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Contents

1 Part 1: Singular points of solutions to analytic differential equations

1.1 Differential equations in the complex domain: basic results

Recall that an *analytic* (also *holomorphic*) function $w = u + iv$ of the complex variable $z = x + iy$ is a differentiable map

$$\mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (u, v)$$

satisfying the *Cauchy - Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.1.1}$$

Introducing the complex combinations

$$z = x + iy, \quad \bar{z} = x - iy$$

one can recast the Cauchy - Riemann equations into the form

$$\frac{\partial w}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\,\frac{\partial w}{\partial y}\right) = 0.$$

Theorem 0. *Any function $w(z)$ analytic on a neighborhood of a point $z = z_0$ admits an expansion into a power series*

$$w(z) = w_0 + w_1(z - z_0) + w_2(z - z_0)^2 + \dots \tag{1.1.2}$$

convergent for $|z - z_0| < \epsilon$ for some sufficiently small $\epsilon > 0$. The coefficients of (1.1.2) *can be computed via the derivatives of $w(z)$ as follows:*

$$w_0 = w(z_0), \quad w_k = \frac{1}{k!}\frac{d^k w(z)}{dz^k}|_{z=z_0}, \quad k = 1, 2, \dots. \tag{1.1.3}$$

Remark 1.1.1 *The function $w(z)$ is sais to be analytic at the point $z = \infty$ if the function $w(1/u)$ is analytic at the point $u = 0$. Such a function can be expanded in a series of the form*

$$w = w_0 + \frac{w_1}{z} + \frac{w_2}{z^2} + \dots \tag{1.1.4}$$

converging for $|z| > R$ for some positive R.

A function $f(w, z)$ of two complex variables is called analytic if two Cauchy - Riemann equations in z and w hold true:

$$\frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial f}{\partial \bar{w}} = 0.$$

Functions of two variables analytic on a polydisk

$$|z| < r, \quad |w| < \rho$$

for some positive r, ρ admit expansion in a convergent double series

$$f(w,z) = \sum_{k,l \geq 0} a_{kl} w^k z^l. \tag{1.1.5}$$

Let us begin with considering a first order ODE of the form

$$w' = f(w,z) \tag{1.1.6}$$

with a function $f(w,z)$ analytic in a neighborhood of the point (w_0, z_0). We are looking for a solution $w = w(z)$ satisfying the initial condition

$$w(z_0) = w_0. \tag{1.1.7}$$

Without loss of generality we assume that $z_0 = w_0 = 0$.

Let us look for a solution in the form of power series

$$w(z) = w_0 + w_1 z + w_2 z^2 + \dots. \tag{1.1.8}$$

Substituting into the equation (1.1.6) one obtains a recursive procedure uniquely determining the coefficients:

$$\begin{aligned}
w_0 &= 0 \\
w_1 &= w'(0) = f(0,0) = a_{00} \\
w_2 &= \frac{1}{2}\left(\frac{d}{dz} f(w,z)\right)_{(0,0)} = \frac{1}{2}\left[f_z(w,z) + f_w(w,z) w'\right]_{(0,0)} = \frac{1}{2}\left(a_{01} + a_{10} a_{00}\right) \\
w_3 &= \frac{1}{6}\left(\frac{d}{dz}\left[f_z(w,z) + f_w(w,z) w'\right]\right)_{(0,0)} \\
&= \frac{1}{6}\left[f_{zz} + 2 f_{wz} w' + w'^2 f_{ww} + w'' f_w\right]_{(0,0)} \\
&= \frac{1}{6}\left[a_{02} + 2 a_{11} a_{00} + 2 a_{20} a_{00}^2 + a_{10}(a_{01} + a_{10} a_{00})\right].
\end{aligned} \tag{1.1.9}$$

In these computation a_{ij} are the coefficients of the Taylor expansion (1.1.5) of the function f.

In general

$$w_k = P_k(a_{ij}) \tag{1.1.10}$$

where P_k is some polynomial with positive coefficients. It remains to prove *convergence* of the series.

Exercise 1.1.2 *Prove that the series*

$$w = z + z^2 + 2!\, z^3 + 3!\, z^4 + \dots + (n-1)!\, z^n + \dots \tag{1.1.11}$$

satisfies the differential equation

$$z^2 w' = w - z.$$

Prove that the series diverges for any $z \neq 0$.

The following classical result, due to Cauchy, establishes convergence of the series solution (1.1.8), (1.1.9) for differential equations with analytic right hand side.

Theorem 1.1.3 *If the function* $f(w,z)$ *is analytic for* $|z| < r$, $|w| < \rho$ *for some* $r > 0$, $\rho > 0$ *then the series* (1.1.8), (1.1.9) *converges to an analytic solution to* (1.1.6) *satisfying the initial condition* (1.1.7) *on the domain*

$$|z| < r_1 \left(1 - e^{-\frac{1}{2M}\frac{\rho_1}{r_1}}\right) \tag{1.1.12}$$

for arbitrary r_1, ρ_1, M *such that*

$$0 < r_1 < r, \quad 0 < \rho_1 < \rho$$

assuming that

$$|f(w,z)| < M \quad \text{for} \quad |w| < \rho_1, \quad |z| < r_1.$$

Let us begin the proof with a definition. Given two power series

$$f(w,z) = \sum a_{kl} w^k z^l, \quad F(w,z) = \sum A_{kl} w^k z^l \tag{1.1.13}$$

such that (i) the series $F(w,z)$ converges for $|z| < r_1$, $|w| < \rho_1$, and (ii) all the coefficients A_{kl} are real positive numbers, we say that $F(w,z)$ is a *majorant* for $f(w,z)$ if

$$|a_{kl}| \leq A_{kl} \quad \text{for all} \quad k,\, l \geq 0.$$

We will use the following notation

$$f(w,z) \preceq F(w,z) \tag{1.1.14}$$

to say that $F(w,z)$ is a majorant of $f(w,z)$. Observe that the power series $f(w,z)$ converges on the same domain $|z| < r_1$, $|w| < \rho_1$ where the series $F(w,z)$ does converge.

Lemma 1.1.4 *Given an arbitrary polynomial* $P(a_{00}, a_{01}, \dots, a_{kl})$ *with positive coefficients, then for any pair* $f(w,z) \preceq F(w,z)$ *the following inequality holds true*

$$|P(a_{00}, a_{01}, \dots, a_{kl})| \leq P(A_{00}, A_{01}, \dots, A_{kl}). \tag{1.1.15}$$

The proof is obvious.

Let us now recall the *Cauchy inequalities* for the Taylor coefficients of an analytic function.

Lemma 1.1.5 *Given a function* $f(w,z)$ *analytic for* $|w| < \rho$, $|z| < r$, *then for any* $0 < r_1 < r$, $0 < \rho_1 < \rho$ *the following equality holds true*

$$\sum_{k,l} |a_{kl}|^2 \rho_1^{2k} r_1^{2l} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |f\left(\rho_1 e^{i\phi}, r_1 e^{i\theta}\right)|^2 d\phi\, d\theta. \tag{1.1.16}$$

Proof: Indeed,

$$f\left(\rho_1 e^{i\phi}, r_1 e^{i\theta}\right) = \sum a_{kl}\rho_1^k r_1^l e^{ik\phi+il\theta},$$
$$\bar f\left(\rho_1 e^{i\phi}, r_1 e^{i\theta}\right) = \sum \bar a_{kl}\rho_1^k r_1^l e^{-ik\phi-il\theta}$$

Multiplying one obtains

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f\,\bar f\,d\phi\,d\theta = \frac{1}{(2\pi)^2}\sum a_{k'l'}\bar a_{k''l''}\rho_1^{k'+k''} r_1^{l'+l''}\int_0^{2\pi} e^{i(k'-k'')\phi}d\phi\int_0^{2\pi} e^{i(l'-l'')\theta}d\theta$$
$$=\sum_{k,l}|a_{kl}|^2\rho_1^{2k}r_1^{2l}$$

where we use the orthogonality

$$\int_0^{2\pi} e^{i(k'-k'')\phi}d\phi = 2\pi\,\delta_{k',k''},\quad \int_0^{2\pi} e^{i(l'-l'')\theta}d\theta = 2\pi\,\delta_{l',l''}.$$

□

Corollary 1.1.6 *Denote*

$$M = \sup_{|w|<\rho_1,\ |z|<r_1}|f(w,z)|.$$

Then the following inequalities hold true for the Taylor coefficients of the function $f(w,z)$

$$|a_{kl}| < \frac{M}{\rho_1^k r_1^l}\quad \forall\, k,\, l\geq 0. \tag{1.1.17}$$

Proof: Indeed,

$$\sum_{k,l}|a_{kl}|^2\rho_1^{2k}r_1^{2l} < M^2.$$

Hence

$$|a_{kl}|^2\rho_1^{2k}r_1^{2l} < M^2.$$

□

Corollary 1.1.7

$$f(w,z)\preceq \frac{M}{\left(1-\frac{z}{r_1}\right)\left(1-\frac{w}{\rho_1}\right)}. \tag{1.1.18}$$

Proof: follows from

$$\sum \frac{M}{\rho_1^k r_1^l}w^k z^l = \frac{M}{\left(1-\frac{z}{r_1}\right)\left(1-\frac{w}{\rho_1}\right)}.$$

□

Proof of Cauchy theorem: consider an auxiliary differential equation

$$W' = \frac{M}{\left(1 - \frac{Z}{r_1}\right)\left(1 - \frac{W}{\rho_1}\right)}. \tag{1.1.19}$$

Let us consider the power series solution to this equation

$$W = \sum_{m \geq 1} B_m Z^m, \quad W(0) = 0.$$

The coefficients B_m can be determined by the procedure (1.1.9) in the form

$$B_m = P_m(b_{00}, b_{01}, \dots)$$

by the *same* polynomials P_m as in (1.1.10). Here

$$b_{kl} = \frac{M}{\rho_1^k r_1^l}$$

are the Taylor coefficients of the rhs of (1.1.19). According to Lemma 1.1.4

$$w(z) \preceq W(Z).$$

It remains to prove convergence of the power series solution to (1.1.19). This equation can be solved explicitly by separation of variables:

$$\left(1 - \frac{W}{\rho_1}\right) dW = M \frac{dZ}{1 - \frac{Z}{r_1}}.$$

This gives

$$\left(1 - \frac{W}{\rho_1}\right)^2 = \frac{2M\, r_1}{\rho_1} \log\left(1 - \frac{Z}{r_1}\right) + K.$$

The initial condition $W(0) = 0$ determines the integration constant

$$K = 1.$$

Finally

$$W = \rho_1 \left[1 - \sqrt{1 + \frac{2M\, r_1}{\rho_1} \log\left(1 - \frac{Z}{r_1}\right)}\right]. \tag{1.1.20}$$

This function has singularities at $Z = r_1$ and

$$\log\left(1 - \frac{Z}{r_1}\right) = -\frac{\rho_1}{2M\, r_1}$$

i.e. at

$$Z_0 = r_1\left(1 - e^{-\frac{1}{2M}\frac{\rho_1}{r_1}}\right)$$

(observe that $0 < Z_0 < r_1$). Hence the Taylor series for the function (1.1.20) converges for

$$|Z| < r_1\left(1 - e^{-\frac{1}{2M}\frac{\rho_1}{r_1}}\right).$$

Due to $w(z) \preceq W(Z)$ the series solution $w(z)$ converges on the same domain. □

Remark 1.1.8 *Uniqueness of the analytic solution easily follows since the procedure* (1.1.9) *determines all the coefficients uniquely.*

Remark 1.1.9 *Let the rhs of the differential equation*

$$w' = f(w, z)$$

be defined on the (punctured) polydisk

$$|w| < \rho, \quad |z| > R$$

for some positive ρ, R. We say that the point $z = \infty$ is a point of analyticity for the differential equation if, after the change of independent variable

$$z = \frac{1}{u}, \quad -u^2 \frac{dw}{du} = f\left(w, 1/u\right)$$

the point $u = 0$ is a point of analyticity, i.e., the function

$$\tilde{f}(w, u) := -u^{-2} f(w, u^{-1})$$

is analytic for

$$|w| < \rho, \quad |u| < \frac{1}{R}.$$

In that case the Cauchy theorem ensures existence, uniqueness, and analyticity of solutions to the differential equation with the initial data

$$w(\infty) = w_0.$$

The solutions are represented as power series in $1/z$:

$$w = w_0 + \frac{w_1}{z} + \frac{w_2}{z^2} + \dots$$

convergent for $|z| > R_1$ for some positive R_1.

I will leave as an exercise to formulate and prove an analogue of the Cauchy theorem for systems of differential equations

$$\begin{aligned} w_1' &= f_1(w_1, \dots, w_n, z) \\ \dots \quad & \dots \quad \dots \quad \dots \\ w_n' &= f_n(w_1, \dots, w_n, z) \end{aligned} \tag{1.1.21}$$

with analytic right hand sides.

The Cauchy theorem can be improved in certain particular cases. As an important example consider a system of linear differential equations

$$\frac{dw_k}{dz} = \sum_{l=1}^{n} a_{kl}(z) w_l + b_k(z), \quad k = 1, \dots, n \tag{1.1.22}$$

with coefficients $a_{kl}(z)$, $b_k(z)$ analytic for $|z - z_0| < r$ for some positive r.

Exercise 1.1.10 *Prove that the solution to the linear system* (1.1.22) *with the initial data*

$$w_1(z_0) = w_1^0, \dots, w_n(z_0) = w_n^0$$

with arbitrary $(w_1^0, \dots, w_n^0) \in \mathbb{C}^n$ exists, is unique, and analytic on the same disk $|z - z_0| < r$.

1.2 Analytic continuation of solutions. Movable and fixed singularities.

The considerations of the previous section describe the local structure of solutions to differential equations near the point of analyticity. Global considerations require working with multivalued solutions to differential equations that we will introduce in this section.

We begin with recalling basic facts about analytic continuation.

Definition 1.2.1 *An* element of analytic function *is a pair (U, f) where $U \subset \bar{\mathbb{C}}$ is a disc and f is an analytic function in U.*

Definition 1.2.2 *An element (V, g) is called* analytic continuation *of (U, f) if*

1) $U \cap V \neq \emptyset$

2) on the intersection $U \cap V$ the identity $g(z) \equiv f(z)$ holds.

Definition 1.2.3 *A* multivalued analytic function $w(z)$ *on a connected domain $\Omega \subset \bar{\mathbb{C}}$ is a set of elements of analytic functions*

$$(U_i, w_{i,j})_{i \in I,\ j \in J} \tag{1.2.1}$$

such that

1) $\bigcup_{i \in I} U_i = \Omega$.

2) For any pair U_{i_1}, U_{i_2} with a non-empty intersection and for any $j_1 \in J$ there exists $j_2 \in J$ such that (U_{i_2}, w_{i_2,j_2}) is analytic continuation of (U_{i_1}, w_{i_1,j_1}).

3) On triple intersections of U_{i_1}, U_{i_2}, U_{i_3} the analytic continuations w_{i_2,j_2} and w_{i_3,j_3} must coincide.

For a multivalued function on $\Omega \subset \bar{\mathbb{C}}$ it is defined an operation of analytic continuation of an element (U_{i_1}, w_{i_1,j_1}) from a point $P \in U_{i_1}$ to another point Q along a curve C. Namely, choose a finite number of elements (U_{i_1}, w_{i_1,j_1}), ..., (U_{i_n}, w_{i_n,j_n}) such that

$$C \subset \bigcup_{k=1}^{n} U_{i_k},$$

$$P \in U_{i_1}, \quad Q \in U_{i_n}$$

and the element $\left(U_{i_{k+1}}, w_{i_{k+1},j_{k+1}}\right)$ is an analytic continuation of an element (U_{i_k}, w_{i_k,j_k})

Exercise 1.2.4 *Prove that the value $w_{i_n,j_n}(Q)$ of the analytic continuation of an element (U_{i_1}, w_{i_1,j_1}) of a multivalued function from $P \in U_{i_1}$ to $Q \in U_{i_n}$ along C does not change with continuous deformations of the curve C.*

In particular, any closed curve

$$C: \ \{z = z(t), \ 0 \le t \le 1, \ z(0) = z(t)\}$$

induces a bijection

$$\mu_C : J \to J \tag{1.2.2}$$

defined by the following condition: for any $j \in J$ the element $\left(U_{i_n}, w_{i_n,\mu_C(j)}\right)$ is the result of analytic continuation of the element $(U_{i_0}, w_{i_0,j})$ from the point $P = z_0$ to the same point $P = z_1$ along the closed curve C. The bijection (1.2.2) depends only on the homotopy class

$$[C] \in \pi_1(\Omega, P)$$

of the closed curve C. In this way one obtains a homomorphism

$$\mu : \pi_1(\Omega, P) \to Aut(J) \tag{1.2.3}$$

$$[C] \mapsto \mu_C. \tag{1.2.4}$$

Definition 1.2.5 *The homomorphism* (1.2.3) *is called* monodromy *of the multivalued analytic function.*

In particular, if the homomorphism μ is trivial,

$$\mu\left(\pi_1(\Omega, P)\right) = \mathrm{id} \in Aut(J)$$

then the multivalued function is actually a collection of J functions analytic in Ω.

An equivalent version of Definition 1.2.3 is

Definition 1.2.6 *A multivalued analytic function on $\Omega \subset \bar{\mathbb{C}}$ is an analytic function on some covering $\hat{\Omega}$ of the domain Ω*

Remark 1.2.7 *The covering $\hat{\Omega}$ has a natural structure of complex variety (of complex dimension one).*

Back to differential equations: let us consider a more general differential equation

$$F(z, w, w') = 0 \tag{1.2.5}$$

where

$$F(z, w, w') = \sum_{k,l} a_{k,l}(z) w^k (w')^l$$

is a polynomial in w, w' with coefficients $a_{kl}(z)$ analytic in z on a domain $D \in \mathbb{C}$.

Theorem 1.2.8 *If $w = w(z)$ is a solution to the differential equation* (1.2.5) *analytic on the disk $U \subset D$ and $(V, \tilde{w})$ with $V \subset D$ is an analytic continuation of the element (U, w) then also the function $\tilde{w}(z)$ satisfies* (1.2.5).

Proof: Indeed,

$$F(z, \tilde{w}(z), \tilde{w}(z)) \equiv 0 \quad \text{for} \quad z \in U \cap V.$$

Due to the "principle of analytic continuation" this function vanishes identically for all $z \in V$.
□

The domain of analyticity and behaviour of solutions to analytic differential equations are determined by their *singularities*. The study of positions of the singularities and their properties is one of the main tasks of the analytic theory of differential equations.

Recall that a point $z_0 \in \bar{\mathbb{C}}$ is an *isolated singularity* of a function $w(z)$ analytic in Ω if there exists a disk U such that

$$U \setminus z_0 \subset \Omega$$

and $z_0 \notin \Omega$.

Let us first subdivide isolated singularities into two big classes.

Definition 1.2.9 *We say that z_0 is a* critical singularity *of a multivalued analytic function $w(z)$ if the analytic continuation along a nontrivial closed loop in $U \setminus z_0$ changes the value of the function, and* non-critical *in the opposite case.*

For example, $z = 0$ is a critical singularity for functions $\sqrt{z}$, $\log z$, but it is a non-critical singularity for the functions $1/z$, $1/z^k$.

We will now consider isolated singularities of analytic differential equations.

Definition 1.2.10 *Let $w(z)$ be a solution to the differential equation*

$$w' = f(w, z)$$

with an isolated singularity at the point $z = z_0$. We call this singularity fixed *if its position does not depend on the choice of the solution $w(z)$; in the opposite case the singularity is called* movable.

Example. Consider the differential equation

$$w' = \frac{1}{2\,w\,z}.$$

The solution can be found explicitly:

$$w = \sqrt{\log \frac{z}{C}}$$

where $C \neq 0$ is an integration constant. It has fixed transcendent singularities at $z = 0$ and $z = \infty$. It also has movable critical singularity at $z = C$.

Observe that the fixed singularities $z = 0$ and $z = \infty$ can be determined looking at the singular points of the coefficients of the differential equation. Localization of movable singularities is often a hard task. The situation simplifies for linear differential equations.

Theorem 1.2.11 *Solutions of linear differential equations with rational coefficients have no movable singularities.*

Proof: Let $z_1, \ldots, z_N \in \bar{\mathbb{C}}$ be all the poles of the coefficients of the system. Any solution defined in a neighborhood of a point $z_0 \in \bar{\mathbb{C}} \setminus \{z_1, \ldots, z_N\}$ can be analytically continued along any curve on the punctured sphere. In this way we obtain a multivalued analytic function defined on all $\bar{\mathbb{C}} \setminus \{z_1, \ldots, z_N\}$. □

Corollary 1.2.12 *All movable singularities of Riccati equation*

$$w' = a(z)w^2 + b(z)w + c(z) \tag{1.2.6}$$

with arbitrary rational coefficients are poles.

Proof: Consider the following linear differential equation

$$y'' + p(z)y' + q(z)y = 0 \tag{1.2.7}$$

with

$$p(z) = -b(z) - \frac{a'(z)}{a(z)}, \quad q(z) = a(z)c(z).$$

The substitution

$$w(z) = -\frac{1}{a(z)} \frac{y'}{y} \tag{1.2.8}$$

into (1.2.7) yields (1.2.6). Since the solutions $y(z)$ to (1.2.7) have no singularities away from the poles of rational functions $p(z)$, $q(z)$ (these are *fixed singularities*), the movable singularities of $w(z)$ are poles (at the zeroes of $y(z)$). □

We want now to prove the following converse statement.

Theorem 1.2.13 *A differential equation of the form*

$$w' = f(w, z)$$

with rational rhs

$$f(w, z) = \frac{\alpha_0(z)w^n + \alpha_1(z)w^{n-1} + \cdots + \alpha_n(z)}{\beta_0(z)w^m + \beta_1(z)w^{m-1} + \cdots + \beta_m(z)} \equiv \frac{P_n(w, z)}{Q_m(w, z)} \tag{1.2.9}$$

has no movable critical singularities iff *it is a Riccati equation.*

Let us begin the proof from the following lemma (useful by itself) describing the local structure of solutions to analytic differential equations near the poles of the rhs.

Lemma 1.2.14 *Let the rhs $f(w,z)$ of the differential equation $w' = f(w,z)$ have a pole at the point (w_0, z_0), i.e., the function $1/f(w,z)$ is analytic near (w_0, z_0) and*

$$\lim_{\substack{w \to w_0 \\ z \to z_0}} \frac{1}{f(w,z)} = 0.$$

Then the differential equation has a solution in the form of a Puiseaux series

$$w = w_0 + a_1(z-z_0)^{1/k} + a_2(z-z_0)^{2/k} + \dots \tag{1.2.10}$$

for some integer $k > 1$ convergent for $|z - z_0| < r$ for some positive r.

Proof: Rewrite the differential equation for the inverse function

$$\frac{dz}{dw} = \frac{1}{f(w,z)}.$$

Applying Cauchy theorem one obtains the solution with the initial data

$$z(w_0) = z_0$$

in the form

$$z = z_0 + b_1(w-w_0) + b_2(w-w_0)^2 + \dots.$$

Observe that

$$b_1 = \left(\frac{1}{f(w,z)}\right)_{(w_0,z_0)} = 0.$$

Denote

$$k := \min\{i \,|\, b_i \neq 0\}.$$

Rewriting the equation

$$z - z_0 = b_k(w-w_0)^k + b_{k+1}(w-w_0)^{k+1} + \dots$$

in the form

$$z - z_0 = b_k(w-w_0)^k \left[1 + c_1(w-w_0) + c_2(w-w_0)^2 + \dots\right]^k$$

where

$$1 + c_1(w-w_0) + c_2(w-w_0)^2 + \dots = \left[1 + \frac{b_{k+1}}{b_k}(w-w_0) + \frac{b_{k+2}}{b_k}(w-w_0)^2 + \dots\right]^{1/k}$$

we apply the analytic version of the implicit function theorem to the equation

$$\zeta = \tilde{b}_1(w-w_0)\left[1 + c_1(w-w_0) + c_2(w-w_0)^2 + \dots\right], \quad \tilde{b}_1 := b_k^{1/k}.$$

One obtains an analytic function $w = w(\zeta)$ for sufficiently small $|\zeta|$:

$$w = w_0 + a_1\zeta + a_2\zeta^2 + \dots, \quad a_1 = \frac{1}{\tilde{b}_1}.$$

The substitution

$$\zeta = (z - z_0)^{1/k}$$

gives the needed solution $w(z)$. □

We see that poles of the rhs of differential equations correspond to algebraic critical singularities of solutions.

Example. The solutions to differential equation

$$w' = \frac{1}{(n+1)\, w^n}$$

have the form

$$w(z) = (z - z_0)^{\frac{1}{n+1}}.$$

For any $n > 0$ it has a movable algebraic critical singularity.

Let us return to the proof of Painlevé theorem. If the degree m of the denominator is positive then, for any $z_0 \in \mathbb{C}$ such that $\beta_0(z_0) \neq 0$ there exists a point (w_0, z_0) such that

$$Q_m(w_0, z_0) = 0.$$

Without loss of generality one can assume that

$$P_n(w_0, z_0) \neq = 0$$

(the polynomials $P_n(w, z)$ and $Q_m(w, z)$ by assumption have no common factors) and

$$\partial_w Q_m(w, z)|_{(w_0, z_)} \neq 0.$$

The last condition ensures local analytic dependence on z_0 of the root $w = w_0$ of the polynomial equation

$$Q_m(w, z_0) = 0.$$

So, according to Lemma 1.2.14 there exists a solution to the differential equation with a critical singularity at $z = z_0$. The possibility of small variations of z_0 proves that this singularity is movable.

We proved that $m = 0$, i.e., the differential equation must have the form

$$w' = \alpha_0(z) w^n + \alpha_1(z) w^{n-1} + \cdots + \alpha_n(z).$$

Put

$$\tilde{w} = \frac{1}{w}.$$

One obtains

$$\tilde{w}' = -\frac{\alpha_0(z) + \alpha_1(z)\tilde{w} + \cdots + \alpha_n(z)\tilde{w}^n}{\tilde{w}^{n-2}}.$$

If $n - 2 > 0$ then this equation has movable critical singularities, i.e., the original equation has movable critical poles. Hence $n = 2$ and we obtain a Riccati equation. □

1.3 Singularities for differential equations not resolved with respect to the derivative. Weierstrass elliptic function.

We consider only a particular class of such equations having the form

$$F(w, w') = 0 \tag{1.3.1}$$

for a polynomial $F(w, w')$. Let us begin with

Example. Consider one-dimensional motion of a point of mass 1 in the field of a polynomial potential $V(x)$. Recall that the equations of the motion have the form

$$\ddot{x} = -\frac{\partial V(x)}{\partial x}. \tag{1.3.2}$$

Conservation of the total energy reduces this second order ODE to the first one

$$\frac{1}{2}\dot{x}^2 + V(x) = E \tag{1.3.3}$$

where E is the integration constant.

Question: when solutions $x = x(t)$ to the differential equation (1.3.3) have no movable critical singularities in the complex t-plane?

Lemma 1.3.1 *The solutions to* (1.3.3) *have no movable critical singularities only if* $\deg V(x) \leq 4$.

Proof: The substitution

$$\tilde{x} = \frac{1}{x}, \quad \frac{dx}{dt} = -\frac{1}{\tilde{x}^2}\frac{d\tilde{x}}{dt}$$

gives

$$\frac{1}{2}\left(\frac{d\tilde{x}}{dt}\right)^2 = \tilde{x}^4\left[E - V\left(\tilde{x}^{-1}\right)\right] = 0. \tag{1.3.4}$$

If the degree of the polynomial $V(x)$ is greater than 4 then the rhs of (1.3.4) has a positive power of $\tilde{x}$ in the denominator.

One arrives at the following question: given a differential equation of the form

$$w'^2 = \frac{P(w)}{w^n}, \quad n \in \mathbb{Z}$$

with a polynomial

$$P(w) = a_0 + a_1 w + \dots,$$

prove existence of movable critical points under the assumption $n > 0$.

To do this we separate the variables

$$\frac{w^{\frac{n}{2}}dw}{\sqrt{P(w)}} = dz.$$

Let us consider the solution with the initial data $w(z_0) = 0$ with arbitrary z_0. It can be found in the form

$$\frac{2}{n+2}(a_0)^{-\frac{1}{2}} w^{\frac{n+2}{2}} \left[1 + c_1 w + c_2 w^2 + \dots\right] = z - z_0$$

where the (convergent) series in the square brackets is defined by the following formula

$$1 + c_1 w + c_2 w^2 + \dots = \left(a_0^{-1} P(w)\right)^{-1/2}.$$

Inverting one obtains the solution $w = w(z)$ with a movable algebraic critical singularity (if $n > 0$)

$$w = b_0(z - z_0)^{\frac{2}{n+2}} + b_1(z - z_0)^{\frac{3}{n+2}} + \dots$$

for some coefficients b_0, b_1, ... □

We will now prove that for $n \leq 4$ all movable singularities of solutions to the equation

$$w'^2 = a_0 w^n + a_1 w^{n-1} + \dots + a_n \tag{1.3.5}$$

are poles. The statement is trivial for $n = 0$, so we start from

Case 1: $n = 1$. General solution to

$$w'^2 = z - a$$

reads

$$w = a + \left(\frac{z - z_0}{2}\right)^2.$$

Case 2: $n = 2$. To solve the differential equation

$$w'^2 = (w - a)(w - b)$$

with $a \neq b$ let us consider the algebraic curve

$$p^2 = (w - a)(w - b).$$

The curve admits a rational parametrization

$$w = \frac{a - b\,s^2}{1 - s^2}$$

$$p = \frac{(a - b)\,s}{1 - s^2}.$$

We have

$$s = \frac{p}{w - a},$$

so

$$p = \frac{dw}{dz} = 2(a - b)\frac{s}{(1 - s^2)^2}\frac{ds}{dz},$$

i.e.

$$\frac{ds}{dz} = \frac{1}{2}(1 - s^2).$$

Integration of this equation gives

$$s = \frac{1 - e^{z-z_0}}{1 + e^{z-z_0}}.$$

Therefore the solutions $w = w(z)$ are represented in the form

$$w = R(e^{z-z_0})$$

for some rational function R.

Case 3. $n = 3$. Doing if necessary an affine change of dependent variable

$$w \mapsto aw + b$$

we reduce the differential equation (1.3.5) to the form

$$w'^2 = 4w^3 - g_2 w - g_3. \qquad (1.3.6)$$

Here g_2, g_3 are some complex number. We will construct a particular solution $w = \wp(z)$ to this differential equation called *Weierstrass elliptic function*. We prove that this solution is meromorphic on the entire complex plane. All other solutions are obtained by a shift of argument

$$w = \wp(z - z_0), \quad z_0 \in \mathbb{C}.$$

Let us fix a pair of complex numbers ω_1, ω_2 such that

$$\mathrm{Im}\, \frac{\omega_2}{\omega_1} > 0. \qquad (1.3.7)$$

Denote

$$\Lambda := \{z = 2m\omega_1 + 2n\omega_2 \,|\, m,\, n \in \mathbb{Z}\} \qquad (1.3.8)$$

the lattice on the plane generated by the vectors $2\omega_1$ and $2\omega_2$. *Elliptic functions* associated with the lattice Λ by definition are meromorphic functions on the complex torus

$$T_{\omega_1,\omega_2} := \mathbb{C}/\Lambda. \qquad (1.3.9)$$

Spelling this definition out we identify elliptic functions with doubly periodic meromorphic functions on the complex plane:

$$f(z + 2\omega_1) = f(z), \quad f(z + 2\omega_2) = f(z). \qquad (1.3.10)$$

Exercise 1.3.2 *Prove that any elliptic function* holomorphic *on $\mathbb{C}$ must be a constant.*

The Weierstrass elliptic function is defined by the following infinite sum

$$\wp(z; \omega_1, \omega_2) := \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus 0} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]. \qquad (1.3.11)$$

Theorem 1.3.3 *The series* (1.3.11) *converges absolutely and uniformly on any compact in $\mathbb{C} \setminus \Lambda$. The sum $\wp(z; \omega_1, \omega_2)$ is a doubly periodic analytic function in $z \in \mathbb{C} \setminus \Lambda$ with double poles at the lattice points.*

Proof: We first prove the following

Lemma 1.3.4 *The series*

$$\sum_{\omega\in\Lambda}{}' \frac{1}{|\omega|^3}$$

converges.

The notation

$$\sum_{\omega\in\Lambda}{}'$$

will be used for summation over all nonzero elements of the lattice Λ.

Proof: For any positive integer n let us evaluate the contribution of the lattice points belonging to the boundary of the parallelogram

$$\Pi_n := \{z = 2\omega_1 s_1 + 2\omega_2 s_2 \,|\, -n \leq s_1,\, s_2 \leq n\}.$$

Denote d the minimal distance from the origin to the boundary points of the parallelogram Π_1. Then the distance from the origin of any lattice point on the boundary $\partial\Pi_n$ is greater or equal to $d\,n$, that is

$$\frac{1}{|\omega|^3} \leq \frac{1}{d^3 n^3}, \quad \omega \in \Lambda \cap \partial\Pi_n.$$

We have $8n$ lattice points on $\partial\Pi_n$, hence

$$\sum_{\omega\in\Lambda\cap\partial\Pi_n}{}' \frac{1}{|\omega|^3} \leq \frac{8}{d^3 n^2}.$$

The estimate for the sum of the series readily follows

$$\sum_{\omega\in\Lambda}{}' \frac{1}{|\omega|^3} \leq \frac{8}{d^3}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{4\pi^2}{3d^3}.$$

□

We are now in a position to prove convergence of the series (1.3.11) on any disk $|z| \leq r$ away from the lattice points. For a given $r > 0$ there is only a finite number of lattice points inside the disk of radius $2r$. Considering only the points $\omega \in \Lambda$ such that $|\omega| \geq 2r$ the following inequalities hold true

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{2\omega\, z - z^2}{\omega^2(\omega - z)^2}\right|$$

$$= \frac{|z(2 - z/\omega)|}{|\omega|^3|1 - z/\omega|^2} \leq \frac{\frac{5}{2}\,r}{\frac{1}{4}|\omega|^3} \leq \frac{10\,r}{|\omega|^3} \quad \text{for} \quad |z| \leq r, \quad z \notin \Lambda.$$

So the convergence readily follows from the Lemma.

For a z inside the disk $|z| < r$, $z \notin \Lambda$ rewrite the series for the $\wp$-function as follows:

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{\omega\in\Lambda,\ 0<|\omega|<2r} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$$
$$+ \sum_{\omega\in\Lambda,\ |\omega|\geq 2r} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right].$$

The second series is a function holomorphic for $|z| < r$. The first (finite) sum, along with the $1/z^2$ term, is a rational function, hence it is meromorphic on the disk $|z| < r$. This prove the statement of the Theorem about the poles.

Let us consider now the derivative

$$\wp'(z;\omega_1,\omega_2) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}.$$

It is clear that this meromorphic function is doubly periodic: the shift $z \mapsto z + 2k\omega_1 + 2l\omega_2$ can be absorbed by a change of the summation indices

$$m \mapsto m + k, \quad n \mapsto n + l.$$

In order to prove double periodicity of the Weierstrass function let us first observe that the difference

$$\wp(z + 2k\omega_1 + 2l\omega_2) - \wp(z)$$

is a constant for any k, $l \in \mathbb{Z}$. Indeed, the derivative of this function is identically equal to 0. To compute the value of the constant $\wp(z + 2\omega_1) - \wp(z)$ it suffices to set $z = -\omega_1$ and then use that $\wp(z)$ is an even function, hence

$$\wp(\omega_1) = \wp(-\omega_1).$$

In a similar way one proves that $\wp(z + 2\omega_2) - \wp(z) = 0$. □

Let us now prove that the Weierstrass function satisfies the differential equation (1.3.6) with

$$g_2 = 60{\sum}' \frac{1}{\omega^4}, \quad g_3 = 140{\sum}' \frac{1}{\omega^6}. \tag{1.3.12}$$

Theorem 1.3.5 *The Weierstrass function $\wp(z;\omega_1,\omega_2)$ satisfies the differential equation*

$$\left(\wp'\right)^2 = 4\wp^3 - g_2\wp - g_3. \tag{1.3.13}$$

Proof: Using the geometric series

$$\frac{1}{\left(1 - \frac{z}{\omega}\right)^2} = 1 + \frac{2z}{\omega} + \frac{3z^2}{\omega^2} + \frac{4z^3}{\omega^3} + \frac{5z^4}{\omega^4} + \dots$$

we derive the Laurent expansion of the Weierstrass function at $z = 0$:

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \dots.$$

Hence

$$\left(\wp'(z)\right)^2 = \frac{4}{z^6} - \frac{2g_2}{5z^2} - \frac{4g_3}{7} + \dots$$

$$\wp^3(z) = \frac{1}{z^6} + \frac{3g_2}{20z^2} + \frac{3}{28}g_3 + \dots.$$

Therefore

$$\wp'^2 - 4\wp^3 = -\frac{g_2}{z^2} - g_3 + O(z^2).$$

This implies that the function

$$\wp'^2 - 4\wp^3 + g_2\wp + g_3 = O(z^2)$$

is holomorphic at $z = 0$. Due to double periodicity it is holomorphic everywhere. Thus it must be equal to a constant. Since the value of this functions at $z = 0$ is equal to zero, the constant is equal to 0. □

It turns out that, choosing in a suitable way the periods $\omega_{1,2}$ of the lattice one can obtain a solution to the Weierstrass differential equation (1.3.13) with arbitrary values of the parameter g_2, g_3 satisfying

$$g_2^3 - 27g_3^2 \neq 0. \tag{1.3.14}$$

Actually the generators of the lattice are defined as the periods of a holomorphic differential over a suitably chosen basis of cycles a_1, $a_2 \in H_1(C;\mathbb{Z})$ on the elliptic curve

$$C: \; p^2 = 4w^3 - g_2 w - g_3, \tag{1.3.15}$$

$$2\omega_1 = \oint_{a_1} \frac{dw}{p}, \quad 2\omega_2 = \oint_{a_2} \frac{dw}{p}.$$

The details of this construction go beyond the scope of this course.

For $g_2^3 - 27g_3^2 = 0$ the polynomial $4w^3 - g_2 - g_3$ has a multiple root. The curve (1.3.15) becomes rational:

$$p^2 = 4(w-a)^2(w-b)$$

for some a, b. The rational parametrization of the curve reads

$$\begin{aligned} w &= b + s^2 \\ p &= 2s(b - a + s^2). \end{aligned}$$

The corresponding differential equation $w'^2 = 4(w-a)^2(w-b)$ integrates in elementary functions.

Case 4. $n = 4$. In order to solve the differential equation

$$w'^2 = a_0 w^4 + 4a_1 w^3 + 6a_2 w^2 + 4a_3 w + a_4 \equiv P_4(w)$$

let us again consider the algebraic curve

$$p^2 = P_4(w). \tag{1.3.16}$$

The substitution

$$\begin{aligned} w &= \tfrac{1}{\tilde{w}} + \alpha \\ p &= -\tfrac{\tilde{p}}{\tilde{w}^2} \end{aligned}$$

yields

$$\tilde{p}^2 \tilde{w}^{-4} = b_4 \tilde{w}^{-4} + b_3 \tilde{w}^{-3} + b_2 \tilde{w}^{-2} + b_1 \tilde{w}^{-1} + P_4(\alpha)$$

with

$$b_k = \frac{1}{k!} P_4^{(k)}(\alpha), \quad k = 1, \dots, 4.$$

If α is a root of the polynomial $P_4(w)$ then one obtains a cubic

$$\tilde{p}^2 = b_4 + b_3 \tilde{w} + b_2 \tilde{w}^2 + b_1 \tilde{w}^3. \tag{1.3.17}$$

One more substitution

$$\tilde{p} = \lambda y, \quad \tilde{w} = \lambda\, x + \mu$$

with

$$\lambda = \frac{4}{b_1}, \quad \mu = -\frac{b_2}{3b_1}$$

reduces equation (1.3.17) to the Weierstrass normal form

$$y^2 = 4x^3 - g_2 x - g_3 \tag{1.3.18}$$

with

$$g_2 = \frac{1}{12}(b_2^2 - 3b_1 b_3) = a_0 a_4 + 3a_2^2 - 4a_1 a_3, \quad g_3 = \frac{1}{432}(9b_1 b_2 b_3 - 2b_2^3 - 27 b_1^2 b_4) = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

If the quartic polynomial $P_4(w)$ had no multiple roots so does the cubic polynomial in the rhs of (1.3.18). Therefore the curve (1.3.18) admits uniformization by Weierstrass elliptic functions:

$$\begin{aligned} \tilde{w} &= \lambda\, \wp(s) + \mu \\ \tilde{p} &= \lambda\, \wp'(s). \end{aligned} \tag{1.3.19}$$

As

$$dz = \frac{dw}{p} = \frac{d\tilde{w}}{\tilde{p}} = ds$$

we finally obtain the general solution in the form

$$w = \alpha + \frac{1}{\lambda\, \wp(z - z_0) + \mu}.$$

It is clearly a meromorphic function in z.

More general result is given by the following theorem of Hermite (1873).

Theorem 1.3.6 *If the polynomial differential equation*

$$F(w, w') = 0$$

has no movable critical points then the genus of the Riemann surface

$$F(w, p) = 0$$

is less or equal to 1. The solutions can be of the following three types: (i) rational functions in z; (ii) rational expressions in exponential functions; (iii) rational expressions in elliptic functions.

Sketch of the *proof*: On the curve $F(w, p) = 0$ consider the abelian differential

$$\omega = \frac{dw}{p} \ (= dz).$$

Since for any abelian differential

$$\#(\text{zeroes}) - \#(\text{poles}) = 2g - 2$$

the differential has a zero (w_0, p_0) for $g > 1$. Choosing a complex coordinate τ near such a point

$$w = w_0 + w_1\tau + w_2\tau^2 + \dots$$
$$p = p_0 + p_1\tau + p_2\tau^2 + \dots$$

one has

$$\omega = \left(a_k\tau^k + a_{k+1}\tau^{k+1} + \dots\right) d\tau$$

for some positive $k \in \mathbb{Z}$, $a_k \neq 0$. So

$$\frac{dz}{d\tau} = a_k\tau^k + a_{k+1}\tau^{k+1} + \dots$$

and

$$z - z_0 = \frac{a_k}{k+1}\tau^{k+1} + \frac{a_{k+1}}{k+2}\tau^{k+2} + \dots$$

or

$$\tau = \left(\frac{(k+1)(z - z_0)}{a_k}\right)^{\frac{1}{k+1}} + \mathcal{O}\left((z - z_0)^{\frac{2}{k+1}}\right).$$

We obtain a movable critical singularity:

$$w = w_0 + w_1\left(\frac{(k+1)(z - z_0)}{a_k}\right)^{\frac{1}{k+1}} + \mathcal{O}\left((z - z_0)^{\frac{2}{k+1}}\right).$$

□

As an application of the Hermite's theorem consider the problem of finding meromorphic functions possessing algebraic addition theorem. We say that the function $f(z)$ satisfies an algebraic addition theorem if there exists an algebraic function $P(u, v, w)$ such that for all x, $y \in \mathbb{C}$ the following identity holds true

$$P(f(x), f(y), f(x + y)) = 0. \tag{1.3.20}$$

Corollary 1.3.7 *Meromorphic functions $f(z)$ possessing algebraic addition theorem are rational functions in z, or in $e^{a\,z}$, or elliptic functions.*

Proof: Differentiating (1.3.20) in x and y obtain

$$P_u f'(x) + P_w f'(x+y) = 0$$
$$P_v f'(y) + P_w f'(x+y) = 0.$$

Hence

$$P_u(f(x), f(y), f(x+y))\, f'(x) - P_v(f(x), f(y), f(x+y))\, f'(y) = 0. \qquad (1.3.21)$$

Eliminating $f(x+y)$ from the equations (1.3.20), (1.3.21) we obtain an equation of the form

$$\Phi(f(x), f'(x), f(y), f'(y)) = 0.$$

Fixing y one arrives at a differential equation of above form by assumption having meromorphic solutions. Due to Hermite theorem the solutions must be rational, trigonometric or elliptic. □

Example (L.Fuchs). Consider the differential equation

$$w'^3 - 3w'^2 - 9w^4 - 12w^2 = 0. \qquad (1.3.22)$$

The associated algebraic curve reads

$$F(p, w) \equiv p^3 - 3p^2 - 9w^4 - 12w^2 = 0. \qquad (1.3.23)$$

This irreducible equation is cubic in p. So the associated Riemann surface has 3 sheets over the complex w-plane.

Step 1: look for the ramification points for the projection $(p, w) \mapsto w$ of the Riemann surface (2.1.19). To this end we have to first solve the system

$$\left.\begin{aligned} F(p,w) &= 0 \\ F_p(p,w) &= 0. \end{aligned}\right\} \qquad (1.3.24)$$

The second equations gives

$$3p^2 - 6p = 0, \quad \text{hence} \quad p = 0 \quad \text{or} \quad p = 2.$$

One obtains

$$\begin{aligned} p = p_1 = 0 \quad &\Rightarrow \quad w = w_1 = 0 \quad \text{or} \quad w = w_{2,3} = \pm\frac{2\,i}{\sqrt{3}} \\ p = p_2 = 2 \quad &\Rightarrow \quad w = w_{4,5} = \pm i\sqrt{\frac{2}{3}}. \end{aligned} \qquad (1.3.25)$$

Let us study the structure of the Riemann surface near the points (p_i, w_j).

1) Near $(p_1, w_1) = (0, 0)$ one has, at the leading order

$$p \simeq \pm 2\, i\, w.$$

Higher order terms are determined uniquely for both branches:

$$p_\pm(w) = \pm 2\, i\, w - \frac{2}{3}\, w^2 + \mathcal{O}(w^3). \tag{1.3.26}$$

One obtains two analytic functions $p = p_\pm(w)$. Hence (p_1, w_1) is a double point of the algebraic curve (2.1.19) (i.e., not a branch point). It is obtained by identification of the two points (1.3.26) of the Riemann surface.

2) Near (p_1, w_2) or (p_1, w_3) one can use the coordinate p as the local parameter:

$$w = \pm \left[\frac{2\, i}{\sqrt{3}} - \frac{i\,\sqrt{3}}{16}\, p^2 + \frac{i}{16\sqrt{3}}\, p^3 \right] + \mathcal{O}(p^4) \quad \text{near} \quad (p, w) = \left(0, \pm\frac{2\, i}{\sqrt{3}}\right). \tag{1.3.27}$$

So, the points (p_1, w_2) and (p_1, w_3) are both second order branch points on the Riemann surface.

3) Near each of the points (p_2, w_4) or (p_2, w_5) both branches of the Riemann surface (2.1.19) are holomorphic:

$$w = i\sqrt{\frac{2}{3}} \pm \frac{i}{2\sqrt{2}}\,(p-2) - i\,\frac{3\sqrt{3}\pm 4}{48\sqrt{2}}(p-2)^2 + \mathcal{O}\left((p-2)^3\right) \tag{1.3.28}$$

or

$$w = -i\sqrt{\frac{2}{3}} \pm \frac{i}{2\sqrt{2}}\,(p-2) + i\,\frac{3\sqrt{3}\pm 4}{48\sqrt{2}}(p-2)^2 + \mathcal{O}\left((p-2)^3\right). \tag{1.3.29}$$

4) At infinity one has to do the substitution

$$w = \frac{1}{u}, \quad p = \frac{1}{q}$$

to arrive at

$$9q^3 - u^4 + 12u^2 q^3 + 3qu^4 = 0. \tag{1.3.30}$$

Near $u = 0$ the curve can be described by a Puiseaux series

$$q = 3^{-2/3}\varepsilon\, u^{4/3} - \frac{\varepsilon^2}{3\cdot 3^{1/3}} u^{8/3} + \mathcal{O}\left(u^{11/3}\right). \tag{1.3.31}$$

Thus at infinity the curve (2.1.19) has a third order branch point.

We obtain on the Riemann surface 3 branch points of orders 2, 2 and 3. The total multiplicity of the branching locus by definition is equal to

$$b = (2-1) + (2-1) + (3-1) = 4.$$

The Riemann - Hurwitz formula for the genus g of a n-sheeted covering of the Riemann sphere with the total multiplicity b of the branched locus gives

$$g = \frac{b}{2} - n + 1 = 0.$$

A simpler way to establish rationality of the curve is to represent it as a quotient of a reducible rational curve

$$\left[(s-1)^2(s+2)+3w^2\right]\left[(s+1)^2(s-2)-3w^2\right]=0 \tag{1.3.32}$$

over the involution

$$(s,w)\mapsto(-s,w).$$

The projection of (1.3.32) to (2.1.19) reads

$$(s,w)\mapsto(p=-1+s^2,w).$$

Further substitution

$$s=2+3t^2$$

gives a rational parametrization of the second component

$$(s+1)^2(s-2)-3w^2=0$$

of the curve (1.3.32)

$$\left.\begin{aligned} p&=3+12t^2+9t^4\\ w&=3t(1+t^2).\end{aligned}\right\} \tag{1.3.33}$$

Now we can integrate the equation:

$$dz=\frac{dw}{p}=\frac{dt}{1+t^2},\qquad t=\tan(z-z_0)$$

that gives the solution in the form

$$w=3\tan^3(z-z_0)+3\tan(z-z_0)=\frac{3\sin(z-z_0)}{\cos^3(z-z_0)}. \tag{1.3.34}$$

Other components of the curve (1.3.32) give nothing new.

Exercise 1.3.8 *Consider a differential equation of the form*

$$F(w',w,z)=0$$

where $F(p,w,z)$ is a polynomial in 3 variables p, w, z. Assume that for any fixed g the genus of the algebraic curve $F(p,w,z)=0$ is equal to zero. Moreover, assume that the solutions to the differential equations have no movable critical singularities. Prove that the solutions to the differential equation expresses rationally through solutions to Riccati equation

$$\frac{du}{dz}=a(z)u^2+b(z)u+c(z)$$

with rational coefficients $a(z)$, $b(z)$, $c(z)$.

At the end of this section we consider an example of application of the above approach to a multi-dimensional mechanical problem.

The Kowalevskaya problem: consider the system of equations of motion of a rigid body with a fixed point

$$\left.\begin{aligned} A_1\,\dot\omega_1 + (A_3 - A_2)\,\omega_2\,\omega_3 &= m\,g\,(c_2\gamma_3 - c_3\gamma_2)\\ A_2\,\dot\omega_2 + (A_1 - A_3)\,\omega_3\,\omega_1 &= m\,g\,(c_3\gamma_1 - c_1\gamma_3)\\ A_3\,\dot\omega_3 + (A_2 - A_1)\,\omega_1\,\omega_2 &= m\,g\,(c_1\gamma_2 - c_2\gamma_1) \end{aligned}\right\} \tag{1.3.35}$$

$$\left.\begin{aligned} \dot\gamma_1 &= \omega_3\,\gamma_2 - \omega_2\,\gamma_3\\ \dot\gamma_2 &= \omega_1\,\gamma_3 - \omega_3\,\gamma_1\\ \dot\gamma_3 &= \omega_2\,\gamma_1 - \omega_1\,\gamma_2 \end{aligned}\right\} \tag{1.3.36}$$

Here $\omega = (\omega_1, \omega_2, \omega_3)$ is the vector of angular velocity of the rigid body, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ the coordinates of the unit vector along the Oz axis; the coordinates of both vectors are given with respect to the co-moving frame having the origin at the fixed point of the rigid body. The positive numbers A_1, A_2, A_3 are the eigenvalues of the inertia tensor, $c = (c_1, c_2, c_3)$ the (constant) coordinates of the baricenter, m is the mass of the body, g is a constant (the gravitational acceleration). What are the values of the parameters A_1, A_2, A_3, c_1, c_2, c_3 for which the solutions $(\omega(t), \gamma(t))$ have no movable critical points in $t \in \mathbb{C}$? The following remarkable theorem was proved by S.Kowalevskaya (1889):

Theorem 1.3.9 *All singularities of solutions to the equations* (1.3.35), (1.3.36) *in the complex t-plane are poles if and only if*

(i) $c_1 = c_2 = c_3 = 0$ *(the case of Euler - Poinsot).*

(ii) $A_1 = A_2$ *and* $c_1 = c_2 = 0$ *(axial symmetry; the Lagrange - Poisson case).*

(iii) $A_1 = A_2 = A_3$ *(spherical symmetry).*

(iv) $A_1 = A_2 = 2A_3$, $c_3 = 0$ *(the Kowalevskaya case).*

In all four cases the dynamical equations (1.3.35), (1.3.36) reduce to a completely integrable Hamiltonian system. That means that, along with the Hamiltonian (the total energy)

$$\begin{aligned} H &= \frac{1}{2}\left(A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2\right) - m\,g\,(c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3)\\ &= \frac{1}{2}\left(\frac{M_1^2}{A_1} + \frac{M_2^2}{A_2} + \frac{M_3^2}{A_3}\right) - m\,g\,(c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3) \end{aligned} \tag{1.3.37}$$

(here we introduce the vector of angular momentum

$$M = (M_1, M_2, M_3), \quad M_i = A_i\omega_i, \quad i = 1,\, 2,\, 3) \tag{1.3.38}$$

and two geometric integrals

$$\begin{aligned}(M, \gamma) &= M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3 \\ (\gamma, \gamma) &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2\end{aligned} \tag{1.3.39}$$

there is an additional independent first integral. Namely, in the Euler - Poinsot case the additional integral is

$$I = (M, M) = M_1^2 + M_2^2 + M_3^2. \tag{1.3.40}$$

In the case of axial symmetry the additional integral is

$$I = M_3. \tag{1.3.41}$$

In the case of spherical symmetry the additional integral is

$$(M, c) = c_1 M_1 + c_2 M_2 + c_3 M_3. \tag{1.3.42}$$

The most nontrivial is the Kowalevskaya case where the additional first integral is quartic:

$$I = \left| A_3(\omega_1 + i\,\omega_2)^2 + m\,g\,(c_1 + i\,c_2)(\gamma_1 + i\,\gamma_2)\right|^2. \tag{1.3.43}$$

Exercise 1.3.10 *On the six-dimensional phase space with the coordinates* $(M_1, M_2, M_3, \gamma_1, \gamma_2, \gamma_3)$ *introduce the following Poisson bracket:*

$$\begin{aligned}\{M_i, M_j\} &= \epsilon_{ijk} M_k \\ \{M_i, \gamma_j\} &= \epsilon_{ijk} \gamma_k \\ \{\gamma_i, \gamma_j\} &= 0\end{aligned} \tag{1.3.44}$$

Here ϵ_{ijk} *is the signature of the permutation* $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$*, summation over repeated index* k *is assumed.*

(i) Prove that the bracket (1.3.44) *satisfies Jacobi identity.*

Hint*: check first that linear functions on the phase space form a Lie algebra with respect to the bracket* (1.3.44)*, namely,*

$$\begin{aligned}&\{(a_1, M) + (\alpha_1, \gamma), (a_2, M) + (\alpha_2, \gamma)\} = (a_3, M) + (\alpha_3, \gamma) \\ &a_3 = a_1 \times a_2, \quad \alpha_3 = a_1 \times \alpha_2 - a_2 \times \alpha_1.\end{aligned} \tag{1.3.45}$$

Here the coefficients a_i*,* α_i *of linear functions are considered as vectors in* $\mathbb{R}^3$*. The Lie algebra* (1.3.45) *is isomorphic to the semidirect product* $so_3 \ltimes \mathbb{R}^3$*.*

(ii) Check that the functions (M, γ) *and* (γ, γ) *are Casimirs of the Poisson bracket, i.e., that they commute with any other function on the phase space.*

(iii) Prove that the equations of rotations of rigid body (1.3.35) *is a Hamiltonian system with the Hamiltonian* (1.3.37)*:*

$$\dot{M}_i = \{H, M_i\}, \quad \dot{\gamma}_i = \{H, \gamma_i\}, \quad i = 1,\, 2,\, 3.$$

(iv) Check involutivity $\{H, I\} = 0$ *in all four cases* (1.3.40) - (1.3.43)*.*

According to the statements of this Exercise the level surface

$$(M,\gamma) = k, \quad (\gamma,\gamma) = 1 \tag{1.3.46}$$

for an arbitrary constant k has a natural structure of symplectic manifolds. The restriction of the flow (1.3.35) to this level surface is a Hamiltonian system with two degrees of freedom. Thus it suffices to have one additional first integral to ensure complete integrability.

In the classical cases (i) - (iii) the equations of motion (1.3.35) can be integrated in elliptic functions. Integration of the case (iv) performed by Kowalevskaya unraveled an important geometric structure behind this integrable case of equations of motion. Namely, consider the common level surface of the four first integrals

$$\left.\begin{aligned} (M,\gamma) &= k \\ (\gamma,\gamma) &= 1 \\ \tfrac{1}{2}\left(\tfrac{M_1^2}{2A_3} + \tfrac{M_2^2}{2A_3} + \tfrac{M_3^2}{A_3}\right) - (c_1\gamma_1 + c_2\gamma_2) &= E \\ \left|A_3(\omega_1 + i\,\omega_2)^2 + m\,g\,(c_1 + i\,c_2)(\gamma_1 + i\,\gamma_2)\right|^2 &= K \end{aligned}\right\} \tag{1.3.47}$$

for generic values of the constants k, E, K. The dynamics of (1.3.35) takes place on this level surface. It turns out that a suitable compactification of this two-dimensional complex algebraic variety has a structure of an abelian variety, i.e., it is isomorphic to a 4-dimensional torus

$$T^4 = \mathbb{C}^2/(\text{integer lattice of rang 4}).$$

The dynamics becomes linear in the natural coordinates on this torus. The solutions to the Kowalevskaya problem can be expressed via theta-functions of two variables. The final form of these expressions was obtained by F.Kötter in 1893.

The quite surprising connection of the property of absence of moving critical points with integrability of the system observed by Kowalevskaya was the starting point for the so-called *Painlevé test* in the modern theory of integrable systems.

1.4 Poincaré method of small parameter

The method of small parameter, proposed by Poincaré (1892) in his studies on celestial mechanics, is also an important tool in the study of properties of solutions of analytic differential equations. The method is applicable to a system of differential equations depending on a parameter ϵ,

$$\frac{d\mathbf{w}}{dz} = \mathbf{f}(z,\mathbf{w},\epsilon) \tag{1.4.1}$$

where $\mathbf{w} = (w_1, \dots, w_n)$ is the vector of depending variables, the vector function $\mathbf{f}$ of the right hand sides is assumed to be analytic in z, w_1, $\dots$, w_n, ϵ for sufficiently small $|\epsilon|$. The system (1.4.1) is considered as a *perturbation* of the so-called *unperturbed* system

$$\frac{d\mathbf{w}}{dz} = \mathbf{f}(z,\mathbf{w},0). \tag{1.4.2}$$

Suppose we know how to solve the unperturbed system. What can be said about solutions of the perturbed one?

Comparing with Cauchy theorem the novelty of the Poincaré method is that, for systems depending analytically on a parameter in certain cases one can establish *global* existence of a solution.

For the sake of simplicity we will formulate it for the case of a system of two first order differential equations, so we redenote $\mathbf{w} = (u, v)$, $\mathbf{f} = (f, g)$.

Theorem 1.4.1 *Consider a system of two differential equations of the form*

$$\left.\begin{array}{l} \frac{du}{dz} = f(z, u, v, \epsilon) \\ \\ \frac{dv}{dz} = g(z, u, v, \epsilon) \end{array}\right\} \tag{1.4.3}$$

with the right hand sides $f(z, u, v, \epsilon)$, $g(z, u, v, \epsilon)$ *analytic in* (z, u, v, ϵ) *for sufficiently small* ϵ. *Assume that for* $\epsilon = 0$ *the general solution*

$$\left.\begin{array}{l} u = \phi(z, C_1, C_2) \\ \\ v = \psi(z, C_1, C_2) \end{array}\right\}, \quad \det \begin{pmatrix} \frac{\partial \phi}{\partial C_1} & \frac{\partial \phi}{\partial C_2} \\ \\ \frac{\partial \psi}{\partial C_1} & \frac{\partial \psi}{\partial C_2} \end{pmatrix} \neq 0 \tag{1.4.4}$$

to the system (1.4.3) *is given. Moreover, assume that the solution* (1.4.4) *is analytic on a neighborhood of a curve*

$$\mathcal{C} : z = z(t), \quad t \in [0, 1], \quad z(0) = z_0, \quad z(1) = z_1 \tag{1.4.5}$$

on the complex plane. Then there exists a unique solution

$$\left.\begin{array}{l} u = \phi(z, C_1, C_2, \epsilon) \\ \\ v = \psi(z, C_1, C_2, \epsilon) \end{array}\right\} \tag{1.4.6}$$

to the full system (1.4.3) *analytic for sufficiently small* ϵ *and also analytic in* z *on a neighborhood of the curve* $\mathcal{C}$.

Proof: Step 1. Construct the solution as a formal series in ϵ

$$\begin{aligned} u &= \phi + \epsilon\, \phi_1 + \epsilon^2 \phi_2 + \dots \\ v &= \psi + \epsilon\, \psi_1 + \epsilon^2 \psi_2 + \dots . \end{aligned} \tag{1.4.7}$$

We will prove that the coefficients are analytic in z near the curve $\mathcal{C}$. They will be determined uniquely by the normalization

$$\phi_i(z_0) = 0, \quad \psi_i(z_0) = 0, \quad i \geq 1. \tag{1.4.8}$$

Let us adopt the following short notations: denote f, g, $\partial f/\partial u$, $\partial f/\partial v$, $\partial f/\partial \epsilon$ etc. for the values of the functions $f(z, u, v, \epsilon)$, $g(z, u, v, \epsilon)$ and their derivatives at

$$(z, u, v, \epsilon) = (z, \phi, \psi, 0).$$

The construction: for the corrections (ϕ_i, ψ_i) one derives from (1.4.3) a recurrent system of inhomogeneous linear differential equations of the form

$$\left.\begin{aligned} \frac{d\phi_k}{dz} &= \frac{\partial f}{\partial u}\,\phi_k + \frac{\partial f}{\partial v}\,\psi_k + F_k \\ \frac{d\psi_k}{dz} &= \frac{\partial g}{\partial u}\,\phi_k + \frac{\partial g}{\partial v}\,\psi_k + G_k \end{aligned}\right\} \tag{1.4.9}$$

where

$$\begin{aligned} &F_k = F_k\left(\frac{\partial^m f}{\partial u^i \partial v^j \partial \epsilon^l}, \phi_a, \psi_b\right), \quad G_k = G_k\left(\frac{\partial^m g}{\partial u^i \partial v^j \partial \epsilon^l}, \phi_a, \psi_b\right) \\ &i + j + l = m \le k, \qquad\qquad a,\, b \le k - 1. \end{aligned} \tag{1.4.10}$$

E.g., for (ϕ_1, ψ_1) we obtain the following system

$$\left.\begin{aligned} \frac{d\phi_1}{dz} &= \frac{\partial f}{\partial u}\,\phi_1 + \frac{\partial f}{\partial v}\,\psi_1 + \frac{\partial f}{\partial \epsilon} \\ \frac{d\psi_1}{dz} &= \frac{\partial g}{\partial u}\,\phi_1 + \frac{\partial g}{\partial v}\,\psi_1 + \frac{\partial g}{\partial \epsilon}, \end{aligned}\right\}$$

so $F_1 = \partial f/\partial\epsilon$, $G_1 = \partial g/\partial\epsilon$. For the second correction one has

$$F_2 = \frac{1}{2}\left[\frac{\partial^2 f}{\partial \epsilon^2} + \frac{\partial^2 f}{\partial u^2}\phi_1^2 + 2\frac{\partial f}{\partial u\,\partial v}\phi_1\psi_1 + \frac{\partial^2 f}{\partial v^2}\psi_1^2\right] + \frac{\partial^2 f}{\partial u\,\partial \epsilon}\phi_1 + \frac{\partial^2 f}{\partial v\,\partial \epsilon}\psi_1$$

$$G_2 = \frac{1}{2}\left[\frac{\partial^2 g}{\partial \epsilon^2} + \frac{\partial^2 g}{\partial u^2}\phi_1^2 + 2\frac{\partial g}{\partial u\,\partial v}\phi_1\psi_1 + \frac{\partial^2 g}{\partial v^2}\psi_1^2\right] + \frac{\partial^2 g}{\partial u\,\partial \epsilon}\phi_1 + \frac{\partial^2 g}{\partial v\,\partial \epsilon}\psi_1$$

etc. Observe that the vector functions

$$\begin{pmatrix} \delta\phi \\ \delta\psi \end{pmatrix}_1 = \begin{pmatrix} \frac{\partial\phi}{\partial C_1} \\ \frac{\partial\psi}{\partial C_1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \delta\phi \\ \delta\psi \end{pmatrix}_2 = \begin{pmatrix} \frac{\partial\phi}{\partial C_2} \\ \frac{\partial\psi}{\partial C_2} \end{pmatrix} \tag{1.4.11}$$

give a fundamental system of solutions to the associated *homogeneous* linear system

$$\left.\begin{aligned} \frac{d}{dz}\delta\phi &= \frac{\partial f}{\partial u}\,\delta\phi + \frac{\partial f}{\partial v}\,\delta\psi \\ \frac{d}{dz}\delta\psi &= \frac{\partial g}{\partial u}\,\delta\phi + \frac{\partial g}{\partial v}\,\delta\psi. \end{aligned}\right\} \tag{1.4.12}$$

due to the assumption (1.4.4). Therefore one can solve the system (1.4.9) iteratively by quadratures using the "variation of constants" method:

$$\begin{pmatrix} \phi_k(z) \\ \psi_k(z) \end{pmatrix} = \int_{z_0}^{z} dz' \begin{pmatrix} \frac{\partial\phi(z)}{\partial C_1} & \frac{\partial\phi(z)}{\partial C_2} \\ \frac{\partial\psi(z)}{\partial C_1} & \frac{\partial\psi(z)}{\partial C_2} \end{pmatrix} \begin{pmatrix} \frac{\partial\phi(z')}{\partial C_1} & \frac{\partial\phi(z')}{\partial C_2} \\ \frac{\partial\psi(z')}{\partial C_1} & \frac{\partial\psi(z')}{\partial C_2} \end{pmatrix}^{-1} \begin{pmatrix} F_k(z') \\ G_k(z') \end{pmatrix}. \tag{1.4.13}$$

The solutions are chosen in such a way that

$$\phi_k(z_0) = \psi_k(z_0) = 0, \quad \forall\, k \ge 1. \tag{1.4.14}$$

By construction every term of the series is analytic on some neighborhood of the curve $\mathcal{C}$.

Step 2: proof of convergence. Denote

$$
\begin{aligned}
F(z, \delta\phi, \delta\psi, \epsilon) &:= f(z, \phi + \delta\phi, \psi + \delta\psi, \epsilon) - f(z, \phi, \psi, 0) \\
G(z, \delta\phi, \delta\psi, \epsilon) &:= g(z, \phi + \delta\phi, \psi + \delta\psi, \epsilon) - g(z, \phi, \psi, 0).
\end{aligned}
\tag{1.4.15}
$$

By definition we have

$$
\left.
\begin{aligned}
\tfrac{d}{dz}\delta\phi &= F(z, \delta\phi, \delta\psi, \epsilon) \\
\tfrac{d}{dz}\delta\psi &= G(z, \delta\phi, \delta\psi, \epsilon).
\end{aligned}
\right\}
\tag{1.4.16}
$$

Let

$$
F = \sum a_{klm}(z)(\delta\phi)^k(\delta\psi)^l\epsilon^m, \quad a_{000}(z) \equiv 0
$$

$$
G = \sum b_{klm}(z)(\delta\phi)^k(\delta\psi)^l\epsilon^m, \quad b_{000}(z) \equiv 0
$$

be the Taylor expansions of the analytic functions $F(z, \delta\phi, \delta\psi, \epsilon)$, $G(z, \delta\phi, \delta\psi, \epsilon)$. Choose a small positive number ρ in such a way that

$$
\left.
\begin{aligned}
|F(z, \delta\phi, \delta\psi, \epsilon)| &< M \\
|G(z, \delta\phi, \delta\psi, \epsilon)| &< M
\end{aligned}
\right\}
\quad
\begin{aligned}
&\text{for} \quad z \in \mathcal{C} \\
&|\delta\phi| < \rho \\
&|\delta\psi| < \rho \\
&|\epsilon| < \rho.
\end{aligned}
$$

From the Cauchy inequalities

$$
|a_{klm}(z)| < \frac{M}{\rho^{k+l+m}}
$$

$$
|b_{klm}(z)| < \frac{M}{\rho^{k+l+m}}
$$

one readily derives the following majorants for the functions F and G:

$$
F(z, \delta\phi, \delta\psi, \epsilon) \preceq \sum \frac{M}{\rho^{k+l+m}}(\delta\phi)^k(\delta\psi)^l\epsilon^m - M
$$

$$
= \frac{M}{\rho}\frac{\delta\phi + \delta\psi + \epsilon}{1 - \frac{\delta\phi+\delta\psi+\epsilon}{\rho}} \preceq \frac{M}{\rho}\frac{(\delta\phi + \delta\psi + \epsilon)\left(1 + \frac{\delta\phi+\delta\psi+\epsilon}{\rho}\right)}{1 - \frac{\delta\phi+\delta\psi+\epsilon}{\rho}},
$$

same for G.

Consider an auxiliary system

$$
\begin{aligned}
\frac{d}{dz}\overline{\delta\phi} &= \frac{M}{\rho}\frac{(\overline{\delta\phi} + \overline{\delta\psi} + \epsilon)\left(1 + \frac{\overline{\delta\phi}+\overline{\delta\psi}+\epsilon}{\rho}\right)}{1 - \frac{\overline{\delta\phi}+\overline{\delta\psi}+\epsilon}{\rho}} \\
\frac{d}{dz}\overline{\delta\psi} &= \frac{M}{\rho}\frac{(\overline{\delta\phi} + \overline{\delta\psi} + \epsilon)\left(1 + \frac{\overline{\delta\phi}+\overline{\delta\psi}+\epsilon}{\rho}\right)}{1 - \frac{\overline{\delta\phi}+\overline{\delta\psi}+\epsilon}{\rho}}.
\end{aligned}
\tag{1.4.17}
$$

Let us find the solution to this system with the initial data

$$\overline{\delta\phi}(z_0) = \overline{\delta\psi}(z_0) = 0 \quad \text{identically in} \quad \epsilon.$$

Due to the symmetry of the system and of the initial condition with respect to the permutation

$$\overline{\delta\phi} \leftrightarrow \overline{\delta\psi}$$

for the solution the identity

$$\overline{\delta\phi}(z) \equiv \overline{\delta\psi}(z)$$

holds true. Denote

$$\Delta := \frac{\overline{\delta\phi} + \overline{\delta\psi} + \epsilon}{\rho} = \frac{2\,\overline{\delta\phi} + \epsilon}{\rho}.$$

For the function $\Delta(z)$ one obtains the following Cauchy problem

$$\frac{d\,\Delta}{dz} = 2\,\frac{M}{\rho}\,\Delta\,\frac{1+\Delta}{1-\Delta}, \quad \Delta(z_0) = \frac{\epsilon}{\rho}. \tag{1.4.18}$$

Separating the variables

$$2\,\frac{M}{\rho}\,dz = \frac{(1-\Delta)\,d\Delta}{\Delta(1+\Delta)} = \left(\frac{1}{\Delta} - \frac{2}{1+\Delta}\right) d\Delta$$

and integrating one obtains the solution in the form

$$\frac{\Delta}{(1+\Delta)^2} = \frac{\epsilon\,\rho}{(\epsilon+\rho)^2} e^{\frac{2M}{\rho}(z-z_0)}.$$

Denote a the right hand side of the equation. For $z = z_0$

$$a(z_0) = \frac{\epsilon\,\rho}{(\epsilon+\rho)^2}.$$

Solving the quadratic equation

$$\Delta = -1 + \frac{1}{2a}\left[1 \pm \sqrt{1-4a}\right]$$

yields

$$\Delta(z_0) = -1 + \frac{(\epsilon+\rho)^2}{2\epsilon\,\rho}\left[1 \pm \frac{\rho-\epsilon}{\rho+\epsilon}\right] = \begin{cases} \frac{\rho}{\epsilon}, & \text{for} \quad + \quad \text{sign} \\ \frac{\epsilon}{\rho}, & \text{for} \quad - \quad \text{sign.} \end{cases}$$

Therefore one must choose the "-" sign. Finally one obtains the solution in the form

$$\Delta(z) = -1 + \frac{(\epsilon+\rho)^2}{2\epsilon\,\rho} e^{-\frac{2M}{\rho}(z-z_0)} \left[1 - \sqrt{1 - \frac{4\,\epsilon\,\rho}{(\epsilon+\rho)^2} e^{\frac{2M}{\rho}(z-z_0)}}\right]$$

$$= \frac{\epsilon\,\rho}{(\epsilon+\rho)^2} e^{\frac{2M}{\rho}(z-z_0)} + 2\frac{\epsilon^2\rho^2}{(\epsilon+\rho)^4} e^{\frac{4M}{\rho}(z-z_0)} + 5\frac{\epsilon^3\rho^3}{(\epsilon+\rho)^6} e^{\frac{6M}{\rho}(z-z_0)} + \dots.$$

This function is analytic for

$$\left|\frac{4\,\epsilon\,\rho}{(\epsilon+\rho)^2} e^{\frac{2M}{\rho}(z-z_0)}\right| < 1,$$

i.e., for sufficiently small $|\epsilon|$ for any z near the compact $\mathcal{C}$. Like in Section 1.1 from analyticity of solutions to the majorant system (1.4.17) one deduces convergence of the "perturbation series" solutions (1.4.7), (1.4.8) to the initial system. □

1.5 Method of small parameter applied to Painlevé classification problem

Painlevé addressed the problem of classification of the second order ordinary differential equations

$$w'' = R(w', w, z) \tag{1.5.1}$$

having no movable critical singularities. His application of the Poincaré method to this problem is based on the following simple corollary from the Theorem 1.4.1.

Theorem 1.5.1 *Under the assumptions of Theorem 1.4.1 consider the case of the closed curve $\mathcal{C}$, $z_1 = z_0$. The solution $(u(z,\epsilon), v(z,\epsilon))$ is single-valued along $\mathcal{C}$ iff the coefficients $\phi_k(z)$, $\psi_k(z)$ of the perturbative expansion (1.4.7) are single-valued for all $k \geq 0$.*

Proof: Let the functions $\phi_k(z)$, $\psi_k(z)$ satisfy

$$\phi_k(z_1) = \phi_k(z_0), \quad \psi_k(z_1) = \psi_k(z_0) \quad \text{for} \quad k < K,$$

$$|\phi_K(z_1) - \phi_K(z_0)|^2 + |\psi_K(z_1) - \psi_K(z_0)|^2 \neq 0.$$

Then

$$\lim_{\epsilon \to 0} \epsilon^{-2K} \left[|\phi(z_1,\epsilon) - \phi(z_0,\epsilon)|^2 + |\psi(z_1,\epsilon) - \psi(z_0,\epsilon)|^2 \right] \neq 0.$$

□

To warm up let us explain how to obtain a simple proof of Theorem 1.2.13 by applying the small parameter method. We will prove first that the rhs of the equation

$$w' = f(w,z)$$

must have no poles. In the opposite case one can locally represent the equation in the form

$$w' = \frac{F(w,z)}{(w - a(z))^k}, \quad F(a(z), z) \neq 0$$

with a positive integer k. After the substitution

$$w = a(z) + u$$

the equation will read as

$$u' = \frac{G(u,z)}{u^k}$$

for some function $G(u,z)$ such that $G(0, z_0) \neq 0$ for some z_0. Observe that after the substitution

$$z = z_0 + \epsilon^m \tau$$
$$u = \epsilon^n v$$

the equation will take the following form

$$\frac{du}{d\tau} = \epsilon^{n-(k+1)m} \frac{G(\epsilon^n v, z_0 + \epsilon^m \tau)}{u^k}.$$

The choice

$$m = 1, \quad n = k + 1$$

yields the following equation

$$\frac{du}{d\tau} = \frac{G(\epsilon^n v, z_0 + \epsilon^m \tau)}{u^k} = \frac{G(0, z_0)}{u^k} + \mathcal{O}(\epsilon)$$

with the rhs depending analytically on the small parameter ϵ. The solutions of the unperturbed equation

$$\frac{du}{d\tau} = \frac{G(0, z_0)}{u^k}$$

have critical singularity at $z = z_0$ if $k > 0$:

$$u = a\, \tau^{\frac{1}{k+1}}, \quad a = [(k+1)G(0, z_0)]^{\frac{1}{k+1}}.$$

Since z_0 can locally take arbitrary values, the singularity will be movable. So, to avoid critical singularities one must have $k = 0$.

The next step is to prove that, for a polynomial rhs the equation

$$w' = a_0(z) w^k + \cdots + a_k(z)$$

has no critical singularities only if $k \leq 2$. The susbtitution

$$\begin{aligned} z &= z_0 + \epsilon^m \tau \\ w &= \epsilon^{-n} u \end{aligned}$$

gives

$$\frac{du}{d\tau} = \epsilon^{m - n(k-1)} \left[a_0 \left(z_0 + \epsilon^m \tau \right) u^k + \epsilon^n a_1 \left(z_0 + \epsilon^m \tau \right) u^{k-1} + \cdots + \epsilon^{n\,k} a_k \left(z_0 + \epsilon^m \tau \right) \right].$$

The choice

$$n = 1, \quad m = k - 1$$

yields the following unperturbed equation

$$\frac{du}{d\tau} = a_0(z_0) u^k.$$

The solution

$$u = \frac{b_0}{\tau^{\frac{1}{k-1}}}, \quad b_0 = [(1-k) a_0(z_0)]^{\frac{1}{1-k}}$$

has a critical singularity if $k > 2$.

Let us return to the Painlevé classification problem. We are looking for the second order ordinary differential equations of the form

$$w'' = R(w', w, z) \tag{1.5.2}$$

admitting no movable critical singularities (the so-called *Painlevé property*). It is assumed that the right hand side $R(w', w, z)$ is a rational function in (w', w) with coefficients that are analytic function in z on some domain $\Omega \subset \mathbb{C}$. There are some obvious classes of equations satisfying Painlevé property. First, second order linear differential equations with rational coefficients are already known to meet the Painlevé condition. Next, second order differential equations of the form

$$\begin{aligned} w'' = 3a(z)\, w\, w' - \left[p(z) + 2\frac{a'(z)}{a(z)}\right] w' - a^2(z)w^3 + \left[a(z)p(z) + 3a'(z)\right] w^2 \\ - \left[q(z) + p(z)\frac{a'(z)}{a(z)} + \frac{a''(z)}{a(z)}\right] w + \frac{r(z)}{a(z)} \end{aligned} \tag{1.5.3}$$

obtained from the linear equation

$$y''' + p(z)y'' + q(z)y' + r(z)y = 0$$

with rational coefficients by substitution

$$w = -\frac{1}{a(z)}\frac{y'}{y}$$

(cf. (1.2.6)). One has to also take into account second order differential equations solved by elliptic functions. E.g., differentiating Weierstrass equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

in z and dividing by $2\wp'$ one arrives at

$$\wp'' = 6\wp^2 - \frac{g_2}{2}. \tag{1.5.4}$$

Remarkably, besides these essentially obvious classes of second order differential equations satisfying Painlevé property there are six new equations discovered by Painlevé and his student B.Gambier. They are the celebrated *Painlevé equations*:

$$w'' = 6w^2 + z \qquad P_I$$

$$w'' = 2w^3 + z\,w + \alpha \qquad P_{II}$$

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{\alpha\, w^2 + \beta}{z} + \gamma\, w^3 + \frac{\delta}{w} \qquad P_{III}$$

$$w'' = \frac{w'^2}{2w} + \frac{3}{2}\, w^3 + 4z\, w^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \qquad P_{IV}$$

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right) w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha\, w + \frac{\beta}{w}\right) + \frac{\gamma\, w}{z} + \frac{\delta\, w(w+1)}{w-1} \qquad P_V$$

$$\begin{aligned} w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right) w'^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right) w' \\ + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \frac{\beta\, z}{w^2} + \frac{\gamma\,(z-1)}{(w-1)^2} + \frac{\delta\, z(z-1)}{(w-z)^2}\right] \end{aligned} \qquad P_{VI}$$

Theorem 1.5.2 *(i) Solutions to the differential equations P_I - P_{VI} have no movable critical singularities.*

(ii) General solutions to these equations are new transcendental functions, i.e., they can not be expressed via algebraic functions, or via solutions to linear differential equations, or via elliptic functions.

(iii) Given a second order differential equation of the form (1.5.2) *with the right hand side depending rationally on w, w' with coefficients analytic in z, suppose that solutions to this equation satisfies Painlevé property. Then solutions to this equation can be expressed via algebraic functions, or elliptic functions, or solutions to linear differential equations, or via solutions to one of the equations P_I - P_{VI}.*

In this section we will give some hints about the proof of the last statement of the theory, namely, we develop an approach to the classification of the second order differential equations satisfying Painlevé property based on the small parameter method.

Proof: Step 1. We prove that the rhs $R(w', w, z)$ must be a polynomial in w' of degree at most 2.

Let us first prove that poles in w' are not allowed in the rhs. Indeed, in the opposite case the equation near the pole reads

$$w'' = \frac{Q(w', w, z)}{[w' - a(w, z)]^k}$$

where k is a positive integer and $Q(a(w, z), w, z)$ is not an identical zero. The substitution

$$u = w' - a(w, z) \tag{1.5.5}$$

represents the equation as a system

$$\frac{dw}{dz} = a(w, z) + u$$

$$\frac{du}{dz} = \frac{Q_1(u, w, z)}{u^k}$$

with some new function $Q_1(u, w, z)$ analytic and non vanishing near $u = 0$. Introducing a small parameter

$$z = z_0 + \epsilon^p Z, \quad w = w_0 + \epsilon^q W, \quad u = \epsilon^r U$$

with arbitrary (z_0, w_0) yields

$$\epsilon^{q-p} \frac{dW}{dZ} = a(w_0 + \epsilon^q W, z_0 + \epsilon^p Z) + \epsilon^r U$$

$$\epsilon^{(k+1)r-p} \frac{dU}{dZ} = \frac{Q_1(\epsilon^r U, w_0 + \epsilon^q W, z_0 + \epsilon^p Z)}{U^k}.$$

The choice

$$q = p = (k+1)r$$

of positive numbers p, q, r gives the following unperturbed system

$$\frac{dW}{dZ} = a(w_0, z_0)$$

$$\frac{dU}{dZ} = \frac{Q_1(0, w_0, z_0)}{U^k}.$$

Integration of the last equation gives critical points for $U(Z)$ if the integer k is positive:

$$U = [(k+1)Q_1(0, w_0, z_0)Z + C]^{\frac{1}{k+1}}.$$

However, from the definition (1.5.5) it follows that the function $u(z)$ cannot have critical points if the solution $w(z)$ has none.

We proved that the rhs of (1.5.2) is a polynomial in w'. Let us show that the degree of this polynomial is at most equal to 2. Indeed, rewriting the differential equation

$$w'' = A_0(w, z)w'^n + A_1(w, z)w'^{n-1} + \cdots + A_n(w, z)$$

as a system

$$\frac{dw}{dz} = u$$

$$\frac{du}{dz} = A_0(w, z)u^n + A_1(w, z)u^{n-1} + \cdots + A_n(w, z)$$

introduce the small parameter as follows

$$z = z_0 + \epsilon^p Z, \quad w = w_0 + \epsilon^q W, \quad u = \epsilon^{-r} U.$$

The system will read

$$\epsilon^{q+r-p} \frac{dW}{dZ} = U$$

$$\epsilon^{r(n-1)-p} \frac{dU}{dZ} = A_0(w_0 + \epsilon^q W, z_0 + \epsilon^p Z)U^n + \mathcal{O}(\epsilon)$$

Choosing

$$p = q + r = r(n-1)$$

gives the following solution to the unperturbed system

$$U = [C - (n-1)A_0(w_0, z_0)Z]^{\frac{1}{n-1}}.$$

This is a critical pole if $n \geq 3$. We arrive at an equation of the form

$$w'' = A_0(w, z)\, w'^2 + A_1(w, z)\, w' + A_2(w, z) \tag{1.5.6}$$

with some rational functions $A_0(w, z)$, $A_1(w, z)$, $A_2(w, z)$.

In these notes we will be unable to reproduce all details of the proof of the Part (iii) of the Painlevé - Gambier theorem. However, to give some flavour of the remaining part of the proof let us jump to considering the particular case of equations of the form (1.5.6) having $A_0 = A_1 = 0$. So, consider an equation of the form

$$w'' = R(w, z) \tag{1.5.7}$$

satisfying the Painlevé property. As before, the function $R(w, z)$ is rational in w.

Step N, $(N >> 1)$. Let us prove that all poles of the rhs in w are simple.

We will now proceed to studying the particular case of equations of the form

$$w'' = a_0(z)w^3 + a_1(z)w^2 + a_2(z)w + a_3(z). \tag{1.5.8}$$

First of all, in the case $a_0(z) \neq 0$ doing the transformation

$$\begin{aligned} z &\mapsto \tilde z = f(z) \\ w &\mapsto \tilde w = \lambda^{-1} w \end{aligned} \tag{1.5.9}$$

with

$$f'(z) = a_0^{1/3}(z), \quad \lambda(z) = \sqrt{2}\, a_0^{-\frac{1}{6}}(z)$$

one can achieve

$$\tilde a_0(\tilde z) \equiv 2.$$

If this is already the case in (1.5.8) then, after the shift

$$w \mapsto w - \frac{1}{6}\, a_1(z)$$

the equation reduces to the form

$$w'' = 2\, w^3 + a(z)w + b(z). \tag{1.5.10}$$

We have now to derive conditions for the functions $a(z)$, $b(z)$ necessary for absence of movable critical singularities.

Introducing a small parameter by means of the substitution

$$w = \epsilon^{-1} W, \quad z = z_0 + \epsilon Z$$

we obtain

$$W'' = 2\, W^3 + \epsilon^2 a_0 W + \epsilon^3 \left[a_0' Z\, W + b_0\right] + \epsilon^4 \left[\frac{1}{2}\, a_0'' Z^2 W + b_0'\, Z\right] + \mathcal{O}(\epsilon^5). \tag{1.5.11}$$

Here we use the following short notations

$$a_0 := a(z_0), \quad a_0' := a'(z)|_{z=z_0} \quad \text{etc.}$$

The unperturbed equation

$$W'' = 2\, W^3$$

has a first integral

$$W'^2 = W^4 + C.$$

The general solution to this equation is given in elliptic functions. However, for the particular case $C = 0$ it has a simple solution

$$W = \phi(Z) \equiv \frac{1}{Z+C} \quad \text{or} \quad W = \phi(Z) \equiv -\frac{1}{Z+C}.$$

Now let us look for the perturbative series solution

$$W = \phi + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \epsilon^4\phi_4 + \dots. \tag{1.5.12}$$

After the substitution one obtains the following recursive equations

$$\begin{aligned}
&\phi_2'' - 6\,\phi^2\,\phi_2 = a_0\,\phi \\
&\phi_3'' - 6\,\phi^2\,\phi_3 = a_0'\,Z\,\phi + b_0 \\
&\phi_4'' - 6\,\phi^2\,\phi_4 = \frac{1}{2}\,a_0''\,Z^2\phi + b_0'\,Z + 6\,\phi\,\phi_2^2 + a_0\phi_2.
\end{aligned} \tag{1.5.13}$$

Choose first

$$\phi = \frac{1}{Z+C}.$$

The following statement will be useful in sequel.

Exercise 1.5.3 *Linear differential equation of the form*

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-1} x\, y' + a_n y = 0, \quad a_1, \dots, a_{n-1}, a_n \in \mathbb{C} \tag{1.5.14}$$

is called Euler equation*. The polynomial*

$$P(k) := k(k-1)\dots(k-n+1) + a_1 k(k-1)\dots(k-n+2) + \dots + a_{n-1}k + a_n \tag{1.5.15}$$

is called the characteristic polynomial *of the Euler equation.*

(i) Given a root k_0 of the characteristic polynomial,

$$P(k_0) = 0, \quad P'(k_0) \neq 0 \tag{1.5.16}$$

prove that

$$y(x) = x^{k_0} \tag{1.5.17}$$

is a solution to the Euler equation.

(ii) If k_0 is a root of multiplicity m, i.e.

$$P(k_0) = 0, \quad P'(k_0) = 0, \dots, P^{(m-1)}(k_0) = 0, \quad P^{(m)} \neq 0 \tag{1.5.18}$$

then prove that the functions

$$y_1 = x^{k_0}, \quad y_2 = x^{k_0}\log x, \dots, y_m = x^{k_0}\log^{m-1} x \tag{1.5.19}$$

are linearly independent solutions to the Euler equations.

(iii) Prove that solutions of the form (1.5.17), (1.5.19) *where k_0 runs through the set of all roots of the characteristic polynomial give a basis in the space of solutions to the Euler equation.*

The linearized equation

$$(Z+C)^2\psi'' - 6\,\psi = 0 \tag{1.5.20}$$

is a particular case of Euler equation. The characteristic equation

$$k(k-1) - 6 = 0$$

has two roots

$$k = -2 \quad \text{and} \quad k = 3.$$

Thus the general solution depending on two arbitrary constants k_1, k_2 reads

$$\psi = \frac{k_1}{(Z+C)^2} + k_2\,(Z+C)^3. \tag{1.5.21}$$

The following statement will be of use for solving inhomogeneous Euler equation.

Exercise 1.5.4 *Consider inhomogeneous Euler equation of the form*

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x\, y' + a_n y = a\, x^\kappa. \tag{1.5.22}$$

(i) Prove that, if κ is not a root of the characteristic polynomial (see Exercise 1.5.3 above), $P(\kappa) \neq 0$, then a particular solution to the inhomogeneous equation is given by the formula

$$y = \frac{a}{P(\kappa)}\, x^\kappa. \tag{1.5.23}$$

(ii) If κ is a simple root *of the characteristic polynomial, i.e.*

$$P(\kappa) = 0, \quad P'(\kappa) \neq 0$$

then prove the following formula for a particular solution:

$$y = \frac{a}{P'(\kappa)} x^\kappa \log x. \tag{1.5.24}$$

Generalize to the case of roots of higher multiplicities.

Applying this method we find general solutions to the linearized equations

$$\phi_2 = -\frac{1}{6}\, a_0\,(Z+C) + \frac{k_1}{(Z+C)^2} + k_2\,(Z+C)^3 \tag{1.5.25}$$

$$\phi_3 = \frac{1}{6}\, a_0'\, C(Z+C) - \frac{1}{4}\left[a_0' + b_0\right](Z+C)^2 + \frac{k_1}{(Z+C)^2} + k_2\,(Z+C)^3. \tag{1.5.26}$$

Integration of the subsequent equation yields logarithmic terms:

$$\phi_4 = \frac{1}{10}(Z+C)^3\left[a_0'' + 2b_0'\right]\log[Z+C] - \frac{1}{50}\left[a_0'' + 2b_0'\right](Z+C)^3 + \frac{1}{4}\,C\,(a_0''+b_0') - \frac{1}{2}C^2 a_0''(Z+C) \tag{1.5.27}$$

(we choose integration constants $k_1 = k_2 = 0$ in (1.5.25)). Cancellation of logarithmic terms implies

$$a''(z_0) + 2b'(z_0) = 0. \tag{1.5.28}$$

Doing similar calculations with the perturbative solution starting with

$$\phi = -\frac{1}{Z+C}$$

yields another constraint:

$$a''(z_0) - 2b'(z_0) = 0. \tag{1.5.29}$$

Therefore

$$a''(z_0) = 0, \quad b'(z_0) = 0.$$

Since z_0 is an arbitrary complex number we finally obtain that

$$a(z) = \lambda\, z + \mu, \quad b(z) = \nu \tag{1.5.30}$$

with some constants λ, μ ν. For $\lambda = 0$ one obtains differential equation solved in elliptic functions. If $\lambda \neq 0$ then, doing if necessary an affine transformation of the independent variable z and a rescaling of w we finally arrive at the Painlevé-II equation

$$w'' = 2\, w^3 + z\, w + \alpha.$$

In a similar way one can deal with equations of the form

$$w'' = a_0(z)w^2 + a_1(z)w + a_2(z), \quad a_0(z) \neq 0. \tag{1.5.31}$$

After the substitution of the form (1.5.9) with

$$\lambda = 6\, a_0^{-\frac{1}{5}}(z), \quad f'(z) = a_0^{2/5}(z)$$

and a suitable shift of the dependent variable one reduces the equation to the form

$$w'' = 6\, w^2 + a(z). \tag{1.5.32}$$

Introducing the small parameter by the substitution

$$z = z_0 + \epsilon\, Z, \quad w = \epsilon^{-2} W$$

one arrives at the following equation

$$\frac{d^2W}{dZ^2} = 6\, W^2 + \epsilon^4 a_0 + \epsilon^5 a_0'\, Z + \frac{1}{2}\, \epsilon^6 a_0''\, Z^2 + \mathcal{O}\left(\epsilon^7\right). \tag{1.5.33}$$

The notations a_0, a_0' etc. are similar to those used above.

For the unperturbed equation choose the particular solution

$$\phi(Z) = \frac{1}{(Z+C)^2}.$$

The coefficients of the perturbative solution

$$W = \phi(Z) + \epsilon^4 \phi_4(Z) + \epsilon^5 \phi_5(Z) + \epsilon^6 \phi_6(Z) + \dots$$

are determined from the following recursive procedure:

$$\begin{aligned}
\phi_4'' - 12\,\phi\,\phi_4 &= a_0\\
\phi_5'' - 12\,\phi\,\phi_5 &= a_0'\,Z\\
\phi_6'' - 12\,\phi\,\phi_6 &= \frac{1}{2}\,a_0''\,Z^2.
\end{aligned}$$

The general solution of the homogeneous linearized equation

$$(Z+C)^2\psi'' - 12\,\psi = 0$$

is obtained in the form

$$\psi = k_1(Z+C)^4 + \frac{k_2}{(Z+C)^3}.$$

Applying the method of Exercise 1.5.4 we obtain single valued solutions ϕ_4 and ϕ_5 but

$$\phi_6 = \frac{1}{14}a_0''(Z+C)^2\log(Z+C) - \frac{1}{3}a_0''C\,(Z+C) - \frac{1}{12}a_0'' + k_1(Z+C)^4 + \frac{k_2}{(Z+C)^3}.$$

This function has no critical point at $Z=-C$ *iff*

$$a''(z_0) = 0.$$

Due to the freedom in the choice of z_0 the function $a(z)$ must be linear,

$$a(z) = \lambda\,z + \mu.$$

In the case $\lambda = 0$ the differential equation (1.5.32) integrates in elliptic functions. Otherwise doing an affine change of the variable z and a rescaling of w (1.5.32) becomes the Painlevé-I equation

$$w'' = 6\,w^2 + z.$$

□

It remains to prove[1] that solutions to the equations P_I - P_{VI} have no movable critical singularities. Actaully we will explain in sequel how to prove that

- solutions to P_I, P_{II} and P_{IV} equations are meromorphic functions on $\mathbb{C}$ having essential singularity at the fixed singularity $z=\infty$;

- solutions to P_{III} and P_V are meromorphic functions on the universal covering of $\mathbb{C}^* = \mathbb{C}\setminus\{z=0\}$;

- solutions to P_{VI} are meromorphic functions on the universal covering of Riemann sphere with three punctures (the fixed singularities) $z=0$, $z=1$ and $z=\infty$.

In all the cases the position of fixed singularities can be easily guessed by looking at the singularities of coefficients of the equation. The proof of meromorphicity is more involved.

[1]One has to also prove that general solutions to the equations P_I - P_{VI} cannot be expressed via known functions. The proof of this statement required some technique of differential Galois theory. It will not be given in the course. Besides, we have to note that some *particular* solutions to these equations can be expressed via known special functions, some of them even via elementary functions. The classification problem of these particular solutions can be difficult. For example the problem remains open for the case of Painlevé-VI equation.

The technique to be used is based on a realization of the Painlevé equations as monodromy preserving deformations of linear differential equations with rational coefficients. In the next section we will begin with explaining the basics of the monodromy theory of solutions to these equations.

2 Part 2: Monodromy of linear differential operators with rational coefficients

2.1 Solutions to linear systems with rational coefficients: the local theory

Let us begin with recalling some basics from the theory of systems of linear differential equations. Consider an n-th order system linear homogeneous system of the form

$$y' = A(z)\, y, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}, \quad A(z) = \begin{pmatrix} a_{11}(z) & \dots & a_{1n}(z) \\ a_{21}(z) & \dots & a_{2n}(z) \\ \cdot & & \cdot \\ \cdot & \dots & \cdot \\ \cdot & & \cdot \\ a_{n1}(z) & \dots & a_{nn}(z) \end{pmatrix}, \quad z \in \mathbb{C}. \tag{2.1.1}$$

The space of solutions defined in a neighborhood of a regular point z_0 of coefficients $a_{ij}(z)$ is a linear space of dimension n. Indeed, there exists a unique solution to (2.1.1) with arbitrary initial data

$$y_1(z_0) = y_1^0, \ y_2(z_0) = y_2^0, \dots, y_n(z_0) = y_n^0, \quad \begin{pmatrix} y_1^0 \\ y_2^0 \\ \cdot \\ \cdot \\ \cdot \\ y_n^0 \end{pmatrix} \in \mathbb{C}^n.$$

Choosing a basis of n linearly independent solutions one can represent an arbitrary solution as a linear combination

$$y(z) = c_1 \begin{pmatrix} y_{11}(z) \\ y_{21}(z) \\ \cdot \\ \cdot \\ \cdot \\ y_{n1}(z) \end{pmatrix} + \dots + c_n \begin{pmatrix} y_{1n}(z) \\ y_{2n}(z) \\ \cdot \\ \cdot \\ \cdot \\ y_{nn}(z) \end{pmatrix} \tag{2.1.2}$$

with some constant coefficients $c_1, \dots, c_n$. The basic vector functions are columns of the *fundamental matrix* for the system (2.1.1)

$$Y(z) = \begin{pmatrix} y_{11}(z) & \dots & y_{1n}(z) \\ \cdot & & \cdot \\ \cdot & \dots & \cdot \\ \cdot & & \cdot \\ y_{n1}(z) & \dots & y_{nn}(z) \end{pmatrix}.$$

Linear independence of the vector functions is equivalent to never vanishing of the determinant of this matrix

$$W(z) := \det Y(z) \neq 0. \tag{2.1.3}$$

Any solution (2.1.2) can be represented in a matrix form as follows

$$y(z) = Y(z)\,c, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{pmatrix} \tag{2.1.4}$$

with an arbitrary constant vector $c \in \mathbb{C}^n$.

The fundamental matrix is a matrix solution to the system (2.1.1):

$$Y' = A(z)\,Y \tag{2.1.5}$$

Conversely, any matrix solution to (2.1.5) satisfying (2.1.3) is a fundamental matrix of the system.

Lemma 2.1.1 *Any two fundamental matrices $Y(z)$, $\tilde{Y}(z)$ for the system* (2.1.5) *are related by a linear transformation of the form*

$$\tilde{Y}(z) = Y(z)\,C \tag{2.1.6}$$

with a nondegenerate constant $n \times n$ matrix C.

Proof: follows from (2.1.4). □

Example 1. Linear systems with constant coefficients

$$y' = A\,y. \tag{2.1.7}$$

A fundamental matrix is given by the matrix exponential function

$$Y(z) = e^{A\,z}. \tag{2.1.8}$$

Let $a_1, \ldots, a_n$ be the eigenvalues of the matrix A (more precisely, the roots of the characteristic polynomial of A). Denote

$$\hat{a} := \operatorname{diag}(a_1, \ldots, a_n)$$

the diagonal matrix. Suppose that the matrix A is diagonalizable. For example this is always the case if the roots of the characteristic polynomial are pairwise distinct. Denote T the matrix of eigenvectors of A. The conjugation by this matrix reduces A to the diagonal form

$$T^{-1} A\,T = \hat{a}.$$

Then a fundamental matrix can be computed as follows

$$Y(z) = T\,e^{\hat{a}\,z}, \quad \text{where} \quad e^{\hat{a}\,z} = \operatorname{diag}\left(e^{a_1 z}, \ldots, e^{a_n z}\right). \tag{2.1.9}$$

It is related to (2.1.30) by the multiplication by T^{-1} on the right. In the presence of multiple roots of the characteristic polynomial

$$\det\left(A - a\cdot \mathrm{Id}\right) = 0$$

the fundamental matrix (2.1.30) involves also polynomials. They can be computed by reducing the matrix A to Jordan normal form and using the following formula for exponential of an elementary Jordan block of size k:

$$e^{J\,z} = e^{a\,z}\begin{pmatrix} 1 & z & \frac{z^2}{2!} & \frac{z^3}{3!} & \dots & \frac{z^{k-1}}{(k-1)!} \\ 0 & 1 & z & \frac{z^2}{2!} & \dots & \frac{z^{k-2}}{(k-2)!} \\ \cdot & & & & \cdot & \\ \cdot & \dots & \dots & \dots & \dots & \cdot \\ \cdot & & & & \cdot & \\ 0 & & \dots & \dots & 1 & z \\ 0 & & & & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} a & 1 & 0 & 0 & \dots & 0 \\ 0 & a & 1 & 0 & \dots & 0 \\ \cdot & & & & & \cdot \\ \cdot & \dots & \dots & \dots & \dots & \cdot \\ \cdot & & & & & \cdot \\ 0 & & \dots & \dots & a & 1 \\ 0 & & & & 0 & a \end{pmatrix}. \tag{2.1.10}$$

Example 2. The system

$$y' = \frac{A}{z}\,y \tag{2.1.11}$$

with constant matrix A can be reduced to (2.1.7) by the substitution

$$\log z = t, \quad \frac{dy}{dt} = A\,y.$$

Thus, for a diagonalizable matrix A the fundamental matrix (2.1.32) becomes

$$Y = T\,z^{\hat a}, \quad \text{where} \quad z^{\hat a} = \mathrm{diag}\left(z^{a_1}, \dots, z^{a_n}\right), \tag{2.1.12}$$

$a_1, \dots, a_n$ are eigenvalues of the matrix A, $T^{-1}A\,T = \hat a$. Observe that the function

$$z^\mu := e^{\mu\,\log z}$$

is a multivalued function on $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ unless μ is an integer. However, the following simple statement holds true.

Lemma 2.1.2 *For an arbitrary $\mu\in\mathbb{C}$ there exists an integer N and a positive ρ such that for an arbitrary choice of the branch of the function z^μ and arbitrary two real numbers $\alpha<\beta$ there exists a constant C such that*

$$|z^\mu| < C\,|z|^N \quad \text{for} \quad |z|<\rho, \quad \alpha < \arg z < \beta. \tag{2.1.13}$$

Proof: Denote $\mu_{\mathrm R}$, $\mu_{\mathrm I}$ respectively the real and imaginary parts of μ. Then

$$|z^\mu| = \left| e^{(\mu_{\mathrm R} + i\,\mu_{\mathrm I})(\log|z| + i\,\arg z)}\right| = e^{-\mu_{\mathrm I}\,\arg z}|z|^{\mu_{\mathrm R}}.$$

We choose

$$N = \max\{k \in \mathbb{Z} \mid k \le \mu_{\rm R}\}$$
$$\rho = 1$$
$$C = \begin{cases} e^{-\mu_{\rm I}\alpha}, & \mu_{\rm I} > 0 \\ e^{-\mu_{\rm I}\beta}, & \mu_{\rm I} < 0 \end{cases}$$

□

To be short we will say that the multivalued function z^μ has a polynomial growth at the singular point $z = 0$. It is important to impose restrictions for the argument: if $z \to 0$ but is not constrained within a sector on the complex plane then the polynomial growth can be violated.

Like in the previous example for a nondiagonalizable matrix A logarithmic terms will appear in the fundamental matrix (cf. the theory of Euler equation above). The solutions will still have a polynomial growth at the origin.

Example 3. Consider now equation

$$y' = \frac{A}{z^r}\, y, \quad r \in \mathbb{Z}, \quad r \ge 2. \tag{2.1.14}$$

This case can also be reduced to a system with constant coefficients by a substitution

$$t = -\frac{1}{r-1}\frac{1}{z^{r-1}}, \quad \frac{dy}{dt} = A\, y. \tag{2.1.15}$$

So the fundamental matrix, for a diagonalizable matrix A can be chosen in the form

$$Y(z) = T\, e^{-\frac{\hat{a}}{r-1}\, z^{1-r}} \tag{2.1.16}$$

in the notations of the previous examples. The matrix entries of this fundamental matrix are single valued functions on $\mathbb{C}^*$ but they have an *essential singularity* at $z = 0$. We leave as an exercise to establish the structure of the fundamental matrix in the case of a nondiagonalizable matrix A.

We will now address the problem of studying the local behaviour of solutions to linear systems of differential equations near a singular point of coefficients analytic on a punctured disk. To be more specific we will assume that the coefficients have a pole at the puncture. Without loss of generality we will assume the puncture to be put at the origin. The system then will be written in the form

$$z^r y' = A(z)\, y \tag{2.1.17}$$

for some integer $r \ge 1$ where the matrix valued function $A(z)$ is analytic on the disc

$$|z| < \rho$$

for some positive ρ and

$$A(0) \ne 0.$$

Definition 2.1.3 *The number $r-1$ is called the* Poincaré rank *of the singular point $z=0$. The singularity of the system* (2.1.17) *will be called* Fuchsian *if $r=1$.*

We have seen, looking at simple examples, that the local behaviour of solutions near the singularity is qualitatively different for the cases $r=1$ (at most polynomial growth in the sense of Lemma 2.1.2) and $r>1$ (exponential growth). We will now analyze the local behaviour for the more general case of systems (2.1.17). We will see that the treatment is different for Fuchsian and non-Fuchsian cases. One can see some difference by applying the method of small parameter to the systems

$$z\,y' = (A_0 + A_1 z + A_2 z^2 + \dots)\,y \tag{2.1.18}$$

or

$$z^r y' = (A_0 + A_1 z + A_2 z^2 + \dots)\,y, \quad r \geq 2. \tag{2.1.19}$$

Introducing the small parameter by rescaling

$$z \mapsto \epsilon z$$

one obtains a regular dependence on ϵ in (2.1.18)

$$z\,y' = (A_0 + \epsilon\,A_1 z + \epsilon^2\,A_2 z^2 + \dots)\,y$$

but a singular one in (2.1.19)

$$z^r y' = \frac{1}{\epsilon^{r-1}}\,(A_0 + \epsilon\,A_1 z + \epsilon^2\,A_2 z^2 + \dots)\,y.$$

An equivalent approach to the study of singularities of the fundamental matrix is based on the theory of gauge transformations of the system (2.1.17).

Given an invertible matrix valued function $G(z)$ defined on the punctured disk, one can transform the system (2.1.1) by the substitution

$$y(z) = G(z)\tilde{y}(z) \tag{2.1.20}$$

to another system of the same type

$$\tilde{y}' = \tilde{A}(z)\,\tilde{y} \tag{2.1.21}$$

with

$$\tilde{A} = G^{-1}A\,G - G^{-1}G'. \tag{2.1.22}$$

The transformations of the form (2.1.22) are called *gauge transformations.*

Definition 2.1.4 *Two systems of the form* (2.1.1), (2.1.21) *are called* strongly equivalent *if there exists a gauge transformation* (2.1.22) *with the matrix valued function $G(z)$ analytic on some disk $|z|<\rho$ with never vanishing determinant. They are called* weakly equivalent *if such a gauge transformation with an analytic matrix valued function exists with the determinant never vanishing for $|z|>0$.*

The proof of the following statement is straightforward.

Lemma 2.1.5 *Given a system of the form* (2.1.1)*, then any weakly equivalent system has a form*

$$z^{\tilde r}\tilde y' = \tilde A(z)\,\tilde y, \quad \tilde A(z) \neq 0 \tag{2.1.23}$$

with some $\tilde r > 0$*, or the new system has no singularity at* $z = 0$*. Any strongly equivalent system has the form* (2.1.23) *with* $\tilde r = r$.

Let us begin with the Fuchsian case.

Definition 2.1.6 *A square matrix* M *is called* resonant *if there exist two eigenvalues* λ_1, λ_2 *of* M *such that the difference* $\lambda_1 - \lambda_2$ *is a positive integer. If such a pair does not exist then the matrix* M *is called* nonresonant.

The main result of the local theory of Fuchsian systems is the following

Theorem 2.1.7 *Any Fuchsian system of the form* (2.1.18) *with nonresonant leading term* A_0 *is strongly equivalent to the "unperturbed" system*

$$z\,\tilde y' = A_0\tilde y. \tag{2.1.24}$$

Proof: We have to find an invertible analytic matrix valued function $G = G(z)$ satisfying the following linear differential equation

$$z\,G' = A(z)\,G - G\,A_0. \tag{2.1.25}$$

The proof will consist of two steps. First we will construct a *formal series solution*

$$G(z) = G_0 + z\,G_1 + z^2 G_2 + \dots \tag{2.1.26}$$

to (2.1.25). At the second step we will prove convergence of the series.

For the sake of technical simplicity the proof of the first part will be done under the additional assumption of diagonalizability of the matrix A_0. Doing if necessary a conjugation

$$A(z) \mapsto T^{-1}A(z)\,T$$

with a suitable invertible constant matrix T we may assume the matrix A_0 to be diagonal,

$$A_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_n) =: \Lambda.$$

Then we choose $G_0 = \mathrm{Id}$. Substituting (2.1.26) to (2.1.25) and collecting the coefficient of z^k we obtain a recursion relation for the coefficients G_1, G_2, ...

$$[\Lambda, G_k] - k\,G_k = -A_k - \sum_{i=1}^{k-1} A_i G_{k-i}, \quad k = 1, 2, \dots \tag{2.1.27}$$

Assuming the coefficients $G_1, \dots, G_{k-1}$ already known we have to determine the matrix G_k. Observe that for a $n \times n$ matrix $M = (M_{ij})$ the commutator $[\Lambda, M] = \Lambda\,M - M\,\Lambda$ has the following entries

$$([\Lambda, M])_{ij} = (\lambda_i - \lambda_j)\,M_{ij}$$

So we can put

$$(G_k)_{ij} = -\frac{1}{\lambda_i - \lambda_j - k}\left(A_k + \sum_{i=1}^{k-1} A_i G_{k-i}\right).$$

Due to the nonresonancy assumption the denominators $\lambda_i - \lambda_j - k$ never vanish.

We have proved, under assumptions of nonresonancy and diagonalizability of A_0, existence of a *formal* gauge transformation between the systems (2.1.18) and (2.1.24). The assumption of diagonalizability is purely technical and can be eliminated due to the following statement.

Exercise 2.1.8 *Let Λ be an arbitrary $n \times n$ matrix. Consider the linear map*

$$Mat(n, \mathbb{C}) \to Mat(n, \mathbb{C}), \quad M \mapsto [\Lambda, M] \tag{2.1.28}$$

of the space $\mathbb{C}^{n^2} = Mat(n, \mathbb{C})$ of all $n \times n$ matrices to itself. Prove that characteristic roots of this map belong to the set of all differences $\lambda_i - \lambda_j$ where $\lambda_1, \ldots, \lambda_n$ are the characteristic roots of the matrix Λ.

We now proceed to the proof of convergence of the series (2.1.26). This will follow from the following general statement.

Lemma 2.1.9 *Consider a system of N linear differential equations*

$$z\, \mathcal{Y}' = \mathcal{A}(z)\mathcal{Y} \tag{2.1.29}$$

with a Fuchsian singularity at the origin and the right hand side analytic for $|z| < r$. Suppose the system has a formal series solution

$$\mathcal{Y}(z) = \mathcal{Y}_0 + z\, \mathcal{Y}_1 + z^2 \mathcal{Y}_2 + \ldots. \tag{2.1.30}$$

Then the series converges for $|z| < r$.

Proof: Denote $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, ... the coefficients of Taylor expansion of the analytic function $\mathcal{A}(z)$,

$$\mathcal{A}(z) = \mathcal{A}_0 + z\, \mathcal{A}_1 + z^2 \mathcal{A}_2 + \ldots. \tag{2.1.31}$$

For the coefficients of the formal series solution to (2.1.29) one obtains the following recursion

$$\mathcal{A}_0 \mathcal{Y}_0 = 0, \quad k\, \mathcal{Y}_k = \mathcal{A}_0 \mathcal{Y}_k + \sum_{i=1}^{k} \mathcal{A}_i \mathcal{Y}_{k-i}, \quad k \geq 1. \tag{2.1.32}$$

Because of convergence of the series (2.1.31) there exist positive constants c and ρ such that

$$\|\mathcal{A}_k\| \leq c\, \rho^k, \quad k = 1, 2, \ldots$$

that is, the inequality

$$|\mathcal{A}_k \mathcal{Y}| \leq c\, \rho^k\, |\mathcal{Y}| \tag{2.1.33}$$

holds true for an arbitrary vector $\mathcal{Y}$. Indeed, it suffices to choose

$$\rho > \frac{1}{r}.$$

Choose a sufficiently big positive integer m in such a way that

$$|\mathcal{A}_0\mathcal{Y}| \leq m\,|\mathcal{Y}|$$

for any vector $\mathcal{Y}$. It is easy to see that the matrix $\mathcal{A}_0 - \kappa\,\mathrm{Id}$ is invertible for any $\kappa > m$ since

$$|\mathcal{A}_0\mathcal{Y} - \kappa\,\mathcal{Y}| \geq |\kappa\,|\mathcal{Y}| - |\mathcal{A}_0\mathcal{Y}|| \geq (\kappa - m)\,|\mathcal{Y}| \neq 0 \quad \forall\,\mathcal{Y} \neq 0.$$

Thus the linear inhomogeneous equation

$$\mathcal{A}_0\mathcal{Y} - \kappa\,\mathcal{Y} = \mathcal{Z}$$

with an arbitrary right hand side $\mathcal{Z}$ for any $\kappa > m$ has a solution. Moreover for the norm of the solution one has an estimate

$$|\mathcal{Y}| \leq \frac{|\mathcal{Z}|}{\kappa - m}.$$

A somewhat stronger estimate

$$|\mathcal{Y}| \leq |\mathcal{Z}|$$

follows if $\kappa \geq m + 1$.

We want to establish estimates for the coefficients determined by (2.1.32)

$$|\mathcal{Y}_k| \leq C\,R^k \tag{2.1.34}$$

for some positive constants C, R. More precisely, we first choose sufficiently large $R > \rho$ in such a way that

$$\frac{c\,\rho}{R - \rho} < 1. \tag{2.1.35}$$

Next, choose the constant C in such a way that the inequalities (2.1.34) hold true for small values of k, namely, for $0 \leq k \leq m$.

Let us now derive (2.1.34) inductively for all k. Rewriting (2.1.32) in the form

$$\mathcal{A}_0\mathcal{Y}_k - k\,\mathcal{Y}_k = \mathcal{Z}_k, \quad \mathcal{Z}_k = -\sum_{i=1}^{k} \mathcal{A}_i\mathcal{Y}_{k-i}$$

and using (2.1.33) and the inductive assumption we obtain

$$|\mathcal{Z}_k| \leq \sum_{i=1}^{k} c\,\rho^i\,C\,R^{k-i} = c\,C\,\rho\,R^{k-1}\sum_{i=0}^{k-1}\left(\frac{\rho}{R}\right)^i < c\,C\,\rho\,R^{k-1}\sum_{i=0}^{\infty}\left(\frac{\rho}{R}\right)^i = C\,R^k\frac{c\,\rho}{R - \rho}.$$

Due to the choice (2.1.35) we derive that

$$|\mathcal{Z}_k| < C\,R^k.$$

Finally, for $k \geq m + 1$ this implies that $|\mathcal{Y}_k| < C\,R^k$.

We proved that the series (2.1.30) converges for

$$|z| < \frac{1}{R}.$$

The last step in the proof of Lemma is in observing that any analytic solution to the linear system (2.1.29) can be analytically extended onto the entire domain of analyticity of coefficients, i.e., the series (2.1.30) converges for all $|z| < r$. □

We can now complete the proof of the Theorem. We can consider the equation (2.1.25) as a system of linear differential equations for the n^2-dimensional vector valued function $G(z)$. We have proved above existence of a formal series solution to this system. Due to Lemma the formal series converges. Since $G(0) = G_0 = \mathrm{Id}$ the matrix $G(z)$ is invertible for sufficiently small $|z|$. □

Corollary 2.1.10 *A Fuchsian system* (2.1.18) *with diagonalizable nonresonant matrix* $A_0 = A(0)$,

$$T^{-1} A_0 T = \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

has a fundamental matrix of the form

$$Y(z) = G(z) z^{\Lambda}, \quad z^{\Lambda} = \operatorname{diag}\left(z^{\lambda_1}, \dots, z^{\lambda_n}\right) \tag{2.1.36}$$

with the matrix $G(z)$ *analytic and invertible for sufficiently small* $|z|$.

What happens in presence of resonances? We give an answer leaving the proofs as an exercise.

Let J be the Jordan normal form of the matrix A_0. Recall that it is an upper triangular matrix determined uniquely up to a permutation of Jordan blocks. We will choose a *good* Jordan form in such a way that the diagonal entries $\lambda_1, \dots, \lambda_n$ (not necessarily distinct) of the matrix J satisfy

$$\operatorname{Re}\lambda_i \geq \operatorname{Re}\lambda_j \quad \text{for} \quad i < j. \tag{2.1.37}$$

Decompose

$$J = \Lambda + R_0 \tag{2.1.38}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

is a diagonal matrix and R_0 is a collection of Jordan blocks with all zeroes on the diagonal. The upper triangular matrix R_0 is nilpotent. The only non-zero entries of this matrices are

$$(R_0)_{ij} \neq 0 \quad \text{only if} \quad i \neq j \quad \text{and} \quad \lambda_i = \lambda_j. \tag{2.1.39}$$

Theorem 2.1.11 *There exist upper triangular matrices* R_1, R_2, … *such that*

$$(R_k)_{ij} \neq 0 \quad \text{only if} \quad \lambda_i - \lambda_j = k \tag{2.1.40}$$

such that the Fuchsian system (2.1.18) *is strongly equivalent to a system with polynomial coefficients of the form*

$$z\,\tilde{y}' = \left(\Lambda + R_0 + z\,R_1 + z^2 R_2 + \dots\right)\tilde{y} \tag{2.1.41}$$

(only finite number of non-zero terms).

Theorem 2.1.12 *The system* (2.1.41) *has a fundamental matrix of the form*

$$\tilde{Y}(z) = z^{\Lambda} z^{R}, \quad R = R_0 + R_1 + \dots. \tag{2.1.42}$$

Hint: Use the following property of the matrices R_k:

$$z^{\Lambda} R_k z^{-\Lambda} = z^k R_k, \quad k = 0,\, 1,\, 2, \dots. \tag{2.1.43}$$

Corollary 2.1.13 *Any system of linear differential equations near a Fuchsian singularity has a fundamental matrix of the form*

$$Y(z) = G(z) z^{\Lambda} z^{R} \tag{2.1.44}$$

where the diagonal matrix Λ *and a nilpotent matrix* R *are as above and the matrix valued function* $G(z)$ *is analytic and invertible on a neighborhood of the point* $z - 0$

Exercise 2.1.14 *Find a fundamental system of the form* (2.1.44) *for the system*

$$z\, y' = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix} y.$$

Since the upper triangular matrix R is nilpotent the function

$$z^R = \mathrm{Id} + R \log z + R^2 \frac{\log^2 z}{2!} + \dots$$

is a polynomial in $\log z$. We arrive at important

Corollary 2.1.15 *Solutions of a Fuchsian system have at most polynomial growth at* $z = 0$.

Definition 2.1.16 *We say that an isolated singularity at* $z = 0$ *of the system* (2.1.1) *is* regular *if all solutions to this system have at most polynomial growth at* $z \to 0$.

From the previous results it follows that

Fuchsian singularity $\Rightarrow$ regular singularity.

The converse statement is false, as it follows from a simple

Counterexample. The system

$$z^2 y' = \begin{pmatrix} 0 & z^2 \\ 1 & -z \end{pmatrix} y \tag{2.1.45}$$

has a singularity at $z = 0$ of Poincaré rank 1. However this is a regular singularity. Indeed, for the components y_1, y_2 one obtains the following system

$$y_1' = y_2$$
$$z^2 y_2' = y_1 - z\, y_2.$$

The substitution $y_2 = y_1'$ into the second equation gives Euler equation for y_1,

$$z^2 y_1'' + z\, y_1' - y_1 = 0.$$

Solving the latter

$$y_1 = c_1\, z + \frac{c_2}{z}$$

yields

$$y_2 = c_1 - \frac{c_2}{z^2}.$$

So the fundamental matrix reads

$$Y(z) = \begin{pmatrix} z & 1/z \\ 1 & -1/z^2 \end{pmatrix}.$$

In general the following statement holds true.

Theorem 2.1.17 *A system of the form* (2.1.1) *with an isolated regular singularity at* $z = 0$ *is weakly equivalent to a Fuchsian system.*

The *proof* of this theorem consists of two parts. First, we establish existence of a fundamental matrix of the form

$$Y(z) = G(z) z^L$$

with some matrix L and a matrix valued function $G(z)$ analytic and invertible on a punctured disk (see Exercise 2.2.6 below). Since the singularity is regular we derive that $G(z)$ has at most a pole at $z = 0$,

$$G(z) = \frac{1}{z^r} M(z)$$

for some $r \geq 0$ and a matrix $M(z)$ analytic for $|z| < r$ and nondegenerate for $|z| > 0$. For the sake of simplicity of notations let us do a shift

$$L \mapsto L - r \cdot \mathrm{Id}$$

in order to recast the fundamental matrix into the form

$$Y(z) = M(z)\, z^L.$$

If $\det M(0) \neq 0$ we are done: the coefficient matrix

$$A(z) = Y'(z) Y^{-1}(z) = M\, \frac{L}{z}\, M^{-1} + M'\, M^{-1}$$

has a simple pole at $z = 0$.

It remains to consider the case of degenerate matrix $M(0)$ (recall that $\det M(z) \neq 0$ for $|z| > 0$). The crucial point in the proof is the following statement, sometimes called *Sauvage's Lemma*.

Lemma 2.1.18 *Given a matrix valued function $M(z)$ analytic for $|z| < r$, invertible for $|z| > 0$ such that* $\det M(0) = 0$. *Then there exist a matrix $P(z)$ polynomially depending on z satisfying*

$$\det P(z) \equiv 1, \tag{2.1.46}$$

a matrix $\tilde{M}(z)$ invertible and analytic on $|z| < r$, and n nonnegative integers $k_1, \ldots, k_n$ such that

$$M(z) = P(z)\, z^K\, \tilde{M}(z), \quad K = \operatorname{diag}(k_1, \ldots, k_n). \tag{2.1.47}$$

In order to complete the proof of the Theorem let us apply to the original system a gauge transformation

$$Y(z) = P(z) z^K\, \tilde{Y}(z). \tag{2.1.48}$$

The new system will have a fundamental matrix of the form

$$\tilde{Y}(z) = \tilde{M}(z)\, z^L \quad \text{with} \quad \tilde{M}(0) \neq 0.$$

So the new system will have a Fuchsian singularity at $z = 0$. It remains to observe that

$$\det\left(P(z) z^K\right) = z^{k_1 + \cdots + k_n}.$$

Therefore (2.1.48) is a weak equivalence of the original system with a Fuchsian system. □

The proof of this Lemma can be found, e.g., in §11 of Chapter IV of Hartman's book.

Exercise 2.1.19 *Find a weak equivalence of the system* (2.1.45) *with a system Fuchsian at* $z = 0$.

At the end of this section we consider the case of scalar linear differential equations of higher order

$$y^{(n)} + a_1(z) y^{(n-1)} + \cdots + a_n(z) y = 0 \tag{2.1.49}$$

with coefficients analytic on a sufficiently small punctured disk

$$0 < |z| < \rho.$$

Exercise 2.1.20 *Let the coefficients satisfy the following conditions near $z = 0$:*

$$z^i a_i(z) \quad \text{is analytic at} \quad z = 0, \quad i = 1, 2, \ldots, n. \tag{2.1.50}$$

Prove that the substitution

$$\begin{aligned} y_1 &= y \\ y_2 &= z\, y_1' \\ &\ldots \\ y_n &= z\, y_{n-1}' \end{aligned} \tag{2.1.51}$$

reduces the equation (2.1.50) *to a system with a Fuchsian singularity at* $z=0$ *of the form*

$$z\,\mathbf{y}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & \dots & 0 & 1 \\ -b_n(z) & -b_{n-1}(z) & \dots & -b_2(z) & -b_1(z) \end{pmatrix} \mathbf{y} \tag{2.1.52}$$

with some functions $b_1(z), \dots, b_n(z)$ *analytic at* $z=0$.

Hint: rewrite the equation (2.1.50) in the form

$$\left(z\frac{d}{dz}\right)^n y + b_1(z)\left(z\frac{d}{dz}\right)^{n-1} y + \dots b_n(z)y = 0. \tag{2.1.53}$$

Remark 2.1.21 *The characteristic equation*

$$\lambda^n + b_1(0)\lambda^{n-1} + \dots + b_n(0) = 0 \tag{2.1.54}$$

for the eigenvalues of the unperturbed matrix for (2.1.52) *can be also obtained from the limiting Euler equation*

$$z^n y_0^{(n)} + \alpha_1 z^{n-1} y_0^{(n-1)} + \dots + \alpha_n y_0 = 0 \tag{2.1.55}$$

where

$$\alpha_i := \lim_{z\to 0} z^i a_i(z), \quad i = 1, \dots, n.$$

Definition 2.1.22 *Under the assumptions* (2.1.50) *we say that the n-th order linear differential equation has a Fuchsian singularity at* $z=0$.

Differently from the case of systems the notions of regular and Fuchsian singularities coincide for scalar differential equations, as it follows from the following theorem, due to L.Fuchs.

Theorem 2.1.23 *All solutions to the scalar differential equation* (2.1.49) *with coefficients* $a_1(z), \dots, a_n(z)$ *analytic on some punctured disk* $0 < |z| < r$ *have regular singularity at* $z=0$ iff *the singularity is Fuchsian.*

Proof: Sufficiency of the Fuchsian condition follows from the reduction of a Fuchsian scalar differential equation to a Fuchsian system given in Exercise 2.1.20, due to Corollary 2.1.15. Let us prove necessity.

We begin from a simple lemma about multivalued functions analytic on the punctured disk.

Lemma 2.1.24 *Any linear differential equation* (2.1.49) *with coefficients analytic on the punctured disk* $0 < |z| < r$ *has a solution of the form* $y_0(z) = g(z)z^\lambda$ *for some complex* λ *with a function* $g(z)$ *analytic on the punctured disk.*

See Lemma 2.2.5 below for the proof.

As all solutions to the equation have regular singularity at $z = 0$, the function $g(z)$ has at most a pole at this point,

$$g(z) = \frac{h(z)}{z^k} \quad \text{for some} \quad k \in \mathbb{Z}, \quad h(z) \quad \text{is analytic for} \quad |z| < r, \quad h(0) \neq 0.$$

Doing a shift $\lambda \mapsto \lambda - k$ we obtain the solution in the form

$$y_0(z) = h(z) z^\lambda.$$

We will now prove the Theorem using induction in n. For $n = 1$ the statement is obvious: the coefficient $a(z)$ of a first order linear differential equation

$$y' + a(z) y = 0$$

reads

$$a(z) = -\frac{y'}{y} = -\frac{\lambda}{z} - \frac{h'(z)}{h(z)}.$$

The second term in this formula is analytic at $z = 0$.

To do the inductive step let us do a substitution

$$u = \frac{y}{y_0}$$

to the equation (2.1.49). We obtain a linear differential equation for the function u that has a trivial solution $u(z) \equiv 1$. Thus the coefficient of u vanishes in this new equation:

$$u^{(n)} + b_1(z) u^{(n-1)} + \cdots + b_{n-1}(z) u' = 0. \tag{2.1.56}$$

We arrive at a differential equation of order $n - 1$ for the function $v = u'$. Solutions to this equation have regular singularities at $z = 0$. From the inductive assumptions it follows existence of limits

$$\lim_{z \to 0} z^i \, b_i(z), \quad i = 1, \dots, n-1.$$

Let us compare the coefficients $b_i(z)$ with the coefficients of the original system. To this end we plug $y = u \, y_0$ into (2.1.49) and collect the coefficients of $u^{(n)}$, $u^{(n-1)}$, ..., u'. After division by y_0 we obtain a triangular transformation

$$b_1 = \binom{n}{1} \frac{y_0'}{y_0} + a_1$$

$$b_2 = \binom{n}{2} \frac{y_0''}{y_0} + \binom{n-1}{1} a_1 \frac{y_0'}{y_0} + a_2$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$b_{n-1} = \binom{n}{n-1} \frac{y_0^{(n-1)}}{y_0} + \binom{n-1}{n-2} a_1 \frac{y_0^{(n-2)}}{y_0} + \cdots + a_{n-1}$$

where we have used the Leibnitz rule for derivating products

$$(u\,v)^{(k)} = u^{(k)}v + \left(\begin{array}{c} k \\ 1 \end{array}\right) u^{(k-1)}v' + \left(\begin{array}{c} k \\ 2 \end{array}\right) u^{(k-2)}v'' + \dots + \left(\begin{array}{c} k \\ 1 \end{array}\right) u'\,v^{(k-1)} + u\,v^{(k)}. \quad (2.1.57)$$

Derivating the function $y_0 = h(z)z^\lambda$, $h(0) \neq 0$ it is easy to see that

$$\frac{y_0^{(k)}}{y_0}$$

has a pole of order k at $z = 0$. Since the functions $b_1, b_2, \dots, b_{n-1}$ have poles of the orders $1, 2, \dots, n-1$ due to the inductive hypothesis, we easily conclude the same order of poles for the functions $a_1, a_2, \dots, a_{n-1}$ respectively. Finally, from the equation for y_0

$$\frac{y_0^{(n)}}{y_0} + a_1 \frac{y_0^{(n-1)}}{y_0} + \dots + a_{n-1}\frac{y_0'}{y_0} + a_n = 0$$

we prove that a_n has a pole of order at most n. □

Example. Bessel equation

$$x^2 y'' + x\,y' + (x^2 - \nu^2)y = 0, \quad \nu \in \mathbb{C}. \quad (2.1.58)$$

The solution has a Fuchsian singularity at $x = 0$. Let us look for a solution in the form

$$y = x^\lambda(c_0 + c_1 x + c_2 x^2 + \dots).$$

We have

$$y' = x^{\lambda-1}\left[\lambda\, c_0 + (\lambda+1)\, c_1 x + (\lambda+2)\, c_2 x^2 + \dots\right].$$

In a similar way

$$y'' = x^{\lambda-2}\sum_{k\geq 0}(\lambda+k)(\lambda+k-1)\, c_k x^k.$$

After the substitution and division by x^λ we obtain

$$\sum_{k\geq 0}\left[(\lambda+k)(\lambda+k-1)\, c_k x^k + (\lambda+k)\, c_k x^k + (-\nu^2 c_k + c_{k-2})x^k\right] = 0.$$

At $k = 0$ we have

$$\lambda(\lambda-1) + \lambda - \nu^2 = 0,$$

so

$$\lambda = \pm\nu^2.$$

Assume that $\nu \neq 0$ (thus no Jordan blocks occur!). Take $\lambda = \nu$. We obtain

$$\left[(\nu+1)^2 - \nu^2\right]\, c_1 = 0 \Rightarrow c_1 = 0,$$

$$\left[(\nu+k)^2 - \nu^2\right] c_k + c_{k-2} = 0. \quad (2.1.59)$$

So

$$c_{\text{odd}} = 0$$

if 2ν is not a negative integer. Choose

$$c_0 = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

Then

$$c_{2k} = \frac{(-1)^k}{2^{2k+\nu} k! \Gamma(\nu+k+1)}$$

where we have used the recursion relation for the gamma-function

$$\Gamma(x+1) = x\,\Gamma(x). \tag{2.1.60}$$

We obtain finally a solution to the Bessel equation in the form

$$J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{2^{2k+\nu}\Gamma(\nu+k+1)} \frac{x^{\nu+2k}}{k!}. \tag{2.1.61}$$

The series converges for all x. The sum of this series is called the *Bessel function* (of the first kind). If $\nu \notin \mathbb{Z}$ then the functions $J_\nu(x)$ and $J_{-\nu}(x)$ give a fundamental system of solutions to Bessel equation. For $\nu = 0$ one can obtain a second solution to Bessel equation linear independent with $J_0(x)$ taking

$$K_0(x) := \frac{d}{d\nu} J_\nu(x)|_{\nu=0}. \tag{2.1.62}$$

Exercise 2.1.25 *Prove the following recursions for Bessel functions*

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x) \tag{2.1.63}$$

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x). \tag{2.1.64}$$

2.2 Monodromy of solutions to linear differential equations, local theory

Recall (see Section 1.2 above) that monodromy describes permutations of branches of multivalued analytic functions. In this Section we will study monodromy of functions defined on a punctured disk $0 < |z| < r$ for a sufficiently small r (that is, analytic functions on the universal covering of this punctured disk). We begin with the following simple

Example. Consider analytic function z^λ, $\lambda \in \mathbb{C}$. From the definition

$$z^\lambda := e^{\lambda(\log|z| + i\arg z + 2\pi\, i\, n)}, \quad n \in \mathbb{Z}$$

it follows that branches of this function are labeled by integers. In order to determine the result of analytic continuation of this function let us consider a closed loop

$$z(t) = z\, e^{2\pi\, i\, t}, \quad 0 \le t \le 1$$

oriented counter-clockwise. We have

$$[z(t)]^\lambda = e^{\lambda(\log|z|+i\arg z+2\pi\, i\, t+2\pi\, i\, n)}. \tag{2.2.1}$$

At $t = 1$ we obtain the result of analytic continuation

$$[z(1)]^\lambda = e^{\lambda(\log|z|+i\arg z+2\pi\, i\,(n+1))} = [z(0)]^\lambda e^{2\pi\, i\,\lambda}.$$

More generally we will adopt the following notations: the result of analytic continuation of a multivalued function $f(z)$ analytic on the punctured disk we will denote

$$f\left(z\, e^{2\pi\, i}\right).$$

For example, for $f(z) = \sqrt{z}$

$$f\left(z\, e^{2\pi\, i}\right) = -f(z);$$

for $f(z) = \log z$

$$f\left(z\, e^{2\pi\, i}\right) = f(z) + 2\pi\, i.$$

Let us study the monodromy of solutions to a system of n linear differential equations

$$y' = A(z)y \tag{2.2.2}$$

with the matrix of coefficients $A(z)$ analytic on the punctured disk $0 < |z| < r$. The following simple statement is the starting point for subsequent considerations.

Lemma 2.2.1 *Given a solution $y(z)$ to the system* (2.2.2)*, then the result $y\left(z\, e^{2\pi\, i}\right)$ of analytic continuation of the solution along the loop* (2.2.1) *is again a solution to the same system.*

Proof: After the substitution $\tilde z = z\, e^{2\pi\, i}$ we obtain for the function $\tilde y := y(\tilde z)$ the system

$$\frac{d\tilde y}{d\tilde z} = A(\tilde z)\tilde y.$$

Because of analyticity of A we have $A(\tilde z) = A(z)$. □

Definition 2.2.2 *The operator of analytic continuation*

$$y(z) \mapsto y\left(z\, e^{2\pi\, i}\right) \tag{2.2.3}$$

acting on the space of solutions to the system of linear differential equations (2.2.2) *is called* the monodromy operator *of* (2.2.2)

As the space of solutions to the system of n linear differential equations is n-dimensional, the monodromy operator is represented by a $n \times n$ matrix M in any basis of solutions. Putting the basic solutions in the columns of a fundamental matrix $Y(z)$ one spells out the definition of the *monodromy matrix* in the form

$$Y\left(z\, e^{2\pi\, i}\right) = Y(z)\, M \tag{2.2.4}$$

Because of nondegeneracy of the fundamental matrix the monodromy matrix does not degenerate.

Example. The monodromy of the fundamental matrix (2.1.36) around a Fuchsian singularity is a diagonal matrix

$$Y(z) = G(z)\, z^{\Lambda} \mapsto Y\left(z\, e^{2\pi\, i}\right) = Y(z)\, M, \quad M = e^{2\pi\, i\, \Lambda} = \mathrm{diag}\left(e^{2\pi\, i\, \lambda_1}, \dots, e^{2\pi\, i\, \lambda_n}\right). \quad (2.2.5)$$

Exercise 2.2.3 *Prove that the monodromy of the fundamental matrix* (2.1.44) *is described by the matrix*

$$M = e^{2\pi\, i\, \Lambda} e^{2\pi\, i\, R}. \quad (2.2.6)$$

Hint: Check that the matrices $e^{2\pi\, i\, \Lambda}$ and $e^{2\pi\, i\, R}$ commute.

Observe that the matrix $e^{2\pi\, i\, R}$ is upper triangular; all its diagonal entries are equal to 1.

Exercise 2.2.4 *Given a non-degenerate complex $n \times n$ matrix M, prove existence of a matrix L such that*

$$M = e^{2\pi\, i\, L}. \quad (2.2.7)$$

The following statement is often useful in the study of local properties of solutions to linear differential equations near a singular point.

Lemma 2.2.5 *There always exists a solution to the linear system* (2.2.2) *near an isolated singularity $z = 0$ that can be represented in the form*

$$y(z) = g(z) z^{\lambda} \quad (2.2.8)$$

for some $\lambda \in \mathbb{C}$ where the vector valued function $g(z)$ is analytic on the punctured disk $0 < |z| < r$. If the solution $y(z)$ grows at most polynomially at $z \to 0$ then the vector valued function $g(z)$ has at most a pole at $z = 0$.

Proof: Let $y(z)$ be an eigenvector of the monodromy operator with an eigenvalue μ. Because of nondegeneracy of the monodromy matrix $\mu \neq 0$. Put

$$\lambda = \frac{1}{2\pi\, i} \log \mu$$

for some choice of a branch of the logarithm. The product

$$g(z) := y(z) z^{-\lambda}$$

is a single-valued analytic function on the punctured disk. It remains to recall that the monodromy operator on the finite dimensional space of solutions always possess an eigenvector.

In order to prove the second part of Lemma it suffices to observe that the function $g(z)$ grows at most polynomially at $z \to 0$ if the function $y(z)$ does so. Since $g(z)$ is analytic on the punctured disk it must have a pole at $z = 0$. □

Exercise 2.2.6 *Prove that a system of the form* (2.2.2) *with an isolated singularity at the origin possesses a fundamental matrix of the form*

$$Y(z) = G(z)\, z^L \tag{2.2.9}$$

for some constant matrix L and a matrix valued function $G(z)$ analytic and nondegenerate on the punctured disk $0 < |z| < r$. Prove that in the case of regular singularity the entries of the matrix $G(z)$ have at most poles at the origin. Deduce that coefficient matrix $A(z)$ must have at most a pole at the origin if the singularity is regular.

2.3 Monodromy of Fuchsian systems and Fuchsian differential equations

We now proceed to the global study of monodromy of differential equations with rational coefficients.

A system of n first order linear differential equations with rational coefficients

$$y' = A(z)\, y \tag{2.3.1}$$

is said to be *Fuchsian* if it can be represented in the form

$$(z - z_0)\, y' = \left[A_0 + A_1(z - z_0) + A_2(z - z_0)^2 + \dots\right] y \tag{2.3.2}$$

near any singular point $z_0 \in \bar{\mathbb{C}}$.

Equivalently, using the simple fraction decomposition of the rational matrix valued function $A(z)$ one obtains a representation of a Fuchsian system in the form

$$\frac{dy}{dz} = \left[\frac{A_1}{z - z_1} + \dots + \frac{A_k}{z - z_k}\right] y \tag{2.3.3}$$

where $A_1, \dots, A_n$ are $n \times n$ constant matrices and the complex numbers $z_1, \dots, z_k$ are pairwise distinct. The singularities of the system (2.3.3) are at these points and, possibly, at infinity.

Lemma 2.3.1 *After the substitution*

$$w = \frac{1}{z}$$

the system (2.3.3) *rewrites*

$$\frac{dy}{dw} = \left[\frac{A_\infty}{w} + \mathcal{O}(1)\right] y \tag{2.3.4}$$

where

$$A_\infty := -(A_1 + \dots + A_k). \tag{2.3.5}$$

Proof: After the substitution one obtains

$$\begin{aligned} -w^2 \frac{dy}{dw} &= w \left[\frac{A_1}{1 - z_1 w} + \dots + \frac{A_k}{1 - z_k w}\right] y \\ &= w \left[A_1 + \dots + A_k + w \left(\frac{z_1 A_1}{1 - z_1 w} + \dots + \frac{z_k A_k}{1 - z_k w}\right)\right] y. \end{aligned} \tag{2.3.6}$$

So

$$\frac{dy}{dw} = \left[\frac{A_\infty}{w} - \left(\frac{z_1 A_1}{1 - z_1 w} + \cdots + \frac{z_k A_k}{1 - z_k w}\right)\right] y.$$

□

Corollary 2.3.2 *The infinite point of the Riemann sphere is a Fuchsian singularity of* (2.3.3) *if* $A_\infty \neq 0$*; the system* (2.3.3) *has no singularity at infinity* iff $A_\infty = 0$.

Exercise 2.3.3 *Prove that after an arbitrary fractional linear transformation*

$$w = \frac{a\,z + b}{c\,z + d}, \quad a\,d - b\,c \neq 0$$

the Fuchsian system (2.3.3) *goes to another Fuchsian system with the singularities at*

$$w_i = \frac{a\,z_i + b}{c\,z_i + d}, \quad i = 1, \ldots, k+1, \quad \text{where} \quad z_{k+1} = \infty$$

and with the residue matrices $A_1, \ldots, A_k, A_{k+1} := A_\infty$.

We are ready to define an important characteristics of a Fuchsian system: the *monodromy representation.* Let us fix a nonsingular point

$$z_0 \in \bar{\mathbb{C}} \setminus \{z_1, \ldots, z_k, z_{k+1}\}.$$

For any closed loop

$$\gamma : [0,1] \to \bar{\mathbb{C}} \setminus \{z_1, \ldots, z_k, z_{k+1}\}, \quad \gamma(0) = \gamma(1) = z_0$$

denote $\hat{M}_\gamma$ the linear transformation

$$y \mapsto \hat{M}_\gamma(y) \tag{2.3.7}$$

the result of analytic continuation of a solution to (2.3.3) along the loop γ. According to Lemma 2.2.1 the result of the analytic continuation defines a linear operator in the space of solutions to (2.3.3) that does not depend on the deformation of the closed loop keeping the end points $\gamma(0) = \gamma(1) = z_0$ fixed. We obtain a map

$$\gamma \mapsto \hat{M}_\gamma$$

of the fundamental group

$$\pi_1\left(\bar{\mathbb{C}} \setminus \{z_1, \ldots, z_k, z_{k+1}\}; z_0\right) \to \mathrm{Aut}(\mathbb{C}^n) \tag{2.3.8}$$

to the group of automorphisms of the linear space of solutions to the system (2.3.3).

Choosing a basis in the space of solutions one can describe the monodromy (2.3.8) by matrices. Namely, analytic continuation of a fundamental matrix $Y(z)$ along the loop γ gives a new fundamental matrix $\hat{M}_\gamma(Y)$ of the same system. The latter is related to Y by the right multiplication by an invertible matrix M_γ:

$$\hat{M}_\gamma(Y(z)) = Y(z)M_\gamma. \tag{2.3.9}$$

Recall that the elements of the fundamental group $\pi_1(X; z_0)$ of a topological space X with respect to the marked point $z_0 \in X$ are the homotopy classes of the continuous loops

$$\gamma : [0,1] \to X, \quad \gamma(0) = \gamma(1).$$

The product $\gamma = \gamma_1\gamma_2$ of two elements in the fundamental group is defined by the consecutive running along the two loops $\gamma_1(t)$ and $\gamma_2(t)$:

$$\gamma(t) = \begin{cases} \gamma_1(2\,t), & 0 \leq t \leq \frac{1}{2} \\ \\ \gamma_2(2\,t-1), & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{2.3.10}$$

for $0 \leq t \leq 1$. The inverse element to γ is defined as the homotopy class of the same loop passed in the opposite direction

$$\gamma^{-1}(t) := \gamma(1-t). \tag{2.3.11}$$

The neutral element is the trivial loop

$$\gamma(t) \equiv z_0 \quad \text{for any} \quad 0 \leq t \leq 1.$$

We are ready to prove the main property of the monodromy map (2.3.8).

Lemma 2.3.4 *The monodromy map* (2.3.8) *is an antihomomorphism of the fundamental group of the punctured Riemann sphere to the group of automorphisms of the space of solutions to* (2.3.3)*, i.e., for any two loops* γ_1*,* γ_2 *on* $\bar{\mathbb{C}} \setminus \{z_1, \dots, z_k, z_{k+1}\}$ *one has*

$$\hat{M}_{\gamma_1\gamma_2} = \hat{M}_{\gamma_2}\hat{M}_{\gamma_1}. \tag{2.3.12}$$

Moreover, for any loop γ

$$\hat{M}_{\gamma^{-1}} = \hat{M}_\gamma^{-1}. \tag{2.3.13}$$

To the homotopy class of the trivial loop it corresponds the identity map of the space of solutions.

Proof: It suffices to prove validity of (2.3.12). Let us do it using the matrix realization (2.3.9) of the monodromy map with respect to a choice of the fundamental matrix $Y(z)$. The analytic continuation of $Y(z)$ along the loop γ_1 transforms $Y(z)$ to $Y(z)M_{\gamma_1}$. The subsequent analytic continuation of the latter matrix along γ_2 gives

$$(Y(z)M_{\gamma_2})\ M_{\gamma_1} = Y(z)M_{\gamma_2}M_{\gamma_1}.$$

This proves that

$$M_{\gamma_1\gamma_2} = M_{\gamma_2}M_{\gamma_1}.$$

□

Definition 2.3.5 *Let $Y(z)$ be a fundamental matrix of the system* (2.3.3). *The antihomomorphism*

$$\pi_1\left(\bar{\mathbb{C}} \setminus \{z_1, \ldots, z_k, z_{k+1}\}; z_0\right) \to GL(n;\mathbb{C}) \tag{2.3.14}$$

$$\gamma \mapsto M_\gamma \tag{2.3.15}$$

mapping the homotopy class of aloop γ to the monodromy matrix M_γ is called the monodromy representation *of the system associated with the chosen fundamental matrix.*

Exercise 2.3.6 *Prove that a change of the fundamental matrix*

$$Y(z) \mapsto \tilde{Y}(z) = Y(z)\, C$$

changes the monodromy representation to the conjugate one:

$$\tilde{M}_\gamma = C^{-1} M_\gamma\, C. \tag{2.3.16}$$

Scalar case. Fuchsian differential equations of order n can be described according to the following statement.

Lemma 2.3.7 *Any Fuchsian differential equation of order n must have the form*

$$\begin{aligned} &y^{(n)} + a_1(z)\, y^{(n-1)} + \cdots + a_n(z)\, y = 0 \\ &a_m(z) = \frac{Q_m(z)}{[P(z)]^m}, \quad m = 1, 2, \ldots, n \\ &P(z) = (z - z_1)(z - z_2)\ldots(z - z_k), \quad z_i \neq z_i \quad \text{for} \quad i \neq j \\ &Q_m(z) \quad \text{is a polynomial of degree} \quad m\,(k-1), \quad m = 1, 2, \ldots, n. \end{aligned} \tag{2.3.17}$$

The regular singularities are at the points $z_1, \ldots, z_k, z_{k+1} = \infty$.

Denote

$$\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}$$

the characteristic exponents of (2.3.17) at the singular point $z = z_i$, $i = 1, \ldots, k, k+1$.

Exercise 2.3.8 *(cf. Exercise 2.3.3). Prove that after a fractional linear transformation*

$$z \mapsto \tilde{z} = \frac{a\,z + b}{c\,z + d}, \quad a\,d - b\,c \neq 0$$

the Fuchsian equation (2.3.17) *transforms to another Fuchsian equation with singularities at*

$$\tilde{z}_i = \frac{a\,z_i + b}{c\,z_i + d}, \quad i = 1, \ldots, k+1$$

with the same characteristic exponents.

Theorem 2.3.9 (Fuchs relation). *The characteristic exponents of a Fuchsian differential equation* (2.3.17) *satisfy*

$$\sum_{i=1}^{k+1}\left[\lambda_1^{(i)}+\cdots+\lambda_n^{(i)}\right]=\frac{n(n-1)}{2}\,(k-1). \tag{2.3.18}$$

Proof: For any system $y_1(z), \ldots, y_n(z)$ of n functions denote

$$W(z)=\det\begin{pmatrix} y_1 & y_2 & \ldots & y_n\\ y_1' & y_2' & \ldots & y_n'\\ y_1'' & y_2'' & \ldots & y_n''\\ \ldots & \ldots & \ldots & \ldots\\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)}\end{pmatrix} \tag{2.3.19}$$

the Wronskian determinant of this system. The following properties of the Wronskian can be easily established:

• If $y_1(z), \ldots, y_n(z)$ is a basis in the space of solutions to an n-th order linear differential equation (2.3.17) then the Wronskian $W(z)$ never vanishes for $z \neq z_1, \ldots, z_k, z_{k+1}$.

• Given two bases $y_1(z), \ldots, y_n(z)$ and $\tilde{y}_1(z), \ldots, \tilde{y}_n(z)$, then the two Wronskians $W(z)$ and $\tilde{W}(z)$ are related by multiplication by a nonzero constant,

$$\tilde{W}(z)=\rho\, W(z), \quad \rho \neq 0.$$

• The analytic continuation of solutions $y_1(z), \ldots, y_n(z)$ along a closed loop $\gamma \in \bar{\mathbb{C}} \setminus \{z_1, \ldots, z_k, z_{k+1}\}$, $\gamma(0)=\gamma(1)=z_0$ results in multiplication of the Wronskian by a nonzero constant

$$W(z_0) \mapsto \mu_\gamma\, W(z_0)$$

independent on the choice of the basepoint z_0.

• Change of the independent variable $z \mapsto \tilde{z}=f(z)$ yields the following transformation of the Wronskian:

$$W(z)=\left(\frac{d\tilde{z}}{dz}\right)^{\frac{n(n-1)}{2}} W(\tilde{z}). \tag{2.3.20}$$

Indeed, one has

$$\begin{aligned}\frac{dy}{dz}&=\frac{d\tilde{z}}{dz}\frac{dy}{d\tilde{z}}\\ \frac{d^2y}{dz^2}&=\left(\frac{d\tilde{z}}{dz}\right)^2\frac{d^2y}{d\tilde{z}^2}+\frac{d^2\tilde{z}}{dz^2}\frac{dy}{d\tilde{z}}\end{aligned}$$

etc., and, more generally,

$$\frac{d^m y}{dz^m}=\left(\frac{d\tilde{z}}{dz}\right)^m\frac{d^m y}{d\tilde{z}^m}+\text{linear combination of }\ \frac{dy}{d\tilde{z}},\ldots,\frac{d^{m-1}y}{d\tilde{z}^{m-1}},\quad m=1,\ldots,n-1.$$

So, up to adding of a linear combination of the previous rows, the rows of the Wronskian will be multiplied by 1, $\frac{d\tilde{z}}{dz}$, $\left(\frac{d\tilde{z}}{dz}\right)^2$ …, $\left(\frac{d\tilde{z}}{dz}\right)^{n-1}$ resp. This gives the transformation law (2.3.20).

We will now study the behavior of the Wronskian at the singularities. Let us assume for simplicity that all characteristic exponents are pairwise distinct and nonresonant. Denote

$$y_1^{(i)} = g_1^{(i)}(z)(z-z_i)^{\lambda_1^{(i)}}, \dots, y_n^{(i)} = g_n^{(i)}(z)(z-z_i)^{\lambda_n^{(i)}}, \quad i = 1, \dots, k,\ k+1 \tag{2.3.21}$$

the bases in the space of solutions associated with these exponents, i.e., the functions $g_1^{(i)}(z)$, ..., $g_n^{(i)}(z)$ are analytic near $z = z_i$ and, moreover, $g_m^{(i)}(z_i) = 1$ for any $m = 1, \dots, n$. Without loss of generality we may assume, doing if necessary a fractional linear transformation, that all the $k+1$ singular points are away from infinity. Then it is easy to see that the Wronskian, calculated for the basis $y_1^{(i)}(z)$, ..., $y_n^{(i)}(z)$ behaves like

$$W(z) = c_i\,(z-z_i)^{\lambda_1^{(i)}+\dots+\lambda_n^{(i)}-\frac{n(n-1)}{2}}\,(1+\mathcal{O}(z-z_i))\,, \quad z \to z_i. \tag{2.3.22}$$

where

$$c_i = \prod_{p<q}\left(\lambda_p^{(i)} - \lambda_q^{(i)}\right) \neq 0.$$

At infinity, according to (2.3.20), one has

$$W(z) = c_\infty z^{n(n-1)}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right), \quad c_\infty \neq 0 \tag{2.3.23}$$

(do a transformation $\tilde{z} = 1/z$).

We now consider the logarithmic differential

$$\omega := d\log W(z). \tag{2.3.24}$$

The differential ω is meromorphic on the Riemann sphere. It has poles at the points z_1, ..., z_{k+1} with residues

$$\mathrm{res}_{z=z_i}\,\omega = \lambda_1^{(i)} + \dots + \lambda_n^{(i)} - \frac{n(n-1)}{2}, \quad i = 1, \dots, k+1$$

and infinity,

$$\mathrm{res}_{z=\infty}\,\omega = -n(n-1).$$

Applying the residue theorem

$$\sum_{P\in\bar{\mathbb{C}}} \mathrm{res}_P\,\omega = 0$$

one obtains (2.3.18). □

Exercise 2.3.10 *Assume as above $z_{k+1} = \infty$. Prove that the substitution*

$$y = (z-z_1)^{\alpha_1}(z-z_2)^{\alpha_2}\dots(z-z_k)^{\alpha_k}\,\tilde{y}$$

transforms the Fuchsian equation (2.3.17) *into another Fuchsian equation for $\tilde{y} = \tilde{y}(z)$ with the same singularities and with the characteristic exponents*

$$\tilde{\lambda}_m^{(i)} = \lambda_m^{(i)} - \alpha_i, \quad m = 1, \dots, n, \quad i = 1, \dots, k.$$

Let us consider in details the case of second order Fuchsian equations with three singularities at $z = z_1$, $z = z_2$, $z = z_3$. The solutions $y = y(z)$ to this equation are conveniently denoted using the so-called *Riemann scheme*

$$y = \mathcal{P}\begin{pmatrix} z_1 & z_2 & z_3 & & \\ \lambda_1^{(1)} & \lambda_1^{(2)} & \lambda_1^{(3)} & ; & z \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \lambda_2^{(3)} & & \end{pmatrix} \tag{2.3.25}$$

where we keep the same notations as above for the characteristic exponents. To motivate this notation we will show that the equation is uniquely determined by the characteristic exponents. So the symbol (2.3.25) denotes a two-dimensional space of solutions to the associated differential equation. Recall that, due to Fuchs relation the characteristic exponents are constrained by the equation

$$\lambda_1^{(1)} + \lambda_2^{(1)} + \lambda_1^{(2)} + \lambda_2^{(2)} + \lambda_1^{(3)} + \lambda_2^{(3)} = 1. \tag{2.3.26}$$

Theorem 2.3.11 (Riemann).*The coefficients of the second order Fuchsian equation with three regular singularities are uniquely determined by the position of singularities and by the characteristic exponents.*

Proof: The equation must have the form

$$y'' + p(z)\, y' + q(z)\, y = 0. \tag{2.3.27}$$

$$p(z) = \frac{a_1}{z - z_1} + \frac{a_2}{z - z_2} + \frac{a_3}{z - z_3}$$

$$q(z) = \frac{1}{(z - z_1)(z - z_2)z - z_3)}\left(\frac{b_1}{z - z_1} + \frac{b_2}{z - z_2} + \frac{b_3}{z - z_3}\right)$$

$$a_i = 1 - (\lambda_1^{(i)} + \lambda_2^{(i)})$$

$$b_i = \lambda_1^{(i)}\lambda_2^{(i)}(z_i - z_j)(z_i - z_k), \quad i = 1,\, 2,\, 3.$$

□

Exercise 2.3.12 *Prove that a second order Fuchsian differential equation with singularities at $z_1, \ldots, z_k$, $z_{k+1} = \infty$ with the prescribed characteristic exponents $\lambda_1^{(i)}$, $\lambda_2^{(i)}$, $i = 1, \ldots, k+1$*

must have the form

$$y'' + p(z)\, y' + q(z)\, y = 0 \tag{2.3.28}$$

$$p(z) = \sum_{i=1}^{k} \frac{1 - (\lambda_1^{(i)} + \lambda_2^{(i)})}{z - z_i}$$

$$q(z) = \frac{1}{P(z)} \left[\sum_{i=1}^{k} \frac{\lambda_1^{(i)} \lambda_2^{(i)}}{z - z_i} P'(z_i) + \lambda_1^{(\infty)} \lambda_2^{(\infty)} z^{k-2} + c_1 z^{k-3} + \dots + c_{k-2} \right]$$

$$\text{where} \quad P(z) = \prod_{i=1}^{k} (z - z_i).$$

Generalize to the case of Fuchsian equations of higher orders.

For $k > 2$ the coefficients $c_1, \ldots, c_{k-2}$ are not determined by the poles and the exponents. They are called *accessory parameters* of the Fuchsian equation.

2.4 Gauss equation and hypergeometric function

$$x(1-x)\, y'' + [\gamma - (\alpha + \beta + 1)\, x]\; y' - \alpha\, \beta\, y = 0. \tag{2.4.1}$$

Riemann scheme for the Gauss equation:

$$y = \mathcal{P} \begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & \alpha & ; & x \\ 1-\gamma & \gamma - \alpha - \beta & \beta & & \end{pmatrix}. \tag{2.4.2}$$

Let

$$\gamma \neq 0,\, -1,\, -2, \ldots.$$

Denote

$$(\alpha)_n := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}, \tag{2.4.3}$$

i.e.

$$(\alpha)_0 = 1, \quad (\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1), \quad n = 1, 2, \ldots.$$

Define the *hypergeometric series* by

$${}_2F_1(\alpha, \beta; \gamma; x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!} \tag{2.4.4}$$

Lemma 2.4.1 *The series* (2.4.4) *converges for* $|x| < 1$*. It gives the unique solution to Gauss equation* (2.4.1) *analytic at* $x = 0$.

The resulting function ${}_2F_1(\alpha,\beta;\gamma;x)$ analytic for $|x| < 1$ here will be also denoted $F(\alpha,\beta;\gamma;x)$ for brevity.

The hypergeometric function can be analytically continued to the entire complex plane with a branch cut from $x = 1$ to $x = \infty$. This gives a possibility to construct a basis of solutions to the hypergeometric equation near $x = 1$ and $x = \infty$.

The substitution

$$y = x^{1-\gamma} u$$

yields a Gauss equation for u of the form

$$x(1-x)\,u'' + [2-\gamma+(\alpha+\beta-2\gamma+3)\,x]\,u' - (\alpha-\gamma+1)(\beta-\gamma+1)\,u = 0. \tag{2.4.5}$$

So the function

$$y = x^{1-\gamma} F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\,x), \quad \gamma \notin \mathbb{Z} \tag{2.4.6}$$

gives another solution to (2.4.1) corresponding to the characteristic exponent $1-\gamma$.

In a similar way at $x = 1$ one obtains the following two solutions

$$y = C_1 F(\alpha,\beta;\alpha+\beta-\gamma+1;\,1-x) + C_2(1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;\,1-x). \tag{2.4.7}$$

At infinity:

$$y = C_1 x^{-\alpha} F(\alpha,1-\gamma+\alpha;1-\beta+\alpha;\,x^{-1}) + C_2 x^{-\beta} F(\beta,1-\gamma+\beta;1-\alpha+\beta;\,x^{-1}) \tag{2.4.8}$$

Exercise 2.4.2 *Derive the following particular cases of the hypergeometric function:*

$$F(-n,\beta;\beta;-x) = (1+x)^n, \quad n \in \mathbb{Z} \tag{2.4.9}$$

$$F(1,1;2;-x) = \frac{\log(1+x)}{x}. \tag{2.4.10}$$

Exercise 2.4.3 *Prove that*

$$\frac{d}{dx} F(\alpha,\beta;\gamma;x) = \frac{\alpha\,\beta}{\gamma} F(\alpha+1,\beta+1;\gamma+1;x). \tag{2.4.11}$$

Exercise 2.4.4 *For the case $\gamma = 1$ prove that the second solution of Gauss equation linearly independent from $F(\alpha,\beta;1;x)$ reads*

$$\begin{aligned} y &= \lim_{\gamma\to 1} (\gamma-1)^{-1}\left[x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;x) - F(\alpha,\beta;\gamma;x)\right] \\ &= F(\alpha,\beta;1;x)\,\log x + \sum_{n=1}^\infty \frac{(\alpha)_n(\beta)_n}{(n!)^2}\,x^n \sum_{k=1}^n \left(\frac{1}{\alpha+k-1} + \frac{1}{\beta+k-1} - \frac{2}{k}\right). \end{aligned} \tag{2.4.12}$$

Monodromy of Gauss equation, "brute force" method: look for three matrices M_0, M_1, M_∞ with the eigenvalues $(1, \nu^{-1})$, $(1, \nu\,\lambda^{-1}\mu^{-1})$ and (λ, μ) respectively satisfying

$$M_\infty M_1 M_0 = \mathrm{Id}. \tag{2.4.13}$$

Here

$$\lambda = e^{2\pi\, i\, \alpha}, \quad \mu = e^{2\pi\, i\, \beta}, \quad \nu = e^{2\pi\, i\, \gamma}. \tag{2.4.14}$$

The nonresonancy will be assumed:

$$\lambda \neq \mu, \quad \nu \neq 1, \quad \nu \neq \lambda\, \mu.$$

Without loss of generality we may assume M_∞ to be diagonal,

$$M_\infty = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Denoting

$$M_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

one obtains a system of equations

$$\begin{aligned}
&\lambda\,(a_0 a_1 + b_1 c_0) = 1 \\
&\mu\,(b_0 c_1 + d_0 d_1) = 1 \\
&a_1 b_0 + b_1 d_0 = 0 \\
&a_0 c_1 + c_0 d_1 = 0. \\
&a_0 + b_0 = 1 + \nu^{-1}, \qquad\qquad a_0 d_0 - b_0 c_0 = \nu^{-1} \\
&a_1 + b_1 = 1 + \nu\,\lambda^{-1}\mu^{-1}, \quad a_1 d_1 - b_1 c_1 = \nu\,\lambda^{-1}\mu^{-1}.
\end{aligned}$$

After some calculations one obtains

$$M_0 = \frac{1}{\lambda - \mu} \begin{pmatrix} \frac{\mu}{\nu}(\lambda - \nu - 1) + 1 & \frac{\lambda}{\nu}(\mu - \lambda) \\ \frac{(\lambda-1)(\mu-1)(\lambda-\nu)(\mu-\nu)}{\lambda\,\nu(\lambda-\mu)} & \frac{\lambda}{\nu}(\nu - \mu + 1) - 1 \end{pmatrix} \tag{2.4.15}$$

$$M_1 = \frac{1}{\lambda - \mu} \begin{pmatrix} \nu - \mu - \frac{\nu}{\lambda} + 1 & \lambda - \mu \\ \frac{(\lambda-1)(\mu-1)(\nu-\mu)(\lambda-\nu)}{\lambda\,\mu(\lambda-\mu)} & \lambda - \nu + \frac{\nu}{\mu} - 1 \end{pmatrix} \tag{2.4.16}$$

Observe that the monodromy matrices M_0, M_∞ are determined non uniquely: there remains an ambiguity doing simultaneous conjugations by a diagonal matrix D:

$$(M_0, M_1) \mapsto \left(D^{-1} M_0 D, D^{-1} M_1 D\right), \quad D = \mathrm{diag}(d_1, d_2).$$

Such a conjugation does not change the diagonal matrix M_∞.

Second method: using Barnes integral representation of hypergeometric functions.

Consider the following integral

$$f(\alpha, \beta; \gamma; z) := \frac{1}{2\pi\, i} \int_{-i\,\infty}^{i\,\infty} \frac{\Gamma(\alpha + s)\Gamma(\beta + s)}{\Gamma(\gamma + s)}\, \Gamma(-s)\, (-z)^s\, ds. \tag{2.4.17}$$

It is assumed that

$$|\arg(-z)| < \pi;$$

the integration path is chosen in such a way that the poles of the function $\Gamma(\alpha+s)\Gamma(\beta+s)$ are *on the left* from the integration path and poles of the function $\Gamma(-s)$ are *on the right* from the integration path. Recall that the function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \quad \mathrm{Re}\, x > 0$$

can be analytically continued to a meromorphic function on the complex plane by using the functional equation

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{2.4.18}$$

It has poles at the non positive integer points

$$x = 0,\ -1,\ -2,\ \dots$$

with the residues

$$\mathrm{res}_{x=-n}\,\Gamma(x) = \frac{(-1)^n}{n!}. \tag{2.4.19}$$

For large $|x|$ the asymptotic behaviour of gamma-function is described by the Stirling formula

$$\log\Gamma(x+a) = \left(x + a - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log 2\pi + o(1)$$
$$\text{for} \quad |x| \to \infty, \quad |\arg(x+a)| \le \pi - \epsilon \quad \text{and} \quad |\arg x| \le \pi - \epsilon \tag{2.4.20}$$

for any small positive ϵ.

From the Stirling formula it follows that, for large $|s|$ on the integration contour the integrand behaves like

$$\mathcal{O}\left[|s|^{\alpha+\beta-\gamma-1} \exp\{-\arg(-z)\cdot \mathrm{Im}\, s - \pi\,|\mathrm{Im} s|\}\right].$$

Therefore the integral converges and, moreover it defines a function analytic in z on the domain $|\arg z| \le \pi - \epsilon$.

Let us prove that the function $f = f(\alpha, \beta; \gamma; z)$ satisfies the Gauss equation in z. Using the identity

$$x\,\Gamma(x) = \Gamma(x+1)$$

we find for the derivatives of the function f the following expressions

$$f' = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,\Gamma(1-s)\,(-z)^{s-1}\,ds$$

$$f'' = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,\Gamma(2-s)\,(-z)^{s-2}\,ds.$$

After the substitution into the Gauss equation one obtains

$$z(1-z)\, f'' + [\gamma - (\alpha+\beta+1)\, z]\, f' - \alpha\,\beta\, f$$
$$= \frac{1}{2\pi\, i}\int_{-i\,\infty}^{i\,\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,(\gamma+s-1)\Gamma(1-s)(-z)^{s-1}\,ds \tag{2.4.21}$$
$$-\frac{1}{2\pi\, i}\int_{-i\,\infty}^{i\,\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,(s+\alpha)(s+\beta)\Gamma(-s)(-z)^{s}\,ds.$$

In the first integral do the substitution

$$s = 1+t.$$

The new integrand reads

$$\frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,(\gamma+s-1)\Gamma(1-s)(-z)^{s-1} =$$
$$= \frac{\Gamma(\alpha+t+1)\Gamma(\beta+t+1)}{\Gamma(\gamma+t+1)}\,(\gamma+t)\Gamma(-t)(-z)^{t} = (\alpha+t)(\beta+t)\frac{\Gamma(\alpha+t)\Gamma(\beta+t)}{\Gamma(\gamma+t)}\Gamma(t)(-z)^{t}.$$

So the two integrals in (2.4.21) cancel.

We now want to prove that the Barnes integral is equal to the hypergeometric function (2.4.4), up to a constant factor.

Lemma 2.4.5 *For $|z|<1$ one has the following identity*

$$f(\alpha,\beta;\gamma;z) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,F(\alpha,\beta;\gamma;z). \tag{2.4.22}$$

Proof: Let us consider the integral of the same expression as in (2.4.17) taken along the half circle C of radius $N+\frac{1}{2}$, $N\in\mathbb{Z}$ with the center at the origin laying to the right of imaginary axes. Using the functional equation (2.4.18) we arrive at the following integral

$$\frac{1}{2\pi\, i}\int_C \frac{\pi\,\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)\Gamma(1+s)\sin\pi s}\,(-z)^s\,ds. \tag{2.4.23}$$

From the Stirling formula we obtain that

$$\frac{\pi\,\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)\Gamma(1+s)\sin\pi s}\,(-z)^s = \mathcal{O}\left(N^{\alpha+\beta-\gamma-1}\right)\frac{(-z)^s}{\sin\pi s}$$

uniformly in $\arg s$ on the contour C when $N\to\infty$. The substitution

$$s = \left(N+\frac{1}{2}\right)\, e^{i\,\theta}$$

yields, for $|z| < 1$,

$$\frac{(-z)^s}{\sin \pi s} = \mathcal{O}\left[\exp\left\{\left(N+\frac{1}{2}\right)\cos\theta\,\log|z| - \left(N+\frac{1}{2}\right)\sin\theta\,\arg(-z) - \left(N+\frac{1}{2}\right)\pi\,|\sin\theta|\right\}\right]$$

$$= \mathcal{O}\left[\exp\left\{\left(N+\frac{1}{2}\right)\cos\theta\,\log|z| - \left(N+\frac{1}{2}\right)\epsilon\,|\sin\theta|\right\}\right]$$

$$= \begin{cases} \mathcal{O}\left[\exp\left\{2^{-\frac{1}{2}}\left(N+\frac{1}{2}\right)\log|z|\right\}\right], & 0 \le |\theta| \le \frac{1}{4}\pi \\ \mathcal{O}\left[\exp\left\{-2^{-\frac{1}{2}}\epsilon\left(N+\frac{1}{2}\right)\right\}\right], & \frac{1}{4}\pi \le |\theta| \le \frac{1}{2}\pi. \end{cases}$$

So, for $\log|z| < 0$ (i.e., for $|z| < 1$) the integrand goes to zero rapidly enough on the arc C. Therefore

$$\int_C \to 0 \quad \text{for} \quad N \to \infty.$$

Let us now consider the following loop integral:

$$\int_{-i\infty}^{i\infty} - \left\{\int_{-i\infty}^{-i\left(N+\frac{1}{2}\right)} + \int_C + \int_{i\left(N+\frac{1}{2}\right)}^{i\infty}\right\}.$$

By Cauchy theorem it is equal to the sum of residues of the integrand at the poles $s = 0,\,1,\,2,\ldots,N$. Let N go to infinity. Then the last three integrals will go to zero under assumption

$$|\arg(-z)| \le \pi - \epsilon \quad \text{and} \quad |z| < 1.$$

It remains to compute the residue

$$\mathrm{res}_{s=n}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,\Gamma(-s)\,(-z)^s$$

for any nonnegative integer n. Applying (2.4.19) we obtain

$$\mathrm{res}_{s=n}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,\Gamma(-s)\,(-z)^s = \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\,\frac{z^n}{n!}.$$

Finally we obtain for the Barnes integral the hypergeometric series

$$f(\alpha,\beta;\gamma;z) = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\,\frac{z^n}{n!} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,F(\alpha,\beta;\gamma;z).$$

□

We are now ready to derive the formula expressing the hypergeometric solution $F(\alpha,\beta;\gamma;z)$ with two solutions (2.4.8) defined near $z = \infty$. Consider the integral

$$\frac{1}{2\pi i}\int_D \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,\Gamma(-s)\,(-z)^s\,ds \tag{2.4.24}$$

where D is the half of a circle of the radius R with the centre at the origin lying on the *left* from the imaginary axis. Like in the previous calculation one can show that the integral

(2.4.24) goes to zero when $R \to \infty$ and $|z| > 1$, assuming that $|\arg(-z)| < \pi$ and the sequence of radii R is chosen in such a way that the distance from the arc D to the poles of the integrand is bounded from below by a positive constant. Computing the residues of the integrand at the poles

$$s = -\alpha - n \quad \text{and} \quad s = -\beta - n, \quad n \in \mathbb{Z}$$

and applying Cauchy theorem one finds

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,\Gamma(-s)\,(-z)^s\,ds \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(1-\gamma+\alpha+n)}{\Gamma(1+n)\Gamma(1-\beta+\alpha+n)}\,\frac{\sin\pi(\gamma-\alpha-n)}{\cos\pi n\,\sin\pi(\beta-\alpha-n)}\,(-z)^{-\alpha-n} \\
&+ \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n)\Gamma(1-\gamma+\beta+n)}{\Gamma(1+n)\Gamma(1-\alpha+\beta+n)}\,\frac{\sin\pi(\gamma-\beta-n)}{\cos\pi n\,\sin\pi(\alpha-\beta-n)}\,(-z)^{-\beta-n}.
\end{aligned}$$

Finally one arrives at the following *connection formula* expressing one solution to Gauss equation as alinear combination of two other solutions:

$$\begin{aligned}
\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,F(\alpha,\beta;\gamma;z) &= \frac{\Gamma(\alpha)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma)}\,(-z)^{-\alpha}F\left(\alpha,1-\gamma+\alpha;1-\beta+\alpha;\frac{1}{z}\right) \\
&+ \frac{\Gamma(\beta)\,\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma)}\,(-z)^{-\beta}F\left(\beta,1-\gamma+\beta;1-\alpha+\beta;\frac{1}{z}\right).
\end{aligned} \tag{2.4.25}$$

The expression (2.4.25) is valid under the assumption

$$|\arg(-z)| < \pi.$$

Exercise 2.4.6 *Compute the integral*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(\alpha+s)\,\Gamma(-s)\,(-z)^s\,ds$$

assuming $|\arg(-z)| < \pi$.

Exercise 2.4.7 (Gauss formula). *Prove that*

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}. \tag{2.4.26}$$

for

$$\mathrm{Re}\,(\gamma-\alpha-\beta) > 0$$

Hint: use Euler integral representation (2.4.37).

Exercise 2.4.8 (Barnes Lemma). *Prove that*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(\alpha+s)\Gamma(\beta+s)\,\Gamma(\gamma-s)\,\Gamma(\delta-s)\,ds$$
$$= \frac{\Gamma(\alpha+\gamma)\,\Gamma(\alpha+\delta)\,\Gamma(\beta+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} \tag{2.4.27}$$

The integration contour is chosen in such a way that poles of $\Gamma(\gamma-s)\,\Gamma(\delta-s)$ *are on the left and poles of* $\Gamma(\alpha+s)\Gamma(\beta+s)$ *are on the right of it. (It is also assumed that the nmbers* α*,* β*,* γ*,* δ *are such that neither of the poles of the first function coincides with any pole of the second function.*

Exercise 2.4.9 *Using Barnes Lemma prove validity of the following connection formula*

$$\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,\Gamma(\alpha)\,\Gamma(\beta)\,F(\alpha,\beta;\gamma;z) =$$
$$= \Gamma(\gamma)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha-\beta)\,F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z) + \tag{2.4.28}$$
$$+\Gamma(\gamma)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)\,\Gamma(\alpha+\beta-\gamma)\,(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z)$$

assuming that

$$|\arg(1-z)|<\pi \quad \text{and} \quad |1-z|<1.$$

Hint:

We finally arrive at the following result. Assume that none of the numbers $1-\gamma$, $\gamma-\alpha-\beta$, $\alpha-\beta$ is an integer. Then we can construct three bases in the space of solutions to Gauss equation as it was explained above (see formulae (2.4.6) - (2.4.8)). We will write the three pairs of basic vectors as three row matrices

$$Y_0(x) := \left(F(\alpha,\beta;\gamma;x),\, x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;x)\right) \tag{2.4.29}$$

$$Y_1(x) := \Big(F(\alpha,\beta;\alpha+\beta-\gamma+1;1-x),\,(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-x)\Big) \tag{2.4.30}$$

$$Y_\infty(x) = \Big(x^{-\alpha}F(\alpha,1-\gamma+\alpha;1-\beta+\alpha;x^{-1}),\,x^{-\beta}F(\beta,1-\gamma+\beta;1-\alpha+\beta;x^{-1})\Big) \tag{2.4.31}$$

In these formula we assume that x belongs to the upper half plane $\operatorname{Im} x>0$. Then in the definition of fractional powers of x and $1-x$ we choose the principal branch of the arguments

$$0<\arg x<\pi, \quad 0<\arg(1-x)<\pi.$$

Theorem 2.4.10 *Under the above assumptions the following connection formulae hold true for* $\operatorname{Im} x > 0$

$$Y_0(x) = Y_1(x) \begin{pmatrix} \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} & \frac{\Gamma(2-\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\,\Gamma(1-\beta)} \\ \frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} & \frac{\Gamma(2-\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)} \end{pmatrix} \tag{2.4.32}$$

$$Y_0(x) = Y_\infty(x) \begin{pmatrix} e^{-i\pi\alpha}\frac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\beta)} & e^{-i\pi(\alpha-\gamma+1)}\frac{\Gamma(2-\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\,\Gamma(1-\alpha)} \\ e^{-i\pi\beta}\frac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\,\Gamma(\alpha)} & e^{-i\pi(\beta-\gamma+1)}\frac{\Gamma(2-\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma+1)\,\Gamma(1-\beta)} \end{pmatrix}. \tag{2.4.33}$$

The formula (2.4.32) remains valid for $|\arg(-x)| < \pi$; the formula (2.4.33) is valid for $|\arg(1-x)| < \pi$.

These connection formulae yield the following expressions for the monodromy matrices in the basis Y_∞:

$$M_0 = \frac{e^{-2i\pi\gamma}}{e^{2i\pi\alpha} - e^{2i\pi\beta}} \times$$
$$\begin{pmatrix} e^{2i\pi(\beta+\alpha)} - e^{2i\pi(\beta+\gamma)} - e^{2i\pi\beta} + e^{2i\pi\gamma} & -\frac{4\pi^2 e^{i\pi(2\beta+\gamma)}\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(1-\alpha)\Gamma(\alpha-\beta)\Gamma(\beta-\gamma+1)\Gamma(\gamma-\alpha)} \\ \frac{4\pi^2 e^{i\pi(2\alpha+\gamma)}\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1-\beta)\Gamma(\beta-\alpha)\Gamma(\alpha-\gamma+1)\Gamma(\gamma-\beta)} & e^{2i\pi(\alpha+\gamma)} - e^{2i\pi(\alpha+\beta)} + e^{2i\pi\alpha} - e^{2i\pi\gamma} \end{pmatrix} \tag{2.4.34}$$

$$M_1 = \frac{1}{e^{2i\pi\alpha} - e^{2i\pi\beta}} \times$$
$$\begin{pmatrix} e^{-2i\pi\alpha}\left(e^{2i\pi(\alpha+\gamma)} - e^{2i\pi(\alpha+\beta)} + e^{2i\pi\alpha} - e^{2i\pi\gamma}\right) & \frac{4\pi^2 e^{i\pi\gamma}\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(1-\alpha)\Gamma(\alpha-\beta)\Gamma(\beta-\gamma+1)\Gamma(\gamma-\alpha)} \\ -\frac{4\pi^2 e^{i\pi\gamma}\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1-\beta)\Gamma(\beta-\alpha)\Gamma(\alpha-\gamma+1)\,\Gamma(\gamma-\beta)} & e^{-2i\pi\beta}\left(e^{2i\pi(\beta+\alpha)} - e^{2i\pi(\beta+\gamma)} - e^{2i\pi\beta} + e^{2i\pi\gamma}\right) \end{pmatrix} \tag{2.4.35}$$

$$M_\infty = \begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{2i\pi\beta} \end{pmatrix}. \tag{2.4.36}$$

The *third method* of computation of the monodromy of Gauss equation is based on the *Euler integral representation* of the hypergeometric functions:

$$F(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t\,x)^{-\alpha}\,dt, \quad \operatorname{Re}\gamma > \operatorname{Re}\beta > 0, \quad |x| < 1. \tag{2.4.37}$$

Exercise 2.4.11 *Prove* (2.4.37).

Hint: expand $(1 - t\,x)^{-\alpha}$ in geometric series and use Euler formula

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y}.$$

for beta-integrals

$$\mathrm{B}(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad \mathrm{Re}\,x > 0, \quad \mathrm{Re}\,y > 0.$$

We will consider here only a very particular case of this representation.

Exercise 2.4.12 *Consider the* complete elliptic integrals

$$\mathrm{K} = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2 s^2)}} \tag{2.4.38}$$

$$i\,\mathrm{K}' = \int_1^{1/k} \frac{ds}{\sqrt{(1-s^2)(1-k^2 s^2)}} \tag{2.4.39}$$

as functions of the complex variable

$$x = k^2, \quad x \neq 0,\ 1,\ \infty.$$

Prove that the functions $y_1 = \mathrm{K}(x)$*,* $y_2 = \mathrm{K}'(x)$ *are two linearly independent solutions to the following Gauss equation*

$$x(1-x)\,y'' + (1-2\,x)\,y' - \frac{1}{4}\,y = 0. \tag{2.4.40}$$

Prove that in the basis y_1*,* y_2 *the monodromy matrices have the following form:*

$$M_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{2.4.41}$$

Derive the formula

$$\mathrm{K} = \frac{\pi}{2}\,F(1/2, 1/2; 1; k^2). \tag{2.4.42}$$

The following books were useful for preparing the present lecture notes.

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