Hodge–GUE correspondence and the discrete KdV equation

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Abstract

We prove the conjectural relationship recently proposed in [16] between certain special cubic Hodge integrals of the Gopakumar–Mariño–Vafa type [27, 36] and GUE correlators.

Contents

1 Introduction

1.1 The Hodge–GUE conjecture

Let $\overline{\mathcal{M}}_{g,k}$ denote the Deligne–Mumford moduli space of stable algebraic curves of genus g with k distinct marked points. Denote by $\mathcal{L}_i$ the i^{th} tautological line bundle over $\overline{\mathcal{M}}_{g,k}$, and by $\mathbb{E}_{g,k}$ the rank g Hodge bundle. Denote $\psi_i := c_1(\mathcal{L}_i)$, $i = 1, \ldots, k$, and $\lambda_j := c_j(\mathbb{E}_{g,k})$, $j = 0, \ldots, g$. The Hodge integrals are integrals over $\overline{\mathcal{M}}_{g,k}$ of the form

$$\int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{i_1} \cdots \psi_k^{i_k} \cdot \lambda_1^{j_1} \cdots \lambda_g^{j_g}, \quad i_1, \ldots, i_k,\, j_1, \ldots, j_g \geq 0.$$

These integrals will be defined to be *zero* unless

$$3g - 3 + k = \sum_{\ell=1}^{k} i_\ell + \sum_{\ell=1}^{g} \ell \cdot j_\ell.$$

We will mainly consider in this paper the following *special cubic Hodge integrals*

$$\int_{\overline{\mathcal{M}}_{g,k}} \Lambda_g(-1)\,\Lambda_g(-1)\,\Lambda_g\left(\frac{1}{2}\right)\,\psi_1^{i_1} \cdots \psi_k^{i_k} \tag{1.1.1}$$

where $\Lambda_g(z) := \sum_{j=0}^{g} \lambda_j\, z^j$ denotes the Chern polynomial of $\mathbb{E}_{g,k}$. Interest to this particular case of cubic Hodge integrals was triggered by the celebrated R. Gopakumar–M. Mariño–C. Vafa conjecture [27, 36] regarding the Chern–Simons/string duality. The special cubic Hodge free energy is the following generating function of Hodge integrals

$$\mathcal{H}_{\rm cubic}(\mathbf{t}; \epsilon) = \sum_{g\geq 0} \epsilon^{2g-2} \sum_{k\geq 0} \frac{1}{k!} \sum_{i_1,\ldots,i_k\geq 0} t_{i_1} \cdots t_{i_k} \int_{\overline{\mathcal{M}}_{g,k}} \Lambda_g(-1)\,\Lambda_g(-1)\,\Lambda_g\left(\frac{1}{2}\right)\,\psi_1^{i_1} \cdots \psi_k^{i_k}. \tag{1.1.2}$$

Here and below, $\mathbf{t} = (t_0, t_1, \ldots)$, and ϵ is a parameter. Denote by $\mathcal{H}_g = \mathcal{H}_g(\mathbf{t})$ the genus g term of $\mathcal{H}_{\rm cubic}(\mathbf{t}; \epsilon)$, i.e.

$$\mathcal{H}_{\rm cubic}(\mathbf{t}; \epsilon) = \sum_{g\geq 0} \epsilon^{2g-2} \mathcal{H}_g(\mathbf{t}). \tag{1.1.3}$$

The exponential

$$e^{\mathcal{H}_{\rm cubic}(\mathbf{t};\epsilon)} =: Z_{\rm cubic}(\mathbf{t}; \epsilon) \tag{1.1.4}$$

is called the special cubic Hodge partition function.

On another hand, let $\mathcal{H}(N)$ denote the space of $N \times N$ Hermitean matrices. Denote by

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re} M_{ij}\, d\mathrm{Im} M_{ij}$$

the standard unitary invariant volume element on $\mathcal{H}(N)$. The GUE partition function of size N with even couplings is defined by

$$Z_N(\mathbf{s};\epsilon) = \frac{(2\pi)^{-N}}{Vol(N)} \int_{\mathcal{H}(N)} e^{-\frac{1}{\epsilon}\operatorname{tr} V(M;\mathbf{s})} dM. \tag{1.1.5}$$

Here, $V(M;\mathbf{s})$ is an **even** polynomial or, more generally, a power series in M

$$V(M;\mathbf{s}) = \frac{1}{2}M^2 - \sum_{j\geq 1} s_{2j}\, M^{2j}, \tag{1.1.6}$$

$\mathbf{s} := (s_2, s_4, s_6, \dots)$, and by $Vol(N)$ we denote the volume of the quotient of the unitary group over the maximal torus $[U(1)]^N$

$$Vol(N) = Vol\left(U(N)/\left[U(1)\right]^N\right) = \frac{\pi^{\frac{N(N-1)}{2}}}{G(N+1)}, \quad G(N+1) := \prod_{n=1}^{N-1} n!.$$

The integral will be considered as a formal saddle point expansion with respect to the small parameter ϵ. Introduce the *'t Hooft coupling parameter* x by

$$x := N\,\epsilon.$$

Expanding the free energy $\mathcal{F}_N(\mathbf{s};\epsilon) := \log Z_N(\mathbf{s};\epsilon)$ in powers of ϵ and replacing the Barnes G-function $G(N+1)$ by its asymptotic expansion yields

$$\mathcal{F}_{\mathrm{even}}(x,\mathbf{s};\epsilon) := \mathcal{F}_N(\mathbf{s})|_{N=\frac{x}{\epsilon}} - \frac{1}{12}\log\epsilon = \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}_g(x,\mathbf{s}). \tag{1.1.7}$$

The GUE free energy $\mathcal{F}_{\mathrm{even}}(x,\mathbf{s};\epsilon)$ with even couplings can be represented [21, 22, 3, 37] in the form

$$\begin{aligned}
&\mathcal{F}_{\mathrm{even}}(x,\mathbf{s};\epsilon)\\
=&\frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x + \zeta'(-1) + \sum_{g\geq 2}\epsilon^{2g-2}\frac{B_{2g}}{4g(g-1)x^{2g-2}}\\
&+\sum_{g\geq 0}\epsilon^{2g-2}\sum_{k\geq 0}\sum_{i_1,\dots,i_k\geq 1} a_g(2i_1,\dots,2i_k)\, s_{2i_1}\dots s_{2i_k}\, x^{2-2g-(k-|i|)},
\end{aligned} \tag{1.1.8}$$

where

$$a_g(2i_1,\dots,2i_k) = \sum_{\Gamma} \frac{1}{\#\,\mathrm{Sym}\,\Gamma} \tag{1.1.9}$$

and the last summation is taken over all connected oriented ribbon graphs Γ (with labelled half edges but unlabelled vertices) of genus g with k vertices of valencies $2i_1, \dots, 2i_k$, $|i| :=$

$i_1 + \cdots + i_k$, and $\#\,\mathrm{Sym}\,\Gamma$ is the order of the symmetry group of Γ. Here and below B_k are the Bernoulli numbers. The exponential

$$e^{\mathcal{F}_{\rm even}(x,\mathbf{s};\epsilon)} =: Z_{\rm even}(x,\mathbf{s};\epsilon) \tag{1.1.10}$$

is called the GUE partition function with **even** couplings. It is convenient to change normalization of the even couplings by introducing

$$\bar{s}_k := \begin{pmatrix} 2k \\ k \end{pmatrix} s_{2k}. \tag{1.1.11}$$

The following statement was formulated in [16].

Conjecture 1.1.1 *The following formula holds true*

$$Z_{\rm even}(x,\bar{\mathbf{s}};\epsilon) = e^{\frac{A(x,\bar{\mathbf{s}})}{\epsilon^2}+\zeta'(-1)} Z_{\rm cubic}\left(\mathbf{t}\left(x+\frac{\epsilon}{2},\bar{\mathbf{s}}\right);\sqrt{2}\,\epsilon\right) Z_{\rm cubic}\left(\mathbf{t}\left(x-\frac{\epsilon}{2},\bar{\mathbf{s}}\right);\sqrt{2}\,\epsilon\right) \tag{1.1.12}$$

where

$$A(x,\bar{\mathbf{s}}) = \frac{1}{2}\sum_{k_1,k_2\geq 1} \frac{k_1\,k_2}{k_1+k_2}\,\bar{s}_{k_1}\,\bar{s}_{k_2} - \sum_{k\geq 1}\frac{k}{1+k}\,\bar{s}_k + x\sum_{k\geq 1}\bar{s}_k + \frac{1}{4} - x \tag{1.1.13}$$

and

$$t_i(x,\bar{\mathbf{s}}) := \sum_{k\geq 1} k^{i+1}\bar{s}_k - 1 + \delta_{i,1} + x\cdot\delta_{i,0}, \qquad i\geq 0. \tag{1.1.14}$$

We refer to the conjectural identity (1.1.12) as a *Hodge–GUE correspondence.*

The Conjecture has been already verified in [16] for genus $g = 0, 1, 2$. In the present paper we prove it for any genus.

Theorem 1.1.2 (Main Theorem) *Conjecture 1.1.1 holds true.*

The proof of the Main Theorem will be given in Sections $2, 3, 4$ below.

Expanding the logarithms of both sides of (1.1.12) near $\bar{\mathbf{s}} = 0$, $x = 1$ one obtains a series of identities for the special cubic Hodge intersection numbers, e.g. $\forall\, g \geq 2$,

$$\begin{aligned} &2^g \sum_{\mu\in\mathbb{Y}} \frac{(-1)^{\ell(\mu)}}{m(\mu)!} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \Lambda_g(-1)\,\Lambda_g(-1)\,\Lambda_g\left(\frac{1}{2}\right) \prod_{i=1}^{\ell(\mu)} \psi_i^{\mu_i+1} \\ &= \frac{1}{2g(2g-1)(2g-2)} \sum_{g'=0}^{g} (2g'-1) \begin{pmatrix} 2g \\ 2g' \end{pmatrix} \frac{E_{2g-2g'}\,B_{2g'}}{2^{2g-2g'}}. \end{aligned} \tag{1.1.15}$$

Here, $\mathbb{Y}$ denotes the set of partitions; for $\mu\in\mathbb{Y}$, $\ell(\mu)$ denotes the length of μ, $m_i(\mu)$ denotes the multiplicity of i in μ, $m(\mu)! := \prod_{i=1}^{\infty} m_i(\mu)!$. And E_k are the Euler numbers, defined via

$$\frac{1}{\cosh z} = \sum_{k=0}^{\infty} \frac{E_k}{k!} z^k.$$

Note that the left hand side of the above identity is actually a finite sum, due to the dimension condition. To the best of our knowledge such identities even the simplest one (1.1.15) did not appear in the literature.

1.2 Three integrable systems and their Virasoro symmetries

Here we will try to explain connections between the main playing characters of Conjecture 1.1.1 and integrable systems. These connections provided motivations for the formulation of the Main Conjecture in [16]. They might also be helpful for the reader for a better understanding of the structure of the proof.

Our point is that the partition functions Z_{even} and Z_{cubic} as functions of coupling parameters are tau functions of certain integrable hierarchies. For the GUE partition function depending on even couplings only it was already observed in 1991 by E. Witten [40]. The corresponding integrable hierarchy is made of the commuting flows for the Volterra lattice equation, also called the *discrete* Korteweg–de Vries (KdV) equation

$$\dot{w}_n = w_n \left(w_{n+1} - w_{n-1} \right), \quad n \in \mathbb{Z}.$$

We will write it in the interpolated version

$$u_t = \frac{e^{u(x+\epsilon)} - e^{u(x-\epsilon)}}{\epsilon}, \tag{1.2.1}$$

$w_n = e^{u(n\,\epsilon, \epsilon\, t)}$. The solution of interest is given by the formula

$$e^{u(x,\mathbf{s};\epsilon)} = \frac{Z_{\text{even}}(x+\epsilon, \mathbf{s}; \epsilon)\, Z_{\text{even}}(x-\epsilon, \mathbf{s}; \epsilon)}{\left[Z_{\text{even}}(x, \mathbf{s}; \epsilon)\right]^2}, \quad t = s_2. \tag{1.2.2}$$

Dependence on other even couplings is governed by the higher flows of the Volterra hierarchy.

For the special cubic Hodge partition function the corresponding integrable hierarchy[1] looks more complicated. The commuting flows of this hierarchy are represented by PDEs with infinite expansions w.r.t. an auxiliary parameter ϵ. The first equation reads

$$\begin{aligned} q_t = {} & q\, q' + \frac{\epsilon^2}{12} \left(q''' - \frac{3}{2} q'\, q'' \right) \\ & - \frac{\epsilon^4}{2880} \left(6\, q^{(5)} - 9\, q'\, q^{(4)} - 45\, q'' q''' + 2\, (q')^2\, q''' + 4\, q'\, (q'')^2 \right) \\ & + \mathcal{O}(\epsilon^6), \qquad {}' = \partial_{t_0}, \ t = t_1. \end{aligned} \tag{1.2.3}$$

The solution to (1.2.3) of interest is given by

$$q(\mathbf{t}; \epsilon) = \epsilon^2 \partial_{t_0}^2 \log Z_{\text{cubic}}(\mathbf{t}; \epsilon).$$

All higher order terms of the ϵ^2-expansion in this and in other equations of the *special cubic Hodge hierarchy* are graded homogeneous differential polynomials in q. See in [14] for the details about the construction of the Hodge hierarchy.

[1] In [42] J. Zhou considered alternative generating functions of the cubic Hodge integrals and showed that they are tau functions of the 2D Toda integrable hierarchy. It would be interesting to establish a direct connection between these results and the constructions of [14] used in the present paper.

It is also worthwhile to mention interesting results of A. Buryak [5] and M. Kazarian [29] about integrable hierarchies involved into the theory of *linear* Hodge integrals.

One more integrable hierarchy appears in this story: this is the celebrated KdV hierarchy where the first equation reads

$$v_t = v\,v' + \frac{\epsilon^2}{12} v''', \qquad ' = \partial_{t_0}. \tag{1.2.4}$$

Due to the remarkable discovery of E. Witten and M. Kontsevich, a particular tau function of the KdV hierarchy is given by the generating function of intersection numbers of psi-classes on the Deligne–Mumford moduli spaces

$$Z_{\mathrm{WK}}(\mathbf{t};\epsilon) := \exp\left(\sum_{g\geq 0} \epsilon^{2g-2} \sum_{k\geq 0} \frac{1}{k!} \sum_{i_1,\dots,i_k\geq 0} \int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{i_1}\cdots\psi_k^{i_k}\, t_{i_1}\cdots t_{i_k}\right). \tag{1.2.5}$$

The solution of (1.2.4) of interest is given in terms of this tau function by

$$v = \epsilon^2 \partial_{t_0}^2 \log Z_{\mathrm{WK}}(\mathbf{t};\epsilon).$$

To establish relationships between the partition functions Z_{even}, Z_{cubic} and Z_{WK}, we will do it in a more general setting, working with arbitrary[2] tau functions τ_{Volterra}, τ_{cubic} and τ_{KdV} of (1.2.1), (1.2.3) and (1.2.4) respectively.

Recall, for the particular case of the Hodge hierarchy (1.2.3) the construction of [14] gives a map

$$\{\text{tau functions of the KdV hierarchy}\} \to \{\text{tau functions of the special cubic Hodge hierarchy}\}.$$

Introduce a function

$$\Phi(z) = 2^{-2z} \frac{\Gamma(1-z)}{\Gamma(1+z)} \sqrt{\frac{\Gamma(1+2z)}{\Gamma(1-2z)}}. \tag{1.2.6}$$

It is a multivalued meromorphic function on the complex plane $z \in \mathbb{C}$ with branch points at nonzero half-integers. With a suitable choice of the branches one has

$$\Phi(z) \to e^{\mp\frac{\pi i}{4}}, \qquad |z|\to\infty, \ \ \pm\operatorname{Re} z > 0.$$

It satisfies the identity

$$\Phi(-z)\,\Phi(z) = 1,$$

so it defines a canonical transformation

$$f(z) \mapsto \Phi(z) f(z)$$

on the Givental symplectic space (see Appendix A below for the details about the Givental's construction). Denote $\widehat{\Phi}$ the quantization[3] of this symplectomorphism acting on the corresponding Fock space.

[2] We consider formal solutions to these PDEs admitting regular expansions in ϵ

$$v(x,\mathbf{t};\epsilon) = \sum_{k\geq 0} \epsilon^k v_k(x;\mathbf{t}).$$

[3] In the quantization procedure we will identify the function $\Phi(z)$ with its asymptotic expansion at $z=\infty$; see Appendix A for the details.

Proposition 1.2.1 *1) For an arbitrary tau function $\tau_{\rm KdV}(\mathbf{t};\epsilon)$ of the KdV hierarchy, the function $\tau_{\rm cubic}(\mathbf{t};\epsilon)$ defined by*

$$\tau_{\rm cubic}(\mathbf{t};\epsilon) := \widehat{\Phi}\,\tau_{\rm KdV}(\mathbf{t};\epsilon) \tag{1.2.7}$$

is a tau function of the special cubic Hodge hierarchy.
2) For $\tau_{\rm KdV}(\mathbf{t};\epsilon) = Z_{\rm WK}(\mathbf{t};\epsilon)$ the corresponding tau function $\tau_{\rm cubic}(\mathbf{t};\epsilon)$ coincides with $Z_{\rm cubic}(\mathbf{t};\epsilon)$.

This proposition is just a reformulation of a part of the results of [14].

Let us now describe another map

{tau functions of the special cubic Hodge hierarchy} $\to$ {tau functions of the Volterra hierarchy}.

First, observe that any solution of the special cubic Hodge hierarchy admitting regular expansion in ϵ can be obtained from another such solution by a shift

$$t_i \mapsto t_i + t_i^0, \quad i \geq 0$$

for some constants t_i^0 that can also depend on ϵ. A similar statement is valid, of course, also for solutions to the KdV and the Volterra hierarchy. We choose a base point $\tau^{\rm vac}_{\rm cubic}(\mathbf{t};\epsilon)$ in the space of tau-functions in such a way that

$$Z_{\rm cubic}(\mathbf{t};\epsilon) = \tau^{\rm vac}_{\rm cubic}(\mathbf{t};\epsilon)|_{t_1\mapsto t_1-1}$$

(which is usually called the "dilaton shift"). More generally, tau function of an arbitrary solution can be represented in the form

$$\tau_{\rm cubic}(\mathbf{t};\epsilon) = \tau^{\rm vac}_{\rm cubic}(\mathbf{t};\epsilon)|_{t_i\mapsto t_i+\bar{t}_i^0,\, i\geq 0}. \tag{1.2.8}$$

Theorem 1.2.2 *1) For any tau function* (1.2.8) *of the special cubic Hodge hierarchy the function $\tau_{\rm Volterra}(x,\bar{\mathbf{s}};\epsilon)$ defined by*

$$\tau_{\rm Volterra}(x,\bar{\mathbf{s}};\epsilon) := c\cdot e^{\frac{A^{\rm vac}(x,\bar{\mathbf{s}}-\bar{\mathbf{s}}^0)}{\epsilon^2}}\tau_{\rm cubic}\left(\mathbf{t}\left(x+\frac{\epsilon}{2},\bar{\mathbf{s}}-\bar{\mathbf{s}}^0\right);\sqrt{2}\,\epsilon\right)\tau_{\rm cubic}\left(\mathbf{t}\left(x-\frac{\epsilon}{2},\bar{\mathbf{s}}-\bar{\mathbf{s}}^0\right);\sqrt{2}\,\epsilon\right) \tag{1.2.9}$$

with

$$A^{\rm vac}(x,\bar{\mathbf{s}}) = \frac{1}{2}\sum_{k_1,\,k_2\geq 1}\frac{k_1k_2}{k_1+k_2}\bar{s}_{k_1}\bar{s}_{k_2} + x\sum_{k\geq 1}\bar{s}_k \tag{1.2.10}$$

and

$$t_i(x,\bar{\mathbf{s}}) = \sum_{k\geq 1}k^{i+1}\bar{s}_k + x\cdot\delta_{i,0} + t_i^0 \tag{1.2.11}$$

for an arbitrary constant c is a tau function of the Volterra hierarchy.

2) For $\tau_{\rm cubic}(\mathbf{t};\epsilon) = Z_{\rm cubic}(\mathbf{t};\epsilon)$, we have $t_i^0 = \delta_{i,1}$. With the choice $c = e^{\zeta'(-1)}$ and $\bar{s}_k^0 = \delta_{k,1}$ the formula (1.2.9) *gives the GUE partition function with even couplings $Z_{\rm even}(x,\bar{\mathbf{s}};\epsilon)$.*

The Main Theorem follows from the above statements. Also an integrable hierarchy conjecture of [14] regarding the relationship between the special cubic Hodge hierarchy (1.2.3) and the Volterra hierarchy readily follows from the first part of the above Theorem 1.2.2.

In a nutshell the tool for proving the above statements including the Main Theorem is in using Virasoro constraints for tau functions of the three hierarchies. For the KdV hierarchy this is well known. Recall [9] that the operators $L_m^{KdV} = L_m^{KdV}\left(\epsilon^{-1}\mathbf{t}, \epsilon\,\partial/\partial\mathbf{t}\right)$, $m \geq -1$ given by

$$L_{-1}^{KdV} := \sum_{i\geq 1} t_i \frac{\partial}{\partial t_{i-1}} + \frac{t_0^2}{2\,\epsilon^2}, \tag{1.2.12}$$

$$L_0^{KdV} := \sum_{i\geq 0} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}, \tag{1.2.13}$$

$$L_m^{KdV} := \frac{\epsilon^2}{2} \sum_{i+j=m-1} \frac{(2i+1)!!\,(2j+1)!!}{2^{m+1}} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{i\geq 0} \frac{(2i+2m+1)!!}{2^{m+1}(2i-1)!!} t_i \frac{\partial}{\partial t_{i+m}}, \quad m \geq 1 \tag{1.2.14}$$

define infinitesimal symmetries of the KdV hierarchy by their linear action on the tau-function

$$\tau_{\mathrm{KdV}}(\mathbf{t};\epsilon) \mapsto \tau_{\mathrm{KdV}}(\mathbf{t};\epsilon) + \delta \cdot L_m^{KdV}\left(\epsilon^{-1}\mathbf{t}, \epsilon\,\partial/\partial\mathbf{t}\right) \tau_{\mathrm{KdV}}(\mathbf{t};\epsilon) + \mathcal{O}(\delta^2), \quad m \geq -1.$$

The Witten–Kontsevich tau function is uniquely specified by the system of *Virasoro constraints*

$$L_m^{KdV}\left(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}\right) Z_{\mathrm{WK}}(\mathbf{t};\epsilon) = 0, \quad m \geq -1$$

where $\tilde{t}_i = t_i - \delta_{i,1}$ (the so-called *dilaton shift*). For the special cubic Hodge hierarchy one can use the operators

$$L_m^{\mathrm{cubic}}\left(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}\right) = \widehat{\Phi} \circ L_m^{KdV}\left(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}\right) \circ \widehat{\Phi}^{-1}, \quad m \geq -1$$

for formulating an analogous system of Virasoro constraints for the special cubic Hodge potential; they were already derived in [41]. Surprisingly, we could not find in the literature any analogue of the Virasoro constraints for the Volterra hierarchy. We obtain them in the following way.

First, let us modify the Virasoro operators $L_m^{\mathrm{cubic}} := L_m^{\mathrm{cubic}}\left(\epsilon^{-1}\mathbf{t}, \epsilon\,\partial/\partial\mathbf{t}\right)$ by introducing

$$\widetilde{L}_m^{\mathrm{cubic}} = \frac{1}{2\pi i} \int_{-\infty}^{0+} L(z)\, e^{m\,z}\, dz, \quad m \geq 0 \tag{1.2.15}$$

where

$$L(z) = \sum_{k\geq -1} \frac{L_k^{\mathrm{cubic}}}{z^{k+2}}$$

and we use in (1.2.15) a Hankel loop integral

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_{-\infty}^{0+} e^t\, t^{-x} dt.$$

The modified operators still satisfy the Virasoro commutation relations.

The following Key Lemma is crucial for completing the proofs.

Lemma 1.2.3 (Key Lemma) *There exist unique operators* $L_m^{\rm even} = L_m^{\rm even}\left(\epsilon^{-1}x, \epsilon^{-1}\mathbf{s}, \epsilon\,\partial/\partial\mathbf{s}\right)$, $m \geq 0$, $\mathbf{s} = (s_2, s_4, \dots)$ *satisfying*

$$e^{-\frac{A^{\rm vac}}{2\epsilon^2}} \circ L_m^{\rm even}\left(\epsilon^{-1}x, \epsilon^{-1}\mathbf{s}, \epsilon\,\partial/\partial\mathbf{s}\right) \circ e^{\frac{A^{\rm vac}}{2\epsilon^2}} = 4^m\, \widetilde{L}_m^{\rm cubic}\left(\epsilon^{-1}\mathbf{t}, \epsilon\,\partial/\partial\mathbf{t}\right)\Big|_{\substack{\mathbf{t} = \mathbf{t}^{\rm vac}(x,\mathbf{s}) \\ \epsilon \mapsto \sqrt{2}\,\epsilon}} \tag{1.2.16}$$

$$A^{\rm vac}(x,\mathbf{s}) = \frac{1}{2}\sum_{k_1\,k_2} \frac{k_1 k_2}{k_1+k_2}\begin{pmatrix} 2k_1 \\ k_1\end{pmatrix}\begin{pmatrix} 2k_2 \\ k_2\end{pmatrix} s_{2k_1} s_{2k_2} + x\sum_k \begin{pmatrix} 2k \\ k\end{pmatrix} s_{2k}$$

$$t_i^{\rm vac}(x,\mathbf{s}) = \sum_k \begin{pmatrix} 2k \\ k\end{pmatrix} k^{i+1} s_{2k} + x\cdot\delta_{i,0}.$$

The uniquely defined linear operators $L_m^{\rm even}$ *satisfy the Virasoro commutation relations*

$$[L_m^{\rm even}\,,\, L_n^{\rm even}] = (m-n)\,L_{m+n}^{\rm even}, \qquad \forall\, m,\, n \geq 0 \tag{1.2.17}$$

and have the following explicit expressions:

$$L_0^{\rm even} = \sum_{k\geq 1} k\, s_{2k}\,\frac{\partial}{\partial s_{2k}} + \frac{x^2}{4\epsilon^2} - \frac{1}{16}, \tag{1.2.18}$$

$$L_n^{\rm even} = \epsilon^2\sum_{k=1}^{n-1}\frac{\partial^2}{\partial s_{2k}\,\partial s_{2n-2k}} + x\,\frac{\partial}{\partial s_{2n}} + \sum_{k\geq 1} k\, s_{2k}\,\frac{\partial}{\partial s_{2k+2n}}, \quad n \geq 1. \tag{1.2.19}$$

We now explain how the operators $L_m^{\rm even}$ are related to symmetries of the Volterra hierarchy.

Theorem 1.2.4 *Let* $\tau_{\rm Volterra}(x,\mathbf{s};\epsilon)$ *be a tau function of the Volterra hierarchy. Introduce the modified tau function* $\tilde{\tau}_{\rm Volterra}(x,\mathbf{s};\epsilon)$ *defined by the equation*

$$\tau_{\rm Volterra}(x,\mathbf{s};\epsilon) = \tilde{\tau}_{\rm Volterra}\left(x+\frac{\epsilon}{2},\mathbf{s};\epsilon\right)\tilde{\tau}_{\rm Volterra}\left(x-\frac{\epsilon}{2},\mathbf{s};\epsilon\right).$$

Then we have

1) *The operators* $L_n^{\rm even}$ *give infinitesimal symmetries of the Volterra hierarchy by means of linear action on the modified tau function*

$$\tilde{\tau}_{\rm Volterra}(x,\mathbf{s};\epsilon) \mapsto \tilde{\tau}_{\rm Volterra}(x,\mathbf{s};\epsilon) + \delta\cdot L_n^{\rm even}\tilde{\tau}_{\rm Volterra}(x,\mathbf{s};\epsilon) + \mathcal{O}(\delta^2). \tag{1.2.20}$$

2) *Denote* $\widetilde{Z}(x,\mathbf{s};\epsilon)$ *the modified GUE partition function associated with* $\tau_{\rm Volterra} = Z_{\rm even}$. *It is annihilated, after the dilaton shift* $s_2 \mapsto s_2 - \frac{1}{2}$ *by the operators* $L_n^{\rm even}$.

Organization of the paper Sections 2, 3, 4 are all towards the proof of the Main Theorem. In Section 5 we present two applications of the Main Theorem, including introduction of a reduced GUE potential and a proof of an integrable hierarchy conjecture proposed in [14].

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2 Virasoro constraints for the cubic Hodge partition function

2.1 Cubic Hodge partition function

Denote by $\mathrm{ch}_i(\mathbb{E}_{g,k})$, $i \geq 0$ the components of the Chern character of $\mathbb{E}_{g,k}$.

Lemma 2.1.1 *The partition function Z_{cubic} has the following alternative expression*

$$Z_{\text{cubic}}(\mathbf{t};\epsilon) = \exp\left(\sum_{g=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i_1,\dots,i_k\geq 0}\int_{\overline{\mathcal{M}}_{g,k}}\psi_1^{i_1}\cdots\psi_k^{i_k}\cdot\Omega_{g,k}\,t_{i_1}\cdots t_{i_k}\right)$$

with

$$\Omega_{g,k} := \exp\left(\sum_{j=1}^{\infty}(2j-2)!\left(2^{-2j+1}-2\right)\mathrm{ch}_{2j-1}(\mathbb{E}_{g,k})\right).$$

Proof Let $x_1,\dots,x_g$ be the Chern roots of $\mathbb{E}_{g,k}$, i.e.

$$\Lambda_g(z) = \prod_{i=1}^{g}(1+z\,x_i), \qquad \mathrm{ch}_m(\mathbb{E}_{g,n}) = \frac{1}{m!}(x_1^m+\cdots+x_g^m).$$

Then we have

$$\begin{aligned}\Omega_{g,k} &= e^{\sum_{j\geq1}(2j-2)!\,((1/2)^{2j-1}-2)\,\mathrm{ch}_{2j-1}} = e^{\sum_{m\geq1}(-1)^{m-1}\,(m-1)!\,((-1)^m+(-1)^m+(1/2)^m)\,\mathrm{ch}_m}\\ &= e^{\sum_{m\geq1}\frac{(-1)^{m-1}}{m}\,((-1)^m+(-1)^m+(1/2)^m)\,(x_1^m+\cdots+x_g^m)} = \Lambda_g(-1)\cdot\Lambda_g(-1)\cdot\Lambda_g\left(\frac{1}{2}\right).\end{aligned}$$

Note that in the above derivations we have used Mumford's relations [38]

$$\mathrm{ch}_{2j}(\mathbb{E}_{g,k}) = 0, \qquad \forall\, j\geq 0.$$

The lemma is proved. □

Lemma 2.1.2 ([23]) *The special cubic Hodge partition function has the following expression*

$$Z_{\text{cubic}}(\mathbf{t};\epsilon) = e^{\sum_{j=1}^{\infty}\frac{B_{2j}}{j\,(2j-1)}\left(-1+2^{-2j}\right)D_j}\,Z_{\mathrm{WK}}(\mathbf{t};\epsilon). \tag{2.1.1}$$

Here, D_j are operators defined by

$$D_j := \frac{\partial}{\partial t_{2j}} - \sum_{i\geq0} t_i\frac{\partial}{\partial t_{i+2j-1}} + \frac{\epsilon^2}{2}\sum_{a=0}^{2j-2}(-1)^a\frac{\partial^2}{\partial t_a\partial t_{2j-2-a}}, \qquad j\geq 1, \tag{2.1.2}$$

and Z_{WK} denotes the Witten–Kontsevich partition function [40, 30]

$$Z_{\mathrm{WK}}(\mathbf{t};\epsilon) := \exp\left(\sum_{g\geq0}\epsilon^{2g-2}\sum_{k\geq0}\frac{1}{k!}\sum_{i_1,\dots,i_k\geq0}\int_{\overline{\mathcal{M}}_{g,k}}\psi_1^{i_1}\cdots\psi_k^{i_k}\,t_{i_1}\cdots t_{i_k}\right).$$

Proof This is just a particular case of the C. Faber–R. Pandharipande [23] algorithm for computing Hodge integrals (see Proposition 2 therein). □

2.2 Virasoro constraints. The first version.

It is known that [9] $Z_{\rm WK}$ satisfies the following system of linear partial differential equations

$$L_m^{KdV}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) Z_{\rm WK}(\mathbf{t};\epsilon) = 0, \quad m \geq -1 \tag{2.2.1}$$

where the operators L_m^{KdV}, $m \geq -1$ have been defined in (1.2.12)–(1.2.14), and $\tilde{t}_i = t_i - \delta_{i,1}$. The operators L_m^{KdV}, $m \geq -1$ satisfy the Virasoro commutation relations

$$\left[L_m^{KdV}\,,\, L_n^{KdV}\right] = (m-n)\, L_{m+n}^{KdV}, \quad \forall\, m,n \geq -1.$$

Equations (2.2.1) are called the *Virasoro constraints* for $Z_{\rm WK}$. Note that solution to (2.2.1) is unique up to a constant factor [9, 33]. It follows from Lemma 2.1.2 and eq. (2.2.1) that

Lemma 2.2.1 ([41]) *The cubic Hodge partition function satisfies*

$$L_m^{\rm cubic}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) Z_{\rm cubic} = 0, \quad \forall\, m \geq -1 \tag{2.2.2}$$

where $\tilde{t}_i = t_i - \delta_{i,1}$, and $L_m^{\rm cubic}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t})$ are linear operators defined by

$$\begin{aligned} L_m^{\rm cubic}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) &= e^G \circ L_m^{KdV}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) \circ e^{-G} \\ G := \sum_{j=1}^\infty \frac{B_{2j}}{j\,(2j-1)} &\left(-1+2^{-2j}\right)\, D_j, \quad m \geq -1 \end{aligned}$$

where the operators D_j are defined in eq. (2.1.2)*.*

Denote $L_m^{\rm cubic} = L_m^{\rm cubic}(\epsilon^{-1}\mathbf{t}, \epsilon\,\partial/\partial\mathbf{t})$. Clearly, the operators $L_m^{\rm cubic}$ satisfy the Virasoro commutation relation

$$[L_m^{\rm cubic}, L_n^{\rm cubic}] = (m-n)\, L_{m+n}^{\rm cubic}, \qquad \forall\, m,n \geq -1.$$

We call equations (2.2.2) the first version of Virasoro constraints for $Z_{\rm cubic}$; we also refer to this version of Virasoro constraints as *Zhou's version.*

Example 2.2.2 *By a straightforward calculation we obtain that [41, 14]*

$$L_{-1}^{\rm cubic} = \sum_{k\geq 1} t_k \frac{\partial}{\partial t_{k-1}} + \frac{t_0^2}{2\epsilon^2} - \frac{1}{16}. \tag{2.2.3}$$

2.3 A Lie algebra lemma

Lemma 2.3.1 *For any basis $\{L_m \,|\, m \geq -1\}$ of an infinite dimensional Lie algebra satisfying*

$$[L_m, L_n] = (m-n)\, L_{m+n}, \qquad \forall\, m, n \geq -1$$

where $[\,,\,]$ denotes the Lie bracket of the Lie algebra, define

$$\widetilde{L}_m := \sum_{k\geq -1} \frac{m^{k+1}}{(k+1)!}\, L_k, \qquad m \geq 0.$$

Then

$$[\widetilde{L}_m\,,\, \widetilde{L}_n] = (m-n)\, \widetilde{L}_{m+n}, \qquad \forall\, m, n \geq 0.$$

Proof

$$\begin{aligned}
[\widetilde{L}_m\,,\, \widetilde{L}_n] &= \left[\sum_{k_1=-1}^{\infty} \frac{m^{k_1+1}}{(k_1+1)!}\, L_{k_1}\,,\, \sum_{k_2=-1}^{\infty} \frac{n^{k_2+1}}{(k_2+1)!}\, L_{k_2} \right] \\
&= \sum_{k_1,k_2=-1}^{\infty} \frac{m^{k_1+1}\, n^{k_2+1}}{(k_1+1)!(k_2+1)!} (k_1-k_2)\, L_{k_1+k_2} \\
&= \sum_{k=-1}^{\infty} \left(\sum_{\substack{k_1+(k_2+1)=k+1 \\ k_1\geq 0, k_2\geq -1}} \frac{m^{k_1+1}\, n^{k_2+1}}{k_1!\,(k_2+1)!} - \sum_{\substack{(k_1+1)+k_2=k+1 \\ k_1\geq -1, k_2\geq 0}} \frac{m^{k_1+1}\, n^{k_2+1}}{(k_1+1)!\, k_2!} \right) L_k \\
&= \sum_{k=-1}^{\infty} \frac{1}{(k+1)!} \left(m\,(m+n)^{k+1} - n\,(m+n)^{k+1} \right) L_k = (m-n)\, \widetilde{L}_{m+n}.
\end{aligned}$$

The lemma is proved. $\Box$

2.4 Virasoro constraints. The second version.

Definition 2.4.1 *Define a set of linear operators $\widetilde{L}_m^{\rm cubic}$ by*

$$\widetilde{L}_m^{\rm cubic} := \sum_{k=-1}^{\infty} \frac{m^{k+1}}{(k+1)!}\, L_k^{\rm cubic}, \quad m \geq 0.$$

Clearly, this definition agrees with (1.2.15). Using Lemma 2.2.1 and Lemma 2.3.1 we obtain

Theorem 2.4.2 *The cubic Hodge partition function satisfies*

$$\widetilde{L}_m^{\rm cubic}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\, \partial/\partial \mathbf{t}) Z_{\rm cubic} = 0, \quad \forall\, m \geq 0 \tag{2.4.1}$$

where $\tilde{t}_i = t_i - \delta_{i,1}$. Moreover,

$$\left[\widetilde{L}_m^{\rm cubic}\,,\, \widetilde{L}_n^{\rm cubic} \right] = (m-n)\, \widetilde{L}_{m+n}^{\rm cubic}, \quad \forall\, m, n \geq 0.$$

We call (2.4.1) the second version of Virasoro constraints for $Z_{\rm cubic}$.

Theorem 2.4.3 *The explicit expressions for $\widetilde{L}_0^{\rm cubic}$, $\widetilde{L}_1^{\rm cubic}$ and $\widetilde{L}_2^{\rm cubic}$ are*

$$\widetilde{L}_0^{\rm cubic} = \sum_{i\geq 1} t_i \frac{\partial}{\partial t_{i-1}} + \frac{t_0^2}{2\epsilon^2} - \frac{1}{16}, \tag{2.4.2}$$

$$\widetilde{L}_1^{\rm cubic} = \frac{1}{2}\sum_{i\geq 0}\sum_{j=0}^{i}\binom{i}{j}(2\,t_{j+1}+t_j)\frac{\partial}{\partial t_i} + \frac{t_0^2}{2\epsilon^2}, \tag{2.4.3}$$

$$\begin{aligned}\widetilde{L}_2^{\rm cubic} = {}& \frac{\epsilon^2}{8}\sum_{i,j\geq 0}\frac{\partial^2}{\partial t_i\partial t_j} + \sum_{i\geq 0}\sum_{j=0}^{i}\binom{i}{j}2^{i-j}(t_{j+1}+t_j)\frac{\partial}{\partial t_i} \\ & -\frac{1}{8}\sum_{i\geq 1}\sum_{j=0}^{i-1}\sum_{r=0}^{i-1-j}(-1)^r\binom{i}{i-1-j-r}2^{i-j-r}t_j\frac{\partial}{\partial t_i} + \frac{t_0^2}{2\epsilon^2} + \frac{1}{16}.\end{aligned} \tag{2.4.4}$$

Proof. Formula (2.4.2) readily follows from the definition 2.4.1, namely, we have $\widetilde{L}_0^{\rm cubic} = L_{-1}^{\rm cubic}$. For $m>0$, the direct calculation of $\widetilde{L}_m^{\rm cubic}$ becomes more complicated, however, we can use the Givental quantization to simplify the computations. The rest of Section 2 is to prove (2.4.3) and (2.4.4).

2.5 Proof of (2.4.3) in Thm. 2.4.3

By Lemma A.0.2 we have

$$e^G = \widehat{\Phi}.$$

So the operators $L_k^{\rm cubic}$ (see in Lemma 2.2.1) have the expressions

$$L_k^{\rm cubic}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) = \widehat{\Phi}\,L_k^{KdV}\left(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}\right)\,\widehat{\Phi}^{-1}, \quad k\geq -1.$$

Now using Lemma A.0.1 we obtain that

$$L_k^{\rm cubic}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) = \left[\widehat{\Phi}\,\left(\widehat{l_k} + \frac{\delta_{k,0}}{16}\right)\,\widehat{\Phi}^{-1}\right]\Big|_{q_i\mapsto\tilde{t}_i,\,\partial_{q_i}\mapsto\partial_{t_i},\,i\geq 0}, \quad k\geq -1 \tag{2.5.1}$$

where $l_k = (-1)^{k+1}z^{3/2}\partial_z^{k+1}z^{-1/2}$, $k\geq -1$. Simplifying (2.5.1) gives

$$L_k^{\rm cubic}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) = \left[\Phi(z)\,l_k\,\Phi(z)^{-1}\right]^\wedge\Big|_{q_i\mapsto\tilde{t}_i,\,\partial_{q_i}\mapsto\partial_{t_i},\,i\geq 0} + \frac{\delta_{k,0}}{16} - \frac{\delta_{k,-1}}{16}, \quad k\geq -1.$$

Here we used the fact that the only possible non-zero cocycle term of the above quantization formula appears when $k=-1$. We arrive at

Lemma 2.5.1 *The operators $\widetilde{L}_m^{\rm cubic}$, $m\geq 0$ have the following expressions*

$$\widetilde{L}_m^{\rm cubic}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) = \left(\sum_{k=-1}^{\infty}\frac{m^{k+1}}{(k+1)!}\,\Phi(z)\,l_k\,\Phi(z)^{-1}\right)^\wedge\Bigg|_{q_i\mapsto\tilde{t}_i,\,\partial_{q_i}\mapsto\partial_{t_i},\,i\geq 0} + \frac{m-1}{16}. \tag{2.5.2}$$

For $m=0$, eq. (2.5.2) gives

$$\widetilde{L}_0^{\mathrm{cubic}}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) = \hat{z}\,|_{q_i\mapsto\tilde{t}_i,\,\partial_{q_i}\mapsto\partial_{t_i},\,i\geq 0} - \frac{1}{16} = \sum_{i\geq 1}\tilde{t}_i\frac{\partial}{\partial t_{i-1}} + \frac{\tilde{t}_0^2}{2\epsilon^2} - \frac{1}{16}$$

which agrees with the previously derived eq. (2.4.2).

Proof of (2.4.3). Recall that

$$l_k = (-1)^{k+1}\,z^{3/2}\,\partial_z^{k+1}\,z^{-1/2}, \qquad k\geq -1.$$

Then we have

$$\sum_{k=-1}^{\infty}\Phi\,\frac{l_k}{(k+1)!}\,\Phi^{-1} = \Phi\,z^{3/2}\,e^{-\partial_z}\,z^{-1/2}\,\Phi^{-1} = z^{3/2}\,(z-1)^{-1/2}\,\frac{\Phi(z)}{\Phi(z-1)}\,e^{-\partial_z}.$$

Noting that

$$\frac{\Phi(z)}{\Phi(z-1)} = \frac{z-\frac{1}{2}}{\sqrt{z(z-1)}} \tag{2.5.3}$$

we arrive at

$$\sum_{k=-1}^{\infty}\Phi(z)\,\frac{l_k}{(k+1)!}\,\Phi(z)^{-1} = z\,\frac{z-1/2}{z-1}e^{-\partial_z}.$$

It remains to compute the following residue

$$-\frac{1}{2}\,\mathrm{Res}_{z=\infty}\,f(-z)\,\frac{z-1/2}{z-1}\,f(z-1)\,\frac{dz}{z}.$$

Write $f(z) = q(z)+p(z)$, where $q(z) = \sum_{i\geq 0} q_i\,z^{-i}$ and $p(z) = \sum_{i\geq 0} p_i\,(-z)^{i+1}$. Then the above residue decomposes into the following four parts

$$\begin{aligned}
\mathrm{I} &= -\frac{1}{2}\,\mathrm{Res}_{z=\infty}\,p(-z)\,\frac{z-1/2}{z-1}\,p(z-1)\,\frac{dz}{z},\\
\mathrm{II} &= -\frac{1}{2}\,\mathrm{Res}_{z=\infty}\,p(-z)\,\frac{z-1/2}{z-1}\,q(z-1)\,\frac{dz}{z},\\
\mathrm{III} &= -\frac{1}{2}\,\mathrm{Res}_{z=\infty}\,q(-z)\,\frac{z-1/2}{z-1}\,p(z-1)\,\frac{dz}{z},\\
\mathrm{IV} &= -\frac{1}{2}\,\mathrm{Res}_{z=\infty}\,q(-z)\,\frac{z-1/2}{z-1}\,q(z-1)\,\frac{dz}{z}.
\end{aligned}$$

Part I obviously vanishes. Part IV gives $\frac{q_0^2}{2\epsilon^2}$. Part II coincides with Part III. So we are left to compute Part III. Using

$$\frac{z-1/2}{z-1} = 1+\frac{1}{2}\sum_{k\geq 1} z^{-k}, \qquad z\to\infty$$

we have

$$\begin{aligned}
\mathrm{III} &= \frac{1}{2}\sum_{i\geq 0} p_i\left(\sum_{j=0}^{i+1} q_j\binom{i+1}{j} + \frac{1}{2}\sum_{j=0}^{i} q_j\sum_{k=1}^{i+1-j}(-1)^k\binom{i+1}{j+k}\right)\\
&= \frac{1}{4}\sum_{i\geq 0} p_i\sum_{j=0}^{i}\binom{i}{j}\,(2\,q_{j+1}+q_j)\,.
\end{aligned}$$

Note that in the last equality we have used the following elementary identity

$$\sum_{k=1}^{i+1-j} (-1)^k \binom{i+1}{j+k} = -\binom{i}{j}. \tag{2.5.4}$$

As a result, we obtain (2.4.3). □

2.6 Proof of (2.4.4) in Thm. 2.4.3

Proof of (2.4.4) We have

$$\sum_{k=-1}^{\infty} 2^{k+1}\, \Phi(z)\, \frac{l_k}{(k+1)!}\, \Phi(z)^{-1} = z^{3/2}\, (z-2)^{-1/2}\, \frac{\Phi(z)}{\Phi(z-2)}\, e^{-2\partial_z}.$$

Noting that

$$\frac{\Phi(z)}{\Phi(z-2)} = \frac{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)}{(z-1)\sqrt{z(z-2)}}$$

we obtain

$$\sum_{k=-1}^{\infty} 2^{k+1}\, \Phi(z)\, \frac{l_k}{(k+1)!}\, \Phi(z)^{-1} = \frac{z\,(z-1/2)\,(z-3/2)}{(z-1)\,(z-2)}\, e^{-2\partial_z}.$$

Computing the following residue

$$-\frac{1}{2}\,\mathrm{Res}_{z=\infty}\, f(-z)\, \frac{z\,(z-1/2)\,(z-3/2)}{(z-1)\,(z-2)}\, f(z-2)\, \frac{dz}{z^2}$$

like it was done above in the proof of (2.4.3) we obtain (2.4.4). □

Theorem 2.4.3 is proved. □

Remark 2.6.1 *Following the above procedure it is easy to derive a generating function for the operators* $\widetilde{L}_m^{\mathrm{cubic}}\left(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\, \partial/\partial\mathbf{t}\right)$*. Namely, the operator*

$$\sum_{m\geq 0} \frac{\lambda^m}{m!} \widetilde{L}_m^{\mathrm{cubic}}\left(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\, \partial/\partial\mathbf{t}\right)$$

depending on the auxiliary parameter λ *is obtained by quantization of the following quadratic Hamiltonian*

$$H(\lambda) = -\frac{1}{2}\,\mathrm{Res}_{z=\infty} \frac{dz}{z} f(-z)\, {}_1F_1\left(\frac{1}{2}-z; 1-z; \lambda\, e^{-\partial_z}\right) f(z).$$

Here

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

is the confluent hypergeometric function.

3 Virasoro constraints for the GUE partition function with even couplings

3.1 The GUE partition function. Toda hierarchy and Virasoro constraints.

Recall that the GUE partition function of size N is defined by

$$Z_N(\mathbf{s}) = \frac{(2\pi)^{-N}}{Vol(N)} \int_{\mathcal{H}(N)} e^{-N \operatorname{tr} V(M;\mathbf{s})} dM \tag{3.1.1}$$

where $V(M;\mathbf{s}) = \frac{1}{2}M^2 - \sum_{j\geq 1} s_j\, M^j$, and $\mathbf{s} = (s_1, s_2, s_3, \dots)$[4]. Introduce

$$x := N\,\epsilon.$$

Expanding the free energy $\mathcal{F}_N(\mathbf{s}) := \log Z_N(\mathbf{s})$ in powers of ϵ yields the GUE free energy

$$\mathcal{F}_{\mathrm{GUE}}(x,\mathbf{s};\epsilon) := \mathcal{F}_N(\mathbf{s})|_{N=\frac{x}{\epsilon}} - \frac{1}{12}\log\epsilon = \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}_g(x,\mathbf{s}). \tag{3.1.2}$$

The GUE free energy $\mathcal{F}_{\mathrm{GUE}}(x,\mathbf{s};\epsilon)$ has the form [21, 22, 3, 37]

$$\mathcal{F}_{\mathrm{GUE}}(x,\mathbf{s};\epsilon) = \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x + \zeta'(-1) + \sum_{g\geq 2}\epsilon^{2g-2}\frac{B_{2g}}{2g(2g-2)x^{2g-2}}$$
$$+ \sum_{g\geq 0}\epsilon^{2g-2}\sum_{k\geq 0}\sum_{i_1,\dots,i_k\geq 1} a_g(i_1,\dots,i_k)\, s_{i_1}\dots s_{i_k}\, x^{2-2g-\left(k-\frac{|i|}{2}\right)}, \tag{3.1.3}$$

$$a_g(i_1,\dots,i_k) = \sum_{\Gamma}\frac{1}{\#\,\mathrm{Sym}\,\Gamma} \tag{3.1.4}$$

where the last summation is taken over all connected oriented ribbon graphs of genus g with k vertices of valencies $i_1, \dots, i_k$. The exponential

$$e^{\mathcal{F}_{\mathrm{GUE}}(x,\mathbf{s};\epsilon)} =: Z_{\mathrm{GUE}}(x,\mathbf{s};\epsilon) \tag{3.1.5}$$

is called the GUE partition function. From (3.1.3), we see that the GUE free energy $\mathcal{F}_{\mathrm{GUE}}(x,\mathbf{s};\epsilon)$ lives in the following Bosonic Fock space

$$\mathcal{B} = \frac{1}{\epsilon^2}\mathbb{C}[\epsilon][[x-1, s_1, s_2, \dots]].$$

Denote $\Lambda = e^{\epsilon\partial_x}$. Define two functions u, v by

$$u = u(x,\mathbf{s};\epsilon) := (\Lambda-1)(1-\Lambda^{-1})\,\mathcal{F}_{\mathrm{GUE}}, \quad v = v(x,\mathbf{s};\epsilon) := \epsilon\frac{\partial}{\partial s_1}(\Lambda-1)\,\mathcal{F}_{\mathrm{GUE}} \tag{3.1.6}$$

and define

$$L = \Lambda + v + e^u\,\Lambda^{-1}.$$

---

[4]In Section 1, we used $\mathbf{s}$ to denote $(s_2, s_4, \dots)$; in this subsection we restore the odd coupling constants $s_1, s_3, \dots$. However, there is no ambiguity as we use the symbol $\mathbf{s}$ just for emphasizing that it is a vector.

Lemma 3.1.1 *The functions v, u satisfy the following equations of the Toda Lattice hierarchy*

$$\epsilon \frac{\partial L}{\partial s_j} = [A_j, L], \quad A_j := \left(L^j\right)_+, \quad \forall\, j \geq 1. \tag{3.1.7}$$

Moreover, Z_{GUE} is the tau-function (cf. Def. 1.2.4 in [15]) of the solution (u, v) to the Toda hierarchy.

Proof of this lemma uses the orthogonal polynomial technique [37], see e.g. [25] (see also [15], esp. Cor. A.2.2, Def. 1.2.4 therein in particular regarding the normalization of the tau-function).

Lemma 3.1.2 *The GUE partition function Z_{GUE} satisfies the following linear PDEs*

$$L_m^{\mathrm{Toda}} Z_{\mathrm{GUE}} = 0, \quad \forall\, m \geq -1. \tag{3.1.8}$$

Here, L_m^{Toda} are linear operators explicitly given by

$$L_m^{\mathrm{Toda}} := \epsilon^2 \sum_{k=1}^{m-1} \frac{\partial^2}{\partial s_k\, \partial s_{m-k}} + 2\, x\, \frac{\partial}{\partial s_m} + \sum_{k\geq 1} k\, s_k\, \frac{\partial}{\partial s_{k+m}} - \frac{\partial}{\partial s_{m+2}}, \quad m \geq 1, \tag{3.1.9}$$

$$L_0^{\mathrm{Toda}} := \sum_{k\geq 1} k\, s_k\, \frac{\partial}{\partial s_k} + \frac{x^2}{\epsilon^2} - \frac{\partial}{\partial s_2}, \tag{3.1.10}$$

$$L_{-1}^{\mathrm{Toda}} := \sum_{k\geq 2} k\, s_k\, \frac{\partial}{\partial s_{k-1}} - \frac{\partial}{\partial s_1} + \frac{x\, s_1}{\epsilon^2}. \tag{3.1.11}$$

Moreover, L_m^{Toda} satisfy the Virasoro commutation relations

$$\left[L_m^{\mathrm{Toda}}, L_n^{\mathrm{Toda}}\right] = (m-n)\, L_{m+n}^{\mathrm{Toda}}, \quad \forall\, m, n \geq -1.$$

This lemma is well-known; see e.g. [37, 35, 25]. Eqs. (3.1.8)–(3.1.11) are called the Virasoro constraints for the GUE partition function. For convenience of the reader we outline the proof of this Lemma.

Proof of Lemma 3.1.2 Recall that the GUE partition function Z_{GUE} can be obtained from the vacuum tau-function [18] of the $\mathbb{P}^1$ Frobenius manifold by shifting times; see in [12] for the details. So the above Virasoro constraints (3.1.8)–(3.1.11) are obtained from e.g. eq. (5.4) of [19] by taking $t^{2,k-1} = k!\,\left(s_k - \frac{1}{2}\delta_{k,2}\right),\ k \geq 1$ and $t^{1,0} = x$, $t^{1,1} = 1$, $t^{1,k} = 0,\ k \geq 2$. □

Lemma 3.1.3 *The GUE free energy $\mathcal{F}_{\mathrm{GUE}}$ satisfies the following property: if $k_1 + \cdots + k_m$ is an odd number, then*

$$\left.\frac{\partial^m \mathcal{F}_{\mathrm{GUE}}}{\partial s_{k_1} \ldots \partial s_{k_m}}\right|_{s_1=s_3=s_5=\cdots=0} \equiv 0. \tag{3.1.12}$$

Proof By using the formulae (3.1.3), (3.1.4) and by noticing that the total valency of any ribbon graph is an even number. □

Lemma 3.1.4 *The following formulae hold true for the GUE free energy $\mathcal{F}_{\rm GUE}$:*

$$\epsilon^2 \frac{\partial^2 \mathcal{F}_{\rm GUE}}{\partial s_1 \partial s_1} = e^u, \tag{3.1.13}$$

$$\epsilon^2 \frac{\partial^2 \mathcal{F}_{\rm GUE}}{\partial s_1 \partial s_3} = e^u \left(v(x)^2 + v(x-\epsilon)^2 + v(x)v(x-\epsilon)\right) + e^u \left(1 + \Lambda + \Lambda^{-1}\right)(e^u), \tag{3.1.14}$$

$$\epsilon^2 \frac{\partial^2 \mathcal{F}_{\rm GUE}}{\partial s_2 \partial s_2} = e^u \left(v(x-\epsilon) + v(x)\right)^2 + e^u \left(\Lambda + \Lambda^{-1}\right)(e^u), \tag{3.1.15}$$

$$\epsilon(\Lambda - 1)\left(\frac{\partial \mathcal{F}_{\rm GUE}}{\partial s_2}\right) = v^2 + (\Lambda + 1)(e^u). \tag{3.1.16}$$

Proof One can use the recursion relations [19] or the matrix resolvent method [15] to obtain these identities. □

3.2 Reduction to even couplings. Discrete KdV hierarchy and Virasoro constraints.

The GUE free energy/partition function with even couplings can be obtained from the GUE free energy/partition function by putting $s_1 = s_3 = s_5 = \cdots = 0$, namely,

$$\mathcal{F}_{\rm even}(x, \mathbf{s}; \epsilon) = \mathcal{F}_{\rm GUE}(x, s_1 = 0, s_2, s_3 = 0, s_4, \dots; \epsilon),$$
$$Z_{\rm even}(x, \mathbf{s}; \epsilon) = Z_{\rm GUE}(x, s_1 = 0, s_2, s_3 = 0, s_4, \dots; \epsilon)$$

where $\mathbf{s} = (s_2, s_4, s_6, \dots)$. It follows from Lemma 3.1.3 and eq. (3.1.6) that

$$v \equiv 0,$$

so we have

$$L = \Lambda + e^u \Lambda^{-1}, \tag{3.2.1}$$

$$u = u(x, \mathbf{s}; \epsilon) = (\Lambda - 1)(1 - \Lambda^{-1}) \mathcal{F}(x, \mathbf{s}; \epsilon). \tag{3.2.2}$$

Note that

$$\mathcal{F}_{\rm even}(x, \mathbf{s}; \epsilon) \in \mathcal{B}^{\rm even} = \frac{1}{\epsilon^2}\mathbb{C}[\epsilon][[x-1, s_2, s_4, \dots]], \quad u(x, \mathbf{s}; \epsilon) \in \epsilon^2 \mathcal{B}^{\rm even}. \tag{3.2.3}$$

It follows from Lemma 3.1.1 the following

Lemma 3.2.1 *The function u satisfies the discrete KdV hierarchy (aka the Volterra hierarchy):*

$$\epsilon \frac{\partial L}{\partial s_{2k}} = \left[(L^{2k})_+, L\right], \quad k \geq 1 \tag{3.2.4}$$

as well as the initial condition

$$e^{u(x, \mathbf{0})} = x. \tag{3.2.5}$$

It should be noted that solution to (3.2.4) and (3.2.5) exists and is unique in $\epsilon^2\mathcal{B}^{\rm even}$. Moreover, one can easily get an analogue of the definition of tau function of the Volterra hierarchy from [15] such that $Z_{\rm even}$ is a particular tau function. The tau function τ of any solution to the Volterra hierarchy is uniquely determined up to a linear function of $\mathbf{s}$ and x. This linear function can further be fixed by the so-called string equation (see below) up to a linear function in x. We omit the details because these are just specializations of the results of [15] to the even couplings.

Example 3.2.2 *The $k=1$ flow of the discrete KdV hierarchy* (3.2.4) *reads as follows*

$$\frac{\partial u}{\partial s_2} = \frac{1}{\epsilon}(\Lambda - \Lambda^{-1})(e^u).$$

Theorem 3.2.3 *Let $Z_{\rm even}(x,\mathbf{s};\epsilon)$ denote the GUE partition function with **even** couplings (see* (1.1.10)*). Define $\widetilde{Z}(x,\mathbf{s};\epsilon)$ by*

$$\log Z_{\rm even}(x,\mathbf{s};\epsilon) = \left(\Lambda^{1/2} + \Lambda^{-1/2}\right)\log \widetilde{Z}(x,\mathbf{s};\epsilon).$$

Then $\widetilde{Z}(x,\mathbf{s};\epsilon)$ satisfies the followings system of equations (which we call the Virasoro constraints):

$$L_n^{\rm even}(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\,\partial/\partial\mathbf{s})\widetilde{Z}(x,\mathbf{s};\epsilon) = 0, \quad n\geq 0 \tag{3.2.6}$$

where $\tilde{s}_{2k} = s_{2k} - \frac{1}{2}\delta_{k,1}$, and $L_n^{\rm even}$ are linear differential operators defined in (1.2.18), (1.2.19)*. Moreover, $L_m^{\rm even}$ satisfy (half of) the Virasoro commutation relations* (1.2.17)*.*

Proof Equations (1.2.17) can be verified straightforwardly. It then suffices to prove (3.2.6) for $n=0,1,2$, because the rest of (3.2.6) can be proved via (1.2.17). Denote $\widetilde{\mathcal{F}} = \log\widetilde{Z}$.

Start with $n=0$. Taking $m=0$ in (3.1.8) of Lemma 3.1.2 we have

$$\sum_{k\geq 1} k\,s_k\,\frac{\partial Z_{\rm GUE}}{\partial s_k} + \frac{x^2}{\epsilon^2}Z_{\rm GUE} - \frac{\partial Z_{\rm GUE}}{\partial s_2} = 0.$$

Taking $s_1 = s_3 = s_5 = \cdots = 0$ in this identity and dividing it by Z we obtain

$$\sum_{k\geq 1} 2k\,s_{2k}\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_{2k}} + \frac{x^2}{\epsilon^2} - \frac{\partial \mathcal{F}_{\rm even}}{\partial s_2} = 0. \tag{3.2.7}$$

Applying $\left(\Lambda^{1/2}+\Lambda^{-1/2}\right)^{-1}$ on both sides of (3.2.7) and dividing by 2 we find

$$\sum_{k\geq 1} k\,s_{2k}\,\frac{\partial \widetilde{\mathcal{F}}}{\partial s_{2k}} + \frac{x^2}{4\epsilon^2} - \frac{1}{16} - \frac{1}{2}\frac{\partial \widetilde{\mathcal{F}}}{\partial s_2} = 0.$$

This proves (3.2.6) with $n=0$.

For $n=1$, taking $m=2$ in (3.1.8) we have

$$2\,x\,\frac{\partial Z_{\rm GUE}}{\partial s_2} + \sum_{k\geq 1} k\,s_k\,\frac{\partial Z_{\rm GUE}}{\partial s_{k+2}} + \epsilon^2\,\frac{\partial^2 Z_{\rm GUE}}{\partial s_1^2} - \frac{\partial Z_{\rm GUE}}{\partial s_4} = 0.$$

Taking $s_1 = s_3 = s_5 = \cdots = 0$ in this identity and dividing it by Z we obtain

$$2\,x\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_2} + \sum_{k\geq 1} 2k\, s_{2k}\, \frac{\partial \mathcal{F}_{\rm even}}{\partial s_{2k+2}} + \epsilon^2\,\left(\frac{\partial^2 \mathcal{F}_{\rm GUE}}{\partial s_1^2} + \left(\frac{\partial \mathcal{F}_{\rm GUE}}{\partial s_1}\right)^2\right)\Bigg|_{s_1=s_3=\cdots=0} - \frac{\partial \mathcal{F}_{\rm even}}{\partial s_4} = 0.$$

Using Lemma 3.1.3 and the identity (3.1.13) we have

$$2\,x\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_2} + \sum_{k\geq 1} 2k\, s_{2k}\, \frac{\partial \mathcal{F}_{\rm even}}{\partial s_{2k+2}} + e^u - \frac{\partial \mathcal{F}_{\rm even}}{\partial s_4} = 0. \tag{3.2.8}$$

Taking $s_1 = s_3 = s_5 = \cdots = 0$ in (3.1.16) we have

$$\epsilon\,\frac{\Lambda - 1}{\Lambda + 1}\left(\frac{\partial \mathcal{F}_{\rm even}}{\partial s_2}\right) = e^u. \tag{3.2.9}$$

Applying $\left(\Lambda^{1/2} + \Lambda^{-1/2}\right)^{-1}$ on both sides of (3.2.8), using (3.2.9), and dividing by 2 we find

$$x\,\frac{\partial \widetilde{\mathcal{F}}}{\partial s_2} + \sum_{k\geq 1} k\, s_{2k}\, \frac{\partial \widetilde{\mathcal{F}}}{\partial s_{2k+2}} - \frac{1}{2}\frac{\partial \widetilde{\mathcal{F}}}{\partial s_4} = 0.$$

This proves (3.2.6) with $n = 1$.

Finally, for $n = 2$, taking $m = 4$ in (3.1.8) we have

$$2\,\epsilon^2 \frac{\partial^2 Z_{\rm GUE}}{\partial s_1\,\partial s_3} + \epsilon^2\,\frac{\partial^2 Z_{\rm GUE}}{\partial s_2 \partial s_2} + 2\,x\,\frac{\partial Z_{\rm GUE}}{\partial s_4} + \sum_{k\geq 1} k\, s_k\, \frac{\partial Z_{\rm GUE}}{\partial s_{k+4}} - \frac{\partial Z_{\rm GUE}}{\partial s_6} = 0.$$

Taking $s_1 = s_3 = \cdots = 0$ in this identity, using Lem. 3.1.3, and dividing it by $Z_{\rm even}$ we obtain

$$\begin{aligned} &2\,\epsilon^2\,\left.\frac{\partial^2 \mathcal{F}_{\rm GUE}}{\partial s_1\,\partial s_3}\right|_{s_1=s_3=\cdots=0} + \epsilon^2\,\left(\frac{\partial^2 \mathcal{F}_{\rm even}}{\partial s_2 \partial s_2} + \left(\frac{\partial \mathcal{F}_{\rm even}}{\partial s_2}\right)^2\right) \\ &+ 2\,x\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_4} + \sum_{k\geq 1} 2k\, s_{2k}\, \frac{\partial \mathcal{F}_{\rm even}}{\partial s_{2k+4}} - \frac{\partial \mathcal{F}_{\rm even}}{\partial s_6} = 0. \end{aligned}$$

Using Lemma 3.1.4 and noticing that $v \equiv 0$ we have

$$\begin{aligned} &2\,e^u\,(1 + \Lambda + \Lambda^{-1})(e^u) + \epsilon^2\,\left(\frac{\partial^2 \mathcal{F}_{\rm even}}{\partial s_2 \partial s_2} + \left(\frac{\partial \mathcal{F}_{\rm even}}{\partial s_2}\right)^2\right) \\ &+ 2\,x\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_4} + \sum_{k\geq 1} 2k\, s_{2k}\, \frac{\partial \mathcal{F}_{\rm even}}{\partial s_{2k+4}} - \frac{\partial \mathcal{F}_{\rm even}}{\partial s_6} = 0 \end{aligned} \tag{3.2.10}$$

as well as

$$\epsilon^2 \frac{\partial^2 \mathcal{F}_{\rm even}}{\partial s_2 \partial s_2} = e^u\,\left(\Lambda + \Lambda^{-1}\right)(e^u), \tag{3.2.11}$$

$$\epsilon\,\frac{\Lambda - 1}{\Lambda + 1}\left(\frac{\partial \mathcal{F}_{\rm even}}{\partial s_4}\right) = e^u\,(1 + \Lambda + \Lambda^{-1})(e^u). \tag{3.2.12}$$

Hence by using eqs. (3.2.9), (3.2.11), (3.2.12) we can rewrite (3.2.10) as

$$\epsilon^2 \left(\frac{\Lambda-1}{\Lambda+1}\left(\frac{\partial \mathcal{F}_{\rm even}}{\partial s_2}\right)\right)^2 + \epsilon\,\frac{\Lambda-1}{\Lambda+1}\left(\frac{\partial \mathcal{F}_{\rm even}}{\partial s_4}\right)$$
$$+\epsilon^2\left(2\frac{\partial^2 \mathcal{F}_{\rm even}}{\partial s_2 \partial s_2} + \left(\frac{\partial \mathcal{F}_{\rm even}}{\partial s_2}\right)^2\right) + 2\,x\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_4} + \sum_{k\geq 1} 2k\, s_{2k}\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_{2k+4}} - \frac{\partial \mathcal{F}_{\rm even}}{\partial s_6} = 0.$$

Using the definition $\mathcal{F}_{\rm even} = (\Lambda^{1/2} + \Lambda^{-1/2})\,\widetilde{\mathcal{F}}$ we obtain

$$2\,\epsilon^2\left(\frac{\partial^2 \mathcal{F}_{\rm even}}{\partial s_2\,\partial s_2} + \left(\Lambda^{1/2}\left(\frac{\partial \widetilde{\mathcal{F}}}{\partial s_2}\right)\right)^2 + \left(\Lambda^{-1/2}\left(\frac{\partial \widetilde{\mathcal{F}}}{\partial s_2}\right)\right)^2\right)$$
$$+\epsilon\,(\Lambda^{1/2} - \Lambda^{-1/2})\left(\frac{\partial \widetilde{\mathcal{F}}}{\partial s_4}\right) + 2\,x\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_4} + \sum_{k\geq 1} 2k\, s_{2k}\,\frac{\partial \mathcal{F}_{\rm even}}{\partial s_{2k+4}} - \frac{\partial \mathcal{F}_{\rm even}}{\partial s_6} = 0.$$

Now applying $\left(\Lambda^{1/2} + \Lambda^{-1/2}\right)^{-1}$ on both sides of the above identity and dividing by 2 we obtain

$$\epsilon^2\left(\frac{\partial^2 \widetilde{\mathcal{F}}}{\partial s_2\,\partial s_2} + \left(\frac{\partial \widetilde{\mathcal{F}}}{\partial s_2}\right)^2\right) + x\,\frac{\partial \widetilde{\mathcal{F}}}{\partial s_4} + \sum_{k\geq 1} k\, s_{2k}\,\frac{\partial \widetilde{\mathcal{F}}}{\partial s_{2k+4}} - \frac{1}{2}\frac{\partial \widetilde{\mathcal{F}}}{\partial s_6} = 0$$

This proves (3.2.6) with $n = 2$. The theorem is proved. □

Remark 3.2.4 *Finding Virasoro constraints for $Z_{\rm even}(x, \mathbf{s})$ in a compact form was an open question [35, 25]. Of course, $Z_{\rm even}$ itself does satisfy certain Virasoro type constraints, but these constraints may contain non-linear terms. E.g. for $L_1^{\rm even}$, the non-linear constraint reads*

$$\left(2\,x\,\frac{\partial}{\partial s_2} + \sum_{k\geq 1} 2k\, s_{2k}\,\frac{\partial}{\partial s_{2k+2}} + e^u - \frac{\partial}{\partial s_4}\right) Z_{\rm even} = 0, \quad u = (\Lambda - 1)\,(1 - \Lambda^{-1})\,\log Z_{\rm even}.$$

The **key** *of our study is the introduction of $\widetilde{Z}$ which linearizes the nonlinear constraints, such that one can write down all the constraints in a closed form. The definition of $\widetilde{Z}$ is so simple, but surprisingly it completely solves the open question.*

Proof of Theorem 1.2.4 Part 2) of the theorem has been proved in Theorem 3.2.3. Part 1) is proved by the fact that any modified tau function $\tilde{\tau}_{\rm Volterra}(x, \mathbf{s}; \epsilon)$ can be obtained from $\widetilde{Z}(x, \mathbf{s}; \epsilon)$ by shifting the $\mathbf{s}$-variables. □

Remark 3.2.5 *Note that the modified tau function $\tilde{\tau}_{\rm Volterra}$ and the solution u to the discrete KdV hierarchy is related by*

$$u = \left(\Lambda^{1/2} - \Lambda^{-1/2}\right)\left(\Lambda - \Lambda^{-1}\right)\log \tilde{\tau}_{\rm Volterra}.$$

The Virasoro symmetries

$$\frac{\partial \tilde{\tau}_{\rm Volterra}}{\partial r_n} := L_n^{\rm even}\,\tilde{\tau}_{\rm Volterra}, \quad n \geq 0$$

also act on u. For example,

$$\frac{\partial u}{\partial r_0} = \sum_{k\geq 1} k\, s_{2k} \frac{\partial u}{\partial s_{2k}} + 1,$$
$$\frac{\partial u}{\partial r_1} = \frac{1}{2}\left(3\Lambda + 3\Lambda^{-1} + 2\right) e^{u(x)} + x\frac{\partial u}{\partial s_2} + \sum_{k\geq 1} k s_{2k} \frac{\partial u}{\partial s_{2k+2}}.$$

4 Proof of the Main Theorem

Proof of Thm. 1.1.2. Let $\mathcal{H}_{\rm cubic}(\mathbf{t};\epsilon)$ denote the cubic Hodge free energy (1.1.2). Define

$$\mathcal{F}^* = \mathcal{F}^*(x,\mathbf{s};\epsilon) = \left(\Lambda^{\frac{1}{2}} + \Lambda^{-\frac{1}{2}}\right) \mathcal{H}_{\rm cubic}\left(\mathbf{t}(x,\mathbf{s}); \sqrt{2}\,\epsilon\right) + \epsilon^{-2} A + \zeta'(-1) \tag{4.0.1}$$

where $\mathbf{s} = (s_2, s_4, s_6, \dots)$, A is the series in $(x,\mathbf{s})$ defined in (1.1.13), $\bar{s}_k = \binom{2k}{k} s_{2k}$ and

$$t_i(x,\mathbf{s}) = \sum_{k\geq 1} k^{i+1} \bar{s}_k - 1 + \delta_{i,1} + x\cdot\delta_{i,0}.$$

Denote $\tilde{t}_i = t_i - \delta_{i,1}$ and $\tilde{\bar{s}}_k = \bar{s}_k - \delta_{k,1}$. Then we have

$$\tilde{t}_i(x,\mathbf{s}) = \sum_{k\geq 1} k^{i+1} \tilde{\bar{s}}_k + x\cdot\delta_{i,0}. \tag{4.0.2}$$

We are to show $\mathcal{F}^* = \mathcal{F}_{\rm even}$. This is precisely the statement of the Conjecture.

Recall that in Thm. 3.2.3 we have proved that the generating series $\widetilde{\mathcal{F}}$ defined by

$$\widetilde{\mathcal{F}}(x,\mathbf{s};\epsilon) = \left(\Lambda^{1/2} + \Lambda^{-1/2}\right)^{-1} \mathcal{F}_{\rm even}(x,\mathbf{s};\epsilon) \tag{4.0.3}$$

satisfies the following system of equations

$$L_n^{\rm even}(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial\mathbf{s})\, e^{\widetilde{\mathcal{F}}(x,\mathbf{s};\epsilon)} = 0, \quad n\geq 0 \tag{4.0.4}$$

where the linear operators $L_n^{\rm even}$ are given by (1.2.18), (1.2.19).

Now we define a series $\widetilde{\mathcal{F}}^*$ by

$$\widetilde{\mathcal{F}}^*(x,\mathbf{s};\epsilon) := \left(\Lambda^{1/2} + \Lambda^{-1/2}\right)^{-1} \mathcal{F}^*(x,\mathbf{s};\epsilon) \overset{(4.0.1)}{=} \mathcal{H}_{\rm cubic}\left(\mathbf{t}(x,\mathbf{s}); \sqrt{2}\epsilon\right) + \epsilon^{-2}\frac{A}{2} + \frac{\zeta'(-1)}{2}. \tag{4.0.5}$$

In order to show that $\mathcal{F}^* = \mathcal{F}_{\rm even}$, it suffices to prove that $\widetilde{\mathcal{F}}^* = \widetilde{\mathcal{F}}$, as the operator

$$\Lambda^{1/2} + \Lambda^{-1/2} = 2\left(1 + \frac{\epsilon^2}{8}\partial_x^2 + \frac{\epsilon^4}{384}\partial_x^4 + \dots\right)$$

is obviously invertible.

4.1 Proof of the Key Lemma

The Key Lemma 1.2.3 builds up a bridge connecting Hodge and GUE, which is crucial in the proof of the Main Theorem.

Proof of the Key Lemma. By (1.2.16) the operators $L_m^{\rm even}$, $m \geq 0$ if exist must be unique. Note that the Key Lemma also gives the explicit form (1.2.18)–(1.2.19) for $L_m^{\rm even}$, so we only need to verify that (1.2.18)–(1.2.19) do satisfy (1.2.16). The operators $L_m^{\rm even}$ given by the formulae (1.2.18)–(1.2.19) satisfy the commutation relations (1.2.17) as it was already proven in Theorem 3.2.3, so it suffices to verify (1.2.16) for $m = 0, 1, 2$.

Let us do it one by one for $m = 0, 1, 2$ in the following three lemmas.

Lemma 4.1.1 *The identity* (1.2.16) *is true for* $m = 0$.

Proof Noting that

$$\frac{\partial A}{\partial \bar{s}_k} = \sum_{k_1 \geq 1} \frac{k\, k_1}{k + k_1} \tilde{\tilde{s}}_{k_1} + x \tag{4.1.1}$$

where A is the series in $(x, \mathbf{s})$ defined in (1.1.13), we have

$$e^{-\frac{A}{2\,\epsilon^2}} \circ L_0^{\rm even}(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial\mathbf{s}) \circ e^{\frac{A}{2\,\epsilon^2}} = \sum_{k\geq 1} k\, \tilde{\bar{s}}_k \frac{\partial}{\partial \bar{s}_k} + \frac{x^2}{4\epsilon^2} - \frac{1}{16} + \frac{1}{2\,\epsilon^2} \sum_{k \geq 1} k\, \tilde{\bar{s}}_k \left(\sum_{k_1 \geq 1} \frac{k\, k_1}{k + k_1} \tilde{\bar{s}}_{k_1} + x \right).$$

Using (4.0.2) and noticing

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t_0}, \tag{4.1.2}$$

$$\frac{\partial}{\partial \bar{s}_k} = \sum_{i \geq 0} k^{i+1} \frac{\partial}{\partial t_i}, \tag{4.1.3}$$

we obtain that

$$\sum_{k \geq 1} k\, \tilde{\bar{s}}_k \frac{\partial}{\partial \bar{s}_k} = \sum_{i \geq 1} \tilde{t}_i \frac{\partial}{\partial t_{i-1}}.$$

Hence

$$e^{-\frac{A}{2\,\epsilon^2}} \circ L_0^{\rm even}(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial\mathbf{s}) \circ e^{\frac{A}{2\,\epsilon^2}} = \sum_{i \geq 1} \tilde{t}_i \frac{\partial}{\partial t_{i-1}} + \frac{t_0^2}{4\epsilon^2} - \frac{1}{16}.$$

The lemma is proved. $\square$

Lemma 4.1.2 *The identity* (1.2.16) *is true for* $m = 1$.

Proof We have

$$
\begin{aligned}
& e^{-\frac{A}{2\epsilon^2}} \circ L_1^{\rm even}(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial\mathbf{s}) \circ e^{\frac{A}{2\epsilon^2}} \\
=& 2x\,\frac{\partial}{\partial\bar s_1} + \sum_{k\geq 1}\frac{2k(2k+1)}{k+1}\tilde{\bar s}_k\,\frac{\partial}{\partial\bar s_{k+1}} + \frac{1}{\epsilon^2}x\,\frac{\partial A}{\partial\bar s_1} + \frac{1}{\epsilon^2}\sum_{k\geq 1}\frac{k(2k+1)}{k+1}\tilde{\bar s}_k\,\frac{\partial A}{\partial\bar s_{k+1}} \\
=& 2x\,\frac{\partial}{\partial\bar s_1} + \sum_{k\geq 1}\frac{2k(2k+1)}{k+1}\tilde{\bar s}_k\,\frac{\partial}{\partial\bar s_{k+1}} + \frac{x}{\epsilon^2}\left(\sum_{k\geq 1}\frac{k}{1+k}\tilde{\bar s}_k + x\right) \\
& + \frac{1}{\epsilon^2}\sum_{k\geq 1}\frac{k(2k+1)}{k+1}\tilde{\bar s}_k\left(\sum_{k_1\geq 1}\frac{(k+1)\,k_1}{k+1+k_1}\tilde{\bar s}_{k_1} + x\right) \\
=& 2\sum_{i\geq 0}\sum_{j=0}^{i}\binom{i}{j}\left(2\tilde t_{j+1} + \tilde t_j\right)\frac{\partial}{\partial t_i} + \frac{t_0^2}{\epsilon^2}
\end{aligned}
\tag{4.1.4}
$$

where we used again (4.1.1), (4.0.2) and (4.1.2), (4.1.3). The lemma is proved. □

Lemma 4.1.3 *The identity* (1.2.16) *is true for* $m = 2$.

Proof By a straightforward calculation we have

$$
\begin{aligned}
& e^{-\frac{A}{2\epsilon^2}} \circ L_2^{\rm even}(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial\mathbf{s}) \circ e^{\frac{A}{2\epsilon^2}} \\
=& 4\,\epsilon^2\,\frac{\partial^2}{\partial\bar s_1\partial\bar s_1} + 6\,x\,\frac{\partial}{\partial\bar s_2} + 4\sum_{k\geq 1}\frac{k(2k+3)(2k+1)}{(k+2)(k+1)}\tilde{\bar s}_k\,\frac{\partial}{\partial\bar s_{k+2}} \\
& + 2\,\frac{\partial^2 A}{\partial\bar s_1^2} + 4\frac{\partial A}{\partial\bar s_1}\frac{\partial}{\partial\bar s_1} + \frac{1}{\epsilon^2}\left(\frac{\partial A}{\partial\bar s_1}\right)^2 + \frac{1}{2\epsilon^2}\left(6\,x\,\frac{\partial A}{\partial\bar s_2} + 4\sum_{k\geq 1}\frac{k(2k+3)(2k+1)}{(k+2)(k+1)}\tilde{\bar s}_k\,\frac{\partial A}{\partial\bar s_{k+2}}\right).
\end{aligned}
$$

It follows from (4.1.2), (4.1.3) and (4.0.2) that

$$
\begin{aligned}
& 4\sum_{k\geq 1}\frac{k(2k+3)(2k+1)}{(k+2)(k+1)}\tilde{\bar s}_k\,\frac{\partial}{\partial\bar s_{k+2}} \\
= \;& 4\sum_{k\geq 1}\frac{k\left(4(k+1)^2-1\right)}{(k+1)}\tilde{\bar s}_k\sum_{i\geq 0}(k+2)^i\frac{\partial}{\partial t_i} \\
= \;& 16\sum_{k\geq 1}\sum_{i\geq 0}k(k+1)\tilde{\bar s}_k\sum_{j=0}^{i}\binom{i}{j}2^{i-j}k^j\frac{\partial}{\partial t_i} - 4\sum_{k\geq 1}\frac{k}{k+1}\tilde{\bar s}_k\sum_{i\geq 0}\sum_{j=0}^{i}\binom{i}{j}(k+1)^j\frac{\partial}{\partial t_i} \\
= \;& 16\sum_{i\geq 0}\sum_{j=0}^{i}\left(\tilde t_j + \tilde t_{j+1} - x\,\delta_{j,0}\right)\binom{i}{j}2^{i-j}\frac{\partial}{\partial t_i} \\
& -4\sum_{k\geq 1}\sum_{i\geq 0}\frac{k}{k+1}\tilde{\bar s}_k\frac{\partial}{\partial t_i} - 4\sum_{i\geq 0}\sum_{j=1}^{i}\binom{i}{j}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}\left(\tilde t_\ell - x\,\delta_{\ell,0}\right)\frac{\partial}{\partial t_i}.
\end{aligned}
\tag{4.1.5}
$$

Also noticing eqs. (4.1.2), (4.1.3), (4.1.1), (4.0.2),

$$\frac{\partial^2 A}{\partial \bar{s}_1^2} = \frac{1}{2}. \tag{4.1.6}$$

and

$$\frac{\partial^2}{\partial \bar{s}_1 \partial \bar{s}_1} = \sum_{i,j\geq 0} \frac{\partial^2}{\partial t_i \partial t_j}, \tag{4.1.7}$$

we find

$$\begin{aligned}
&e^{-\frac{A}{2\,\epsilon^2}} \circ L_2^{\text{even}}(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial \mathbf{s}) \circ e^{\frac{A}{2\,\epsilon^2}} \\
=&4\,\epsilon^2 \sum_{i,j\geq 0} \frac{\partial^2}{\partial t_i \partial t_j} + 6\,x\,\frac{\partial}{\partial \bar{s}_2} + 16 \sum_{i\geq 0}\sum_{j=0}^{i} \left(\tilde{t}_j + \tilde{t}_{j+1} - x\,\delta_{j,0}\right) \binom{i}{j} 2^{i-j} \frac{\partial}{\partial t_i} \\
&- 4 \sum_{k\geq 1}\sum_{i\geq 0} \frac{k}{k+1} \tilde{\bar{s}}_k \frac{\partial}{\partial t_i} - 4 \sum_{i\geq 0}\sum_{j=1}^{i} \binom{i}{j} \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} \left(\tilde{t}_\ell - x\,\delta_{\ell,0}\right) \frac{\partial}{\partial t_i} \\
&+ 1 + 4 \left(\sum_{k\geq 1} \frac{k}{1+k} \tilde{\bar{s}}_k + x \right) \frac{\partial}{\partial \bar{s}_1} + \frac{1}{\epsilon^2} \left(\sum_{k\geq 1} \frac{k}{1+k} \tilde{\bar{s}}_k + x \right)^2 + \frac{3\,x}{\epsilon^2} \left(\sum_{k\geq 1} \frac{2\,k}{2+k} \tilde{\bar{s}}_k + x \right) \\
&+ \frac{2}{\epsilon^2} \sum_{k\geq 1} \frac{k(2k+3)(2k+1)}{(k+2)(k+1)} \tilde{\bar{s}}_k \left(\sum_{k_1\geq 1} \frac{(k+2)\,k_1}{k+2+k_1} \tilde{\bar{s}}_{k_1} + x \right) \\
=&16 \left[\frac{\epsilon^2}{4} \sum_{i,j\geq 0} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{i\geq 0}\sum_{j=0}^{i} \binom{i}{j} 2^{i-j} (\tilde{t}_{j+1} + \tilde{t}_j) \frac{\partial}{\partial t_i} \right. \\
&\left. -\frac{1}{8} \sum_{i\geq 1}\sum_{j=0}^{i-1}\sum_{r=0}^{i-1-j} (-1)^r \binom{i}{i-1-j-r} 2^{i-j-r} \tilde{t}_j \frac{\partial}{\partial t_i} + \frac{t_0^2}{4\epsilon^2} + \frac{1}{16} \right].
\end{aligned}$$

In the last equality we have used the following elementary identity

$$\sum_{j=\ell}^{i} \binom{i}{j} \binom{j-1}{\ell-1} = \sum_{r=\ell}^{i} (-1)^{r-\ell} \binom{i}{r} 2^{i-r}, \quad \forall\, \ell, i \in \mathbb{Z},\ i \geq \ell \geq 0.$$

The lemma is proved. □

The Key Lemma is proved. □

Theorem 2.4.2 along with the Key Lemma immediately implies the following theorem.

Theorem 4.1.4 *The function $\widetilde{\mathcal{F}}^*(x, \mathbf{s}; \epsilon)$ satisfies the following Virasoro constraints*

$$L_n^{\text{even}}\left(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\,\partial/\partial \mathbf{s}\right) e^{\widetilde{\mathcal{F}}^*(x,\mathbf{s};\epsilon)} = 0, \qquad n \geq 0 \tag{4.1.8}$$

where $\tilde{s}_{2k} = s_{2k} - \frac{1}{2}\delta_{k,1}$.

4.2 End of the proof of the Main Theorem

Genus expansion. By definition the special cubic Hodge free energy $\mathcal{H}_{\rm cubic}(\mathbf{t};\epsilon)$ and the GUE free energy $\mathcal{F}_{\rm even}(x,\mathbf{s};\epsilon)$ with even couplings have the following genus expansions:

$$\mathcal{H}_{\rm cubic}(\mathbf{t};\epsilon) = \sum_{g=0}^{\infty} \epsilon^{2g-2}\, \mathcal{H}_g(\mathbf{t}), \tag{4.2.1}$$

$$\mathcal{F}_{\rm even}(x,\mathbf{s};\epsilon) = \sum_{g=0}^{\infty} \epsilon^{2g-2}\, \mathcal{F}_g(x,\mathbf{s}). \tag{4.2.2}$$

Recall that it was proven in [16] the genus $0,1,2$ parts of Conjecture 1.1.1, i.e.

$$\mathcal{F}_0(x,\mathbf{s}) = \mathcal{H}_0(\mathbf{t}(x,\mathbf{s})) + A, \tag{4.2.3}$$

$$\mathcal{F}_1(x,\mathbf{s}) = 2\,\mathcal{H}_1(\mathbf{t}(x,\mathbf{s})) + \frac{1}{8}\,\frac{\partial^2\mathcal{H}_0(\mathbf{t}(x,\mathbf{s}))}{\partial x^2} + \zeta'(-1), \tag{4.2.4}$$

$$\mathcal{F}_2(x,\mathbf{s}) = 4\,\mathcal{H}_2(\mathbf{t}(x,\mathbf{s})) + \frac{1}{4}\,\frac{\partial^2\mathcal{H}_1(\mathbf{t}(x,\mathbf{s}))}{\partial x^2} + \frac{1}{384}\,\frac{\partial^4\mathcal{H}_0(\mathbf{t}(x,\mathbf{s}))}{\partial x^4}. \tag{4.2.5}$$

From (4.0.3) and (4.0.5), we see that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^*$ also have genus expansions:

$$\widetilde{\mathcal{F}}(x,\mathbf{s};\epsilon) =: \sum_{g=0}^{\infty} \epsilon^{2g-2}\, \widetilde{\mathcal{F}}_g(x,\mathbf{s}), \tag{4.2.6}$$

$$\widetilde{\mathcal{F}}^*(x,\mathbf{s};\epsilon) =: \sum_{g=0}^{\infty} \epsilon^{2g-2}\, \widetilde{\mathcal{F}}^*_g(x,\mathbf{s}). \tag{4.2.7}$$

Noting that

$$\widetilde{\mathcal{F}}_0(x,\mathbf{s}) = \frac{\mathcal{F}_0(x,\mathbf{s})}{2}, \quad \widetilde{\mathcal{F}}^*_0(x,\mathbf{s}) = \frac{\mathcal{H}_0\left(\mathbf{t}(x,\mathbf{s})\right)}{2} + \frac{A}{2}$$

and using (4.2.3) we obtain the following identity

$$\widetilde{\mathcal{F}}_0(x,\mathbf{s}) = \widetilde{\mathcal{F}}^*_0(x,\mathbf{s}). \tag{4.2.8}$$

Similarly, we have

$$\widetilde{\mathcal{F}}_1(x,\mathbf{s}) = \widetilde{\mathcal{F}}^*_1(x,\mathbf{s}), \quad \widetilde{\mathcal{F}}_2(x,\mathbf{s}) = \widetilde{\mathcal{F}}^*_2(x,\mathbf{s}). \tag{4.2.9}$$

Let us proceed to higher genera.

Recall two crucial lemmas which were proven in [14] and [16], respectively.

Lemma 4.2.1 ([14]) *There exist functions $H_g(z,z_1,z_2,\dots,z_{3g-2})$, $g\ge 1$ of independent variables z, z_1, z_2, $\dots$ such that*

$$\mathcal{H}_g(\mathbf{t}) = H_g\left(v(\mathbf{t}), \frac{\partial v(\mathbf{t})}{\partial t_0}, \dots, \frac{\partial^{3g-2} v(\mathbf{t})}{\partial t_0^{3g-2}}\right), \quad g\ge 1 \tag{4.2.10}$$

and that

$$\sum_{j=1}^{3g-2} j\, z_j \frac{\partial H_g}{\partial z_j} = (2g-2)\, H_g.$$

Here $v(\mathbf{t}) := \frac{\partial^2 \mathcal{H}_0(\mathbf{t})}{\partial t_0^2}$ *is the unique series solution to*

$$v = t_0 + \sum_{i\geq 1} t_i \frac{v^i}{i!}, \quad v(\mathbf{t}) = t_0 + \dots . \tag{4.2.11}$$

Lemma 4.2.2 ([16]) *There exist functions* $F_g(z, z_1, \dots, z_{3g-2})$, $g \geq 1$ *of independent variables* $v, v_1, v_2, \dots$ *such that*

$$\mathcal{F}_g(x, \mathbf{s}) = F_g\left(u(x,\mathbf{s}), \frac{\partial u(x,\mathbf{s})}{\partial x}, \dots, \frac{\partial^{3g-2} u(x,\mathbf{s})}{\partial x^{3g-2}}\right), \quad g \geq 1 \tag{4.2.12}$$

and that

$$\sum_{j=1}^{3g-2} j\, z_j \frac{\partial F_g}{\partial z_j} = (2g-2)\, F_g.$$

Here $u(x, \mathbf{s}) := \frac{\partial^2 \mathcal{F}_0(x,\mathbf{s})}{\partial x^2} = \log w(x, \mathbf{s})$, *and* $w(x, \mathbf{s})$ *is the unique series solution to*

$$w = x + \sum_{k\geq 1} k\, \bar{s}_k\, w^k, \qquad w(x, \mathbf{s}) = x + \dots . \tag{4.2.13}$$

Lemma 4.2.3 *For any* $g \geq 2$, *there exist functions* $\widetilde{F}_g(z, z_1, \dots, z_{3g-2})$ *and* $\widetilde{F}_g^*(z, z_1, \dots, z_{3g-2})$ *of independent variables* $z, z_1, \dots, z_{3g-2}$ *such that*

$$\widetilde{\mathcal{F}}_g(x, \mathbf{s}) = \widetilde{F}_g\left(u(x,\mathbf{s}), \frac{\partial u(x,\mathbf{s})}{\partial x}, \dots, \frac{\partial^{3g-2} u(x,\mathbf{s})}{\partial x^{3g-2}}\right), \tag{4.2.14}$$

$$\widetilde{\mathcal{F}}_g^*(x, \mathbf{s}) = \widetilde{F}_g^*\left(u(x,\mathbf{s}), \frac{\partial u(x,\mathbf{s})}{\partial x}, \dots, \frac{\partial^{3g-2} u(x,\mathbf{s})}{\partial x^{3g-2}}\right) \tag{4.2.15}$$

with $u(x, \mathbf{s})$ *defined as in Thm. 4.2.2.*

Proof Observe that, as in [16], under the substitution

$$t_i(x, \mathbf{s}) = \sum_{k\geq 1} k^{i+1} \bar{s}_k - 1 + \delta_{i,1} + x \cdot \delta_{i,0}$$

we have $v(\mathbf{t}(x, \mathbf{s})) = u(x, \mathbf{s})$, where $v(\mathbf{t})$ is defined as in Thm. 4.2.1. The lemma is then proved by Lemmata 4.2.1, 4.2.2 and the defining eqs. (4.0.3), (4.0.5). □

End of the proof of the Main Theorem. We have proved that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^*$ satisfy the same set of PDEs (4.1.8) (or say (4.0.4)), and have the same structures (4.2.6)–(4.2.7), (4.2.14)–(4.2.15). We are left to prove certain uniqueness.

Similarly as in [9, 33], one can deduce that eqs. (4.1.8), or eqs. (4.0.4) determine $\widetilde{\mathcal{F}}^*$, or $\widetilde{\mathcal{F}}$ in the ring $\mathcal{B}^{even}$ up to an additive arbitrary function in x. However, we already know that both $\widetilde{\mathcal{F}}^*$ and $\widetilde{\mathcal{F}}^*$ have genus expansions (4.2.6)–(4.2.7). So we must have

$$\widetilde{\mathcal{F}} - \widetilde{\mathcal{F}}^* = K(x; \epsilon) = \sum_{g\geq 0} \epsilon^{2g-2} K_g(x)$$

where $K_g(x)$ are functions in x only. We are going to show that $\forall\, g \geq 0$, $K_g(x)$ vanishes. Note that the genus $0, 1, 2$ parts were proven (cf. (4.2.8), (4.2.9)). So we are left to consider $g \geq 3$.

Theorem 4.2.4 *For any $g \geq 3$, $K_g(x) \equiv 0$.*

Proof Since $g \geq 3$, we know from Lem. 4.2.3 that there exist a function $Q_g(z, z_1, \ldots, z_{3g-2})$ of independent variables $z, z_1, \ldots, z_{3g-2}$ such that

$$\widetilde{\mathcal{F}}_g(x, \mathbf{s}) - \widetilde{\mathcal{F}}^*_g(x, \mathbf{s}) = Q_g\left(u(x, \mathbf{s}), \frac{\partial u(x, \mathbf{s})}{\partial x}, \ldots, \frac{\partial^{3g-2} u(x, \mathbf{s})}{\partial x^{3g-2}}\right) = K_g(x).$$

It follows that

$$\frac{\partial Q_g\left(u(x, \mathbf{s}), \frac{\partial u(x,\mathbf{s})}{\partial x}, \ldots, \frac{\partial^{3g-2} u(x,\mathbf{s})}{\partial x^{3g-2}}\right)}{\partial \bar{s}_k} \equiv 0, \quad \forall\, k \geq 1. \tag{4.2.16}$$

Recall that the function $u(x, \mathbf{s})$ satisfies the following PDEs [16]:

$$\frac{\partial u}{\partial \bar{s}_k} = k\, e^{ku}\, u_x, \quad k \geq 1.$$

Substituting these flows in the l.h.s. of (4.2.16) and dividing the resulting expression by e^{ku}, we obtain a polynomial in k of degree $3g-1$. Since for any $k \in \mathbb{Z}$ this polynomial must vanish, all its coefficients must vanish. Looking at these coefficients from the highest degree ($= 3g-1$) to the lowest degree ($= 1$), and using its triangular nature (can be deduced easily) we obtain

$$\frac{\partial Q_g}{\partial z_j} = 0, \quad j = 1, \ldots, 3g-2; \quad \frac{\partial Q_g}{\partial z} = 0.$$

So Q_g must be a constant. However, noting that

$$\sum_{j=1}^{3g-2} j\, z_j \frac{\partial Q_g}{\partial z_j} = (2g-2)\, Q_g,$$

we find that Q_g must vanish. As a result, $K_g(x) \equiv 0$. The theorem is proved. □

We arrive at $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}^*$, which implies $\mathcal{F}_{\text{even}} = \mathcal{F}^*$. The Main Theorem is proved. □

Proof of Theorem 1.2.2 Part 1) of the theorem follows from the Main Theorem by shifting the time variables. Part 2) of the theorem is a reformulation of the Main Theorem. □

5 Two applications of the Main Theorem

Several applications of the Main Conjecture proven in the present paper have been given in [16]. In this section we present two more new applications.

5.1 Application I. On the reduced GUE free energy

In this subsection, we introduce a reduced GUE free energy, which will provide a new understanding of the relationship between the intersection numbers of psi-classes on the $\overline{\mathcal{M}}_{g,k}$ ("topological gravity") and matrix integrals ("matrix gravity").

Recall that the following formula has been obtained in [16] as a consequence of the Hodge–GUE conjecture: $\forall\, g \geq 2$,

$$F_g(v, v_1, \dots, v_{3g-2}) = \frac{v_{2g-2}}{2^{2g}(2g)!} + \frac{D_0^{2g-2}\left[H_1(v; v_1)\right]}{2^{2g-3}(2g-2)!} + \sum_{m=2}^{g} \frac{2^{3m-2g}}{(2g-2m)!} D_0^{2(g-m)}\left[H_m(v, v_1, \dots, v_{3m-2})\right] \tag{5.1.1}$$

where $D_0 := v_1\,\partial_v + \sum_{k\geq 1} v_{k+1}\,\partial_{v_k}$. Noticing that, for any $m \geq 2$, H_m is a rational function of $v_1, v_2, \dots$, which does not contain v explicitly [14], and that

$$H_1 = \frac{1}{24}\log v_1 - \frac{1}{16} v$$

we find that for any $g \geq 2$, F_g is also a rational function in $v_1, v_2, \dots$, which does not contain v explicitly, i.e.

$$\frac{\partial F_g}{\partial v} = 0, \quad g \geq 2.$$

Introduce a gradation for these rational functions by assigning

$$\widetilde{\deg}\, v_k = 1, \quad \forall\, k \geq 1.$$

Then both H_g and F_g with $g \geq 2$ decompose into homogeneous parts w.r.t. to $\widetilde{\deg}$

$$H_g = \sum_{d=1-g}^{2g-2} H_g^{[d]}, \quad F_g = \sum_{d=1-g}^{2g-2} F_g^{[d]}, \quad g \geq 2.$$

Definition 5.1.1 *Define the lowest degree GUE free energy $\mathcal{F}^{\rm red}$ by*

$$\begin{aligned}
&\mathcal{F}^{\rm red}(x, \mathbf{s}; \epsilon) := \sum_{g\geq 0} \epsilon^{2g-2} \mathcal{F}_g^{\rm red}(x, \mathbf{s})\\
&\mathcal{F}_0^{\rm red}(x, \mathbf{s}) := \mathcal{F}_0(x, \mathbf{s})\\
&\mathcal{F}_1^{\rm red}(x, \mathbf{s}) := \mathcal{F}_1(x, \mathbf{s})\\
&\mathcal{F}_g^{\rm red}(x, \mathbf{s}) := F_g^{[1-g]}\left(\frac{\partial u(x, \mathbf{s})}{\partial x}, \dots, \frac{\partial^{3g-2} u(x, \mathbf{s})}{\partial x^{3g-2}}\right), \quad g \geq 2
\end{aligned}$$

where $u(x, \mathbf{s})$ is defined as in Lem. 4.2.2.

Now let

$$\mathcal{F}^{\rm WK}(\mathbf{t}; \epsilon) = \sum_{g\geq 0} \epsilon^{2g-2} \mathcal{F}_g^{\rm WK}(\mathbf{t})$$

denote the free energy of the Witten–Kontsevich correlators, where

$$\mathcal{F}_g^{\rm WK}(\mathbf{t}) := \sum_{k\geq 0} \frac{1}{k!} \sum_{i_1, \dots, i_k \geq 0} t_{i_1} \cdots t_{i_k} \int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{i_1} \cdots \psi_k^{i_k}.$$

We know that for $g \geq 2$,

$$\mathcal{F}_g^{\rm WK}(\mathbf{t}) = H_g^{[1-g]}\left(\frac{\partial v(\mathbf{t})}{\partial t_0}, \dots, \frac{\partial^{3g-2} v(\mathbf{t})}{\partial t_0^{3g-2}}\right)$$

where $v(\mathbf{t})$ is defined in Lem. 4.2.1. By comparing the lowest degree part of the identity (5.1.1) and noticing that the operator D_0 does not change the degree defined by $\widetilde{\deg}$, we arrive at

Corollary 5.1.2 *The following formula holds true*

$$\mathcal{F}^{\rm red}(x,\mathbf{s};\epsilon)+\epsilon^{-2}\Big(-\frac{1}{2}\sum_{k_1,k_2\geq 1}\frac{k_1\,k_2}{k_1+k_2}\,\bar s_{k_1}\,\bar s_{k_2}+\sum_{k\geq 1}\frac{k}{1+k}\,\bar s_k-x\sum_{k\geq 1}\bar s_k-\frac{1}{4}+x\Big)$$
$$=\sum_{g\geq 0}\epsilon^{2g-2}\,2^g\,\mathcal{F}_g^{\rm WK}\left(\mathbf{t}(x,\mathbf{s})\right)+\zeta'(-1) \tag{5.1.2}$$

where $\bar s_k:=\binom{2k}{k}s_{2k}$ *and*

$$t_i(x,\mathbf{s}):=\sum_{k\geq 1}k^{i+1}\bar s_k-1+\delta_{i,1}+x\cdot\delta_{i,0},\quad i\geq 0.$$

5.2 Application II. Proof of the integrable hierarchy conjecture

The Main Theorem confirms the validity of the following Integrable Hierarchy Conjecture originally proposed in [14].

Theorem 5.2.1 *Let* $\mathcal{H}_{\rm cubic}(\mathbf{t};b;\epsilon)$ *denote the following generating series of special cubic Hodge integrals*

$$\mathcal{H}_{\rm cubic}(\mathbf{t};b;\epsilon)=\sum_{g\geq 0}\epsilon^{2g-2}\sum_{k\geq 0}\frac{1}{k!}\sum_{i_1,\dots,i_k\geq 0}t_{i_1}\cdots t_{i_k}\int_{\overline{\mathcal{M}}_{g,k}}\Lambda_g(-2b)\,\Lambda_g(-2b)\,\Lambda_g(b)\,\psi_1^{i_1}\cdots\psi_k^{i_k}$$

where b *is a non-zero parameter. Define*

$$U(\mathbf{t};b;\epsilon)=\left(\Lambda^{1/2}-\Lambda^{-1/2}\right)\left(\Lambda-\Lambda^{-1}\right)\mathcal{H}_{\rm cubic}\left(\mathbf{t};b;\frac{\epsilon}{\sqrt{b}}\right),\quad \Lambda:=e^{\epsilon\,\partial_{t_0}}. \tag{5.2.1}$$

Then U *satisfies the following discrete KdV equation (which is equivalent to* (1.2.1) *by a rescaling):*

$$\frac{\partial U}{\partial t}=\frac{1}{2\epsilon}\left(e^{U(t_0+\epsilon)}-e^{U(t_0-\epsilon)}\right) \tag{5.2.2}$$

where

$$\partial_t:=\sum_{i\geq 0}2^i b^i\,\partial_{t_i}.$$

Proof Let us first prove (5.2.2) for $b=\frac{1}{2}$. According to the Main Theorem, under the following substitution of time variables

$$t_i(x,\mathbf{s}):=\sum_{k\geq 1}k^{i+1}\bar s_k-1+\delta_{i,1}+x\cdot\delta_{i,0},\quad i\geq 0,$$

the following identity holds true

$$\mathcal{F}_{\rm even}(x,\mathbf{s};\epsilon)=\left(\Lambda^{\frac{1}{2}}+\Lambda^{-\frac{1}{2}}\right)\,\mathcal{H}_{\rm cubic}\left(\mathbf{t}(x,\mathbf{s});\frac{1}{2};\sqrt{2}\,\epsilon\right)+\epsilon^{-2}\,A+\zeta'(-1) \tag{5.2.3}$$

where $\mathcal{F}_{\rm even}$ is the GUE free energy with even couplings, A is defined in (1.1.13). Define

$$u(x,\mathbf{s};\epsilon)=(\Lambda-1)\,(1-\Lambda^{-1})\mathcal{F}_{\rm even}(x,\mathbf{s};\epsilon).$$

Noting that

$$\partial_{t_0} = \partial_x \quad \Rightarrow \quad \Lambda = e^{\epsilon\, \partial_x}$$

and applying $(\Lambda - 1)\,(1 - \Lambda^{-1})$ on both sides of (5.2.3) we obtain

$$u(x, \mathbf{s}; \epsilon) = U\left(\mathbf{t}(x, \mathbf{s}); 1/2; \epsilon\right).$$

Since u satisfies

$$\frac{\partial u}{\partial \bar{s}_1} = \frac{1}{2\epsilon}(\Lambda - \Lambda^{-1})\, e^u$$

and since

$$\frac{\partial}{\partial \bar{s}_1} = \sum_{i\geq 0} \frac{\partial}{\partial t_i}$$

we have

$$\frac{\partial U}{\partial t} = \frac{1}{2\epsilon}\left(e^{U(t_0+\epsilon)} - e^{U(t_0-\epsilon)}\right).$$

Let us now consider a general b, $b \neq 0$. Denote $v(\mathbf{t})$ the unique series solution to

$$\sum_{i\geq 0} t_i \frac{v^i}{i!} = v.$$

It was shown in [14] that there exist functions $H_g(z, z_1, \ldots, z_{3g-2}; b)$ such that

$$\mathcal{H}_g(\mathbf{t}; b) = H_g\left(v(\mathbf{t}), \frac{\partial v(\mathbf{t})}{\partial t_0}, \ldots, \frac{\partial^{3g-2} v(\mathbf{t})}{\partial t_0^{3g-2}}; b\right), \quad g \geq 1.$$

Expand

$$U(\mathbf{t}; b; \epsilon) = \sum_{g\geq 0} \epsilon^{2g} U^{[2g]}(\mathbf{t}; b).$$

Eq. (5.2.2) is now proved by noticing

$$U^{[0]}(\mathbf{t}; b) = 2\, b\, v(\mathbf{t})$$

as well as observing the following identity [17, 14, 28]

$$-b\frac{\partial H_g(v, v_1, \ldots, v_{g-2}; b)}{\partial b} + \sum_{k=0}^{3g-2} v_k \frac{\partial H_g(v, v_1, \ldots, v_{3g-2}; b)}{\partial v_k} = \frac{1}{24}\delta_{g,1} - (g-1) H_g, \quad \forall\, g \geq 1.$$

The theorem is proved. $\square$

Theorem 5.2.1 can be equivalently described as follows: the Hodge hierarchy [14] associated with

$$\Lambda_g(-2b)\, \Lambda_g(-2b)\, \Lambda_g(b), \quad b \neq 0$$

is *normal Miura equivalent* to the discrete KdV hierarchy (up to a simple rescaling).

Remark 5.2.2 *At a first glance, it looks not easy to use eq.* (5.2.2) *for computing Hodge integrals. However, actually, the single discrete KdV equation* (5.2.2) *contains the full information about the special cubic Hodge integrals via the quasi-triviality approach [18, 13, 34]! This is because the construction of the so-called Hodge hierarchy [14] associated to* $\Lambda_g(-2b)\,\Lambda_g(-2b)\,\Lambda_g(b)$ *tells that* $V := \epsilon^2\,\partial_{t_0}^2\mathcal{H}(\mathbf{t};b;\epsilon)$ *gives the quasi-triviality transformation of this hierarchy. The algorithm of computing* $\mathcal{H}_g$ *from the quasi-triviality approach is given in [34, 17].*

It would be interesting to study the so-called double ramification counterpart of the special cubic Hodge hierarchy in the framework of the conjectural DR/DZ correspondence formulated by A. Buryak [6]. We plan to do it in a subsequent publication.

A Appendix. Givental quantization

Denote by $\mathcal{V}$ the space of Laurent polynomials in z with coefficients in $\mathbb{C}$. Define a symplectic bilinear form ω on $\mathcal{V}$ by

$$\omega(f,g) := -\mathrm{Res}_{z=\infty}\, f(-z)\, g(z)\, \frac{dz}{z^2} = -\omega(g,f), \quad \forall\, f,g\in\mathcal{H}.$$

The pair $(\mathcal{V},\omega)$ is called a Givental symplectic space. For any $f\in\mathcal{V}$, write

$$f = \sum_{i\ge 0} q_i\, z^{-i} + \sum_{i\ge 0} p_i\,(-z)^{i+1}.$$

Then $\{q_i,\, p_i\}|_{i=0}^{\infty}$ gives a system of canonical coordinates for $(\mathcal{V},\omega)$. The canonical quantization in these coordinates yields operators of the form

$$\widehat{p_i} = \epsilon\frac{\partial}{\partial q_i}, \quad \widehat{q_i} = \frac{1}{\epsilon}\, q_i$$

on the Fock space of formal power series in q_i. For any infinitesimal linear symplectic transformation A on $(\mathcal{V},\omega)$, i.e. A satisfies

$$\omega(A\,f,g) + \omega(f,A\,g) = 0, \quad \forall\, f,g\in\mathcal{V},$$

the Hamiltonian associated to A is

$$H_A(f) = \frac{1}{2}\omega(f,A\,f) = -\frac{1}{2}\mathrm{Res}_{z=\infty}\, f(-z)\, A\, f(z)\, \frac{dz}{z^2}.$$

This Hamiltonian is a quadratic function on $\mathcal{V}$, and its quantization is defined via

$$\widehat{p_ip_j} = \epsilon^2\,\frac{\partial^2}{\partial q_i\partial q_j}, \quad \widehat{p_iq_j} = q_j\,\frac{\partial}{\partial q_i}, \quad \widehat{q_iq_j} = \frac{1}{\epsilon^2}\, q_iq_j.$$

Denote the quantization of H_A by $\widehat{A}$. We have, for any two infinitesimal symplectic transformations A,B,

$$\left[\widehat{A}\,,\,\widehat{B}\right] = \widehat{[A,B]} + \mathcal{C}\left(H_A\,,\,H_B\right),$$

where $\mathcal{C}$ is the so-called 2-cocycle term satisfying

$$\mathcal{C}(p_i p_j, q_k q_l) = -\mathcal{C}(q_k q_l, p_i p_j) = \delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k},$$

and $\mathcal{C} = 0$ for all other pairs of quadratic monomials in p, q.

Denote $D = -z\,\partial_z\,z^{-1}$ and put

$$l_k := z^{1/2}\,D^{k+1}\,z^{1/2} = (-1)^{k+1} z^{3/2} \partial_z^{k+1} z^{-1/2}, \quad k \geq -1. \tag{A.0.1}$$

Then we have

Lemma A.0.1 ([26]) *The operators l_k are infinitesimal symplectic transformations on $\mathcal{V}$; moreover,*

$$L_k^{KdV}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\,\partial/\partial\mathbf{t}) = \widehat{l_k}\Big|_{q_i \mapsto \tilde{t}_i,\, \partial_{q_i} \mapsto \partial_{t_i},\, i \geq 0} + \frac{\delta_{k,0}}{16}, \quad k \geq -1$$

where $\tilde{t}_i := t_i - t_i^0$.

Lemma A.0.2 ([26]) *The multiplication operators z^{1-2j}, $j \geq 1$ are infinitesimal symplectic transformations on $\mathcal{V}$; moreover, the operators D_j defined in* (2.1.2) *satisfy*

$$D_j = \widehat{z^{1-2j}}\Big|_{q_i \mapsto \tilde{t}_i,\, \partial_{q_i} \mapsto \partial_{t_i},\, i \geq 0}, \quad j \geq 1. \tag{A.0.2}$$

Consider now quantization $\widehat{\Phi}$ of the symplectomorphism $f(z) \mapsto \Phi(z) f(z)$ where the function $\Phi(z)$ was defined by eq. (1.2.6) above. It will be defined by

$$\widehat{\Phi} := e^{(\log\Phi(z))^\wedge}$$

where we replace $\log\Phi(z)$ by its asymptotic expansion at $|z| \to \infty$, $\mathrm{Re}\, z \neq 0$. The latter has the form, up to an inessential piecewise constant term

$$\log\Phi(z) \sim \sum_{k=1}^{\infty} \frac{B_{2k}}{k(2k-1)} \frac{2^{-2k}-1}{z^{2k-1}}.$$

Using (2.1.2) we immediately arrive at

Lemma A.0.3 *We have*

$$\widehat{\Phi} = e^{\sum_{k=1}^{\infty} \frac{B_{2k}}{k(2k-1)} D_k \left(2^{-2k}-1\right)}. \tag{A.0.3}$$

Remark A.0.4 *The function $\Phi(z)$ is analytic near $z = 0$, $\Phi(0) = 1$ and*

$$\log\Phi(z) = -2z\log 2 - 2\sum_{k=1}^{\infty} \frac{2^{2k}-1}{2k+1} \zeta(2k+1)\, z^{2k+1}, \quad |z| < \frac{1}{2}. \tag{A.0.4}$$

One can define another quantum operator $\hat{\Phi}_0$ by quantizing the series (A.0.4). *Geometric interpretation of this quantum operator remains an interesting open question.*

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