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CLASSICAL DOUBLE, *R*-OPERATORS, AND NEGATIVE FLOWS OF INTEGRABLE HIERARCHIES

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Using the classical double $\mathcal{G}$ of a Lie algebra $\mathfrak{g}$ equipped with the classical R-operator, we define two sets of functions commuting with respect to the initial Lie–Poisson bracket on $\mathfrak{g}^*$ and its extensions. We consider examples of Lie algebras $\mathfrak{g}$ with the "Adler–Kostant–Symes" R-operators and the two corresponding sets of mutually commuting functions in detail. Using the constructed commutative Hamiltonian flows on different extensions of $\mathfrak{g}$, we obtain zero-curvature equations with $\mathfrak{g}$-valued U–V pairs. The so-called negative flows of soliton hierarchies are among such equations. We illustrate the proposed approach with examples of two-dimensional Abelian and non-Abelian Toda field equations.

Keywords: classical R-operator, integrable hierarchy

1. Introduction

The theory of hierarchies of integrable partial differential equations is based on the possibility to represent each of the equations of the hierarchy in the so-called zero-curvature form $U_t - V_x + [U, V] = 0$ with the corresponding U–V-pair taking values in some infinite-dimensional Lie algebra $\mathfrak{g}$ (e.g., in the algebra of matrix-valued Laurent polynomials of a single complex parameter λ). There are several approaches to constructing zero-curvature equations starting from Lie algebras. The first of them is based on interpreting the zero-curvature equation as one Lax (Euler–Arnold) equation written on the centrally extended algebra of $\mathfrak{g}$-valued functions of x [1]. In this approach, $\partial_x - U$ plays the Lax operator role, and the second operator in the Lax pair is V.

In an alternative approach of [2], [3], the zero-curvature equations are interpreted as the compatibility conditions for two auxiliary Lax (Euler–Arnold) equations. The commutativity of these Lax flows is guaranteed by the Lie–Poisson commutativity of the corresponding Hamiltonians. In this approach, the elements U and V in the zero-curvature equations coincide with algebra-valued gradients of commuting Hamiltonians obtained using the Adler–Kostant–Symes (AKS) scheme. In more detail, these Hamiltonians coincide with the restrictions of Casimir functions of $\mathfrak{g}$ to the spaces dual to the subalgebras $\mathfrak{g}_\pm$, where $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$. Such an approach allows constructing two types of integrable equations associated with the Lie algebra $\mathfrak{g}$, namely, integrable equations with the elements U and V belonging to the same Lie subalgebras $\mathfrak{g}_+$ or $\mathfrak{g}_-$. But the approach in [3] does not cover all known integrable equations. In particular, it does not work for integrable equations (sometimes called *negative flows* of integrable hierarchies[1]) with

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[1] In this paper, we cannot cite or review all the papers on the subject. The interested reader may consult, for example, the recent papers [4]–[6].

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U–V-pairs where the U-operator belongs to $\mathfrak{g}_+$ and the V-operator belongs to $\mathfrak{g}_-$. In [7], such equations were included in the general scheme by showing that the restrictions of the Casimir functions of $\mathfrak{g}$ to the spaces dual to the subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ commute not only inside each group but also between the groups. This allows constructing negative flows of integrable hierarchies as a consequence of the commutativity of Lax flows generated by "positive" and "negative" Hamiltonians. In [8], [9], it was proposed to generalize the above scheme to the case of Lie algebras $\mathfrak{g}$ that have a general classical R-operator not always related to the decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ (i.e., not always associated with the AKS scheme). It was shown that the restrictions of the Casimir functions of $\mathfrak{g}$ to the subalgebras $\mathfrak{g}_{R_\pm}$, where $\mathfrak{g}_{R_\pm} = \operatorname{Im} R_\pm$, commute not only inside each group but also between the groups. This observation allows obtaining two sets of mutually commuting functions on $\mathfrak{g}^*$ and three types of zero-curvature equations, in particular, those corresponding to negative flows of soliton hierarchies [9]. We note that the corresponding commutativity does not follow from the standard R-matrix scheme [10] on $\mathfrak{g}$.

Nevertheless, it turns out that the scheme proposed in [8], [9] is still not the most general approach for generating commutative flows on $\mathfrak{g}^*$ and hence not the most general approach to the construction of soliton hierarchies with $\mathfrak{g}$-valued U–V pairs. In particular, it does not include the infinite-component Toda hierarchy and does not produce the corresponding auxiliary Lax equations [11].

Here, we propose a more general approach for constructing commuting flows and zero-curvature equations with $\mathfrak{g}$-valued U–V pairs. For this, we consider commuting flows not on $\mathfrak{g}^*$ but on $\mathcal{G}^*$, where $\mathcal{G}$ is the classical double of $\mathfrak{g}$. We use the fact that a classical R-operator on $\mathfrak{g}$ induces a natural R-operator $\mathcal{R}$ on $\mathcal{G} = \mathfrak{g} \oplus \mathfrak{g}$ [12]. This R-operator $\mathcal{R}$ on $\mathcal{G}$ proves to be always of the AKS type, regardless of the form of the original operator R on $\mathfrak{g}$. It hence follows that $\mathcal{G}_{\mathcal{R}} = \mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{\mathcal{R}_-}$, where $\mathcal{G}_{\mathcal{R}}$ is a linear space $\mathcal{G}$ equipped with the so-called $\mathcal{R}$-bracket [10]. Moreover, it turns out that $\mathcal{G}_{\mathcal{R}_+} \simeq \mathfrak{g}$ and $\mathcal{G}_{\mathcal{R}_-} \simeq \mathfrak{g}_R$, where the algebra $\mathfrak{g}_R$ is a linear space $\mathfrak{g}$ equipped with the R-bracket [12] and $\mathcal{G}_{\mathcal{R}_\pm} \equiv \operatorname{Im} \mathcal{R}_\pm$.

Therefore, our first observation is that using the standard R-matrix scheme [10] applied to the Lie algebra $\mathcal{G}$ equipped with the R-operator $\mathcal{R}$, we can obtain a set of commuting flows on extensions of $\mathfrak{g}$ by some Lie algebra $\mathfrak{a}$, where $\mathfrak{a} = \mathfrak{g}_R/J_R$ and J_R is an ideal in $\mathfrak{g}_R$. In the particular case where $J_R = \mathfrak{g}_R$, we in a simple way rederive the result in [8] about the commutativity of the restrictions of the Casimir functions of $\mathfrak{g}$ to the subalgebras $\mathfrak{g}_{R_\pm}$. We thus show that the results in [8] fit into the general R-matrix scheme. In the case $J_R = [\mathfrak{g}_R, \mathfrak{g}_R]$, we obtain an important generalization of the abovementioned result, namely, we prove commutativity of the restrictions of the Casimir functions of $\mathfrak{g}$ to the subalgebras $\mathfrak{g}_{R_\pm}$ shifted using the respective constant elements $c_\mp \in [\mathfrak{g}_{R_\mp}, \mathfrak{g}_{R_\mp}]$. We note that the obtained functions in this case commute with respect to the Lie–Poisson bracket on $\mathfrak{g}$ "shifted" by the constant element $c_+ - c_-$.

A consideration of commutative families on more complicated quotients (with a non-Abelian $\mathfrak{a}$) might also be useful in the theory of soliton equations. Indeed, our second simple observation suggests that whatever quotient of $\mathcal{G}_{\mathcal{R}}$ is considered, from the Lax equations $\dot{L} = [L, M]$ corresponding to the commuting Hamiltonians (the Casimir functions restricted to this quotient), M-operators can be chosen to take values in $\mathfrak{g}$. It hence follows that zero-curvature equations with $\mathfrak{g}$-valued U–V-pairs can be obtained as a consistency condition for the Lax equations on $\mathfrak{g} \ominus \mathfrak{a}$.

We illustrate the above method with the example of Abelian and non-Abelian Toda field equations (see [13]–[15] and the references therein) that are naturally obtained in the framework of the above scheme if $\mathfrak{g}$ is a loop algebra equipped with various gradings. The corresponding quotient algebra in this case is the simplest non-Abelian extension of $\mathfrak{g}$ obtained in the framework of the above construction. In the case where $\mathfrak{g} = gl(\!(\infty)\!)$ equipped with the natural decomposition into a sum of two subalgebras coinciding with the upper triangular and strictly lower triangular matrices, we recover the results in [16] for the Lie–Poisson structure and Lie-theoretical interpretation of the infinite-component Toda field equations, its U–V pair, auxiliary Lax pairs, etc. [11].

In conclusion, for completeness, we also consider the prolongation of the second- and third-order Poisson structures existing in the cases of certain R-operators on $\mathfrak{g}$ to the classical double. It turns out that the quadratic and cubic structures are always prolonged to $\mathcal{G}$ if they exist on $\mathfrak{g}$. Nevertheless, their use in the soliton theory is restricted because the quotient spaces described above, the Poisson spaces of the linear $\mathcal{R}$-bracket on $\mathcal{G}$, are generally not Poisson subspaces of the quadratic and cubic brackets.

The structure of this paper is as follows. In Sec. 2, we introduce the main definitions and notation. In Sec. 3, we use the classical double to obtain commutative families on $\mathfrak{g}^*$ and its extensions. In Sec. 4, we use the obtained results to construct zero-curvature equations with $\mathfrak{g}$-valued U–V pairs and illustrate this approach with examples of Abelian and non-Abelian Toda field equations. Finally, in Sec. 5, we consider the prolongation of the second- and third-order Poisson structures to the double.

2. Definitions and notation

2.1. Lie algebras and classical $\boldsymbol{R}$-operators. Let $\mathfrak{g}$ be a Lie algebra (finite- or infinite-dimensional) with the Lie bracket $[\,\cdot\,,\,\cdot\,]$, and let $R\colon \mathfrak{g} \to \mathfrak{g}$ be a linear operator. The operator R is called a classical R-operator if it satisfies the modified Yang–Baxter equation [10]

$$R\big([R(X),Y]+[X,R(Y)]\big)-[R(X),R(Y)]=[X,Y]$$

for all $X,Y\in\mathfrak{g}$. Using a classical R-operator, we can define another bracket on $\mathfrak{g}$ by the formula [10]

$$[X,Y]_R=[R(X),Y]+[X,R(Y)],\qquad X,Y\in\mathfrak{g}. \tag{1}$$

We let $\mathfrak{g}_R$ denote the linear space $\mathfrak{g}$ equipped with the Lie bracket $[\,\cdot\,,\,\cdot\,]_R$. We also use the notation $R_\pm \equiv R\pm\mathrm{Id}$ hereafter.

It is known [10] that the images $\mathfrak{g}_{R_\pm}=\operatorname{Im}R_\pm$ of the maps $R_\pm$ define Lie subalgebras $\mathfrak{g}_{R_\pm}\subset\mathfrak{g}$. As is easily seen from the definition, $\mathfrak{g}_{R_+}+\mathfrak{g}_{R_-}=\mathfrak{g}$, but this sum is generally not a direct sum of vector spaces, i.e., $\mathfrak{g}_{R_+}\cap\mathfrak{g}_{R_-}\neq 0$ in the general case.

Remark 1. The situation is much simpler in the case of a Lie algebra $\mathfrak{g}$ with the so-called AKS decomposition into a direct sum of two Lie subalgebras: $\mathfrak{g}=\mathfrak{g}_+ + \mathfrak{g}_-$. Indeed, if $P_\pm$ are the projection operators on the subalgebras $\mathfrak{g}_\pm$, then $R=P_+-P_-$ is a classical R-matrix [10]. It is easy to see that in this case, $R_\pm=\pm 2P_\pm$ are proportional to the projection operators $R_\pm$ on the subalgebras $\mathfrak{g}_\pm$. It also follows that $\mathfrak{g}_{R_\pm}\equiv\mathfrak{g}_\pm$ and $\mathfrak{g}_{R_+}\cap\mathfrak{g}_{R_-}=0$. It is also known that $\mathfrak{g}_R=\mathfrak{g}_+\ominus\mathfrak{g}_-$ in this case [10].

2.2. Classical double. We now consider the “double” of the Lie algebra $\mathfrak{g}$, i.e., the direct sum algebra $\mathcal{G}=\mathfrak{g}\oplus\mathfrak{g}$. We identify the elements of $\mathcal{X}\in\mathcal{G}$ with vector columns $\mathcal{X}=\left(\begin{smallmatrix}X_1\\X_2\end{smallmatrix}\right)$, where $X_i\in\mathfrak{g}$. The bracket of two elements $\mathcal{X},\mathcal{Y}\in\mathcal{G}$ is given by the standard formula

$$[\mathcal{X},\mathcal{Y}]=\begin{pmatrix}[X_1,Y_1]\\ [X_2,Y_2]\end{pmatrix}.$$

The following construction was developed in [12].

Theorem 2.1. 1. *Given an arbitrary classical R-operator on $\mathfrak{g}$, the operator defined on the double by the formula*

$$\mathcal{R}=\begin{pmatrix}R & -R_-\\ R_+ & -R\end{pmatrix},$$

is a classical R-operator on $\mathcal{G}$.

2. *The corresponding R-bracket $[\,\cdot\,,\,\cdot\,]_{\mathcal{R}}$ on $\mathcal{G}$ has the form*

$$[\mathcal{X},\mathcal{Y}]_{\mathcal{R}} = \begin{pmatrix} [X_1,Y_1]_R - \big([X_1,R_-(Y_2)] + [R_-(X_2),Y_1]\big) \\ -[X_2,Y_2]_R + \big([X_2,R_+(Y_1)] + [R_+(X_1),Y_2]\big) \end{pmatrix}.$$

3. *The R-matrix $\mathcal{R}$ is of the AKS type.*

For the reader's convenience, we sketch the proof of statement 3 in the theorem. We consider the operators

$$\mathcal{R}_+ = \mathcal{R} + \mathrm{Id} = \begin{pmatrix} R_+ & -R_- \\ R_+ & -R_- \end{pmatrix}, \qquad \mathcal{R}_- = \mathcal{R} - \mathrm{Id} = \begin{pmatrix} R_- & -R_- \\ R_+ & -R_+ \end{pmatrix}.$$

We let $\mathcal{G}_{\mathcal{R}_\pm} = \operatorname{Im} R_\pm$ denote the corresponding Lie subalgebras. It is easy to see that

$$\mathcal{R}_+(\mathcal{X}) = \begin{pmatrix} R_+(X_1) - R_-(X_2) \\ R_+(X_1) - R_-(X_2) \end{pmatrix}, \qquad \mathcal{R}_-(\mathcal{X}) = \begin{pmatrix} R_-(X_1 - X_2) \\ R_+(X_1 - X_2) \end{pmatrix}.$$

It hence follows that $\mathcal{G}_{\mathcal{R}_+} \equiv \mathcal{G}_{\mathrm{d}} \simeq \mathfrak{g}$ and $\mathcal{G}_{\mathcal{R}_-} \simeq \mathfrak{g}_R$, where

$$\mathcal{G}_{\mathrm{d}} = \left\{ \begin{pmatrix} X \\ X \end{pmatrix} \middle| X \in \mathfrak{g} \right\}, \qquad \mathcal{G}_{\mathcal{R}_-} = \left\{ \begin{pmatrix} R_-(X) \\ R_+(X) \end{pmatrix} \middle| X \in \mathfrak{g} \right\}.$$

It is easy to see that $\operatorname{Ker} \mathcal{R}_+ = \operatorname{Im} \mathcal{R}_-$ and $\operatorname{Ker} \mathcal{R}_- = \operatorname{Im} \mathcal{R}_+$. Hence, the decomposition $\mathcal{G} = \mathcal{G}_{\mathcal{R}_+} + \mathcal{G}_{\mathcal{R}_-}$ is a decomposition into a direct sum of vector spaces, and the operator $\mathcal{R}$ is of the AKS type, i.e., $\mathcal{R} = \mathcal{P}_{\mathcal{G}_{\mathcal{R}_+}} - \mathcal{P}_{\mathcal{G}_{R_-}}$. This also follows from the easily proved identities $\mathcal{R}_\pm^2 = 2\mathcal{R}_\pm$ and $\mathcal{R}_+\mathcal{R}_- = 0$, which imply that $\mathcal{R}_\pm$ are proportional to projection operators and their images do not intersect. Therefore, we have $\mathcal{G}_{\mathcal{R}} = \mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{R_-}$. For the $\mathcal{R}$-bracket on the double, this identity means that $[\mathcal{X},\mathcal{Y}]_{\mathcal{R}} = 2([\mathcal{X}_+,\mathcal{Y}_+] - [\mathcal{X}_-,\mathcal{Y}_-])$, where $\mathcal{X}_\pm \equiv \mathcal{R}_\pm(\mathcal{X})$ or, more explicitly,

$$[\mathcal{X},\mathcal{Y}]_{\mathcal{R}} = \begin{pmatrix} \big[\big(R_+(X_1) - R_-(X_2)\big), \big(R_+(Y_1) - R_-(Y_2)\big)\big] \\ \big[\big(R_+(X_1) - R_-(X_2)\big), \big(R_+(Y_1) - R_-(Y_2)\big)\big] \end{pmatrix} - \begin{pmatrix} [R_-(X_1 - X_2), R_-(Y_1 - Y_2)] \\ [R_+(X_1 - X_2), R_+(Y_1 - Y_2)] \end{pmatrix}.$$

Remark 2. In the case of the AKS R-operators, all the formulas in this subsection are substantially simplified. In particular, the action of the R-operator $\mathcal{R}$ on the element $\mathcal{X}$ is given by the formula

$$\mathcal{R}(\mathcal{X}) = \begin{pmatrix} X_1^+ - X_1^- + 2X_2^- \\ 2X_1^+ - X_2^+ + X_2^- \end{pmatrix},$$

and R-bracket (1) is written as

$$[\mathcal{X},\mathcal{Y}]_{\mathcal{R}} = \begin{pmatrix} [X_1^+,Y_1^+] - [X_1^-,Y_1^-] - ([X_1,Y_2^-] + [X_2^-,Y_1]) \\ -[X_2^+,Y_2^+] + [X_2^-,Y_2^-] + ([X_2,Y_1^+] + [X_1^+,Y_2]) \end{pmatrix},$$

where $X_i = X_i^+ + X_i^-$, $Y_i = Y_i^+ + Y_i^-$, $X_i^\pm = P_\pm(X)$, and $Y_i^\pm = P_\pm(Y)$, $i = 1, 2$.

2.3. Dual spaces, Lie–Poisson brackets, and invariant functions. Let $\mathfrak{g}^*$ be the space dual to the Lie algebra $\mathfrak{g}$ and $\langle\,\cdot\,,\,\cdot\,\rangle\colon \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{C}$ be the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Let $\{X_i \mid i\in I\}$ be a basis in the Lie algebra $\mathfrak{g}$, where the set I is finite in the case of a finite-dimensional Lie algebra and countable in the infinite-dimensional case. Let $\{X_i^* \mid i\in I\}$, $\langle X_j^*, X_i\rangle=\delta_{ij}$, be a basis in the dual space $\mathfrak{g}^*$. Let $L=\sum_{i\in I} L_i X_i^*\in\mathfrak{g}^*$ be a generic element of $\mathfrak{g}^*$ and L_i be the coordinate functions on $\mathfrak{g}^*$. We consider the standard Lie–Poisson bracket between $F_1, F_2\in C^\infty(\mathfrak{g}^*)$ on $\mathfrak{g}^*$:

$$\{F_1(L),F_2(L)\}=\langle L,[\nabla F_1,\nabla F_2]\rangle,\qquad \nabla F_k(L)=\sum_{i\in I}\frac{\partial F_k(L)}{\partial L_i}X_i,$$

where $\nabla F_k(L)$ is a so-called algebra-valued gradient of F_k, $k=1,2$. Moreover, the R-operator provides the so-called R-bracket on $\mathfrak{g}^*$ [10]:

$$\{F_1(L),F_2(L)\}_R=\langle L,[\nabla F_1,\nabla F_2]_R\rangle. \tag{2}$$

We consider the space $\mathcal{G}^*$ dual to the double. We define its elements $\mathcal{L}\in\mathfrak{g}^*\oplus\mathfrak{g}^*$ as the vector columns $\mathcal{L}=\left(\begin{smallmatrix}L_1\\ L_2\end{smallmatrix}\right)$, where $L_1,L_2\in\mathfrak{g}^*$, and the pairing between $\mathcal{L}\in\mathcal{G}^*$ and $\mathcal{X}\in\mathcal{G}$ is given by

$$\langle\mathcal{L},\mathcal{X}\rangle=\langle L_1,X_1\rangle+\langle L_2,X_2\rangle.$$

It also defines the standard Lie–Poisson bracket on $\mathcal{G}^*$,

$$\{F_1(\mathcal{L}),F_2(\mathcal{L})\}=\langle\mathcal{L},[\widetilde{\nabla}F_1,\widetilde{\nabla}F_2]\rangle, \tag{3}$$

and the R-bracket on $\mathcal{G}^*$ corresponding to the R-operator $\mathcal{R}$,

$$\{F_1(\mathcal{L}),F_2(\mathcal{L})\}_{\mathcal{R}}=\langle\mathcal{L},[\widetilde{\nabla}F_1,\widetilde{\nabla}F_2]_{\mathcal{R}}\rangle,\qquad \widetilde{\nabla}F=\begin{pmatrix}\nabla_1 F\\ \nabla_2 F\end{pmatrix}, \tag{4}$$

where $\nabla_{1,2}F$ is the algebra-valued gradient of F with respect to the variable $L_{1,2}$.

Let R^* denote the operator adjoint to R,

$$R^*\colon\mathfrak{g}^*\to\mathfrak{g}^*,\qquad \langle R^*(L),X\rangle\equiv\langle L,R(X)\rangle.$$

It is easy to see that the operators adjoint to $\mathcal{R}_\pm$ have the form

$$\mathcal{R}_+^*(\mathcal{L})=\begin{pmatrix}R_+^*(L_1)+R_+^*(L_2)\\ -\big(R_-^*(L_1)+R_-^*(L_2)\big)\end{pmatrix},\qquad \mathcal{R}_-^*(\mathcal{L})=\begin{pmatrix}R_-^*(L_1)+R_+^*(L_2)\\ -\big(R_-^*(L_1)+R_+^*(L_2)\big)\end{pmatrix}. \tag{5}$$

We use these explicit formulas to construct Poisson-commuting functions.

In what follows, we also need the explicit form of the Casimir functions on $\mathcal{G}$. Let $I(L)\in I^G(\mathfrak{g}^*)$ be a Casimir function of $\mathfrak{g}$, i.e., $\{I(L),F(L)\}=0$ for all $F(L)\in S(\mathfrak{g}^*)$. Let $\{I_k(L)\}_{k\in K}$ denote the set of generators of the ring of Casimir functions on $\mathfrak{g}^*$. Here, the set of labels K is infinite if the Lie algebra is infinite-dimensional.

Lemma 2.1. *The ring of Casimir functions of $\mathcal{G}$ is generated by the functions*

$$I_{k,1}(\mathcal{L})\equiv I_k(L_1),\qquad I_{k,2}(\mathcal{L})\equiv I_k(L_2),\quad k\in K. \tag{6}$$

The proof is straightforward.

We briefly comment on quadratic Casimir functions and commuting functions obtained using them. Let $(\,\cdot\,,\,\cdot\,)$ be an invariant form on $\mathfrak{g}$. Using this form, we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$. We then have the obvious second-order Casimir functions (or generating functions of formal Casimir functions in the case of loop algebras)

$$I_{2,1}=\frac{1}{2}(L_1,L_1),\qquad I_{2,2}=\frac{1}{2}(L_2,L_2).$$

3. Classical double and commuting flows

To pass to the main construction considered in this paper, we recall the following theorem, which can be obtained from the general theory of R-brackets [10] applied to the classical double.

Theorem 3.1. 1. *The Casimir functions $I_{k,\epsilon}(\mathcal{L})$, $\epsilon = 1,2$, of the Lie–Poisson brackets of $\mathcal{G}$ commute with respect to the brackets $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{R}}$ on $\mathcal{G}^*$.*

2. *The Hamiltonian flows*

$$\frac{d}{dt_k^\epsilon}F(\mathcal{L}) = \{F(\mathcal{L}), I_{k,\epsilon}\}_{\mathcal{R}}, \quad k \in K, \quad \epsilon = 1,2,$$

generated by the functions $I_{k,\epsilon}$ can be written in the Euler–Arnold form

$$\frac{d\mathcal{L}}{dt_k^\epsilon} = \operatorname{ad}^*_{\mathcal{R}_+\widetilde{\nabla} I_{k,\epsilon}}\mathcal{L}. \tag{7}$$

We briefly consider the commuting flows on $\mathfrak{g}$ and its extensions that can be obtained using the theory of the classical double. For this, we recall that the projection on a quotient algebra is a canonical homomorphism, which allows obtaining the following corollary.

Corollary 3.1. *Let J be an ideal in $\mathcal{G}_{\mathcal{R}}$. Let $\pi\colon \mathcal{G} \to \mathcal{G}/J$ denote the projection on the quotient algebra. Let π^* be the dual map. Then*

1. *the functions $I_{k,\epsilon}(\pi^*(\mathcal{L}))$ commute with respect to the brackets $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{R}}$ on $(\mathcal{G}/J)^*$, and*
2. *the Hamiltonian flows corresponding to $I_{k,\epsilon}(\pi^*(\mathcal{L}))$ can be written in the Euler–Arnold form*

$$\pi^*\left(\frac{d\mathcal{L}}{dt_k^\epsilon}\right) = \operatorname{ad}^*_{M_{k,\epsilon}}\pi^*(\mathcal{L}), \tag{8}$$

where

$$M_{k,\epsilon} = \mathcal{R}_+\bigl(\pi\widetilde{\nabla} I_{k,\epsilon}(\pi^*(\mathcal{L}))\bigr), \quad k \in K, \quad \epsilon = 1,2.$$

We assume that there exist nontrivial ideals $J_{R_\pm} \subset \mathfrak{g}_{R_\pm}$ such that the quotients $\mathfrak{g}_{R_\pm}/J_{R_\pm}$ are finite-dimensional. In such a case, applying Corollary 3.1 and taking the quotient by the ideal $J = J_{R_+} + J_{R_-}$, we obtain a Poisson-commuting set of functions on the space dual to finite-dimensional extensions of $\mathfrak{g}$. Indeed, we have $\mathcal{G}_{\mathcal{R}}/J = (\mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{\mathcal{R}_-})/J \simeq \mathfrak{g} \ominus \mathfrak{a}$, where $\mathfrak{a} \simeq \mathcal{G}_{\mathcal{R}_-}/J$.

Remark 3. For the quotient algebras described above, the M-operators in Lax equations (8) have the forms

$$M_{k,1} = \begin{pmatrix} R_+\bigl(\nabla I_{k,1}(\pi^*(\mathcal{L}))\bigr) \\ R_+\bigl(\nabla I_{k,1}(\pi^*(\mathcal{L}))\bigr)\end{pmatrix}, \qquad M_{k,2} = -\begin{pmatrix} R_-\bigl(\nabla I_{k,2}(\pi^*(\mathcal{L}))\bigr) \\ R_-\bigl(\nabla I_{k,2}(\pi^*(\mathcal{L}))\bigr)\end{pmatrix},$$

i.e., they belong to the diagonal subalgebra and can be identified with elements of $\mathfrak{g}$. We use this fact when constructing zero-curvature equations with values in $\mathfrak{g}$. We note that the corresponding Lax equations (8), dynamical variables, Poisson brackets, etc., belong to the double of $\mathfrak{g}$. We also note that the operator π is absent from the above formulas for the M-operators because the ideal J in this case was chosen such that $\mathcal{R}_+\pi = \mathcal{R}_+$.

Example 1. We consider the case of AKS R-operators $R = P_+ - P_-$. We write the quotient algebras $\mathcal{G}_\mathcal{R}/J$ and the corresponding dual spaces in explicit form. In this case, we have $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ and $\mathfrak{g}_{R_\pm} = \mathfrak{g}_\pm$, and $J_{R_\pm} \equiv J_\pm$ are ideals in $\mathfrak{g}_\pm$. Elements of the corresponding quotient $\mathcal{G}_\mathcal{R}/J$, where and $J = J_+ + J_-$, are given by

$$\mathcal{X} = \begin{pmatrix} X_1^+ + X_1^{-'} \\ X_2^- + X_2^{+'} \end{pmatrix}, \qquad X_1^{-'} \in \mathfrak{g}_-/J_-, \qquad X_2^{+'} \in \mathfrak{g}_+/J_+.$$

The corresponding dual space consists of the elements

$$\mathcal{L} = \begin{pmatrix} L_1^+ + L_1^{-'} \\ L_2^- + L_2^{+'} \end{pmatrix},$$

where $L_1^{-'} \in (\mathfrak{g}_-/J_-)^*$, $L_1^+ \in (\mathfrak{g}_+)^*$, $L_2^- \in (\mathfrak{g}_-)^*$, and $L_2^{+'} \in (\mathfrak{g}_+/J_+)^*$.

In the next subsection, we consider the cases of $\mathfrak{a} \simeq \mathcal{G}_{\mathcal{R}_-}/J = 0$ and of an Abelian $\mathfrak{a}$ that lead to commutative algebras of integrals on $\mathfrak{g}^*$ itself.

3.1. Dual *R*-matrix commutativity. We consider a consequence of general Theorem 3.1 that we need for constructing Poisson-commuting sets on $\mathfrak{g}^*$ with respect to the standard Lie–Poisson brackets $\{\,\cdot\,,\,\cdot\,\}$. The following theorem holds.

Theorem 3.2. 1. *The functions $I_k(R^*_\pm(L))$ on $\mathfrak{g}^*$ generate an Abelian subalgebra in $C^\infty(\mathfrak{g}^*)$ with respect to the Lie–Poisson brackets $\{\,\cdot\,,\,\cdot\,\}$ on $\mathfrak{g}^*$:*

$$\begin{aligned} \big\{I_k\big(R^*_+(L)\big), I_l\big(R^*_+(L)\big)\big\} &= 0, \\ \big\{I_k\big(R^*_-(L)\big), I_l\big(R^*_-(L)\big)\big\} &= 0, \\ \big\{I_k\big(R^*_+(L)\big), I_l\big(R^*_-(L)\big)\big\} &= 0. \end{aligned}$$

2. *The Hamiltonian equations corresponding to the Hamiltonians $I_k^{R_\pm}(L)$ can be written in the Euler–Arnold form*

$$\frac{dL}{dt_k^\pm} = \mathrm{ad}^*_{M_k^\pm} L, \qquad M_k^\pm = \nabla I_k(R^*_\pm(L)). \tag{9}$$

Proof. We project the functions $I_{k,\epsilon}(\mathcal{L})$, $\epsilon = 1, 2$, on the space dual to the quotient algebra $\mathcal{G}_\mathcal{R}/\mathcal{G}_{R_-}$, which coincides with the space dual to the subalgebra $\mathcal{G}_{R_+}$. Using formulas (5) and (6), we obtain the expressions for the projected Casimir functions:

$$I_{k,1}\big(P^*_{\mathcal{G}_{R_+}}(\mathcal{L})\big) = I_k\big(R^*_+(L_1 + L_2)\big), \qquad I_{k,2}\big(P^*_{\mathcal{G}_{R_+}}(\mathcal{L})\big) = I_k\big(R^*_-(L_1 + L_2)\big),$$

where we take into account that $P^*_{\mathcal{G}_{R_+}} = \mathcal{R}^*_+/2$ and assume that the I_k are homogeneous functions of L.

We note that because $\mathcal{R}_+$ is a projection operator and the corresponding R-operator $\mathcal{R}$ is of the AKS type, it is easy to derive the equality

$$\big\{F\big(\mathcal{R}^*_+(\mathcal{L})\big), G\big(\mathcal{R}^*_+(\mathcal{L})\big)\big\} = \big\{F\big(\mathcal{R}^*_+(\mathcal{L})\big), G\big(\mathcal{R}^*_+(\mathcal{L})\big)\big\}_\mathcal{R}.$$

Using the fact that the projection on a quotient algebra is a canonical homomorphism, we obtain

$$\big\{F\big(\mathcal{R}^*_\pm(\mathcal{L})\big), G\big(\mathcal{R}^*_\pm(\mathcal{L})\big)\big\}_\mathcal{R} = \big(\{F(\mathcal{L}), G(\mathcal{L})\}_\mathcal{R}\big)\big|_{\mathcal{L}=\mathcal{R}^*_\pm(\mathcal{L})}.$$

Hence, setting $F = I_{k,\epsilon}$ and $G = I_{l,\epsilon'}$, we obtain

$$\big\{I_{k,\epsilon}\big(\mathcal{R}_+^*(\mathcal{L})\big), I_{l,\epsilon'}\big(\mathcal{R}_+^*(\mathcal{L})\big)\big\} = \big(\{I_{k,\epsilon}(\mathcal{L}), I_{l,\epsilon'}(\mathcal{L})\}_{\mathcal{R}}\big)\big|_{\mathcal{L}=\mathcal{R}_+^*(\mathcal{L})}, \quad \epsilon, \epsilon' = 1, 2.$$

On the other hand, $\{I_{k,\epsilon}(\mathcal{L}), I_{l,\epsilon'}(\mathcal{L})\}_{\mathcal{R}} = 0$ by Theorem 3.1.

Using the explicit form of the functions $I_{k,\epsilon}(\mathcal{L})$, we finally obtain

$$\big\{I_k\big(R_+^*(L_1+L_2)\big), I_l\big(R_+^*(L_1+L_2)\big)\big\} = 0,$$
$$\big\{I_k\big(R_-^*(L_1+L_2)\big), I_l\big(R_-^*(L_1+L_2)\big)\big\} = 0,$$
$$\big\{I_k\big(R_+^*(L_1+L_2)\big), I_l\big(R_-^*(L_1+L_2)\big)\big\} = 0.$$

Now, to complete the proof of statement 1 in the theorem, it remains to note that elements of the form $L \equiv L_1 + L_2$ belong to the subspace $(\mathcal{G}_{\mathrm{d}})^*$ and the corresponding coordinate functions constitute a Lie algebra isomorphic to $(\mathfrak{g}, \{\,\cdot\,,\,\cdot\,\})$ with respect to the initial Lie–Poisson brackets on $\mathcal{G}$.

Statement 2 in the theorem can be proved using statement 2 in Theorem 3.1. It can also be proved by noting that any Hamiltonian equation on $\mathfrak{g}^*$ is rewritten in the Euler–Arnold form. The theorem is proved.

Remark 4. In [8], the above theorem was proved directly without any appeal to the classical double. Nevertheless, the proof using the classical double is simpler and makes Theorem 3.2 understandable from the standpoint of the general R-matrix scheme.

Example 2. We now consider the case $R = P_+ - P_-$ in more detail and present the explicit form of the Lie algebra $\mathcal{G}_{\mathcal{R}_+}$ realized as a quotient algebra. We have

$$\operatorname{Ker}\mathcal{R}_+ = \operatorname{Im}\mathcal{R}_- = \begin{pmatrix} X_1^- \\ X_2^+ \end{pmatrix}.$$

The corresponding quotient algebra $\mathcal{G}_{\mathcal{R}}/\operatorname{Im}\mathcal{R}_-$ can be identified with the linear space consisting of the elements $\mathcal{X} = \begin{pmatrix} X_1^+ \\ X_2^- \end{pmatrix}$. This space is isomorphic to $\mathcal{G}_{\mathcal{R}_+}$, and the isomorphism is given by the map $\mathcal{R}_+$:

$$\begin{pmatrix} X_1^+ \\ X_2^- \end{pmatrix} \to \begin{pmatrix} X_1^+ - X_2^- \\ X_1^+ - X_2^- \end{pmatrix}.$$

The corresponding dual space consists of the elements $\mathcal{L} = \begin{pmatrix} L_1^+ \\ L_2^- \end{pmatrix}$. We note that such elements can be identified with the elements $L = L_1^+ + L_2^-$ of the linear space $\mathfrak{g}^*$. Moreover, the corresponding Lie–Poisson brackets of these elements on $\mathcal{G}_{\mathcal{R}}/\operatorname{Im}\mathcal{R}_-$ coincide with the standard Lie–Poisson bracket of the element $L = L^+ + L^-$ on $\mathfrak{g}^*$.

The Casimir functions $I_{k,1}(\mathcal{L})$ and $I_{k,2}(\mathcal{L})$ restricted to the dual space of the quotient algebra coincide with the respective functions $I_k(L_1^+)$ and $I_k(L_2^-)$. After the identification described above, they pass into the functions $I_k(L^+)$ and $I_k(L^-)$ on $\mathfrak{g}^*$.

3.2. Shift of the argument and commutative algebras. We consider commutative subalgebras of functions on $\mathfrak{g}^*$ generalizing the commutative algebras constructed in the preceding subsection. These subalgebras depend on additional parameters obtained using the theory of the classical double. The method that allows introducing additional parameters into commutative subalgebras is a generalization of the so-called shift of the argument. The following theorem holds.

Theorem 3.3. *Let $c_\pm$ be constant elements of $\mathfrak{g}^*_{R_\pm}$ such that*

$$c_\pm \perp ([\mathfrak{g}_{R_+}, \mathfrak{g}_{R_+}] \cup [\mathfrak{g}_{R_-}, \mathfrak{g}_{R_-}]),$$

and let $I_k(L)$ and $I_l(L)$ be Casimir functions of $\mathfrak{g}$. Then the following statements hold:

1. *The equalities*

$$\{I_k(R^*_+(L)+c_-), I_l(R^*_+(L)+c_-)\}_c = 0,$$
$$\{I_k(R^*_-(L)+c_+), I_l(R^*_-(L)+c_+)\}_c = 0,$$
$$\{I_k(R^*_+(L)+c_-), I_l(R^*_-(L)+c_+)\}_c = 0$$

hold, where $\{\,\cdot\,,\,\cdot\,\}_c$ is the shifted bracket

$$\{F_1(L), F_2(L)\}_c = \langle L, [\nabla F_1, \nabla F_2]\rangle + \langle c_- - c_+, [\nabla F_1, \nabla F_2]\rangle. \tag{10}$$

2. *The corresponding Hamiltonian equations can be written as Euler–Arnold equations*

$$\frac{dL}{dt^\pm_k} = \mathrm{ad}^*_{\nabla I_k(R^*_\pm(L)+c_\mp)}(L + c_- - c_+). \tag{11}$$

Proof. To prove statement 1 in this theorem, we take into account that $\mathcal{G}_\mathcal{R} = \mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{\mathcal{R}_-}$. Hence, $[\mathcal{G}_\mathcal{R}, \mathcal{G}_\mathcal{R}] = [\mathcal{G}_{\mathcal{R}_+}, \mathcal{G}_{\mathcal{R}_+}] \ominus [\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$. We explicitly describe the ideal $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$. We have

$$\mathcal{G}_{\mathcal{R}_-} = \left\{ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \middle|\, X_1 \in \mathfrak{g}_{R_-},\ X_2 \in \mathfrak{g}_{R_+} \right\}.$$

It hence follows that the element $C = (c_1, c_2) \in \mathcal{G}^*$ is orthogonal to $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ if $c_1 \perp [\mathfrak{g}_{R_-}, \mathfrak{g}_{R_-}]$ and $c_2 \perp [\mathfrak{g}_{R_+}, \mathfrak{g}_{R_+}]$. On the other hand, it follows from the explicit form of the elements of $\mathcal{G}^*_{\mathcal{R}_-}$ that $c_2 = -c_1 = -c$ and that $C = (c, -c)$ is an element of the space dual to the Lie subalgebra $\mathcal{G}_{\mathcal{R}_-}$. This element is orthogonal to $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ if $c \perp ([\mathfrak{g}_{R_+}, \mathfrak{g}_{R_+}] \cup [\mathfrak{g}_{R_-}, \mathfrak{g}_{R_-}])$. Therefore, taking the quotient of the Lie algebra $\mathcal{G}_\mathcal{R}$ by the ideal $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$, taking into account that the projection on the quotient algebra is a canonical homomorphism, and applying Theorem 3.1 to the Casimir functions $I_{k,\epsilon}$ and $I_{l,\epsilon'}$, we obtain

$$\{I_k(R^*_+(L_1+L_2)+c), I_l(R^*_+(L_1+L_2)+c)\} = 0,$$
$$\{I_k(R^*_-(L_1+L_2)+c), I_l(R^*_-(L_1+L_2)+c)\} = 0,$$
$$\{I_k(R^*_+(L_1+L_2)+c), I_l(R^*_-(L_1+L_2)+c)\} = 0.$$

As a result, we obtain a commutative subalgebra with a shift element c symmetrically entering both the "positive" and the "negative" integrals, i.e., c has components belonging both to $\mathfrak{g}^*_{R_-}$ and $\mathfrak{g}^*_{R_+}$. We note that the shift of the parts of the Lax matrices belonging to $\mathfrak{g}^*_{R_\pm}$ by a constant element of the same space $\mathfrak{g}^*_{R_\pm}$ can be eliminated by a change of variables. But this leads to changing the Poisson brackets. Making such a shift and setting $c_\pm = \pm R^*_\pm(c)$ and $L = L_1 + L_2$, we obtain statement 1 in the theorem.

Statement 2 is proved analogously to statement 2 in the preceding theorem. The theorem is proved.

Example 3. As in the preceding examples, we consider the case of AKS R-operators $R = P_+ - P_-$. We describe the quotient algebras $\mathcal{G}_\mathcal{R}/[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ and the corresponding dual spaces explicitly. We have $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}] = [\mathfrak{g}_+, \mathfrak{g}_+] \ominus [\mathfrak{g}_-, \mathfrak{g}_-]$. The elements of the corresponding quotients $\mathcal{G}_\mathcal{R}/[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ are written as

$$\mathcal{X} = \begin{pmatrix} X_1^+ + X_1^{-'} \\ X_2^- + X_2^{+'} \end{pmatrix},$$

where $X_1^+ \in \mathfrak{g}_+$, $X_1^{-'} \in \mathfrak{g}_-/[\mathfrak{g}_-, \mathfrak{g}_-]$, $X_2^- \in \mathfrak{g}_-$, and $X_2^{+'} \in \mathfrak{g}_+/[\mathfrak{g}_+, \mathfrak{g}_+]$. The corresponding dual space consists of the elements

$$\begin{pmatrix} L_1^+ + L_1^{-'} \\ L_2^- + L_2^{+'} \end{pmatrix}, \qquad L_1^{-'} \in (\mathfrak{g}_-/[\mathfrak{g}_-, \mathfrak{g}_-])^*, \qquad L_2^{+'} \in (\mathfrak{g}_+/[\mathfrak{g}_+, \mathfrak{g}_+])^*.$$

The elements $L_1^{-'}$ and $L_2^{+'}$ are constant with respect to the Poisson brackets on $\mathcal{G}_\mathcal{R}/[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$, and we can set $c_- = L_1^{-'}$ and $c_+ = L_2^{+'}$.

The Casimir functions restricted to the dual space of the quotient algebra are the functions $I_k(L_1^+ + L_1^{-'})$ and $I_l(L_2^- + L_2^{+'})$. Using the same arguments as in Example 2 and the making the same identification, we can write these functions as $I_k(L^+ + c_-)$ and $I_l(L^- + c_+)$, where L is a generic element of $\mathfrak{g}^*$. By Theorem 3.3 proved above, they commute with respect to the brackets $\{\,\cdot\,,\,\cdot\,\}_c$ on $\mathfrak{g}^*$:

$$\{I_k(L^+ + c_-), I_l(L^+ + c_-)\}_c = 0,$$

$$\{I_k(L^- + c_+), I_l(L^- + c_+)\}_c = 0,$$

$$\{I_k(L^+ + c_-), I_l(L^- + c_+)\}_c = 0,$$

where the shifted bracket $\{\,\cdot\,,\,\cdot\,\}_c$ is defined using formula (10).

4. Integrable hierarchies and negative flows

4.1. Doubles, *R*-operators, and negative flows of soliton hierarchies. Using the results in the preceding section, we can construct a hierarchy of integrable equations in partial derivatives admitting zero-curvature representations. We use one of the Lie algebraic approaches to the theory of soliton equations [3]. It is based on the interpretation of zero-curvature conditions as a consistency condition for two auxiliary commuting Lax flows on the space dual to an infinite-dimensional Lie algebra.

For simplicity, we formulate the result for the particular case of the graded infinite-dimensional Lie algebra $\mathfrak{g}$ of $\mathfrak{s}$-valued Laurent polynomials of one complex parameter λ or their subalgebras. Here, $\mathfrak{s}$ is a simple Lie algebra in some matrix realization. We note that the dual space $\mathfrak{g}^*$ can be identified with the space of formal $\mathfrak{s}$-valued power series.

The following theorem holds.

Theorem 4.1. *Let $\mathfrak{g}$ be the Lie algebra defined above. Let $\mathcal{G}$ denote its double and $\mathcal{G}^*$ denote the corresponding dual space. Let J be an ideal in $\mathcal{G}_{\mathcal{R}_-}$ of finite codimension. Let $\pi\colon \mathcal{G} \to \mathcal{G}/J$ denote the natural projection on the quotient algebra and $\pi^*\colon (\mathcal{G}/J)^* \to \mathcal{G}^*$ denote the dual map. If the Hamiltonians $I_{k,\epsilon}(\pi^*(\mathcal{L}))$ on $\mathcal{G}^*$ are finite polynomials, then the $\mathfrak{g}$-valued functions $M_{k,+} = R_+\nabla I_{k,1}\big(\pi^*(\mathcal{L})\big)$ and $M_{l,-} = R_-\nabla I_{l,2}\big(\pi^*(\mathcal{L})\big)$ satisfy the zero-curvature equations with values in $\mathfrak{g}$*

$$\frac{\partial M_{k,\pm}}{\partial t_l^\pm} - \frac{\partial M_{l,\pm}}{\partial t_k^\pm} + [M_{k,\pm}, M_{l,\pm}] = 0, \tag{12}$$

$$\frac{\partial M_{k,\pm}}{\partial t_l^\mp} - \frac{\partial M_{l,\mp}}{\partial t_k^\pm} + [M_{k,\pm}, M_{l,\mp}] = 0. \tag{13}$$

Proof. We note that using the commutativity of two Hamiltonian flows generated by the Hamiltonians $I_{k,\epsilon}(\pi^*(\mathcal{L}))$ and $I_{l,\epsilon'}(\pi^*(\mathcal{L}))$ (here $\epsilon, \epsilon' \in \{+,-\}$) and two Euler–Arnold equations (8), we can easily derive the equation

$$\operatorname{ad}^*_{[\![M_{k,\epsilon}, M_{l,\epsilon'}]\!]}\, \pi^*(\mathcal{L}) = 0, \tag{14}$$

where

$$[\![M_{k,\epsilon}, M_{l,\epsilon'}]\!] \equiv \frac{\partial M_{k,\epsilon}}{\partial t_l^{\epsilon'}} - \frac{\partial M_{l,\epsilon'}}{\partial t_k^{\epsilon}} + [M_{k,\epsilon}, M_{l,\epsilon'}].$$

In the considered case, the coadjoint representation of $\mathfrak{g}$ (and hence of $\mathcal{G}$) can be identified with the adjoint representation. Therefore, it follows from Eq. (14) that the expression $[\![M_{k,\epsilon}, M_{l,\epsilon'}]\!]$ belongs to the kernel of $\operatorname{ad}_{\pi^*(\mathcal{L})}$. Because the underlying finite-dimensional Lie algebra is semisimple, it follows that this kernel is spanned by the expressions $(\pi^*(\mathcal{L}))^k$, where the associative multiplication on the double is componentwise. Because J is an ideal in $\mathcal{G}_{\mathcal{R}_-}$ of finite codimension, $\mathcal{G}_{\mathcal{R}} = \mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{\mathcal{R}_-}$, $\mathcal{G}_{\mathcal{R}_+} \simeq \mathfrak{g}$, and the dual space $\mathcal{G}^*_{\mathcal{R}_+}$ can be written in component form as

$$\mathcal{R}^*_+(\mathcal{L}) = \begin{pmatrix} R^*_+(L_1 + L_2) \\ -R^*_-(L_1 + L_2) \end{pmatrix},$$

we find that the two components of $\pi^*(\mathcal{L})$ and hence of $\bigl(\pi^*(\mathcal{L})\bigr)^k$ are semi-infinite formal Laurent power series (each in its own direction). On the other hand, $I_{k,\epsilon}\bigl(\pi^*(\mathcal{L})\bigr)$ and $I_{l,\epsilon'}\bigl(\pi^*(\mathcal{L})\bigr)$ are (by the theorem condition) finite polynomials. Hence, their matrix gradients $M_{k,\epsilon}$ and $M_{l,\epsilon'}$ and also $[\![M_{k,\epsilon}, M_{l,\epsilon'}]\!]$ belong to the algebra $\mathfrak{g}$ of $\mathfrak{s}$-valued Laurent polynomials (here $\mathfrak{g}$ is realized as a diagonal subalgebra in $\mathfrak{g} \oplus \mathfrak{g}$). Hence, $[\![M_{k,\epsilon}, M_{l,\epsilon'}]\!]$ cannot be equal to a linear combination of the expressions $(\pi^*(\mathcal{L}))^k$ except in the case of the trivial linear combination. This means that $[\![M_{k,\epsilon}, M_{l,\epsilon'}]\!] = 0$. The theorem is proved.

Remark 5. Using the same arguments, we can show that this theorem also holds for more complicated infinite-dimensional Lie algebras, for example, for the quasigraded Lie algebras responsible for the integrability of the Landau–Lifshitz equation and its various generalizations [7] or for the graded algebras of the type A_∞, C_∞, and D_∞. In the latter case, instead of the condition that the Hamiltonians are polynomials, less rigid conditions must be imposed.

Remark 6. We note that Eqs. (12) and (13) define three types of integrable hierarchies: two "small" hierarchies associated with the Lie subalgebras $\mathfrak{g}_{R_\pm}$ defined by Eqs. (12) and one "large" hierarchy associated with the whole Lie algebra $\mathfrak{g}$ including both types of Eqs. (12) and (13). Equations (13) contain a U–V pair with the U-operator taking values in $\mathfrak{g}_{R_+}$ and the V-operator taking the values in $\mathfrak{g}_{R_-}$. They can be interpreted as the "negative flows" of the integrable hierarchy associated with $\mathfrak{g}_{R_\pm}$.

4.2. Case of graded Lie algebras. In this subsection, we demonstrate how the general scheme described above for producing U–V pairs satisfying zero-curvature equations works for concrete Lie algebras. We concentrate on the simplest possible examples associated with graded Lie algebras.

4.2.1. Quotients of the double and invariant functions. We consider the example of $\mathbb{Z}$-graded algebras and the quotient algebras of the corresponding double. By the definition of graded Lie algebras, we have

$$\mathfrak{g} = \sum_{j\in\mathbb{Z}} \mathfrak{g}_j, \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

Using the grading property, we easily obtain the decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, where $\mathfrak{g}_+ = \sum_{j\ge 0} \mathfrak{g}_j$ and $\mathfrak{g}_- = \sum_{j<0} \mathfrak{g}_j$ are Lie subalgebras. Let $P_\pm$ denote the projection operators on the Lie subalgebras $\mathfrak{g}_\pm$.

Hence, $R = P_+ - P_-$ is a classical R-operator [17]. In the standard way [17], we find that $J_{+k} = \sum_{j>k} \mathfrak{g}_j$ and $J_{-l} = \sum_{j>l} \mathfrak{g}_{-l}$ are ideals in $\mathfrak{g}_\pm$ and in $\mathfrak{g}_R = \mathfrak{g}_+ \ominus \mathfrak{g}_-$. Hence, we can consider the quotient algebra $\mathfrak{g}_R/(J_{+k} \ominus J_{-l})$ and the quotient algebra $\mathcal{G}_{\mathcal{R}}/(J_{+k} \ominus J_{-l})$ of the corresponding "double." The elements of the last quotient algebra have the form

$$\begin{pmatrix} X_1^+ + X_1^{-'} \\ X_2^- + X_2^{+'} \end{pmatrix}, \quad X_1^+ \in \mathfrak{g}_+, \quad X_1^{-'} \in \sum_{j=1}^{l} \mathfrak{g}_{-j}, \quad X_2^- \in \mathfrak{g}_-, \quad X_2^{+'} \in \sum_{j=0}^{k} \mathfrak{g}_j.$$

The corresponding elements of the dual space have the explicit form

$$\mathcal{L} = \begin{pmatrix} L_1^+ + L_1^{-'} \\ L_2^- + L_2^{+'} \end{pmatrix}, \quad L_1^+ \in \mathfrak{g}_+^*, \quad L_1^{-'} \in \sum_{j=1}^{l} \mathfrak{g}_{-j}^*, \quad L_2^- \in \mathfrak{g}_-^*, \quad L_2^{+'} \in \sum_{j=0}^{k} \mathfrak{g}_j^*.$$

We note that the corresponding components $L_1 = L_1^+ + L_1^{-'}$ and $L_2 = L_2^- + L_2^{+'}$ of the Lax operator are semi-infinite (i.e., infinite in only one direction).

We assume that there exists an invariant bilinear form $(\,\cdot\,,\,\cdot\,)$ on $\mathfrak{g}$ such that $(\mathfrak{g}_i, \mathfrak{g}_j) \sim \delta_{i+j,0}$. In this case, we can identify the spaces $\mathfrak{g}^*$ and $\mathfrak{g}$ and construct the second-order Casimir functions using the formulas

$$I_{2,1}^0 = \frac{1}{2} \sum_{i \in \mathbb{Z}} (L_1^{(i)}, L_1^{(-i)}), \qquad I_{2,2}^0 = \frac{1}{2} \sum_{i \in \mathbb{Z}} (L_2^{(i)}, L_2^{(-i)}),$$

where $L_{1,2}^{(\pm i)} \subset \mathfrak{g}_{\mp i}$. We note that on the quotient algebra described above, all these expressions are finite polynomials if the space $\mathfrak{g}_i$ is finite-dimensional.

We consider several examples.

4.2.2. General non-Abelian Toda systems and graded Lie algebras. We consider the above construction in the case where $k = 1$ and $l = 1$. We then have

$$I_{2,1}^0 = \frac{1}{2}(L_1^{(0)}, L_1^{(0)}) + (L_1^{(1)}, L_1^{(-1)}), \qquad I_{2,2}^0 = \frac{1}{2}(L_2^{(0)}, L_2^{(0)}) + (L_2^{(1)}, L_2^{(-1)}),$$

and the Lax matrix is

$$\mathcal{L} = \begin{pmatrix} L_1^+ + L_1^{(-1)} \\ L_2^- + L_2^{(0)} + L_2^{(1)} \end{pmatrix}, \qquad L_1^+ = \sum_{i=0}^{\infty} L_1^{(i)}, \qquad L_2^- = \sum_{i=1}^{\infty} L_1^{(-i)}.$$

We note that $L_1^{(-1)}$ is a central element because $L_1^{(-1)} \in (\mathfrak{g}_-/[\mathfrak{g}_-, \mathfrak{g}_-])^*$.

The M-operators corresponding to the integrals $I_{2,1}^0$ and $I_{2,2}^0$ have the form

$$M_{2,1}^0 = \mathcal{R}_+ \nabla I_{2,1}^0 = \begin{pmatrix} \bar{L}_1^{(0)} + \bar{L}_1^{(1)} \\ \bar{L}_1^{(0)} + \bar{L}_1^{(1)} \end{pmatrix}, \quad M_{2,2}^0 = \mathcal{R}_+ \nabla I_{2,2}^0 = -\begin{pmatrix} \bar{L}_2^{(-1)} \\ \bar{L}_2^{(-1)} \end{pmatrix},$$

where

$$\bar{L}_1^{(0)} = \frac{1}{2} \nabla (L_1^{(0)}, L_1^{(0)}) \in \mathfrak{g}_0, \qquad \bar{L}_1^{(1)} = P_+ \nabla (L_1^{(1)}, L_1^{(-1)}) \in \mathfrak{g}_1,$$

$$\bar{L}_2^{(-1)} = P_- \nabla (L_2^{(1)}, L_2^{(-1)}) \in \mathfrak{g}_{-1}.$$

Their components, namely, the operators

$$U = \bar{L}_1^{(0)} + \bar{L}_1^{(1)}, \qquad V = \bar{L}_2^{(-1)}, \tag{15}$$

are the U–V pairs of Abelian and non-Abelian Toda field equations, as is shown below.

We consider the zero-curvature equation for the U–V pair:

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0.$$

This equation yields the equations for the homogeneous components:

$$\frac{\partial \bar{L}_1^{(1)}}{\partial t} = 0, \qquad \frac{\partial \bar{L}_1^{(0)}}{\partial t} = -[\bar{L}_1^{(1)}, \bar{L}_2^{(-1)}], \qquad \frac{\partial \bar{L}_2^{(-1)}}{\partial x} = [\bar{L}_1^{(0)}, \bar{L}_2^{(-1)}]. \tag{16}$$

The first of these equations is satisfied automatically because $\bar{L}_1^{(1)}$ is obtained from the central element $L_1^{(-1)}$ and hence $\bar{L}_1^{(1)} = C^{(1)} = \text{const}$. We solve the last equation in (16). Because of the grading, it is easy to see that $\mathfrak{g}_0 \subset \mathfrak{g}$ is a subalgebra. Let G_0 denote the corresponding Lie group. Let $g_0 \in G_0$. It can be shown by direct verification that the substitution

$$\bar{L}_2^{(-1)} = g_0 C^{(-1)} g_0^{-1}, \qquad \bar{L}_1^{(0)} = (\partial_x g_0) g_0^{-1}, \qquad g_0 = g_0(x, t),$$

where $C^{(-1)}$ is a constant element of the space $\mathfrak{g}_{-1}$, solves the last equation in (16). The second equation in (16) after this solution is substituted becomes

$$\partial_t((\partial_x g_0) g_0^{-1}) = -[C^{(1)}, g_0 C^{(-1)} g_0^{-1}] \tag{17}$$

and is the so-called non-Abelian Toda field equation [15].

4.3. Loop algebras and the standard Toda system. The main example of the above construction is connected with loop algebras. Let $\mathfrak{s}$ be the Lie algebra of a simple Lie group G. Let $\mathfrak{g} = \mathfrak{s} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$ denote the loop algebra. We assume that $\mathfrak{s}$ is equipped with an automorphism $\sigma\colon \mathfrak{s} \to \mathfrak{s}$ of the order p. There is a natural decomposition [18] $\mathfrak{s} = \sum_{i=0}^{p-1} \mathfrak{s}_i$ such that

$$\mathfrak{s}_k = \{X \in \mathfrak{s} \mid \sigma(X) = e^{2\pi i k/p} X\}.$$

In particular, $\mathfrak{s}_0$ is the subalgebra stable under the action of the automorphism σ.

We extend this grading to the loop space $\mathfrak{g}$, by definition setting

$$\deg \lambda = p, \qquad \deg X \otimes q(\lambda) = \deg X + \deg q(\lambda).$$

In this case, we obtain

$$\mathfrak{g}_j = \{X(\lambda) \in \mathfrak{s} \otimes \mathrm{Pol}(\lambda, \lambda^{-1}) | \deg X(\lambda) = j\}.$$

In particular, $\mathfrak{g}_0 = \mathfrak{s}_0$. The corresponding group $G_0 \subset G$ hence coincides with the Lie group of the Lie subalgebra $\mathfrak{s}_0$. Equation (17) is written for the generic element of this group.

We consider the most interesting example of such a situation corresponding to the case of an Abelian group G_0. Let $\mathfrak{g}$ be a loop algebra with the principal grading. In more detail, let $\sigma\colon \mathfrak{s} \to \mathfrak{s}$ be a Coxeter automorphism. Let h denote the Coxeter number of $\mathfrak{s}$. We have the corresponding $\mathbb{Z}_h$-grading of $\mathfrak{s}$

$$\mathfrak{s} = \sum_{i=0}^{h-1} \mathfrak{s}_i.$$

The subalgebra $\mathfrak{s}_0$ coincides with the Cartan subalgebra $\mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{H_{\alpha_i} \mid i = 1, \ldots, \operatorname{rank}\mathfrak{g}\}$. Moreover, $\mathfrak{s}_k = \mathrm{Span}_{\mathbb{C}}\{X_\alpha \mid \alpha \in \Delta,\ |\alpha| = k \mod h\}$. Here, H_{α_i}, X_α is a Cartan–Weil basis of $\mathfrak{s}$, Δ is the set of all roots, and $|\alpha|$ denotes the root height. In particular, $H_{\alpha_i} = [X_{\alpha_i}, X_{-\alpha_i}]$, where α_i are simple roots.

We describe the subspaces $\mathfrak{g}_i$ explicitly. By definition, we have

$$\mathfrak{g}_0 = \mathfrak{h}, \qquad \mathfrak{g}_k = \sum_{|\alpha|=k} \mathfrak{s}_\alpha + \lambda \sum_{|\alpha|=h-k} \mathfrak{s}_{-\alpha}, \quad k = 1, \ldots, h-1,$$

where $\mathfrak{s}_\alpha = \mathrm{Span}_{\mathbb{C}}\{X_\alpha\}$. The other graded subspaces are $\mathfrak{g}_{k+nh} = \lambda^n \mathfrak{g}_k$, $k = 1, \ldots, h-1$. We consider the corresponding U–V-pair given by (15):

$$U = (\partial_x g_0) g_0^{-1} + C^{(1)}, \qquad V = g_0 C^{(-1)} g_0^{-1}.$$

In this case, the group G_0 is Abelian and coincides with the Cartan subgroup. It is therefore easy to parameterize its element g_0 as $g_0 = \exp\{\sum_{i=1}^{\operatorname{rank}\mathfrak{s}} \phi_i H_{\alpha_i}\}$ and obtain

$$U = \sum_{i=1}^{\operatorname{rank}\mathfrak{s}} \partial_x \phi_i H_{\alpha_i} + \sum_{\alpha_i \in P} c_{\alpha_i}^{(1)} X_{\alpha_i} + \lambda c_{-\theta}^{(1)} X_{-\theta},$$

$$V = \sum_{\alpha_i \in P} c_{\alpha_i}^{(-1)} e^{-\alpha_i(\phi)} X_{-\alpha_i} + \lambda^{-1} c_\theta^{(-1)} e^{-\theta(\phi)} X_\theta,$$

where P is the set of simple roots, θ is the highest root, and H_{α_i} is the basic element in the Cartan subalgebra corresponding to the simple root α_i. In this U–V pair, it is easy to recognize the U–V pair of the finite-component Toda field equation [14]. The corresponding Eqs. (17) become

$$\partial_t \partial_x \phi_i = c_{\alpha_i}^{(1)} c_{-\alpha_i}^{(-1)} e^{-\alpha_i(\phi)} + a_i c_{-\theta}^{(1)} c_\theta^{(-1)} e^{\theta(\phi)}, \qquad \phi = \sum_{i=1}^{\operatorname{rank}\mathfrak{s}} \phi_i H_{\alpha_i}, \tag{18}$$

where the constants a_i are defined from the decomposition of $H_\theta = [X_\theta, X_{-\theta}]$

$$H_\theta = \sum_{i=1}^{\operatorname{rank}\mathfrak{s}} a_i H_{\alpha_i}.$$

It is easy to see from the explicit form of (18) that the coefficients $c_{-\alpha_i}^{(-1)}$, $c_{\alpha_i}^{(1)}$, $c_\theta^{(-1)}$, and $c_{-\theta}^{(1)}$ are redundant, and if they are nonzero, then they can be eliminated from the equations by a rescaling.

Remark 7. The R-operator used in this paper is non-skew-symmetric. There is an alternative approach to the finite-component two-dimensional Toda lattice based on a skew-symmetric R-operator [1], [19].

4.4. Infinite-component Toda system. Now let $\mathfrak{g} = gl((\infty))$. We recall that this is the Lie algebra of infinite matrices $M = (M_{ij})_{i,j \in \mathbb{Z}}$, where $M_{ij} = 0$ for $|i-j| \gg 1$. This situation can be regarded as the $n \to \infty$ limit of the case $\mathfrak{g} = gl(n)$ in the preceding section. But it deserves a more careful consideration.

The basis in the algebra $gl((\infty))$ consists of the elements X_{ij}, $i, j \in \mathbb{Z}$, with the standard commutation relations $[X_{ij}, X_{kl}] = \delta_{kj} X_{il} - \delta_{il} X_{kj}$. In terms of this basis, we have the graded subspaces of the natural $\mathbb{Z}$-grading $\mathfrak{g}_k = \mathrm{Span}_{\mathbb{C}}\{X_{ij} \mid j - i = k\}$. There exists a natural invariant bilinear form $(\cdot\,,\cdot)$ on $gl((\infty))$ such that $(X_{ij}, X_{kl}) = \delta_{kj}\delta_{il}$. Using this form, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ such that $\mathfrak{g}_k^* = \mathfrak{g}_{-k}$.

We consider the classical double of $gl((\infty))$. We apply the construction in Sec. 4.2.1 to the corresponding dual space and its quotient spaces by the ideals of the form J_{+k} and J_{-l}, where k and l are fixed positive integers. The elements of the spaces dual to these quotients have the form

$$\mathcal{L} = \begin{pmatrix} L_1^+ + L_1^{-'} \\ L_2^- + L_2^{+'} \end{pmatrix} \in [\mathcal{G}_\mathcal{R}/(J_{+k} \ominus J_{-l})]^*,$$

where

$$L_1^+ = \sum_{i=0}^{\infty} L_1^{(i)}, \qquad L_2^- = \sum_{i=1}^{\infty} L_2^{(-i)},$$

$$L_1^{-'} = \sum_{j=1}^{l} L_1^{(-i)}, \qquad L_2^{+'} = \sum_{j=0}^{k} L_1^{(i)}, \qquad L_s^{(i)} \in gl((\infty))_{-i}, \quad s = 1, 2.$$

In particular, the Lax matrix $\mathcal{L}$ of the infinite-component Toda system corresponds to the case $k = l = 1$. We therefore consider only this case in what follows.

We let the same symbols X_{ij} denote the natural basis in the space dual to $gl((\infty))$. The Lax operator can be described in this basis by the coordinates $l_1^{(m)}(i)$, $l_2^{(m)}(j)$, as

$$L_s^{(m)} = \sum_{i\in\mathbb{Z}} l_s^{(m)}(i-m) X_{i,i-m}, \quad s = 1, 2.$$

Just as before, the coordinates $l_1^{(-1)}(i)$ are Casimir functions of the Lie–Poisson bracket on the space dual to the quotient $\mathcal{G}_\mathcal{R}/(J_{+1} \ominus J_{-1})$. We can therefore set them equal to constants, $l_1^{(-1)}(i) = c_i$. Hence, $L_1^{(-1)} = \sum_{i\in\mathbb{Z}} c_i X_{i-1,i}$. Using the invariant form $(\,\cdot\,,\,\cdot\,)$ on $gl((\infty))$, we obtain two quadratic Hamiltonians $I_{2,1}^0$ and $I_{2,2}^0$ on the double of $gl((\infty))$ with the explicit form

$$I_{2,s}^0 = \frac{1}{2}\sum_{i\in\mathbb{Z}} (l_s^{(0)}(i))^2 + \sum_{i\in\mathbb{Z}} l_s^{(1)}(i) l_s^{(-1)}(i), \quad s = 1, 2,$$

on the quotient space under consideration. The flows generated by these Hamiltonians are written in the Lax form

$$\frac{\partial \mathcal{L}}{\partial t_s} = [\widetilde{M}_{2,s}^0, \mathcal{L}], \quad s = 1, 2,$$

where the M-operators with values in the double of $gl((\infty))$ are

$$\widetilde{M}_{2,1}^0 = \mathcal{R}_+ \widetilde{\nabla} I_{2,1}^0 = \begin{pmatrix} L_1^{(0)} + L_1^{(-1)} \\ L_1^{(0)} + L_1^{(-1)} \end{pmatrix}, \qquad \widetilde{M}_{2,2}^0 = \mathcal{R}_+ \widetilde{\nabla} I_{2,2}^0 = \begin{pmatrix} L_2^{(1)} \\ L_2^{(1)} \end{pmatrix}.$$

Although $\mathfrak{g}$ is not a loop algebra and the functions $I_{2,s}^0$ are not finite polynomials in this case, similarly to the proof of Theorem 4.1, we can prove that the corresponding M-operators satisfy the zero-curvature condition. Therefore, we can write a $gl((\infty))$-valued U–V-pair satisfying the zero-curvature equation:

$$U = L_1^{(0)} + L_1^{(-1)}, \qquad V = L_2^{(1)}, \qquad L_s^{(i)} \in gl((\infty))_{-i}, \quad s = 1, 2.$$

It yields the equations

$$\partial_x v_i = v_i(u_{i+1} - u_i), \qquad \partial_t u_i = c_{i-1} v_{i-1} - c_i v_i, \quad i \in \mathbb{Z}, \tag{19}$$

where $u_i \equiv l_1^{(0)}(i)$, $v_i \equiv l_2^{(1)}(i)$, $x = t_1$, and $t = t_2$. By the substitution $u_i = \partial_x \phi_i$, $v_i = e^{\phi_{i+1}-\phi_i}$, Eqs. (19) reduce to the usual infinite-component Toda system [11]:

$$\partial_{xt}^2 \phi_i = c_{i-1} e^{\phi_i - \phi_{i-1}} - c_i e^{\phi_{i+1}-\phi_i}, \quad i \in \mathbb{Z}. \tag{20}$$

Just as before, if the constants c_i are nonzero, then they can be eliminated by a rescaling.

Remark 8. Equation (20) does not have the same form as Eqs. (18), which corresponds to $gl(n)$, because we use a different basis in the Cartan subalgebra in this subsection: $H_i \equiv X_{ii}$ instead of $H_{\alpha_i} = X_{ii} - X_{i-1i-1}$.

4.5. Lie–Poisson bracket for the infinite-component Toda system. In this subsection, we explicitly describe the R-operator Lie–Poisson bracket corresponding to the natural AKS decomposition used in the preceding subsection. We begin with the Lie brackets and then use the fact that the Lie–Poisson brackets of the coordinate functions repeat the Lie brackets of the basis elements of the algebra.

For convenience, we introduce the basis

$$X^{(i)}(m) \equiv X_{m,i+m}, \quad i, m \in \mathbb{Z},$$

in the algebra $gl((\infty))$. The commutation relations in this basis become

$$[X^{(i)}(m), X^{(j)}(n)] = \delta_{i+n-m,0} X^{(i+j)}(m) - \delta_{m-n+j,0} X^{(i+j)}(n),$$

and the R-operator is then written as $R = P_+ - P_-$, where P_+ and P_- are the projection operators on the Lie subalgebras respectively generated by $X^{(i)}(m)$, $i \geq 0$, and $X^{(j)}(n)$, $j < 0$. The R-bracket on $gl((\infty))$ can be written as

$$[X^{(i)}(m), X^{(j)}(n)]_R = 2\big(1 - \sigma(i) - \sigma(j)\big)\big(\delta_{i+n-m,0} X^{(i+j)}(m) - \delta_{m-n+j,0} X^{(i+j)}(n)\big),$$

where $\sigma(i) = 1$ if $i < 0$ and $\sigma(i) = 0$ if $i \geq 0$.

For the double of $gl((\infty))$, i.e., for the direct sum $gl((\infty)) \oplus gl((\infty))$, we obtain the R-bracket written for the basic elements $X_s^{(i)}(m)$, $s = 1, 2$:

$$\begin{aligned}
[X_1^{(i)}(m), X_1^{(j)}(n)]_R &= 2\big(1 - \sigma(i) - \sigma(j)\big)\big(\delta_{i+n-m,0} X_1^{(i+j)}(m) - \delta_{m-n+j,0} X_1^{(i+j)}(n)\big), \\
[X_2^{(i)}(m), X_2^{(j)}(n)]_R &= 2\big(\sigma(i) + \sigma(j) - 1\big)\big(\delta_{i+n-m,0} X_2^{(i+j)}(m) - \delta_{m-n+j,0} X_2^{(i+j)}(n)\big), \\
[X_1^{(i)}(m), X_2^{(j)}(n)]_R &= 2\big(\sigma(i) - 1\big)\big(\delta_{i+n-m,0} X_1^{(i+j)}(m) - \delta_{m-n+j,0} X_1^{(i+j)}(n)\big) + \\
&\quad + 2\sigma(j)\big(\delta_{i+n-m,0} X_2^{(i+j)}(m) - \delta_{m-n+j,0} X_2^{(i+j)}(n)\big).
\end{aligned}$$

The Lie–Poisson brackets for the coordinate functions $l_s^{(i)}(m)$ follow from these commutation relations:

$$\begin{aligned}
\{l_1^{(i)}(m), l_1^{(j)}(n)\}_R &= 2\big(1 - \sigma(i) - \sigma(j)\big)\big(\delta_{i+n-m,0} l_1^{(i+j)}(m) - \delta_{m-n+j,0} l_1^{(i+j)}(n)\big), \\
\{l_2^{(i)}(m), l_2^{(j)}(n)\}_R &= 2\big(\sigma(i) + \sigma(j) - 1\big)\big(\delta_{i+n-m,0} l_2^{(i+j)}(m) - \delta_{m-n+j,0} l_2^{(i+j)}(n)\big), \\
\{l_1^{(i)}(m), l_2^{(j)}(n)\}_R &= 2\big(\sigma(i) - 1\big)\big(\delta_{i+n-m,0} l_1^{(i+j)}(m) - \delta_{m-n+j,0} l_1^{(i+j)}(n)\big) + \\
&\quad + 2\sigma(j)\big(\delta_{i+n-m,0} l_2^{(i+j)}(m) - \delta_{m-n+j,0} l_2^{(i+j)}(n)\big).
\end{aligned}$$

The Lie–Poisson brackets of the Toda system are obtained using the specialization $l_1^{(-1)}(i) = c_i$; $l_1^{(k)}(i) = 0$, $k < -1$; and $l_2^{(j)}(i) = 0$, $j > 1$. These brackets coincide with the first Poisson structure of the two-dimensional Toda hierarchy found in [16].

5. Quadratic and cubic Poisson structures on the double

In this section, we discuss the prolongation of the second- and third-degree Poisson brackets from $\mathfrak{g}$ to its classical double $\mathcal{G}$ and the consistency of the corresponding brackets.

5.1. Quadratic Poisson structure. It is known that for some classical R-operators on $\mathfrak{g}$, in addition to the linear R-bracket, a second-degree Poisson bracket, important in the theory of classical integrable systems, can also be defined.

Hereafter, we assume that there exists an identification between $\mathfrak{g}$ and $\mathfrak{g}^*$. Moreover, we assume that the Lie algebra $\mathfrak{g}$ also has the structure of an associative algebra. The following theorem holds [20], [21].

Theorem 5.1. *Let the classical R-operator and its skew-symmetric part $(R-R^*)/2$ satisfy the modified classical Yang–Baxter equation on $\mathfrak{g}$. Then*

1. *the formula*

$$\{F_1,F_2\}_2=\langle L,[R(L\nabla F_1+\nabla F_1 L),\nabla F_2]\rangle-\langle L,[R(L\nabla F_2+\nabla F_2 L),\nabla F_1]\rangle \tag{21}$$

   defines a Poisson bracket on $\mathfrak{g}$,

2. *the Casimir functions of $\mathfrak{g}$ mutually commute with respect to bracket* (21),
3. *the Hamiltonian equations of motion with respect to the Casimir functions I_k on $\mathfrak{g}$ and* (21) *are written in the Lax form*

$$\frac{dL}{dt_k}=[R(L\nabla I_k+\nabla I_k L),L],$$

   and

4. *Poisson brackets* (21) *and* (2) *are compatible.*[2]

It turns out that this theorem can be extended to the double of $\mathfrak{g}$.

Theorem 5.2. *Let the classical R-operator and its skew-symmetric part $(R-R^*)/2$ satisfy the modified classical Yang–Baxter equation on $\mathfrak{g}$. Then*

1. *the formula*

$$\{F_1(\mathcal{L}),F_2(\mathcal{L})\}_2=\langle \mathcal{L},[\mathcal{R}(\mathcal{L}\widetilde{\nabla} F_1+\widetilde{\nabla} F_1\mathcal{L}),\nabla F_2]\rangle-\langle \mathcal{L},[\mathcal{R}(\mathcal{L}\widetilde{\nabla} F_2+\widetilde{\nabla} F_2\mathcal{L}),\nabla F_1]\rangle \tag{22}$$

   defines a Poisson bracket on $\mathcal{G}$,

2. *the Casimir functions of $\mathcal{G}$ mutually commute with respect to bracket* (22),
3. *the Hamiltonian equations of motion with respect to the Casimir functions $I_{k,\epsilon}$ on $\mathcal{G}$ and* (22) *are written in the Lax form*

$$\frac{d\mathcal{L}}{dt_k^{\epsilon}}=[R(\mathcal{L}\widetilde{\nabla} I_{k,\epsilon}+\widetilde{\nabla} I_{k,\epsilon}\mathcal{L}),\mathcal{L}],\quad \epsilon=1,2,$$

   and

4. *Poisson brackets* (22) *and* (3) *are compatible.*

[2] We recall that two Poisson brackets $\{\,\cdot\,,\,\cdot\,\}_1$ and $\{\,\cdot\,,\,\cdot\,\}_2$ on the same space are said to be compatible if an arbitrary linear combination $a_1\{\,\cdot\,,\,\cdot\,\}_1+a_2\{\,\cdot\,,\,\cdot\,\}_2$ is a Poisson bracket.

Remark 9. Lie–Poisson bracket (22) can be written more explicitly in terms of the operators R and $R_{\pm}$ as

$$\begin{aligned}\{F_1(L_1, L_2), F_2(L_1, L_2)\}_2 = {} & \langle L_1, [R(L_1\nabla_1 F_1 + \nabla_1 F_1 L_1) - R_-(L_2\nabla_2 F_1 + \nabla_2 F_1 L_2), \nabla_1 F_2]\rangle + \\ & + \langle L_2, [R_+(L_1\nabla_1 F_1 + \nabla_1 F_1 L_1) - R(L_2\nabla_2 F_1 + \nabla_2 F_1 L_2), \nabla_2 F_2]\rangle - \\ & - \langle L_1, [R(L_1\nabla_1 F_2 + \nabla_1 F_2 L_1) - R_-(L_2\nabla_2 F_2 + \nabla_2 F_2 L_2), \nabla_1 F_1]\rangle - \\ & - \langle L_2, [R_+(L_1\nabla_1 F_2 + \nabla_1 F_2 L_1) - R(L_2\nabla_2 F_2 + \nabla_2 F_2 L_2), \nabla_2 F_1]\rangle.\end{aligned}$$

Proof of the theorem. We first note that if $\mathfrak{g}$ is an associative algebra, then $\mathcal{G}$ is also an associative algebra and has the natural structure of a direct sum of associative algebras. To prove the theorem, it suffices to apply Theorem 5.1 and prove that the conditions in Theorem 5.1 on the classical R-operator on $\mathfrak{g}$ imply that the classical R-operator $\mathcal{R}$ and its skew-symmetric part $(\mathcal{R} - \mathcal{R}^*)/2$ satisfy the modified classical Yang–Baxter equation on the double $\mathcal{G}$. This statement for $\mathcal{R}$ follows automatically from the results in [12]. It remains to show that $(\mathcal{R} - \mathcal{R}^*)/2$ satisfies the modified classical Yang–Baxter equation on $\mathcal{G}$. We show this directly. We have

$$\mathcal{A} \equiv \frac{1}{2}(\mathcal{R} - \mathcal{R}^*) = \frac{1}{2}\begin{pmatrix} A & -S \\ S & -A \end{pmatrix}, \qquad A = R - R^*, \qquad S = R + R^*.$$

Substituting this expression in the modified classical Yang–Baxter equation, we find that it is equivalent to the three conditions

$$\begin{aligned} & A\big([A(X), Y] + [X, A(Y)]\big) - [A(X), A(Y)] = 4[X, Y], \\ & -S\big([S(X), Y]\big) - A\big([X, S(Y)]\big) + [A(X), S(Y)] = 0, \\ & S\big([A(X), Y] + [X, A(Y)]\big) - [S(X), S(Y)] = 0, \end{aligned} \tag{23}$$

which must be satisfied for any $X, Y \in \mathfrak{g}$. The first of these equations follows from the conditions in the theorem. Using the definition of the operators A and S, we can easily show that (23) are equivalent to the three equations

$$R\big([R(X), Y] + [X, R(Y)]\big) - [R(X), R(Y)] = [X, Y], \tag{24a}$$

$$R^*\big([R^*(X), Y]\big) - R^*\big([X, R(Y)]\big) + [R^*(X), R(Y)] = [X, Y], \tag{24b}$$

$$R\big([R^*(X), Y] + [X, R^*(Y)]\big) - [R^*(X), R^*(Y)] = -[X, Y] \tag{24c}$$

for all $X, Y \in \mathfrak{g}$. Equation (24a) is the modified classical Yang–Baxter equation for R. Equation (24b) is derived using the modified classical Yang–Baxter equation and the condition for the existence of a nondegenerate invariant form on $\mathfrak{g}$. Finally, Eq. (24c) is derived using the modified classical Yang–Baxter equation for A. The theorem is proved.

Remark 10. Theorem 5.2 means that if a quadratic Poisson structure exists on $\mathfrak{g}$, then it can always be extended to the double $\mathcal{G}$. In particular, such an extension exists for skew-symmetric R-operators on $\mathfrak{g}$ because the corresponding operator $\mathcal{R}$ on $\mathcal{G}$ is also skew-symmetric.

5.2. Cubic Poisson structure. In this subsection, we describe the cubic Poisson structure on $\mathfrak{g}$ and its prolongation to $\mathcal{G}$. We use the following theorem from [20], [21], which holds under the same assumptions about the Lie algebra $\mathfrak{g}$ as in the preceding subsection.

Theorem 5.3. *Let R be a classical R-operator. Then*

1. *the formula*

$$\{F_1(L), F_2(L)\}_3 = \langle L, [R(L\nabla F_1 L + L\nabla F_1 L), \nabla F_2]\rangle - \\ - \langle L, [R(L\nabla F_2 L + L\nabla F_2 L), \nabla F_1]\rangle \tag{25}$$

   defines a Poisson bracket on $\mathfrak{g}$,

2. *the Casimir functions of $\mathfrak{g}$ mutually commute with respect to bracket* (25),

3. *the Hamiltonian equations of motion with respect to the Casimir functions I_k of $\mathfrak{g}$* and (25) *are written in the Lax form*

$$\frac{dL}{dt_k} = [R(L\nabla I_k L), L],$$

   and

4. *if the skew-symmetric part $(R - R^*)/2$ of R satisfies the modified classical Yang–Baxter equation on $\mathfrak{g}$, then Poisson brackets* (25), (21), *and* (2) *are compatible.*

A similar statement also holds for the classical double of $\mathfrak{g}$.

Corollary 5.1. *Let R be a classical R-operator. Then*

1. *the formula*

$$\{F_1(\mathcal{L}), F_2(\mathcal{L})\}_3 = \langle \mathcal{L}, [\mathcal{R}(\mathcal{L}\widetilde{\nabla} F_1 \mathcal{L} + \mathcal{L}\widetilde{\nabla} F_1 \mathcal{L}), \widetilde{\nabla} F_2]\rangle - \\ - \langle \mathcal{L}, [\mathcal{R}(\mathcal{L}\widetilde{\nabla} F_2 \mathcal{L} + \mathcal{L}\widetilde{\nabla} F_2 \mathcal{L}), \widetilde{\nabla} F_1]\rangle \tag{26}$$

   defines a Poisson bracket on $\mathcal{G}$,

2. *the Casimir functions on $\mathcal{G}$ mutually commute with respect to bracket* (26).

3. *the Hamiltonian equations of motion with respect to the Casimir functions $I_{k,\epsilon}$ on $\mathcal{G}$ and* (22) *are written in the Lax form*

$$\frac{d\mathcal{L}}{dt_k^\epsilon} = [\mathcal{R}(\mathcal{L}\widetilde{\nabla} I_{k,\epsilon}\mathcal{L}), \mathcal{L}],$$

   and

4. *if the skew-symmetric part $(R - R^*)/2$ of R satisfies the modified classical Yang–Baxter equation on $\mathfrak{g}$, then Poisson brackets* (26), (22), *and* (4) *are compatible.*

Remark 11. Lie–Poisson bracket (26) can be written more explicitly in terms of the operators R and $R_\pm$ as

$$\begin{aligned}\{F_1(L_1, L_2), F_2(L_1, L_2)\}_3 = {} & \langle L_1, [R(L_1\nabla_1 F_1 L_1) - R_-(L_2\nabla_2 F_1 L_2), \nabla_1 F_2]\rangle + \\ & + \langle L_2, [R_+(L_1\nabla_1 F_1 L_1) - R(L_2\nabla_2 F_1 L_2), \nabla_2 F_2]\rangle - \\ & - \langle L_1, [R(L_1\nabla_1 F_2 L_1) - R_-(L_2\nabla_2 F_2 L_2), \nabla_1 F_1]\rangle - \\ & - \langle L_2, [R_+(L_1\nabla_1 F_2 L_1) - R(L_2\nabla_2 F_2 L_2), \nabla_2 F_1]\rangle.\end{aligned}$$

Proof of the corollary. We note that using the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ as linear spaces and $\mathfrak{g}$-modules, we can also identify the spaces $\mathcal{G}$ and $\mathcal{G}^*$ as linear spaces and $\mathcal{G}$-modules. As explained above, the double $\mathcal{G}$ of an associative algebra $\mathfrak{g}$ always has the structure of an associative algebra. To prove the corollary, it only remains to apply Theorem 5.3 to the classical double $\mathcal{G}$ and take into account that, as proved in Theorem 5.2, the antisymmetric part $(\mathcal{R} - \mathcal{R}^*)/2$ satisfies the modified classical Yang–Baxter equation on $\mathcal{G}$ if $(R - R^*)/2$ satisfies the modified classical Yang–Baxter equation on $\mathfrak{g}$.

Remark 12. The second- and third-degree Poisson structures exist and are compatible with the linear Poisson structure (when R and $(R - R^*)/2$ satisfy the modified classical Yang–Baxter equation). They produce commuting Hamiltonian flows written in the Lax form. Nevertheless, their use in soliton theory is not as straightforward and universal as the use of the linear Poisson bracket. Indeed, to obtain the phase space of a soliton equation, as explained above, we must restrict ourself to certain linear subspaces coinciding with the quotient algebras of the corresponding linear R-bracket. On the other hand, these linear subspaces, generally speaking, are not quotient spaces by the ideals of the quadratic and cubic brackets. To restrict these brackets to the corresponding subspaces, we must apply an additional Dirac reduction. This procedure was considered in [16] for the infinite-component Toda system.

At the end of this section, we explain why the second- and third-order Poisson structures exist and are compatible with the linear Poisson structure in the case of an infinite-component Toda system before the restriction to the quotient spaces of the linear bracket. Because the infinite-component Toda system is connected with the double of $gl((\infty))$, we must prove the following proposition.

Proposition 5.1. *On the double of $gl((\infty))$ equipped with the AKS R-operator $R = P^+ - P^-$, where P^+ and P^- are projection operators on the subalgebras of upper triangular and strictly lower triangular matrices, there exist quadratic and cubic Poisson structures compatible with the linear Poisson bracket and given by the respective formulas* (22) *and* (26).

Proof. We note that the double of $gl((\infty))$ is obviously an associative algebra. Hence, to apply Theorem 5.2 and Corollary 5.1 to the considered case, it suffices to show that $(R - R^*)/2$ satisfies the modified classical Yang–Baxter equation on $gl((\infty))$. In our case, $R = P^+ - P^-$, where P^+ and P^- are projection operators on the respective subalgebras of upper triangular and strictly lower triangular matrices We also note that this R-operator can be written as $R = P_+ + P_0 - P_-$, where P_+ is the projection operator on the Lie subalgebra of strictly upper triangular matrices, P_0 is the projection on the Lie subalgebra of diagonal matrices, and $P_- \equiv P^-$. Using the explicit form of the invariant pairing, i.e., the bilinear form on $gl((\infty))$, we can easily show that $P_\pm^* = P_\mp$, $P_0^* = P_0$, and consequently $(R - R^*)/2 = P_+ - P_-$. It follows from the results in [22] (also see [9]) that if $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_0 + \mathfrak{g}_-$ is a triangular decomposition of a Lie algebra $\mathfrak{g}$, i.e., $\mathfrak{g}_\pm$ and $\mathfrak{g}_0$ are closed Lie subalgebras and $\mathfrak{g}_0$-modules, then any operator of the form $P_+ + R_0 - P_-$, where $P_\pm$ are projection operators on $\mathfrak{g}_\pm$, is a solution of the modified classical Yang–Baxter equation on $\mathfrak{g}$ if R_0 is a solution of the modified classical Yang–Baxter equation on $\mathfrak{g}_0$. Moreover, if the subalgebra $\mathfrak{g}_0$ is Abelian, then any operator R_0 (including the trivial one) is a solution of the modified Yang–Baxter equation on $\mathfrak{g}_0$. The operator $P_+ - P_-$ in this case is therefore a solution of the modified Yang–Baxter equation on $\mathfrak{g}$.

On the other hand, it is obvious that the decomposition of the algebra $\mathfrak{g} = gl((\infty))$ into the strictly upper-triangular, strictly lower-triangular, and diagonal parts is a triangular decomposition with an Abelian (diagonal) part $\mathfrak{g}_0$. Hence, the considered R-operator $(R - R^*)/2 = P_+ - P_-$ is indeed a solution of the modified Yang–Baxter equation on $gl((\infty))$, and by Theorem 5.2 and Corollary 5.1, there exist quadratic and cubic Poisson structures on $gl((\infty))$ and its double. The proposition is proved.

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