unity e over a field k (we will consider only the cases $k = \mathbb{R}, \mathbb{C}$) and $<,>$ is a k-bilinear symmetric non-degenerate invariant form on A, that is,

$$<x \cdot y, z> = <x, y \cdot z>$$

for arbitrary vectors x, y, z in A.

Definition Frobenius structure $(\cdot, e, <,>, E, d)$ on the manifold M is a structure of a Frobenius algebra on the tangent spaces $T_v M = (A_v, <,>_v)$ depending (smoothly, analytically, etc.) on the point $v \in M$. It must satisfy the following axioms.

FM1. The curvature of the metric $<,>_v$ on M (not necessarily positive definite) vanishes. Denote ∇ the Levi-Civita connection for the metric. The unity vector field e must be flat, $\nabla e = 0$.

FM2. Let c be the 3-tensor $c(x, y, z) := <x \cdot y, z>$, $x, y, z \in T_v M$. The 4-tensor $(\nabla_w c)(x, y, z)$ must be symmetric in $x, y, z, w \in T_v M$.

FM3. A linear vector field $E \in \mathrm{Vect}(M)$ (called Euler vector field) must be fixed on M, that is, $\nabla\nabla E = 0$, such that

$$\mathrm{Lie}_E(x \cdot y) - \mathrm{Lie}_E x \cdot y - x \cdot \mathrm{Lie}_E y = x \cdot y$$
$$\mathrm{Lie}_E <,> = (2 - d) <,>$$

for some number $d \in k$ called "charge."

The last condition (also called quasihomogeneity) means that the derivations $Q_{\mathrm{Func}(M)} := E$, $Q_{\mathrm{Vect}(M)} := \mathrm{id} + \mathrm{ad}_E$ define on the space $\mathrm{Vect}(M)$ of vector fields on M a structure of graded Frobenius algebra over the graded ring of functions $\mathrm{Func}(M)$.

Flatness of the metric $<,>$ implies local existence of a system of flat coordinates $v^1, \ldots, v^n$ on M. Usually, they are chosen in such a way that

$$e = \frac{\partial}{\partial v^1}$$

is the unity vector field. In such coordinates, the problem of local classification of Frobenius manifolds reduces to the WDVV associativity equations [1], [3], [4]. Namely, $\eta_{\alpha\beta}$ is the constant Gram matrix of the metric in these coordinates

$$\eta_{\alpha\beta} := \left\langle \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta} \right\rangle$$

The structure constants of the Frobenius algebra $A_v = T_v M$

$$\frac{\partial}{\partial v^\alpha} \cdot \frac{\partial}{\partial v^\beta} = c^\gamma_{\alpha\beta}(v) \frac{\partial}{\partial v^\gamma} \qquad [6]$$

can be locally represented by third derivatives [2] of a function $F(v)$ satisfying [1], [3], [4]. The function $F(v)$ is called "potential" of the Frobenius manifold. It is defined up to adding of an at most quadratic polynomial in $v^1, \ldots, v^n$.

A generalization of the above definition to the case of Frobenius supermanifolds can be found in Manin (1999). For the more general class of the so-called F-manifolds, the requirement of the existence of a flat invariant metric has been relaxed.

Deformed Flat Connection

One of the main geometrical structures of the theory of Frobenius manifolds is the deformed flat connection. This is a symmetric affine connection on $M \times \mathbb{C}^*$ defined by the following formulas:

$$\begin{aligned} &\tilde{\nabla}_x y = \nabla_x y + z x \cdot y, \quad x, y \in TM, z \in \mathbf{C}^* \\ &\tilde{\nabla}_{\mathrm{d}/\mathrm{d}z} y = \partial_z y + E \cdot y - \frac{1}{z} \mathcal{V} y \\ &\tilde{\nabla}_x \frac{\mathrm{d}}{\mathrm{d}z} = \tilde{\nabla}_{\mathrm{d}/\mathrm{d}z} \frac{\mathrm{d}}{\mathrm{d}z} = 0 \end{aligned} \qquad [7]$$

where, as above, ∇ is the Levi-Civita connection for the metric $<,>$ and

$$\mathcal{V} := \frac{2 - d}{2} - \nabla E \qquad [8]$$

is an operator on the tangent bundle TM antisymmetric with respect to $<,>$,

$$<\mathcal{V} x, y> = - <x, \mathcal{V} y>$$

Observe that the unity vector field e is an eigenvector of this operator with the eigenvalue

$$\mathcal{V} e = -\frac{d}{2} e$$

The connection $\tilde{\nabla} = \tilde{\nabla}(z)$ is not metric but it satisfies

$$\nabla <x, y> = <\tilde{\nabla}(-z) x, y> + <x, \tilde{\nabla}(z) y>$$
$$x, y \in TM$$

for any $z \in \mathbb{C}^*$. As it was discovered in Dubrovin (1992), vanishing of the curvature of the connection $\tilde{\nabla}$ is essentially equivalent to the axioms of Frobenius manifold.

Definition A "deformed flat function" $f(v;z)$ on a domain in $M \times \mathbb{C}^*$ is defined by the requirement of horizontality of the differential df

$$\tilde{\nabla} df = 0 \qquad [9]$$

Due to vanishing of the curvature of $\tilde{\nabla}$ locally there exist n independent deformed flat functions $f_1(v;z), \ldots, f_n(v;z)$ such that their differentials, together with the flat 1-form dz, span the cotangent plane $T^*_{(v;z)}(M \times \mathbb{C}^*)$. They will be called "deformed flat coordinates." The global analytic properties of deformed flat coordinates can be derived, for the case of semisimple Frobenius manifolds, from the results of the section "Moduli of semisimple Frobenius manifolds" discussed later.

One can relax the definition of Frobenius manifold dropping the last axiom FM3. The potential $F(v)$ in this case satisfies [1] and [3] but not [4]. In this case, the deformed flat connection $\tilde{\nabla}$ is just a family of affine flat connections on M depending on the parameter $z \in \mathbb{C}$ given by the first line in [7]. The curvature and torsion of this family of connections vanishes identically in z. The deformed flat functions of $\tilde{\nabla}$ defined as in [9] can be chosen in the form of power series in z. The flatness equations written in the flat coordinates on M yield a recursion equation for the coefficients of these power series

$$\tilde{\nabla} df = 0, \quad f = \sum_{p \geq 0} \theta_p(v) z^p$$
$$\partial_\lambda \partial_\mu f = z c^\nu_{\lambda\mu}(v) \partial_\nu f$$

$$\begin{aligned} &\partial_\lambda \partial_\mu \theta_0(v) = 0 \\ &\partial_\lambda \partial_\mu \theta_{p+1}(v) = c^\nu_{\lambda\mu}(v) \partial_\nu \theta_p(v) \end{aligned} \quad p \geq 0 \qquad [10]$$

Thus, $f(v;0)$ is just an affine linear function of the flat coordinates $v^1, \ldots, v^n$; the dependence on z can be considered as a deformation of the affine structure. This motivates the name "deformed flat coordinates." The coefficients of the expansions of the deformed flat coordinates are the leading terms of the ε-expansion of the Hamiltonian densities of the integrable hierarchies associated with the Frobenius manifolds (see below).

Intersection Form of a Frobenius Manifold

Another important geometric structure on M is the intersection form of the Frobenius manifold. It is a symmetric bilinear form on the cotangent bundle T^*M defined by the formula

$$(\omega_1, \omega_2) = i_E \omega_1 \cdot \omega_2, \quad \omega_1, \omega_2 \in T^*M \qquad [11]$$

Here the multiplication law on the cotangent planes is defined by means of the isomorphism.

$$<,>: TM \to T^*M$$

The discriminant $\Sigma \subset M$ is a proper analytic (for an analytic M) subset where the intersection form degenerates. One can introduce a new metric on the open subset $M \backslash \Sigma$ taking the inverse of the intersection form. A remarkable result of the theory of Frobenius manifolds is vanishing of the curvature of this new metric. Moreover, the new flat metric together with the following new multiplication:

$$x * y := x \cdot y \cdot E^{-1}$$

defines on $M \backslash \Sigma$ a structure of an almost-dual Frobenius manifold (Dubrovin 2004). In the original flat coordinates $v^1, \ldots, v^n$ the coordinate expressions for the new metric and for the associated Levi-Civita connection ∇^*, called the Gauss–Manin connection, read

$$\begin{aligned} g^{\alpha\beta}(v) &:= (dv^\alpha, dv^\beta) = E^\gamma(v) c^{\alpha\beta}_\gamma(v) \\ \nabla^{*\alpha}\, dv^\beta &= \Gamma^{\alpha\beta}_\gamma(v)\, dv^\gamma \\ \Gamma^{\alpha\beta}_\gamma(v) &:= -g^{\alpha\nu}(v)\Gamma^\beta_{\nu\gamma}(v) = c^{\alpha\epsilon}_\gamma(v)\left(\frac{1}{2} - \mathcal{V}\right)^\beta_\epsilon \end{aligned} \qquad [12]$$

The pair $(\,,\,)$ and $<\,,\,>$ of bilinear forms on T^*M possesses the following property crucial for understanding the relationships between Frobenius manifolds and integrable systems: they form a flat pencil. That means that on the complement to the subset

$$\Sigma_\lambda := \{v \in M \mid \det(g^{\alpha\beta}(v) - \lambda \eta^{\alpha\beta}) = 0\}$$

The inverse to the bilinear form

$$(\,,\,)_\lambda := (\,,\,) - \lambda <,> \qquad [13]$$

defines a metric with vanishing curvature. Flat functions $p = p(v;\lambda)$ for the flat metric are determined from the system

$$(\nabla^* - \lambda \nabla)\, dp = 0 \qquad [14]$$

They are called "periods" of the Frobenius manifold. The periods $p(v;\lambda)$ are related to the deformed flat functions $f(v;z)$ by the suitably regularized Laplace-type integral transform

$$p(v;\lambda) = \int_0^\infty e^{-\lambda z} f(v;z) \frac{dz}{\sqrt{z}} \qquad [15]$$

Choosing a system of n independent periods, one obtains a system of flat coordinates $p^1(v;\lambda), \ldots, p^n(v;\lambda)$ for the metric $(\,,\,)_\lambda$ on $M \backslash \Sigma_\lambda$,

$$\left(dp^i(v;\lambda), dp^j(v;\lambda)\right)_\lambda = G^{ij} \qquad [16]$$

for some constant nondegenerate matrix G^{ij}.

The structure of a flat pencil on the Frobenius manifold M gives rise to a natural Poisson pencil (= bi-Hamiltonian structure) on the infinite-dimensional "manifold" $\mathcal{L}(M)$ consisting of smooth maps of a circle to M (the so-called loop space). In the flat coordinates $v^1, \ldots, v^n$ for the metric $<,>$ the Poisson pencil has the form

$$\begin{aligned}\{v^\alpha(x), v^\beta(y)\}_1 &= \eta^{\alpha\beta}\delta'(x-y)\\ \{v^\alpha(x), v^\beta(y)\}_2 &= g^{\alpha\beta}(v(x))\delta'(x-y)\\ &\quad + \Gamma^{\alpha\beta}_\gamma(v(x))v^\gamma_x\delta(x-y)\end{aligned} \qquad [17]$$

By definition of the Poisson pencil, the linear combination $a_1\{,\}_1 + a_2\{,\}_2$ of the Poisson brackets is again a Poisson bracket for arbitrary constants a_1, a_2. Choosing a system of n independent periods $p^i(v;\lambda), i=1,\ldots,n$, as a new system of dependent variables, one obtains a reduction of the Poisson bracket $\{,\}_\lambda := \{,\}_2 - \lambda\{,\}_1$ for a given λ to the canonical form

$$\{p^i(v(x);\lambda), p^j(v(y);\lambda)\}_\lambda = G^{ij}\delta'(x-y) \qquad [18]$$

Under an additional assumption of existence of tau function (Dubrovin 1996, Dubrovin and Zhang), one can prove that any Poisson pencil on $\mathcal{L}(M)$ of the form [17] with a nondegenerate matrix $(\eta^{\alpha\beta})$ comes from a Frobenius structure on M.

Canonical Coordinates on Semisimple Frobenius Manifolds

Definition The Frobenius manifold M is called semisimple if the algebras T_vM are semisimple for v belonging to an open dense subset in M.

Any n-dimensional semisimple Frobenius algebra over $\mathbb{C}$ is isomorphic to the orthogonal direct sum of n copies of one-dimensional algebras. In this section, all the manifolds will be assumed to be complex analytic.

Near a semisimple point, the roots $u_i = u_i(v)$, $i=1,\ldots,n$, of the characteristic equation

$$\det(g^{\alpha\beta}(v) - \lambda\eta^{\alpha\beta}) = 0 \qquad [19]$$

can be used as local coordinates. The vectors $\partial/\partial u_i, i=1,\ldots,n$, are basic idempotents of the algebras T_vM

$$\frac{\partial}{\partial u_i}\cdot\frac{\partial}{\partial u_j} = \delta_{ij}\frac{\partial}{\partial u_i}$$

We call $u_1,\ldots,u_n$ "canonical coordinates." Observe that we violate the indices convention labeling the canonical coordinates by subscripts. We will never use summation over repeated indices when working in the canonical coordinates. Actually, existence of canonical coordinates can be proved without using [4] (see details in Dubrovin (1992)).

Choosing locally branches of the square roots

$$\psi_{i1}(u) := \sqrt{<\partial/\partial u_i, \partial/\partial u_i>}, \quad i=1,\ldots,n \qquad [20]$$

we obtain a transition matrix $\Psi = (\psi_{i\alpha}(u))$,

$$\frac{\partial}{\partial v^\alpha} = \sum_{i=1}^n \frac{\psi_{i\alpha}(u)}{\psi_{i1}(u)}\frac{\partial}{\partial u_i} \qquad [21]$$

from the basis $\partial/\partial v^\alpha$ to the orthonormal basis

$$\begin{aligned}\langle f_i, f_j\rangle &= \delta_{ij}\\ f_1 &= \psi_{11}^{-1}(u)\frac{\partial}{\partial u_1}\\ f_2 &= \psi_{21}^{-1}(u)\frac{\partial}{\partial u_2}, \ldots\\ f_n &= \psi_{n1}^{-1}(u)\frac{\partial}{\partial u_n}\end{aligned} \qquad [22]$$

The matrix $\Psi(u)$ satisfies orthogonality condition

$$\Psi^*(u)\Psi(u) \equiv \eta, \quad \eta = (\eta_{\alpha\beta}), \quad \eta_{\alpha\beta} := \left\langle \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right\rangle$$

In this formula Ψ^* stands for the transposed matrix. The lengths [20] coincide with the first column of this matrix.

Denote $V(u) = (V_{ij}(u))$ the matrix of the antisymmetric operator $\mathcal{V}$ [8] with respect to the orthonormal frame

$$V(u) := \Psi(u)\mathcal{V}\Psi^{-1}(u) \qquad [23]$$

The antisymmetric matrix $V(u) = (V_{ij}(u))$ satisfies the following system of commuting time-dependent Hamiltonian flows on the Lie algebra so(n) equipped with the standard Lie–Poisson brackets $\{V_{ij}, V_{kl}\} = V_{il}\delta_{jk} - V_{jl}\delta_{ik} + V_{jk}\delta_{il} - V_{ik}\delta_{jl}$:

$$\frac{\partial V}{\partial u_i} = \{V, H_i(V;u)\}, \quad i=1,\ldots,n \qquad [24]$$

with quadratic Hamiltonians

$$H_i(V;u) = \frac{1}{2}\sum_{j\neq i}\frac{V_{ij}^2}{u_i - u_j} \qquad [25]$$

The matrix $\Psi(u)$ satisfies

$$\begin{aligned}\frac{\partial\Psi}{\partial u_i} &= V_i(u)\Psi,\\ V_i(u) &:= \mathrm{ad}_{E_i}\mathrm{ad}_U^{-1}(V(u)), \ i=1,\ldots,n\end{aligned} \qquad [26]$$

Here the matrix unity E_i has the entries $(E_i)_{ab} = \delta_{ai}\delta_{ib}, U = \mathrm{diag}(u_1,\ldots,u_n)$. Conversely, given a solution to [24] and [26], one can reconstruct the

Frobenius manifold structure by quadratures (Dubrovin 1998). The reconstruction depends on a choice of an eigenvector of the constant matrix $\mathcal{V} = \Psi^{-1}(u)\ V(u)\Psi(u)$.

The system [24] coincides with the equations of isomonodromic deformations (*see* Isomonodromic Deformations) of the following linear differential operator with rational coefficients:

$$\frac{dY}{dz} = \left(U + \frac{\mathcal{V}}{z}\right)Y \qquad [27]$$

The latter is nothing but the last component of the deformed flat connection [7] written in the orthonormal frame [22]. Other components of the horizontality equations yield

$$\partial_i Y = (zE_i + V_i(u))Y, \quad i = 1, \ldots, n \qquad [28]$$

The compatibility conditions of the system [27] and [28] coincide with [24].

The integration of [24], [26] and, more generally, the reconstruction of the Frobenius structure can be reduced to a solution of a certain Riemann–Hilbert problem (*see* Riemann–Hilbert Problem).

The isomonodromic tau function of the semisimple Frobenius manifold is defined by

$$d \log \tau_I(u) = \sum_{i=1}^{n} H_i(V(u); u) du_i \qquad [29]$$

It is an analytic function on a suitable unramified covering of the semisimple part of M.

Alternatively, eqns [24] can be represented as the isomonodromy deformations of the dual Fuchsian system

$$[U - \lambda]\frac{d\phi}{d\lambda} = \left(\frac{1}{2} + V\right) \qquad [30]$$

The latter comes from the Gauss–Manin system for the periods $p = p(v; \lambda)$ of the Frobenius manifold written in the canonical coordinates [22].

Moduli of Semisimple Frobenius Manifolds

All n-dimensional semisimple Frobenius manifolds form a finite-dimensional space. They depend on $n(n-1)/2$ essential parameters. To parametrize the Frobenius manifolds one can choose, for example, the initial data for the isomonodromy deformation equations [24]. Alternatively, they can be parametrized by monodromy data of the deformed flat connection according to the following construction.

The first part of the monodromy data is the spectrum $(V, <\,,\,>, \hat{\mu}, R)$ of the Frobenius manifold associated with the Poisson pencil. Here V is an n-dimensional linear space equipped with a symmetric nondegenerate bilinear form $<\,,\,>$. Two linear operators on V, a semisimple operator $\hat{\mu}: V \to V$, and a nilpotent operator $R: V \to V$ must satisfy the following properties. First, the operator $\hat{\mu}$ is antisymmetric:

$$\hat{\mu}^* = -\hat{\mu} \qquad [31]$$

and the operator R satisfies

$$R^* = -e^{-\pi i \hat{\mu}} R\, e^{\pi i \hat{\mu}} \qquad [32]$$

Here the adjoint operators are defined with respect to the bilinear form $<\,,\,>$. The last condition to be imposed onto the operator R can be formulated in a simple way by choosing a basis $e_1, \ldots, e_n$ of eigenvectors of the semisimple operator $\hat{\mu}$,

$$\hat{\mu} e_\alpha = \mu_\alpha e_\alpha, \quad \alpha = 1, \ldots, n$$

We require the existence of a decomposition

$$R = R_0 + R_1 + R_2 + \cdots \qquad [33]$$

where for any integer $k \geq 0$ the linear operator R_k satisfies

$$R_k e_\alpha \in \operatorname{span}\{e_\beta \,|\, \mu_\beta = \mu_\alpha + k\} \ \ \forall \alpha = 1, \ldots, n \qquad [34]$$

In the nonresonant case, such that none of the differences of the eigenvalues of $\hat{\mu}$ being equal to a positive integer, all the matrices $R_1, R_2, \ldots$, are equal to zero. Observe a useful identity

$$z^{\hat{\mu}} R\, z^{-\hat{\mu}} = R_0 + zR_1 + z^2 R_2 + \cdots \qquad [35]$$

More generally, for any operator $A: V \to V$ commuting with $e^{2\pi i \hat{\mu}}$ a decomposition is defined as

$$\begin{aligned} A &= \oplus_{k \in \mathbb{Z}} [A]_k \\ z^{\hat{\mu}} A\, z^{-\hat{\mu}} &= \sum_{k \in \mathbb{Z}} z^k [A]_k \end{aligned} \qquad [36]$$

In particular, $[R]_k = R_k, k \geq 0$, $[R]_k = 0$, $k < 0$.

One has to also choose an eigenvector e of the operator $\hat{\mu}$ such that $R_0 e = 0$; denote $-d/2$ the corresponding eigenvalue

$$e \in V, \qquad \hat{\mu} e = -\frac{d}{2} e, \qquad R_0 e = 0 \qquad [37]$$

The second part of the monodromy data is a pair of linear operators

$$C: V \to \mathbb{C}^n, \qquad S: \mathbb{C}^n \to \mathbb{C}^n$$

The space $\mathbb{C}^n$ is assumed to be equipped with the standard complex Euclidean structure given by the sum of squares. The properties of the operators S, C depend on the choice of an unordered set

$u^0 = (u_1^0, \ldots, u_n^0)$ of n pairwise distinct complex numbers and on a choice of a ray ℓ_+ on an auxiliary complex z-plane starting at the origin such that

$$\operatorname{Re} z\left(u_i^0 - u_j^0\right) \neq 0, \quad i \neq j, \ z \in \ell_+ \qquad [38]$$

Let us order the complex numbers in such a way that

$$e^{z(u_i^0 - u_j^0)} \to 0, \quad i < j, |z| \to \infty, \ z \in \ell_+ \qquad [39]$$

The operator S must be upper triangular

$$\begin{aligned} S &= (S_{ij}), \quad S_{ij} = 0, \ i > j \\ S_{ii} &= 1, \qquad\quad i = 1, \ldots, n \end{aligned} \qquad [40]$$

The operator C must satisfy

$$C^* S\, C = e^{\pi i \hat\mu} e^{\pi i R} \qquad [41]$$

Here the adjoint operator C^* is understood as follows:

$$C^* : \mathbb{C}^n \overset{\simeq}{\to} \mathbb{C}^{n*} \to V^* \xrightarrow{<,>^{-1}} V$$

The group of diagonal $n \times n$ matrices

$$D = \operatorname{diag}(\pm 1, \ldots, \pm 1)$$

acts on the pairs (S, C) by

$$S \mapsto DSD, \qquad C \mapsto DC$$

One is to factor out the action of this diagonal group. Besides, the operator C is defined up to a left action of certain group of linear operators depending on the spectrum.

For the generic (i.e., nonresonant) case where $e^{2\pi i \hat\mu}$ has simple spectrum, the operator C is defined up to left multiplication by any matrix commuting with $e^{2\pi i \hat\mu}$. In this situation, the monodromy data $(\hat\mu, R, S, C)$ are locally uniquely determined by the $n(n-1)/2$ entries of the matrix S. Therefore, near a generic point, the variety of the monodromy data is a smooth manifold of the dimension $n(n-1)/2$. At nongeneric points, the variety can get additional strata.

The monodromy data S, C are determined at an arbitrary semisimple point of a Frobenius manifold in terms of the analytic properties of horizontal sections of the deformed flat connection $\tilde\nabla$ [7] in the complex z-plane (the so-called "Stokes matrix" and the "central connection matrix" of the operator [27]). Locally, they do not depend on the point of the semisimple Frobenius manifold (the isomonodromicity property).

We will now describe the reconstruction procedure giving a parametrization of semisimple Frobenius manifolds in terms of the monodromy data $(\hat\mu, R, S, C)$.

Conversely, to reconstruct the Frobenius manifold near a semisimple point with the canonical coordinates $u_1^0, \ldots, u_n^0$, one is to solve the following boundary-value problem. Let

$$\ell = (-\ell_-) \cup \ell_+$$

be the oriented line on the complex z-plane chosen as in [38]. Here the ray ℓ_- is the opposite to ℓ_+. Denote Π_R/Π_L the right/left half-planes with respect to ℓ. To reconstruct the Frobenius manifold, one is to find three matrix-valued functions $\Phi_0(z; u)$, $\Phi_R(z; u)$, and $\Phi_L(z; u)$:

$$\Phi_0(z; u) : V \to \mathbb{C}^n$$
$$\Phi_{R/L}(z; u) : \mathbb{C}^n \to \mathbb{C}^n$$

for u close to u^0 such that $\Phi_0(z; u)$ is analytic and invertible for $z \in \mathbb{C}$, $\Phi_R(z; u)/\Phi_L(z; u)$ are analytic and invertible for $z \in \Pi_R/\Pi_L$ resp., and continuous up to the boundary $\ell \backslash 0$ and

$$\Phi_{R/L}(z; u) \sim 1 + O(1/z), \quad |z| \to \infty, \ z \in \Pi_R/\Pi_L$$

The boundary values of the functions $\Phi_0(z; u)$, $\Phi_R(z; u)$, and $\Phi_L(z; u)$ must satisfy the following boundary-value problem (as above $U = \operatorname{diag}(u_1, \ldots, u_n)$):

$$\Phi_R(z; u) = \Phi_L(z; u) e^{zU} S e^{-zU}, \quad z \in \ell_+ \qquad [42]$$

$$\Phi_R(z; u) = \Phi_L(z; u) e^{zU} S^* e^{-zU}, \quad z \in \ell_- \qquad [43]$$

$$\begin{aligned} \Phi_0(z; u) z^{\hat\mu} z^R &= \Phi_R(z; u) e^{zU} C, \quad z \in \Pi_R \\ \Phi_0(z; u) z^{\hat\mu} z^R &= \Phi_L(z; u) e^{zU} SC, \quad z \in \Pi_L \end{aligned} \qquad [44]$$

Here $z^{\hat\mu} := e^{\hat\mu \log z}$, $z^R := e^{R \log z}$ are considered as $\operatorname{Aut}(V)$-valued functions on the universal covering of $\mathbb{C} \backslash 0$; the branch cut in the definition of $\log z$ is chosen to be along ℓ_-.

The solution of the above boundary-value problem [42]–[44], if exists, is unique. It can be reduced to a certain Riemann–Hilbert problem, that is, to a problem of factorization of an analytic $n \times n$ nondegenerate matrix-valued function on the annulus

$$G(z; u), \quad r < |z| < R, \quad \det G(z; u) \neq 0$$

depending on the parameter $u = (u_1, \ldots, u_n)$ in a product

$$G(z; u) = G_0(z; u)^{-1} G_\infty(z; u) \qquad [45]$$

of two matrix-valued functions $G_0(z; u)$ and $G_\infty(z; u)$ analytic for $|z| < R$ and $r < |z| \leq \infty$ resp., with nowhere-vanishing determinant.

Existence of a solution to the Riemann–Hilbert problem for a given $u=(u_1,\dots,u_n), u_i \neq u_j$ for $i \neq j$, means triviality of certain n-dimensional vector bundle over the Riemann sphere with the transition functions given by $G(z;u)$. Existence of the solution for $u=u^0$ implies solvability of the Riemann–Hilbert problem for u sufficiently close to u^0. From these arguments, it can be deduced that the matrices $\Phi_0(z;u), \Phi_{R/L}(z;u)$ are analytic in $(z;u)$ for u sufficiently close to u^0. Moreover, they can be analytically continued in u to the universal covering of the space of configurations of n distinct points on the complex plane:

$$(\mathbb{C}^n \backslash \cup_{i\neq j}\{u_i = u_j\})/S_n \qquad [46]$$

The resulting functions are meromorphic on the universal covering, according to the results of B Malgrange and T Miwa. The structure of the global analytic continuation is given (Dubrovin 1999) in terms of a certain action of the braid group

$$B_n = \pi_1\left((\mathbb{C}^n \backslash \cup_{i\neq j}\{u_i = u_j\})/S_n\right)$$

on the monodromy data.

Examples of Frobenius Manifolds

Example 0 Trivial Frobenius manifold, $M=A_0$ a graded Frobenius algebra, $F(v)=(1/6) < e, v\cdot v\cdot v >$ is a cubic polynomial.

First nontrivial examples appeared in the setting of 2D topological field theories (Dijkgraaf *et al.* 1991, Witten 1991) (*see* Topological Quantum Field Theory: Overview). Mathematical formalization of these ideas gives rise to the following two classes of examples.

Example 1 Frobenius structure on the base of an isolated hypersurface singularity. The construction (Hertling 2002, Sabbah 2002) uses the K Saito theory of periods of primitive forms. For the example of A_n singularity $f(x)=x^{n+1}$ the Frobenius structure on the base of universal unfolding

$$M_{A_n} = \{f_s(x) = x^{n+1} + s_1 x^{n-1} + \cdots + s_n \,|\, s_1,\dots,s_n \in \mathbb{C}\}$$

is constructed as follows (Dijkgraaf *et al.* 1991):

$$e = \frac{\partial}{\partial s_n}$$
$$E = \frac{1}{n+1}\sum (k+1) s_k \frac{\partial}{\partial s_k}$$
$$d = \frac{n-1}{n+1}$$

The multiplication is introduced by identifying the tangent space $T_s M$ with the quotient algebra

$$T_s M_{A_n} = \mathbb{C}[x]/(f_s'(x))$$

The metric has the form

$$<\partial_{s_i}, \partial_{s_j}> = -(n+1)\,\mathrm{res}_{x=\infty} \frac{\partial f_s(x)/\partial s_i \partial f_s(x)/\partial s_j}{f_s'(x)}\, dx$$

The flat coordinates $v_\alpha = v_\alpha(s)$ can be found from the expansion of the solution to the equation $f_s(x) = k^{n+1}$,

$$x = k - \frac{1}{n+1}\left(\frac{v_n}{k} + \frac{v_{n-1}}{k^2} + \cdots + \frac{v_1}{k^n}\right) + O\left(\frac{1}{k^{n+2}}\right)$$

The potentials of the Frobenius manifolds M_{A_n} for $n=1,2,3$ read

$$\begin{aligned} F_{A_1} &= \tfrac{1}{6} v_1^3 \\ F_{A_2} &= \tfrac{1}{2} v_1^2 v_2 + \tfrac{1}{72} v_2^4 \\ F_{A_3} &= \tfrac{1}{2} v_1 v_2^2 + \tfrac{1}{2} v_1^2 v_3 + \tfrac{1}{16} v_2^2 v_3^2 + \tfrac{1}{960} v_3^5 \end{aligned} \qquad [47]$$

The space of polynomials M_{A_n} can be identified with the orbit space of $\mathbb{C}/W(A_n)$ of the Weyl group of the type A_n. More generally (Dubrovin 1996), the orbit space $M_W := \mathbb{C}^n/W$ of an arbitrary irreducible finite Coxeter group $W \subset O(n)$ carries a natural structure of a polynomial semisimple Frobenius manifold. Conversely, all irreducible polynomial semisimple Frobenius manifolds with positive degrees of the flat coordinates can be obtained by this construction (Hertling 2002). Generalizations for the orbit spaces of certain infinite groups were obtained in Dubrovin and Zhang (1998b) and Bertola (2000).

Example 2 Gromov–Witten (GW) invariants (*see* Topological Sigma Models). Let X be a smooth projective variety. We will assume for simplicity that $H^{\text{odd}}(X)=0$. To every such variety, one can associate a bunch of rational numbers. They are expressed in terms of intersection theory of certain cycles on the moduli spaces $X_{g,m,\beta}$ of stable genus g and degree β curves on X with m marked points (see details in Kontsevich and Manin (1994)):

$$\begin{aligned} X_{g,m,\beta} := \{ & f : (C_g, x_1, \dots, x_m) \to X, \\ & f_*[C_g] = \beta \in H_2(X;Z)\} \end{aligned} \qquad [48]$$

Denote $n := \dim H^*(X;\mathbb{C})$. Choosing a basis $\phi_1 = 1$, $\phi_2, \dots, \phi_n$ we define the numbers

$$\begin{aligned} & <\tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m})>_{g,\beta} \\ & := \int_{[X_{g,m,\beta}]^{\text{virt}}} ev_1^*(\phi_{\alpha_1}) \wedge c_1^{p_1}(\mathcal{L}_1) \\ & \quad \wedge \cdots \wedge ev_m^*(\phi_{\alpha_m}) \wedge c_1^{p_m}(\mathcal{L}_m) \end{aligned} \qquad [49]$$

for arbitrary non-negative integers $p_1, \ldots, p_m$. Here the evaluation maps $ev_i, i = 1, \ldots, m$, are given by

$$ev_i : X_{g,m,\beta} \to X, \qquad f \mapsto f(x_i)$$

The so-called tautological line bundles $\mathcal{L}_i$ over $X_{g,m,\beta}$ by definition have the fiber $T^*_{x_i} C_g, i = 1, \ldots, m$ (see the article Moduli Spaces: An Introduction regarding the construction of the so-called virtual fundamental class $[X_{g,m,\beta}]^{\text{virt}}$). The numbers [49] can be defined for an arbitrary compact symplectic manifold X where one is to deal with the intersection theory on the moduli spaces of pseudoholomorphic curves fixing a suitable almost-complex structure on X. They depend only on the symplectic structure on X. In particular, the numbers

$$< \tau_0(\phi_{\alpha_1}) \ldots \tau_0(\phi_{\alpha_m}) >_{g,\beta} \qquad [50]$$

are called the genus g and degree β GW invariants of X. In certain cases, they admit an interpretation in terms of enumerative geometry of the variety X (Kontsevich and Manin 1994). The numbers [49] with some of $p_i > 0$ are called "gravitational descendents."

One can form a generating functions of the numbers [49]

$$\mathcal{F}^X_g = \sum_m \sum_{\beta \in H_2(X;Z)} \frac{1}{m!} t^{\alpha_1, p_1} \ldots t^{\alpha_m, p_m} < \tau_{p_1}(\phi_{\alpha_1}) \ldots \tau_{p_m}(\phi_{\alpha_m}) >_{g,\beta} \qquad [51]$$

(summation over repeated indices $1 \le \alpha_1, \ldots, \alpha_m \le n$ will always be assumed). Here $t^{\alpha,p}$ are indeterminates labeled by pairs (α, p) with $\alpha = 1, \ldots, n$, $p = 0, 1, 2, \ldots$. (Usually one is to insert in the definition of $\mathcal{F}^X_g$ elements q^β of the Novikov ring $\mathbb{C}[H_2(X;Z)]$. However, due to the divisor axiom (Kontsevich and Manin 1994) and these insertions can be compensated by a suitable shift in the space of couplings $\boldsymbol{t} = (t^{\alpha,p})$.) We finally introduce the full generating function called total GW potential (it is also called the free energy of the topological sigma model with the target space X)

$$\mathcal{F}^X(\boldsymbol{t}; \epsilon) = \sum_{g \ge 0} \epsilon^{2g-2} \mathcal{F}^X_g \qquad [52]$$

Restricting the genus-zero generating function onto the so-called small phase space

$$F^X(v) := \mathcal{F}^X_0(t^{\alpha,0} = v^\alpha, t^{\alpha,p>0} = 0) \qquad v = (v^1, \ldots, v^n) \qquad [53]$$

one obtains a solution to the WDVV associativity equations. This solution defines a structure of (formal) Frobenius manifold on $H^*(X)$ with the bilinear form η given by the Poincaré pairing

$$\eta_{\alpha\beta} = \int_X \phi_\alpha \wedge \phi_\beta$$

the unity

$$e = \frac{\partial}{\partial v^1}$$

and the Euler vector field

$$E = \sum_{\alpha=1}^{n} [(1 - q_\alpha) v^\alpha + r_\alpha] \frac{\partial}{\partial v^\alpha}$$

Here the numbers q_α, r_α are defined by the conditions

$$\phi_\alpha \in H^{2q_\alpha}(X), \qquad c_1(X) = \sum_\alpha r_\alpha \phi_\alpha$$

The resulting Frobenius manifold will be denoted M_X. The corresponding n-parameter family of n-dimensional algebras on the tangent spaces $T_v M_X$ is also called "quantum cohomology" $\text{QH}^*(X)$. At the point $v_{\text{cl}} \in M_X$ of classical limit, the algebra $T_{v_{\text{cl}}} M_X$ coincides with the cohomology ring $H^*(X)$. In all known examples, the series [53] actually converges in a neighborhood of the point v_{cl}. Therefore, one obtains a genuine Frobenius structure on a domain $M_X \subset H^*(X;\mathbb{C})/2\pi i H_2(X;Z)$. However, a general proof of convergence is still missing.

In particular, for $d = 1$, the quantum cohomology of complex projective line $\boldsymbol{P}^1$ is a two-dimensional Frobenius manifold with the potential, unity, and the Euler vector field

$$F(u, v) = \tfrac{1}{2} u v^2 + e^u, \quad e = \frac{\partial}{\partial v}, \quad E = v \frac{\partial}{\partial v} + 2 \frac{\partial}{\partial u}$$

For $d = 2$ one has a three-dimensional Frobenius manifold $QH^*(\boldsymbol{P}^2)$ with

$$F(v_1, v_2, v_3) = \tfrac{1}{2} v_1^2 v_3 + \tfrac{1}{2} v_1 v_2^2 + \sum_{k \ge 1} N_k \frac{v_3^{3k-1}}{(3k-1)!} e^{k v_2}, \quad e = \frac{\partial}{\partial v_1}, \quad E = v_1 \frac{\partial}{\partial v_1} + 3 \frac{\partial}{\partial v_2} - v_3 \frac{\partial}{\partial v_3} \qquad [54]$$

where N_k = number of rational curves on $\boldsymbol{P}^2$ passing through $3k - 1$ generic points. WDVV [5] yields (Kontsevich and Manin 1994) recursion relations for the numbers N_k starting from $N_1 = 1$. The closed analytic formula for the function [54] is still unknown.

Only for certain very exceptional X the Frobenius manifold M_X is semisimple (e.g., for $X = \boldsymbol{P}^d$). The general geometrical reasons of the semisimplicity of M_X are still to have been understood.

For the case X = Calabi–Yau manifold, the Frobenius manifold $QH^*(X)$ is never semisimple. This Frobenius structure can be computed in terms of the mirror symmetry construction (*see* Mirror Symmetry: A Geometric Survey).

Frobenius Manifold and Integrable Systems

The identities in the cohomology ring generated by the cocycles $ev_i^*(\phi_\alpha)$ and $\psi_i := c_1(\mathcal{L}_i)$ can be recast into the form of differential equations for the generating function [52]. The variable $x := t^{1,0}$ corresponding to $\phi_1 = 1$ plays a distinguished role in these differential equations. According to the idea of Witten (1991), the differential equations for the generating functions can be written as a hierarchy of systems of n evolutionary PDEs ($n = \dim H^*(X)$) for the unknown functions

$$w_\alpha = \langle\langle \tau_0(\phi_\alpha)\tau_0(\phi_1)\rangle\rangle = \epsilon^2 \frac{\partial^2 \mathcal{F}^X(t,\epsilon)}{\partial t^{1,0}\partial t^{\alpha,0}} \qquad [55]$$

The variable $x = t^{1,0}$ is the spatial variable of the equations of the hierarchy. The remaining parameters (coupling constants) $t^{\alpha,p}$ of the generating function play the role of the time variables. Witten suggested to use the two-point correlators

$$h_{\alpha,p} = \langle\langle \tau_{p+1}(\phi_\alpha)\tau_0(\phi_1)\rangle\rangle = \epsilon^2 \frac{\partial^2 \mathcal{F}^X(t,\epsilon)}{\partial t^{1,0}\partial t^{\alpha,p}} \qquad [56]$$

as the densities of the Hamiltonians of the flows of the hierarchy.

Existence of such a hierarchy can be proved for the case of GW invariants (and their descendents) of complex projective spaces $\boldsymbol{P}^d$ (the results of Givental (2001) along with Dubrovin and Zhang (2005) can be used). For $d = 0$ one obtains, according to the celebrated result by Kontsevich conjectured by Witten (*see* Topological Gravity, Two-Dimensional), the tau function of the solution to the KdV hierarchy (*see* Korteweg–de Vries Equation and Other Modulation Equations) specified by the initial condition,

$$u(x)\,|_{t=0} = x$$

For $d = 1$ the hierarchy in question is the extended Toda lattice (see details in Dubrovin and Zhang (2004); *see also* Toda Lattices). For all other $d \geq 2$, the needed integrable hierarchy is a new one. It can be associated (Dubrovin and Zhang) with an arbitrary n-dimensional semisimple Frobenius manifold M. The equations of the hierarchy have the form

$$w_t^i = A_j^i(w)w_x^j + \epsilon^2\Big[B_j^i(w)w_{xxx}^i + C_{jk}^i(w)w_x^j w_{xx}^k + D_{jkl}^i(w)w_x^j w_x^k w_x^l\Big] + O(\epsilon^4), \quad i = 1,\ldots,n \qquad [57]$$

The coefficients of ϵ^{2g} are graded homogeneous polynomials in u_x, u_{xx}, etc., of the degree $2g+1$,

$$\deg \mathrm{d}^m u/\mathrm{d}x^m = m$$

The construction of the hierarchy is done in two steps. First, we construct the leading approximation (Dubrovin 1992). The equation of the hierarchy specifying the dependence on $t = t^{\alpha,p}$ at $\epsilon = 0$ reads

$$\frac{\partial v}{\partial t^{\alpha,p}} = \partial_x(\nabla\theta_{\alpha,p+1}(v)) \qquad \alpha = 1,\ldots,n, \quad p \geq 0 \qquad [58]$$

The functions $\theta_{\alpha,p}(v), v \in M$, are the coefficients of expansion [10] of the deformed flat functions normalized by $\theta_{\alpha,0} = v_\alpha$. The solution $v = v(x,t)$ of interest is determined from the implicit function equations

$$v = xe + \sum_{\alpha,p} t^{\alpha,p}\nabla\theta_{\alpha,p}(v) \qquad [59]$$

Next, one has to find solution

$$\Delta\mathcal{F} = \sum_{g\geq 1} \epsilon^{2g-2}\mathcal{F}_g(v; v_x,\ldots,v^{(3g-2)}) \qquad [60]$$

of the following universal loop equation (closely related with the Virasoro conjecture of Eguchi and Xiong (1998)):

$$\begin{aligned}
&\sum_{r\geq 0}\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,r}}\partial_x^r\left(\frac{1}{E(v)-\lambda}\right)^\gamma \\
&+\sum_{r\geq 1}\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,r}}\sum_{k=1}^{r}\binom{r}{k}\partial_x^{k-1}\partial_e p_\alpha G^{\alpha\beta}\partial_x^{r-k+1}\partial^\gamma p_\beta \\
&= -\frac{1}{16}\mathrm{tr}(\mathcal{U}-\lambda)^{-2} + \frac{1}{4}\mathrm{tr}\left[(\mathcal{U}-\lambda)^{-1}\mathcal{V}\right]^2 \\
&+\frac{\epsilon^2}{2}\sum\left(\frac{\partial^2\Delta\mathcal{F}}{\partial v^{\gamma,k}\partial v^{\rho,l}} + \frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,k}}\frac{\partial\Delta\mathcal{F}}{\partial v^{\rho,l}}\right) \\
&\times \partial_x^{k+1}\partial^\gamma p_\alpha G^{\alpha\beta}\partial_x^{l+1}\partial^\rho p_\beta \\
&+\frac{\epsilon^2}{2}\sum\frac{\partial\Delta\mathcal{F}}{\partial v^{\gamma,k}}\partial_x^{k+1} \\
&\times\left[\nabla\frac{\partial p_\alpha(v;\lambda)}{\partial\lambda}\cdot\nabla\frac{\partial p_\beta(v;\lambda)}{\partial\lambda}\cdot v_x\right]^\gamma G^{\alpha\beta}
\end{aligned} \qquad [61]$$

Here $\mathcal{U}=\mathcal{U}(v)$ is the operator of multiplication by $E(v), p_\alpha = p_\alpha(v;\lambda), \alpha = 1,\ldots,n$, is a system of flat coordinates [16] of the bilinear form [13]. The substitution

$$v_\alpha \mapsto w_\alpha = v_\alpha + \epsilon^2 \partial_x \partial_{t^{\alpha,0}} \Delta\mathcal{F}(v; v_x, v_{xx}, \ldots; \epsilon^2) \qquad \alpha = 1,\ldots,n \qquad [62]$$

transforms [58] to [57]. The terms of the expansion [60] are not polynomial in the derivatives. For example (Dubrovin and Zhang 1998a),

$$\mathcal{F}_1 = \frac{1}{24}\sum_{i=1}^{n} \log u_i' + \log \frac{\tau_I(u)}{J^{1/24}(u)}$$
$$J(u) = \det\left(\frac{\partial v^\alpha}{\partial u_i}\right) = \pm \prod_{i=1}^{n} \psi_{i1}(u) \qquad [63]$$

(the canonical coordinates have been used) where $\tau_I(u)$ is the isomonodromic tau function [29]. The transformation [62] applied to the solution [59] expresses higher-genus GW invariants of a variety X with semisimple quantum cohomology $QH^*(X)$ via the genus-zero invariants. For the particular case of $X = \boldsymbol{P}^2$, the formula [63] yields (Dubrovin and Zhang 1998a)

$$\frac{\phi''' - 27}{8(27 + 2\phi' - 3\phi'')} = -\frac{1}{8} + \sum_{k\geq 1} k N_k^{(1)} \frac{e^{kz}}{(3k)!}$$

Here

$$\phi(z) = \sum_{k\geq 0} N_k \frac{e^{kz}}{(3k-1)!}$$

is the generating function of the genus-zero GW invariants of $\boldsymbol{P}^2$ (see [54]) and $N_k^{(1)} =$ the number of elliptic plane curves of the degree k passing through $3k$ generic points.

See also: Bi-Hamiltonian Methods in Soliton Theory; Functional Equations and Integrable Systems; Integrable Systems: Overview; Isomonodromic Deformations; Korteweg–de Vries Equation and Other Modulation Equations; Mirror Symmetry: A Geometric Survey; Moduli Spaces: An Introduction; Painlevé Equations; Riemann–Hilbert Problem; Toda Lattices; Topological Gravity, Two-Dimensional; Topological Quantum Field Theory: Overview; Topological Sigma Models.

Further Reading

Bertola M (2000) Frobenius manifold structure on orbit space of Jacobi groups. I, II. *Differential Geometry and Applications* 13: 19–41, 213–233.

Dijkgraaf R, Verlinde H, and Verlinde E (1991) Topological strings in $d < 1$. *Nuclear Physics* B 352: 59–86.

Dubrovin B (1992) Integrable systems in topological field theory. *Nuclear Physics* B 379: 627–689.

Dubrovin B (1996) Geometry of 2D topological field theories. In: Francaviglia M and Greco S (eds.) *Integrable Systems and Quantum Groups, Montecatini Terme, 1993*, Springer Lecture Notes in Math., vol. 1620, pp. 120–348.

Dubrovin B (1998) Geometry and analytic theory of Frobenius manifolds. *Proceedings of the International Congress of Mathematicians*, vol. II (Berlin, 1998). *Doc. Math.* 1998, Extra Vol. II, 315–326.

Dubrovin B (1999) Painlevé transcendents in two-dimensional topological field theory. In: Conte R (ed.) *The Painlevé Property: 100 Years Later*, CRM Ser. Math. Phys. pp. 287–412. New York: Springer.

Dubrovin B (2004) On almost duality for Frobenius manifolds. In: Buchstaber VM and Krichever IM (eds.) *Geometry, Topology, and Mathematical Physics*, Amer. Math. Soc. Transl. Ser. 2, vol. 212, pp. 75–132. Providence, RI: American Mathematical Society.

Dubrovin B and Zhang Y (1998a) Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation. *Communications in Mathematical Physics* 198: 311–361.

Dubrovin B and Zhang Y (1998b) Extended affine Weyl groups and Frobenius manifolds. *Compositio Math.* 111: 167–219.

Dubrovin B and Zhang Y (2004) Virasoro symmetries of the extended Toda hierarchy. *Communications in Mathematical Physics* 250: 161–193.

Dubrovin B and Zhang Y (2001) Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants, math/0108160.

Dubrovin B and Zhang Y (2005) Integrable hierarchies of the topological type (to appear).

Eguchi T and Xiong CS (1998) Quantum cohomology at higher genus: topological recursion relations and Virasoro conditions. *Advances in Theoretical and Mathematical Physics* 2: 219–229.

Givental A (2001) Gromov–Witten invariants and quantization of quadratic Hamiltonians. *Moscow Mathematical Journal* 1: 551–568, 645.

Hertling C (2002) Frobenius Manifolds and Moduli Spaces for Singularities, *Cambridge Tracts in Mathematics,* vol. 151. Cambridge: Cambridge University Press.

Hitchin N (1997) Frobenius manifolds. With Notes by David Calderbank. In: *Gauge Theory and Symplectic Geometry (Montreal, PQ, 1995).* , NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. vol. 488, pp. 69–112. Dordrecht: Kluwer Academic Publishers.

Kontsevich M and Manin Yu (1994) Gromov–Witten classes, quantum cohomology and enumerative geometry. *Communications in Mathematical Physics* 164: 525–562.

Manin Yu (1999) *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces,* American Mathematical Society Colloquium Publications, vol. 47. Providence, RI: American Mathematical Society.

Sabbah C (2002) Déformations isomonodromiques et variétés de Frobenius. *Savoirs Actuels (Les Ulis).* Paris: CNRS Éditions.

Witten E (1991) Two-dimensional gravity and intersection theory on moduli space. *Surv. of Differential Geometry* 1: 243–310.