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Hamiltonian PDEs and Frobenius manifolds

B. A. Dubrovin

Abstract. In the first part of this paper the theory of Frobenius manifolds is applied to the problem of classification of Hamiltonian systems of partial differential equations depending on a small parameter. Also developed is a deformation theory of integrable hierarchies including the subclass of integrable hierarchies of topological type. Many well-known examples of integrable hierarchies, such as the Korteweg–de Vries, non-linear Schrödinger, Toda, Boussinesq equations, and so on, belong to this subclass that also contains new integrable hierarchies. Some of these new integrable hierarchies may be important for applications. Properties of the solutions to these equations are studied in the second part. Consideration is given to the comparative study of the local properties of perturbed and unperturbed solutions near a point of gradient catastrophe. A *Universality Conjecture* is formulated describing the various types of critical behaviour of solutions to perturbed Hamiltonian systems near the point of gradient catastrophe of the unperturbed solution.

Given an n-dimensional manifold M^n, denote by

$$\mathscr{L}(M^n) = \{S^1 \to M^n\}$$

the space of loops with values in M^n. The main objects of our study are Hamiltonian vector fields on $\mathscr{L}(M^n)$ depending on the small parameter ε. They will be called vector fields on the extended loop space $\mathscr{L}(M^n) \otimes \mathbb{R}[[\varepsilon]]$. More specifically, we will study systems of evolutionary partial differential equations (PDEs) with one spatial variable x represented in the form

$$u^i_t = A^i_j(u)u^j_x + \varepsilon\left(B^i_j(u)u^j_{xx} + \frac{1}{2}\,C^i_{jk}(u)u^j_x u^k_x\right) + O(\varepsilon^2), \qquad i = 1, \dots, n. \tag{1}$$

Here $u = (u^1, \dots, u^n)$ are local coordinates on M^n. This manifold will be assumed to have a trivial topology (an n-dimensional ball), although we will use non-linear changes of variables in this ball.

It is assumed that the terms of order ε^k in the expansions in (1) are polynomials in the derivatives u_x, …, $u^{(k+1)}$ of degree $k+1$, where the degree is defined by

$$\deg u^{(m)} = m, \qquad m = 1, 2, \dots .$$

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It will be required that (1) is a Hamiltonian system with respect to a *local Poisson bracket*

$$u_t^i = \{u^i(x), H\} = \sum_{k\geqslant 0} \varepsilon^k \sum_{m=0}^{k+1} A_{k,m}^{ij}\bigl(u; u_x, \dots, u^{(m)}\bigr)\,\partial_x^{k-m+1} \frac{\delta H}{\delta u^j(x)}\,, \tag{2}$$

$$\{u^i(x), u^j(y)\} = \sum_{k\geqslant 0} \varepsilon^k \sum_{m=0}^{k+1} A_{k,m}^{ij}\bigl(u(x); u_x(x), \dots, u^{(m)}(x)\bigr)\,\delta^{(k-m+1)}(x-y), \tag{3}$$

$$\deg A_{k,m}^{ij}\bigl(u; u_x, \dots, u^{(m)}\bigr) = m,$$

with *local Hamiltonian*

$$H = \sum_{k\geqslant 0} \varepsilon^k \int h_k\bigl(u; u_x, \dots, u^{(k)}\bigr)\,dx, \qquad \deg h_k\bigl(u; u_x, \dots, u^{(k)}\bigr) = k. \tag{4}$$

Note that $\delta(x)$ in (3) is the Dirac delta function. The meaning of this notation is clear from the explicit expression (2). The integral in (4) is understood in the sense of formal variational calculus. In other words the integral of a differential polynomial $h = h(u; u_x, \dots, u^{(m)})$ is defined as the equivalence class modulo total derivatives:

$$h(u; u_x, \dots, u^{(m)}) \sim h(u; u_x, \dots, u^{(m)}) + \partial_x\bigl(f(u; u_x, \dots, u^{(m-1)})\bigr),$$

$$\partial_x = \sum_{k\geqslant 0} u^{i(k+1)} \frac{\partial}{\partial u^{i(k)}}\,, \quad \text{where } u^{i(k)} := \frac{d^k u^i}{dx^k}\,.$$

Furthermore, $\delta H/\delta u^j(x)$ is the Euler–Lagrange operator

$$\frac{\delta H}{\delta u^j(x)} = \frac{\partial h}{\partial u^j} - \partial_x \frac{\partial h}{\partial u_x^j} + \partial_x^2 \frac{\partial h}{\partial u_{xx}^j} - \cdots \quad \text{for } H = \int h\,dx.$$

The coefficients of the Poisson bracket as well as the Hamiltonian densities are assumed to be polynomials in the derivatives at every order in ε. The antisymmetry and Jacobi identity must be satisfied as identities for formal power series in ε. The bracket (3) defines a Lie algebra structure $\mathscr{G}_{\text{loc}}$ on the space of all local functionals

$$\{F, G\} = \int \frac{\delta F}{\delta u^i(x)}\, A^{ij} \frac{\delta G}{\delta u^j(x)}\,dx, \tag{5}$$

$$A^{ij} := \sum_{k\geqslant 0} \varepsilon^k \sum_{m=0}^{k+1} A_{k,m}^{ij}\bigl(u; u_x, \dots, u^{(m)}\bigr)\,\partial_x^{k-m+1},$$

$$F = \sum_{k\geqslant 0} \varepsilon^k \int f_k\bigl(u; u_x, \dots, u^{(k)}\bigr)\,dx, \qquad G = \sum_{l\geqslant 0} \varepsilon^l \int g_l\bigl(u; u_x, \dots, u^{(l)}\bigr)\,dx,$$

$$\deg f_k\bigl(u; u_x, \dots, u^{(k)}\bigr) = k, \qquad \deg g_l\bigl(u; u_x, \dots, u^{(l)}\bigr) = l.$$

The full ring of functions on the infinite-dimensional 'manifold' $\mathscr{L}(M^n) \otimes \mathbb{C}[[\varepsilon]]$ is defined as a suitably completed symmetric tensor algebra over $\mathscr{G}_{\text{loc}}$.

The above formulae define a class of functions, vector fields, and Poisson brackets on the infinite-dimensional 'manifold' $\mathscr{L}(M^n)\otimes\mathbb{C}[[\varepsilon]]$. In order to develop a geometric approach to the study of these objects we will now introduce a class of admissible 'changes of coordinates' on our 'manifold'. They will be given in terms of the so-called *generalized Miura transformations*

$$u^i \mapsto \tilde{u}^i = \sum_{k\geqslant 0} \varepsilon^k F_k^i\bigl(u; u_x, \dots, u^{(k)}\bigr), \tag{6}$$

$$\deg F_k^i\bigl(u; u_x, \dots, u^{(k)}\bigr) = k, \qquad \det\left(\frac{\partial F_0^i(u)}{\partial u^j}\right) \neq 0.$$

The coefficients $F_k^i\bigl(u; u_x, \dots, u^{(k)}\bigr)$ must be differential polynomials. It is easy to see that the transformations (6) form a group. Indeed, to invert the transformation (6) one has to solve a system of differential equations for u^1, …, u^n. The needed solution is obtained as the WKB expansion in the small parameter ε. It is an easy exercise to prove that the class of evolution PDEs (1), the Poisson brackets (3), as well as the class of local Hamiltonians (4) is invariant with respect to the group of generalized Miura transformations.

We will say that two objects of our theory (that is, two systems of evolution PDEs (1), two local Poisson brackets (3), or two local Hamiltonians (4)) are *equivalent* if they are related by a generalized Miura transformation.

The main problems of our research are
- the problem of classification of general Hamiltonian systems of PDEs,
- application to the study of integrable PDEs,
- new approaches to studying the properties of solutions.

Let us begin with classifying the Poisson brackets.

Theorem 1. *Under the assumption*

$$\det\bigl(A_{0,0}^{ij}(u)\bigr) \neq 0 \tag{7}$$

any Poisson bracket of the form (3) *is equivalent to the following standard Poisson bracket*:

$$\{\tilde{u}^i(x), \tilde{u}^j(y)\} = \eta^{ij}\,\delta'(x-y), \qquad \eta^{ij} = \eta^{ji} = \text{const}, \quad \det\bigl(\eta^{ij}\bigr) \neq 0. \tag{8}$$

In the proof of this theorem we use the theory of Poisson brackets of hydrodynamic type developed by Novikov and the author in 1983. According to this theory the leading term

$$g^{ij}(u) := A_{0,0}^{ij}(u)$$

of the Poisson bracket defines a (contravariant) metric of vanishing curvature on the manifold M^n. We also use triviality of the Poisson cohomology of the bracket (8) proved by Getzler in 2001.

Next in the realization of our programme is the classification of *bi-Hamiltonian* structures of the form (3), (7). Recall that the systems

$$u_t^i = \{u^i(x), H_1\}_1 = \{u^i(x), H_2\}_2, \qquad i = 1, \dots, n,$$

Hamiltonian with respect to two *compatible* Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ of the above form can always be included into a maximal Abelian subalgebra called an *integrable hierarchy* in the Lie algebra of Hamiltonian vector fields (this can be proved).

Theorem 2. *Under the additional assumptions of strong non-degeneracy and semisimplicity, every bi-Hamiltonian system is defined by*

1) *a pencil of Poisson brackets of hydrodynamic type*

$$\{v^i(x),v^j(y)\}_2-\lambda\,\{v^i(x),v^j(y)\}_1=\bigl(g_2^{ij}(v(x))-\lambda\,g_1^{ij}(v(x))\bigr)\delta'(x-y)\\ +\bigl(\Gamma^{ij}_{k\,1}(v)-\lambda\,\Gamma^{ij}_{k\,2}(v)\bigr)v_x^k\,\delta(x-y); \qquad (9)$$

2) *a collection of n functions of one variable*

$$c_1(w^1),\dots,c_n(w^n),$$

called central invariants.

The proof of this theorem is based on

• the *quasi-triviality* theorem: every bi-Hamiltonian structure becomes equivalent to a dispersionless one (9) if one extends the class of admissible transformations (6), allowing rational dependence of the coefficients $F_k^i(u;u_x,\dots)$ for $k\geqslant 1$ on the jets u_x, $u_{xx},\dots$.

• calculation of the *bi-Hamiltonian cohomology*, that is, of the deformations of a pair of acyclic anticommuting differentials on the multivectors on $\mathscr{L}(M^n)\otimes\mathbb{R}[[\epsilon]]$ defined by the bi-Hamiltonian structure.

Let us outline the construction of the central invariants. To every Poisson bracket (3) we assign a series of matrices depending on an auxiliary parameter p:

$$\pi^{ij}(u;p)=\sum_{k\geqslant 0}A^{ij}_{k,0}(u)p^k. \qquad (10)$$

Recall that the degree in the derivatives (in x of u) in the coefficients $A^{ij}_{k,0}$ is equal to zero, and therefore these only depend on u. To a pair of Poisson brackets we assign a characteristic equation

$$\det\bigl(\pi_2^{ij}(u;p)-\lambda\,\pi_1^{ij}(u;p)\bigr)=0. \qquad (11)$$

Let $\lambda^1(u;p),\dots,\lambda^n(u;p)$ be the roots of this equation:

$$\lambda^i(u;p)=\sum_{k\geqslant 0}\lambda^i_k(u)p^k,$$

where

$$\lambda^i_0(u)=w^i(u), \qquad \lambda^i_k(u)=0 \quad \text{for odd } k.$$

The conditions of semisimplicity and strong non-degeneracy imply that the leading terms $w^1(u),\ \dots,\ w^n(u)$ of these expansions are pairwise distinct and non-constant. From this one can derive that these functions can be used as local coordinates on M^n. Put

$$c_i=\frac{1}{3}\,\frac{\lambda^i_2(u)}{\langle dw^i,dw^i\rangle_1}\,, \qquad i=1,\dots,n. \qquad (12)$$

It turns out that for every $i = 1, \ldots, n$ the function c_i depends only on one coordinate w^i. Moreover, two bi-Hamiltonian structures with the same dispersionless limit (9) are equivalent if and only if they have the same central invariants.

Example 3. The bi-Hamiltonian structure of the Korteweg–de Vries (KdV) equation

$$u_t + u\,u_x + \frac{\varepsilon^2}{12}\,u_{xxx} = 0, \tag{13}$$

is

$$\{u(x), u(y)\}_2 - \lambda\{u(x), u(y)\}_1 = \big(u(x) - \lambda\big)\delta'(x-y) + \frac{1}{2}\,u_x\delta(x-y) + \frac{1}{8}\,\varepsilon^2\delta'''(x-y). \tag{14}$$

The canonical transformation

$$u = v - \frac{\varepsilon^2}{12}\,(\log v')'' + \varepsilon^4\left(\frac{v^{\mathrm{IV}}}{288 v'^2} - \frac{7v''v'''}{480v'^3} + \frac{v''^3}{90v'^4}\right)'' + O\big(\varepsilon^6\big), \tag{15}$$

rational in the derivatives $v' = v_x$, $v'' = v_{xx}$, $\ldots$. transforms the dimensionless bi-Hamiltonian structure

$$\{v(x), v(y)\}_2 - \lambda\{v(x), v(y)\}_1 = \big(v(x) - \lambda\big)\delta'(x-y) + \frac{1}{2}\,v_x\delta(x-y) \tag{16}$$

into (14). Here $w = u$ and the unique central invariant is equal to the constant $c_1 = 1/24$.

Example 4. The bi-Hamiltonian structure of the Camassa–Holm equation

$$u_t - \varepsilon^2 u_{txx} = \frac{3}{2}\,u u_x - \varepsilon^2\left[u_x u_{xx} + \frac{1}{2}\,u u_{xxx}\right] \tag{17}$$

is given by the formula

$$\{u(x), u(y)\}_2 - \lambda\{u(x), u(y)\}_1 = \big(u(x) - \lambda\big)\delta'(x-y) + \frac{1}{2}\,u_x\delta(x-y) + \lambda\frac{\varepsilon^2}{8}\,\delta'''(x-y). \tag{18}$$

The dispersionless limits for (14) and (18) coincide. However, the central invariant of the bi-Hamiltonian structure (18) is equal to

$$c_1 = \frac{1}{24}\,w, \qquad w = u.$$

Therefore the KdV and Camassa–Holm hierarchies are inequivalent.

The theory of central invariants describes the structure of the space of infinitesimal deformations of bi-Hamiltonian structures of hydrodynamic type. It remains an open problem to prove the vanishing of higher obstructions to the deformation; that is, the problem of the existence of a bi-Hamiltonian structure with a given dispersionless limit and given central invariants. We will now consider a particular subclass of the so-called *integrable hierarchies of topological type* associated with semisimple Frobenius manifolds.

Frobenius manifolds M^n correspond to a particular class of Poisson pencils of hydrodynamic type. The characteristic feature of Frobenius manifolds is the existence of a commutative and associative multiplication on the tangent bundle

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = c_{ij}^k(u) \frac{\partial}{\partial u^k},$$

and also of a zero curvature metric defined in the flat coordinates $u^1, \dots, u^n$ by a constant symmetric non-singular matrix

$$\left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle = \eta_{ij}.$$

The local existence of a potential $F(u)$ such that

$$\left\langle \frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k} \right\rangle = \frac{\partial^3 F(u)}{\partial u^i \, \partial u^j \, \partial u^k}$$

is required. Moreover, the existence of a flat unit vector field e and a linear Euler vector field E satisfying

$$[e, E] = e, \quad E\,F = (3 - d)F + \text{quadratic terms}$$

are also required. Here d is a constant.

A remarkable property of Frobenius manifolds (and characteristic, under certain additional assumptions) is the existence of a flat pencil of metrics

$$\begin{aligned} (du^i, du^j)_1 &= \eta^{ij}, \\ (du^i, du^j)_2 &= \mathrm{i}_E \, du^i \cdot du^j. \end{aligned} \tag{19}$$

Thus on the loop space $\mathscr{L}(M^n)$ there arises a bi-Hamiltonian structure of hydrodynamic type and, therefore, an integrable hierarchy. We will not enter into details concerning the construction of this hierarchy here but simply write explicitly one of the equations of the hierarchy:

$$\mathbf{u}_t + \mathbf{u} \cdot \mathbf{u}_x = 0, \qquad \mathbf{u} = (u^1, \dots, u^n) \in M^n \simeq T_{\mathbf{u}} M^n. \tag{20}$$

In this formula the Frobenius manifold is locally identified with its tangent space due to the existence of the flat metric.

Frobenius manifolds also possess many other remarkable properties. In particular, semisimple Frobenius manifolds (for which the algebra on the tangent plane $T_u M^n$ at a generic point $u \in M^n$ is semisimple) can be described in terms of isomonodromy deformations of certain linear differential operators with rational coefficients. There is also a remarkable connection between the theory of semisimple Frobenius manifolds and the theory of reflection groups. Of particular importance for our study is the existence of a *tau-function* for the integrable hierarchies associated with Frobenius manifolds. This is the main motivation for considering the particular subclass of integrable hierarchies we are now going to explain.

The main question to be addressed is the *reconstruction problem*: for which Frobenius manifolds can the system (20) be considered as the zero dispersion limit of an integrable hierarchy on $\mathscr{L}(M^n)\otimes\mathbb{C}[[\varepsilon]]$? And if such an ε-extension exists, how may they all be classified?

Theorem 2 above says that every such hierarchy is uniquely determined by its dispersionless limit along with the collection of central invariants. The characteristic feature of *integrable hierarchies of topological type* is that

- the dispersionless bi-Hamiltonian structure is described via the flat pencil of metrics of the form (19) associated with a semisimple Frobenius manifold;
- all central invariants are constant and equal to each other.

Theorem 5. *For any semisimple Frobenius manifold there exists a unique integrable hierarchy of topological type associated with this manifold with the central invariants*

$$c_1 = c_2 = \cdots = c_n = \frac{1}{24}\,.$$

The clue to the proof of this theorem lies in the invariance of integrable hierarchies of topological type with respect to Virasoro symmetries acting linearly on the tau-function.

Table 1. List of examples of Frobenius manifolds and the associated integrable hierarchies of topological type

$n=1$	$F=\frac{1}{6}\,v^3$	KdV [1]
$n=2$	$F=\frac{1}{2}\,uv^2+u^4$	Boussinesq
$n=2$	$F=\frac{1}{2}\,uv^2+e^u$	Toda
$n=2$	$F=\frac{1}{2}\,uv^2+\frac{1}{2}\,u^2\left(\log u-\frac{3}{2}\right)$	NLS
$n=2$	$F=\frac{1}{2}\,uv^2-\mathrm{Li}_3(e^{-u})$	Ablowitz–Ladik
$n=3$	$F=\frac{1}{2}\,(uw^2+u^2v)+\frac{1}{6}\,v^2w^2+\frac{1}{60}\,w^5$	A_3 Drinfeld–Sokolov hierarchy, intersection theory on the moduli spaces of spin 3 curves
$n=3$	$F=\frac{1}{2}\,(uv^2+vw^2)-\frac{1}{24}\,w^4+4we^u$	generalized Toda lattice associated with difference Lax operator of bidegree (2,1); orbifold Gromov–Witten invariants for a curve with one second-order singularity
$n=3$	$F=\frac{1}{2}\,(\tau v^2+vu^2)-\frac{i\pi}{48}\,u^4E_2(\tau)$	higher corrections to elliptic Whitham asymptotics, the KdV case
$n=4$	$F=\frac{i}{4\pi}\,\tau v^2-2uvw+u^2\log\left[\frac{\pi}{u}\,\frac{\theta_1'(0\vert\tau)}{\theta_1(2w\vert\tau)}\right]$	higher corrections to elliptic Whitham asymptotics, the NLS/Toda case

[1] In this table we write KdV, Boussinesq, Toda, NLS to refer to the integrable hierarchies associated with the Korteweg–de Vries equation, Boussinesq equation, Toda lattice equations, and the non-linear Schrödinger equation, respectively.

Let us now study the properties of the solutions to the equations constructed. It is natural to ask how these properties depend on the choice of a Frobenius manifold; how they change with variation of the truncation order in ε; and what part of these properties continues to hold for non-integrable perturbations.

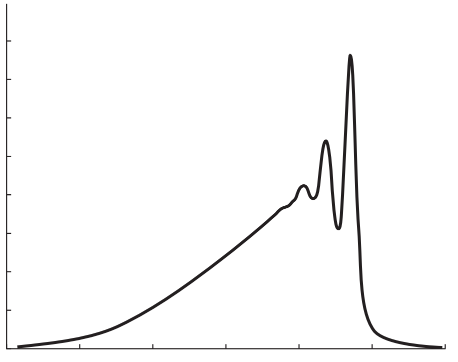

Figure 1. Critical behaviour of solutions to KdV

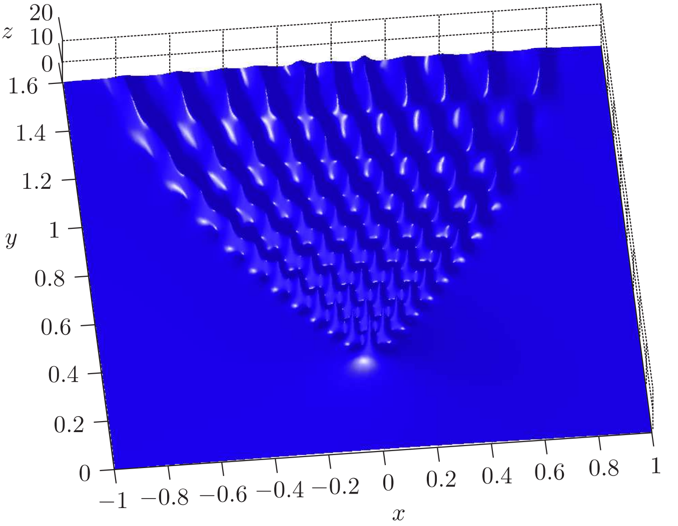

Figure 2. Critical behaviour of solutions to the focusing non-linear Schrödinger equaion $i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$; the graph of $u = |\psi|^2$ is displayed

For small times the contribution of higher ε-corrections is small. Solutions of the dispersionless system and of its perturbation begin to diverge near the point of *gradient catastrophe* where the derivatives u_x, u_t become large. It turns out that the behaviour of solutions to Hamiltonian evolution PDEs is qualitatively different from what happens with solutions to dissipative systems: namely, instead of a shock wave, rapid oscillations with the period ε occur (see Fig. 1).

The first impression is that the behaviour of solutions to different PDEs looks completely different. For example, it is difficult to find any similarity between the critical behaviour shown in Fig. 1 (the KdV case) and Fig. 2 (the NLS case).

Nevertheless, there are various reasons to expect that only a finite number of types of critical behaviour may occur in generic solutions to generic Hamiltonian PDEs according to the *Universality Conjecture* we next discuss.

As one can see from Table 2, for $n = 1, 2$ the types of critical behaviour for generic solutions to the unperturbed systems are described by algebraic functions known from singularity theory, such as the bifurcation diagram of the A_3 singularity, the Whitney singularity, and also Thom's elliptic umbilic singularity. For solutions to the perturbed systems these algebraic functions are replaced by certain particular solutions to Painlevé equations and their generalizations. Let us describe these solutions.

Table 2. Types of critical behaviour of solutions to low-order Hamiltonian PDEs

Number of dependent variables	**Dispersionless system**	**Perturbed system**
$n = 1$	for $t < 0$ solution to $x = ut - \frac{1}{6}u^3$	a special solution $U(X,T)$ to the ODE P_I^2 $X = UT - \frac{1}{6}U^3 - \left[\frac{1}{24}(U'^2 + 2UU'') + \frac{U^{\mathrm{IV}}}{240}\right]$
$n = 2$ hyperbolic case	for $t < 0$ solution to the system in the characteristic variables $x_+ = r_+,$ $x_- = r_+ r_- - \frac{1}{6}r_-^3$	same function $U(X,T)$, $r_+ = x_+ + U''(x_-, x_+),$ $r_- = U(x_-, x_+)$
$n = 2$ elliptic case	for $z \neq 0$ the solution to complex quadratic equation $z = \frac{1}{2}w^2$	the *tritronquée* solution $W_0(Z)$ to the ODE P_I $W'' = 6W^2 - Z$

We will begin with the equation

$$X = TU - \left[\frac{1}{6}U^3 + \frac{1}{24}(U'^2 + 2UU'') + \frac{1}{240}U^{\mathrm{IV}}\right]. \tag{21}$$

This is an ODE for the function $U = U(X)$ depending on the parameter T. In the theory of Painlevé equations it is known as the higher order analogue of the Painlevé-I equation (see below). It is known that for all values of the parameter T any solution to (21) is a meromorphic function of the complex variable X. The particular solution we are interested in has no poles on the whole real line X for all real values of the parameter T (the existence of this solution was only proved by

Claeys and Vanlessen in 2006). We will denote by $U(X,T)$ such a solution uniquely defined for all $(X,T)\in\mathbb{R}^2$ (see Fig. 3).

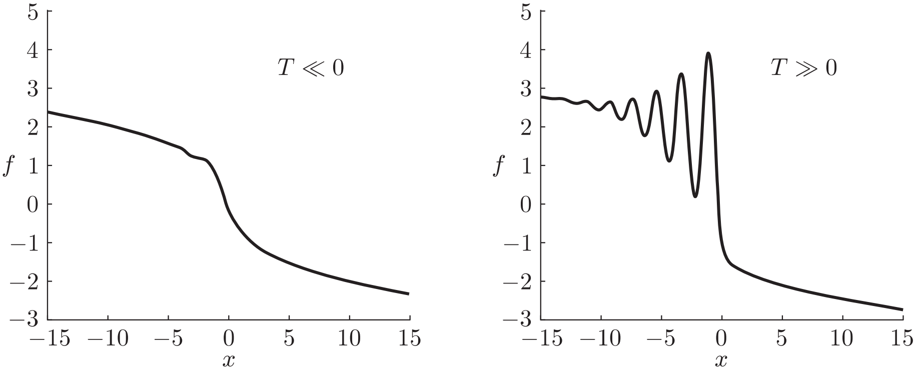

Figure 3. The solution $U(X,T)$ to the ODE (21) for two values of the parameter T

We are now ready to formulate the Universality Conjecture for scalar Hamiltonian PDEs.

Conjecture 6. *Let us consider a generic Hamiltonian perturbation of the equation*

$$v_t + a(v)v_x = 0, \qquad a'(v)\neq 0. \tag{22}$$

A generic solution to the perturbed equation near its critical point (x_0,t_0,v_0) *may be represented in the following form*:

$$u \simeq v_0 + \left(\frac{\varepsilon^2 c_0}{\kappa^2}\right)^{1/7} U\left(\frac{x-a_0(t-t_0)-x_0}{(\kappa\, c_0^3\varepsilon^6)^{1/7}}\,,\ \frac{a_0'(t-t_0)}{(\kappa^3 c_0^2\varepsilon^4)^{1/7}}\right) + O(\varepsilon^{4/7}), \tag{23}$$

where $a_0=a(v_0)$, $a_0'=a'(v_0)$, c_0 *and* κ *are some constants, and* $U(X,T)$ *is the solution to the ODE* (21) *described above.*

A proof of this conjecture for solutions to the KdV equation with rapidly decreasing analytic initial data has recently been obtained by Claeys and Grava.

As follows from Table 2, the same special function also describes the critical behaviour of generic solutions to a perturbed Hamiltonian hyperbolic system of the second order. For the critical behaviour of solutions to Hamiltonian perturbations of elliptic systems (for example, for the focusing non-linear Schrödinger equation) another special function is needed. We shall now describe this function.

It appears as a particular solution to the classical Painlevé-I equation ($\mathrm{P_I}$)

$$W'' = 6W^2 - Z. \tag{24}$$

As above, all solutions to this equation are meromorphic functions of the complex variable Z. The asymptotic distribution of poles of a generic solution to P_I was thoroughly studied by Boutroux in 1913. Boutroux proved that the lines of poles of a generic solution to $P_{\rm I}$ accumulate along the five rays in the complex plane

$$\arg Z = \frac{2\pi n}{5}, \qquad n = 0, \pm 1, \pm 2. \tag{25}$$

Boutroux' main discovery was the proof of the existence of special solutions for which the lines of poles asymptotically truncate along three consecutive rays of the form (25). These solutions, called *tritronquée*, are determined uniquely for every triple of consecutive rays.

Let us denote by $W_0(Z)$ the particular *tritronquée* solution associated with the triple of rays (25) with $n = 0, \pm 1$. By definition this solution has at most a finite number of poles in the sector $|\arg Z| < 4\pi/5 - \delta$ for an arbitrary positive δ. The following conjecture, due to Grava, Klein, and the author, claims that there are no poles in this sector.

Conjecture 7. *The tritronquée solution $W_0(Z)$ is an analytic function for all complex Z satisfying*

$$|\arg Z| < \frac{4\pi}{5}. \tag{26}$$

From the graph shown in Fig. 4 it follows that the solution under consideration has no poles in the sector (26).

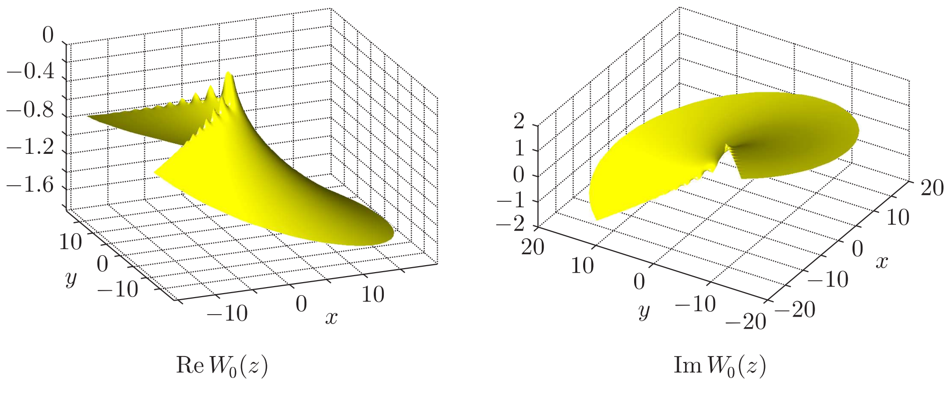

Figure 4. The graph of the real (left) and imaginary (right) parts of the *tritronquée* solution $W_0(Z)$ to $P_{\rm I}$ in the sector $|\arg Z| < 4\pi/5$

We are now ready to formulate the Universality Conjecture describing the critical behaviour of solutions to Hamiltonian perturbations of systems of elliptic type. By definition the unperturbed system possesses a pair of complex conjugate Riemann invariants w and $\bar{w}$. The characteristic directions z and $\bar{z}$ are also complex conjugate. One can conclude from Table 2 that the critical points of the unperturbed solution are isolated. Moreover, in a neighborhood of a critical point the solution has a singularity described by the square root of a complex quantity.

Conjecture 8. *A generic solution to a generic Hamiltonian perturbation of an arbitrary quasi-linear second order system of elliptic type near a critical point can* *be represented in the form*

$$w \simeq w^0 + \alpha\varepsilon^{2/5} W_0(\varepsilon^{-4/5} z) + O(\varepsilon^{4/5}), \qquad z = \beta_+ x + \beta_- t + z_0, \tag{27}$$

where $\alpha \neq 0$, $\beta_\pm$, z_0 *are some complex constants such that*

$$|\arg z| < \frac{4\pi}{5} \quad \text{for small } |t - t_0|$$

for all $x \in \mathbb{R}$.

This conjecture, first formulated by Grava, Klein, and the author in the study of the critical behaviour of solutions to the focusing non-linear Schrödinger equation, so far remains open.

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Part 1: classification problems

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B. A. Dubrovin
Steklov Mathematical Institute,
Russian Academy of Sciences;
International School for Advanced Studies (SISSA)
E-mail: dubrovin@sissa.it

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Translated by THE AUTHOR