Notes on integrable systems and Toda lattice

Tamara Grava

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1 A short review of tensors

Let us consider an N dimensional smooth manifold P over the real numbers.

Definition 1.1 A tensor of type (p,q) in a given coordinate chart $(U, x = (x^1, \dots, x^N))$ of the manifold P is described by a set of N^{p+q} real numbers

$$A^{i_1,\dots i_p}_{j_1,\dots j_p}(x).$$

In another coordinate chart $(\tilde{U}, \tilde{x} = (\tilde{x}^1, \dots, \tilde{x}^N))$ the tensor is described by

$$\tilde{A}^{i'_1,\dots i'_p}_{j'_1,\dots j'_p}(\tilde{x})$$

and if $U \cap \tilde{U}$

$$\tilde{A}^{i'_1,\dots i'_p}_{j'_1,\dots j'_p}(\tilde{x}) = \frac{\partial \tilde{x}^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial \tilde{x}^{i_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial \tilde{x}^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial \tilde{x}^{j'_q}} A^{i_1,\dots i_p}_{j_1,\dots j_p}(x)$$

where we sum over repeated indices.

For example $(1,0)$ tensors are associated to vector fields. Indeed

$$\tilde{X}^{i'} = \frac{\partial \tilde{x}^{i'}}{\partial x^i} X^i$$

which is exactly the law of transformation of the vector field

$$X = X^i \frac{\partial}{\partial x^i} = \left[X^i \frac{\partial \tilde{x}^{i'}}{\partial x^i} \right] \frac{\partial}{\partial \tilde{x}^{i'}}.$$

In the same way $(0,1)$ tensors are one forms $\omega = \omega_i dx^i$ Under a change of coordinates $\tilde{x} = \tilde{x}(x)$ we have

$$\omega = \left[\omega_i \frac{\partial x^i}{\partial \tilde{x}^{i'}} \right] d\tilde{x}^{i'}.$$

Given two tensors of type (p,q) their linear combination is still a tensor of type (p,q). Therefore, the tensors of type (p,q) form a linear space $\mathcal{T}^p_q(P)$. Such space can be identified with the tensor product of p copies of the tangent space $T_x(P)$ and q copies of the cotangent space $T^*_x(P)$, here we identify a point of the space with the system of coordinates $x = (x^1, \dots, x^N)$ at the point. In this way a basis of the space $\mathcal{T}^p_q$

$$\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} dx^{j_1} \otimes \cdots \otimes dx^{j_q},$$

and a decomposition of a tensor A of type (p,q) with respect to this basis gives

$$A = A^{i_i \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} dx^{j_1} \otimes \cdots \otimes dx^{j_q}.$$

There are two important linear operations that can be done on tensors: product of tensors and contraction of tensor. Given a tensor A of type (p,q) and a tensor B of type (r,s) the tensor product $A \otimes B$ is the tensor of type $(p+r, q+s)$ such that

$$(A \otimes B)^{i_1 \dots i_p k_1 \dots k_r}_{j_1 \dots j_q n_1 \dots n_s} = A^{i_1 \dots i_p}_{j_1 \dots j_q} B^{k_1 \dots k_r}_{n_1 \dots n_s}$$

For example if we consider the $(1,1)$ tensor A and the vector v their tensor product is

$$(A \otimes v)^{ik}_j = A^i_j v^k.$$

The contraction of tensors transform a tensor of type (p, q) to a tensor of type $(p-1, q-1)$. It depends on the choice of one upper index i_k and one lower j_l:

$$C^{i_k}_{j_l}(A)^{i_1 \dots i_p}_{j_1 \dots j_q} = A^{i_1 \dots i_{k-1} s\, i_{k+1} \dots i_p}_{j_1 \dots j_{l-1} s\, j_{l+1} \dots j_q}$$

where it is summed over the repeated index s.

Example 1.2 Let v be a vector and $A = A^i_j$ a $(1,1)$ tensor. Their tensor product $A \otimes v$ is a $(2,1)$ tensor. The operation of contraction gives a $(1,0)$ tensor

$$C^k_j(A \otimes v)^{ik}_j = A^i_s v^s.$$

For this reason we can think of a $(1,1)$ tensor as a linear operation on the tangent space $A : TP \to TP$ such that $v \to Av$ where $(Av)^i = A^i_j v^j$. The adjoint operator A^* can be identified with linear operations from the cotangent space $T^*P \to T^*P$

$$\omega \to A^*\omega, \quad (A^*\omega)_j = A^i_j \omega_i$$

Example 1.3 A $(0,2)$ tensor $\omega = \omega_{ij}$ can be realised as a bilinear form on the space $TP \otimes TP$

$$(\omega, v, u) \to \omega_{ij} v^i u^j := \omega(v, u).$$

An important subspace of $\mathcal{T}_q(P)$ and $\mathcal{T}^p(P)$ is the space of antisymmetric $(p,0)$ and $(0,q)$ tensors. The operation Alt defined as

$$\mathrm{Alt}(a_{i_1,\dots i_p}) = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sign}(\sigma)\, a_{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(p)}}$$

produces an antisymmetric tensor and the same applies to $(0,q)$ tensors. The subspace of antisymmetric $(p,0)$ tensors is denoted as $\Lambda^p TP \subset TP \otimes \cdots \otimes TP$. Combining the operation of tensor product with the operation of alternation, one obtains the operation of wedge product $\wedge$. Namely the space $\Lambda^p(T_xP)$, can be identified with the antisymmetric product of p copies of the tangent space T_xP. In particular, $\Lambda^1 TP = TP$. If $(x^1, ..., x^N)$ is a local system of coordinates at x, then $\Lambda^p T_x P$ admits a linear basis consisting of the elements

$$\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_p}}.$$

A smooth p-vector field A is by definition a section of $\Lambda^p TP$, in local coordinates takes the form

$$A(x) = \sum_{i_1 < \cdots < i_p} A^{i_1,\dots,i_p} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_p}} = \frac{1}{p!} \sum_{i_1,\dots,i_p} A^{i_1,\dots,i_p} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_p}}$$

The coefficients $A^{i_1,\dots,i_p}(x)$ are smooth function of x and antisymmetric with respect to the change of the indices and transform as a $(p,0)$ tensor.

In the same way an antisymmetric tensor $(0,q)$ is a q-differential antisymmetric form ω or a section of $\Lambda^q T^*P$. In local coordinates it takes the form

$$\omega = \sum_{i_1<\cdots<i_q} \omega_{i_1,\dots i_q}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_q} = \frac{1}{q!} \sum_{i_1,\dots,i_q} \omega_{i_1,\dots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}, \tag{1.1}$$

where $\omega_{i_1,\cdots_q}$ is a smooth antisymmetric tensor of type $(0,q)$. The form $d\omega$ is the $q+1$ antisymmetric tensor defined as

$$(d\omega)_{j_1,\dots j_{q+1}} = \sum_{m=1}^{q+1} (-1)^{m+1} \frac{\partial \omega_{j_1\dots\widehat{j}_m\dots j_{q+1}}(x)}{\partial x^{j_m}}.$$

The pullback of a q form ω is defined as follows. Let $f : M \to P$ be a smooth map and let ω be a q-form on P defined by (1.1). Then the form $f^*\omega$ on M is defined as

$$f^*\omega = \sum_{i_1<\cdots<i_q} \omega_{i_1,\dots i_q}(x(y)) dx^{i_1}(y) \wedge \cdots \wedge dx^{i_q}(y),$$

where the map f in local coordinates takes the form

$$y = (y^1, \dots, y^N) \xrightarrow{f} (x^1(y), \dots x^N(y)).$$

Exercise 1.4 Show that the operation of exterior differentiation commutes with the operation of pullback. Namely if $f : M \to P$ is a smooth map and ω is a differential form on P then

$$f^*(d\omega) = f^*(d\omega).$$

Given a k-vector field A and k-form α, the pairing $\langle \alpha, A \rangle$ is the function

$$\langle \alpha,\, A \rangle = \sum_{i_1<\cdots<i_k} \alpha_{i_1,\dots i_k} A^{i_1,\dots,i_k}.$$

Exercise 1.5 Show that the above definition of $\langle \alpha, A \rangle$ does not depend on the choice of local coordinates.

A smooth k-vector field A defines a $\mathbb{R}$-multilinear skew symmetric map from $\mathcal{C}^\infty(P) \times \cdots \times \mathcal{C}^\infty(P)$ (k-times) to $\mathcal{C}^\infty(P)$ by the formula

$$A(f_1, \dots, f_k) = \langle A, df_1 \wedge \cdots \wedge df_k \rangle \tag{1.2}$$

For example for a 1-vector field X we have

$$X(f) = \langle X, df\rangle = X^i \frac{\partial f}{\partial x^i}$$

and for a 2-vector field π we have

$$\langle \pi, df_1 \wedge df_2\rangle = \pi(f_1, f_2) = \pi^{ij} \frac{\partial f_1}{\partial x^i}\frac{\partial f_2}{\partial x^j}.$$

Exercise 1.6 Show that a $\mathbb{R}$-multilinear skew-symmetric map $A : \mathcal{C}^\infty(P) \times \cdots \times \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P)$ arises from a smooth k-vector field by formula (1.2) if and only if A is skew symmetric and (1.2) satisfies the Leibnitz rule

$$A(fg, f_2, \ldots, f_k) = fA(g, f_2, \ldots, f_k) + gA(f, f_2, \ldots, f_k).$$

A map that satisfies the above condition is said multi-derivation, and the above exercise shows that multi-derivations are identified with multi-vector fields.

The Lie derivative of a tensor follows the notion of Lie derivative of a function along a vector field X. Indeed let Φ_t the one parameter flux generated by the vector field X, then we have

$$L_X f(x) := \frac{d}{dt}\phi_t^* f(x)|_{t=0} = \frac{d}{dt} f(\phi_t(x))|_{t=0} = X^i(x)\frac{\partial f(x)}{\partial x^i}.$$

Similarly, the Lie derivative of a vector Y along the vector X is given by

$$L_X Y(x) = \frac{d}{dt}\phi_t^* Y(\phi_t(x))|_{t=0} = \frac{d}{dt}\left(Y^i(\phi_t(x))\frac{\partial}{\partial \phi_t^i(x)}\right).$$

We have

$$Y^i(\phi_t(x)) = Y^i(x^1 + tX^1(x) + O(t^2), \ldots, x^N + X^N(x)t + O(t^2)) = Y^i(x) + tX^k \frac{\partial}{\partial x^k} Y^i + O(t^2)$$

and

$$\frac{\partial}{\partial \phi_t^i(x)} = \frac{\partial x^s}{\partial \phi_t^i(x)}\frac{\partial}{\partial x^s} = \left(\delta^{is} - t\frac{\partial X^i}{\partial x^s} + O(t^2)\right)\frac{\partial}{\partial x^s}$$

so that the Lie derivative takes the standard form

$$L_X(Y) = \left(X^k \frac{\partial}{\partial x^k} Y^s - Y^s \frac{\partial X^i}{\partial x^s}\right)\frac{\partial}{\partial x^s}.$$

In a similar way one obtains the Lie derivative of a tensor T of type (p, q)

$$\begin{aligned} L_X(T)^{i_1,\ldots i_p}_{j_1\ldots,j_q} = X^s \frac{\partial}{\partial x^s} T^{i_1,\ldots i_p}_{j_1\ldots,j_q} - \frac{\partial X^{i_1}}{\partial x^s} T^{s i_2\ldots i_p}_{j_1\ldots,j_q} - \cdots - \frac{\partial X^{i_p}}{\partial x^s} T^{i_1\ldots i_{p-1}s}_{j_1\ldots,j_q} \\ + \frac{\partial X^s}{\partial x^{j_1}} T^{i_1\ldots i_p}_{s j_2\ldots j_q} + \ldots \frac{\partial X^s}{\partial x^{j_q}} T^{i_1\ldots i_p}_{j_1\ldots j_{q-1}s}\,. \end{aligned}$$

We conclude remarking that the Lie derivative is a linear operation that satisfies Leibniz rule with respect to product (tensor or wedge product) so that

$$L_X(T \otimes S) = L_X(T) \otimes S + T \otimes L_X(S).$$

2 Poisson Manifolds

In this section we introduce the concept of Poisson bracket and Poisson manifold.

Definition 2.1 A manifold P is said to be a Poisson manifold if P is endowed with a Poisson bracket $\{.\,.\}$, that is a Lie algebra structure defined on the space $\mathcal{C}^\infty(P)$ of smooth functions over P

$$\begin{aligned} \mathcal{C}^\infty(P) \times \mathcal{C}^\infty(P) &\to \mathcal{C}^\infty(P) \\ (f,g) &\mapsto \{f,g\} \end{aligned} \tag{2.1}$$

so that $\forall f,g,h \in \mathcal{C}^\infty(P)$ the bracket $\{\,.\,,\,.\,\}$

- is antisymmetric:
$$\{g,f\} = -\{f,g\}, \tag{2.2}$$
- bilinear
$$\begin{aligned} \{af+bh,g\} &= a\{f,g\} + b\{h,g\}, \\ \{f,ag+bh\} &= a\{f,g\} + b\{f,h\}, \quad a,\, b \in \mathbb{R} \end{aligned} \tag{2.3}$$
- satisfies Jacobi identity
$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0; \tag{2.4}$$
- it satisfies Leibnitz identity with respect to the product of function
$$\{f\,g,h\} = g\,\{f,h\} + f\,\{g,h\}. \tag{2.5}$$

From the exercise 1.6 a bilinear antisymmetric map that satisfies the Leibniz rule can be identified with a bi-vector, namely an antisymmetric $(2,0)$ tensor. Let us denote this tensor by π. Then we have

$$\{f,g\} = \pi(f,g) = \langle \pi, df \wedge dg \rangle = \pi^{ij}(x) \frac{\partial f(x)}{\partial x^i} \frac{\partial g(x)}{\partial x^j},$$

where $x = (x^1, \ldots, x^N)$ is a system of coordinates. In particular

$$\{x^i, x^j\} = \pi^{ij}(x), \quad i,\, j = 1, \ldots, N = \dim P. \tag{2.6}$$

In order to satisfy the Jacobi identity we need to impose the condition

$$\{x^i, \{x^j, x^k\}\} + \{x^j, \{x^k, x^i\}\} + \{x^k, \{x^j, x^i\}\} = 0, \quad 1 \leq i < j < k \leq N,$$

which give the relation

$$\frac{\partial \pi^{ij}(x)}{\partial x^s}\pi^{sk}(x) + \frac{\partial \pi^{ki}(x)}{\partial x^s}\pi^{sj}(x) + \frac{\partial \pi^{jk}(x)}{\partial x^s}\pi^{si}(x) = 0, \quad 1 \leq i < j < k \leq N.$$

We summarise the above considerations with the following theorem.

Theorem 2.2 *1) Given a Poisson manifold P, and a system of local coordinates over P, then the matrix $\pi^{ij}(x)$ defined in (2.6) is antisymmetric and satisfies*

$$\frac{\partial \pi^{ij}(x)}{\partial x^s}\pi^{sk}(x) + \frac{\partial \pi^{ki}(x)}{\partial x^s}\pi^{sj}(x) + \frac{\partial \pi^{jk}(x)}{\partial x^s}\pi^{si}(x) = 0, \quad 1 \leq i < j < k \leq N. \tag{2.7}$$

Furthermore the Poisson bracket of two smooth functions is calculated according to

$$\{f, g\} = \pi^{ij}(x)\frac{\partial f(x)}{\partial x^i}\frac{\partial g(x)}{\partial x^j}. \tag{2.8}$$

2) Given a change of coordinates

$$\tilde{x}^k = \tilde{x}^k(x), \quad k = 1, \ldots, N,$$

then the matrices $\pi^{ij}(x) = \{x^i, x^j\}$ e $\tilde{\pi}^{kl}(\tilde{x}) = \{\tilde{x}^k, \tilde{x}^l\}$ satisfy the rule of transformation of a tensor of type (2,0):

$$\tilde{\pi}^{kl}(\tilde{x}) = \pi^{ij}(x)\frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}. \tag{2.9}$$

3) Viceversa, given a smooth manifold P and an antisymmetric tensor (2,0) $\pi^{ij}(x)$ such that (2.7) is satisfied, then (2.8) defines over P a Poisson bracket.

Definition 2.3 If the rank of the matrix π^{ij} is equal to $N = \dim P$, the Poisson bracket is non degenerate.

It immediately follows that non degenerate Poisson bracket exists only on even dimensional manifolds.

Definition 2.4 Given a Poisson bracket $\{\,,\,\}$, the set of functions that commutes with any other functions of $\mathcal{C}^\infty(P)$, namely

$$\{f \in \mathcal{C}^\infty(P) \,|\, \{f, h\} = 0, \forall h \in \mathcal{C}^\infty(P)\}$$

are called *Casimirs* of the Poisson bracket.

For a nondegenerate Poisson bracket, the only Casimir is zero.

For a given $f \in \mathcal{C}^\infty(P)$, and f not a Casimir of the Poisson bracket, the map

$$g \to \{f, g\}$$

is a derivation. It immediately follows that there is a unique vector field X_f such that

$$X_f(g) = \{f, g\}.$$

In particular in local coordinates we have

$$X_f = \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

In this way the Poisson bracket defines a homomorphism

$$\begin{aligned} \mathcal{C}^\infty(P) &\to TP \\ f &\to X_f = \{f\,,.\} \end{aligned}$$

so that

$$[X_f, X_g] = X_{\{f,g\}}.$$

Definition 2.5 A diffeomorphism $\phi : P \to P$ that preserves the Poisson bracket

$$\phi^*\{f, g\} = \{\phi^* f, \phi^* g\}, \quad \phi^* f(x) = f(\phi(x)), \tag{2.10}$$

is called a Poisson diffeomorphism.

Let ϕ_t with $t \geq 0$ be a one parameter group of diffeomorphism of P generated by the smooth vector field X, namely

$$X = \frac{d}{dt} \phi_t(x)|_{t=0}.$$

The infinitesimal version of the relation (2.10) can be obtained by differentiating at $t = 0$ the relation $\{\phi_t^* f, \phi_t^* g\} = \phi_t^* \{f, g\}$, which gives

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}. \tag{2.11}$$

In this case the vector field X is called *Poisson vector field.*

Lemma 2.6 *A vector field X on a Poisson manifold (P, π), is a Poisson vector field iff*

$$L_X \pi = 0. \tag{2.12}$$

Proof. By the Leibniz rule we have $X(\{f,g\}) = L_X(\{f,g\}) = L_X(\langle\pi, df \wedge dg\rangle) = \langle L_X\pi, df \wedge dg\rangle + \langle\pi, dL_X f \wedge dg\rangle + \langle\pi, df \wedge dL_X g\rangle = \langle L_X\pi, df \wedge dg\rangle + \{X(f), g\} + \{f, X(g)\}$, namely

$$X(\{f,g\}) = \langle L_X\pi, df \wedge dg\rangle + \{X(f), g\} + \{f, X(g)\},$$

which implies, by (2.11), the statement of the Lemma. □

It can be easily verified that any Hamiltonian vector field X_h, is a Poisson vector field, indeed (2.11) is nothing but the Jacobi identity. Every Hamiltonian vector field is a Poisson vector field, while the contrary is not true in general.

Definition 2.7 A $2n$-dimensional P manifold is called symplectic manifold if it is endowed with a close non degenerate 2-form ω.

In local coordinates one has

$$\omega = \sum_{i<j,1}^{n} \omega_{ij} dx^i \wedge dx^j,$$

where $\wedge$ stands for the exterior product. We recall that the form ω is closed if $d\omega = \sum_{ijk=1}^{n} \frac{\partial}{\partial x^k}\omega_{ij} dx^k \wedge dx^i \wedge dx^j = 0$, which implies that

$$\frac{\partial}{\partial x^k}\omega_{ij} + \frac{\partial}{\partial x^i}\omega_{jk} + \frac{\partial}{\partial x^j}\omega_{ki} = 0, \quad i \neq j \neq k.$$

Lemma 2.8 *A Poisson manifold $\{P, \pi\}$ with non degenerate Poisson bracket π , is a symplectic manifold, with $\omega_{ij} = (\pi^{ij})^{-1}$.*

Indeed the Jacobi identity is equivalent to the closure of the 2-form ω. For a symplectic manifold (P, ω) we have the map from $TP \to TP^*$

$$X \to \omega(X, .), \quad \omega(X, .) = \omega_{ij} X^i dx^j.$$

Therefore we have the identities

$$\{f, g\} = -\omega(X_f, X_g) = X_f(g) = -\langle df, X_g\rangle.$$

The *classical Darboux theorem* says that in the neighbourhood of every point of (P, ω) $\dim P = 2n$, there is a local systems of co-ordinates $(q^1, \ldots q^n, p_1, \ldots, p_n)$ called Darboux coordinates or canonical coordinates such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \tag{2.13}$$

In such coordinated the Poisson bracket takes the form

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$

with Hamiltonian vector field

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} \right)$$

and Poisson tensor π

$$\pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The existence of Darboux coordinates is related to the vanishing of the second group of the so called Poisson cohomology $H^*(U,\pi)$, where U is an open neighbourhood of P. If the Poisson bracket is non-degenerate, the Poisson cohomology coincides with the de-Rham cohomology and Darboux theorem is equivalent to the vanishing of the second de-Rham cohomology group in an open set. In order to have global Darboux coordinates one needs the vanishing of the Poisson cohomology group $H^2(P,\pi)$. There are many tools for computing de Rham cohomology groups, and these groups have probably been computed for most ÒfamiliarÓ manifolds. However, when π is not symplectic, then $H^*(P,\pi)$ does not vanish even locally [10] and it is much more difficult to compute it then the de Rham cohomology. There are few Poisson (non-symplectic) manifolds for which Poisson cohomology has been computed [7]. The Poisson cohomology $H^*(P,\pi)$ can have infinite dimension even when P is compact, and the problem of determining whether $H^*(P,\pi)$ is finite dimensional or not is already a difficult open problem for most Poisson structures that we know of. In the case of linear Poisson structures, Poisson cohomology is intimately related to Lie algebra cohomology, also known as Chevalley - Eilenberg cohomology, [4].

2.1 Hamiltonian systems

Given a Poisson manifold (P,π), $\dim P = N$, and a function $H \in \mathcal{C}^\infty(P)$, an Hamiltonian system in local coordinates $(x^1,\ldots,x^N)$ is a set of N first order ODEs defined by

$$\frac{d}{dt} x^i := \dot{x}^i = \{x^i, H\},$$

with initial condition $x^i(t=0) = x^i_0$. For a symplectic manifold (P,ω), $\dim P = 2n$, the Hamilton equations in Darboux coordinates takes the form

$$\begin{aligned} \dot{q}^i &= \{q^i, H\} = \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= \{p_i, H\} = \frac{\partial H}{\partial q^i}, \quad i = 1,\ldots,n \end{aligned} \tag{2.14}$$

with initial conditions $q^i(t=0) = q_0^i$, $p_i(t=0) = p_i^0$.

Definition 2.9 A function $F \in \mathcal{C}^\infty(P)$ is said to be a conserved quantity for the Hamiltonian system (2.14) if

$$\frac{dF}{dt} = \{F, H\} = 0.$$

Namely conserved quantities Poisson commute with the Hamiltonian. We remark that if $F_1, \ldots, F_m$ are conserved quantities, then any function of $g = g(F_1, \ldots, F_m)$ is a conserved quantity.

Next we will show that couples of commuting vector fields define an action of an Abelian group.

Lemma 2.10 *Let $(P, \{\,.\,,\,.\,\})$ be a nondegenerate Poisson bracket. Consider the Hamiltonians $F, H \in \mathcal{C}^\infty(P)$, and their Hamiltonian flows*

$$\frac{dx^i}{dt} = \{x^i, H\}, \quad i = 1, \ldots, N \tag{2.15}$$

$$\frac{dx^i}{ds} = \{x^i, F\}, \quad i = 1, \ldots, N. \tag{2.16}$$

The common solution $x(t,s)$ of (2.15) and (2.16) with initial data $x(0,0) = x_0 \in P$ exists for sufficiently small t and s if

$$\{F, H\} = 0.$$

Proof. By definition the common solution must satisfy the relation

$$\frac{d}{ds}\frac{dx^i(t,s)}{dt} = \frac{d}{dt}\frac{dx^i(t,s)}{ds}.$$

Taking the derivative with respect to s of equation (2.15) and with respect to t of equation (2.16). One has

$$\frac{d}{ds}\frac{dx^i}{dt} = \frac{d}{ds}\{x^i, H\} = \{\frac{d}{ds}x^i, H\} + \{x^i, \frac{d}{ds}H\} = \{\{x^i, F\}, H\}, \quad i = 1, \ldots, N$$

$$\frac{d}{dt}\frac{dx^i}{ds} = \frac{d}{dt}\{x^i, F\} = \{\frac{d}{dt}x^i, H\} + \{x^i, \frac{d}{dt}H\} = \{\{x^i, H\}, F\}, \quad i = 1, \ldots, N$$

Subtracting the two terms and applying Jacobi identity one arrives to

$$\frac{d}{ds}\frac{dx^i}{dt} - \frac{d}{dt}\frac{dx^i}{ds} = \{\{x^i, F\}, H\} + \{\{H, x^i\}, F\} = \{\{H, F\}, x^i\} = 0, \quad i = 1, \ldots, N.$$

which is equal to zero by the commutativity of F and H. $\square$

We remark that the converse statement is also true (see Dubrovin pg.20).

2.2 Integrable systems and Liouville-Arnold theorem

We start with the definition of a canonical transformation. A diffeomorphism of P to itself defines a change of coordinates $x \to \Phi(x)$. Let us notice that we need $2n$ functions to define Φ.

Definition 2.11 A change of coordinates $x \to \Phi(x)$ is a canonical transformation if $\Phi^*\omega = \omega$ where Φ^* is the pullback of the symplectic form ω through Φ.

Since $\omega = dW$ and the pullback commutes with differentiation (see exercise 1.4), one has

$$\omega - \Phi^*\omega = dW - \Phi^*(dW) - dW = d(W - \Phi^*W) = 0.$$

Namely the form $d(W - \Phi^*W)$ is exact, and by Poincare theorem there is locally a function S defined on an open set of P so that

$$W - \Phi^*W = dS. \tag{2.17}$$

The function S is called the generating function of the canonical transformation. In other words a canonical change of coordinates is defined by one function. Canonical transformations are used to make a suitable change of coordinates that reduce an integrable Hamiltonian system to a "trivial" evolution. We first introduce the concept of integrable system.

Definition 2.12 A Hamiltonian system defined on a $2n$ dimensional Poisson manifold P with non degenerate Poisson bracket and with Hamiltonian $H \in \mathcal{C}^\infty(P)$ is called completely integrable if there are n independent conserved quantities $H = H_1, \dots, H_n$ in involution, namely

$$\{H_j, H_k\} = 0, \quad j, k = 1, \dots, n \tag{2.18}$$

and the gradients $\nabla H_1, \dots \nabla H_n$ are linearly independent.

Let us consider the level surface

$$M_E = \{(p,q) \in P \,|\, H_1(p,q) = E_1, \quad H_2(p,q) = E_2, \quad H_n(p,q) = E_n\} \tag{2.19}$$

for some constants $E = (E_1, \dots, E_n)$. Here without loss of generality we assume that (q,p) are canonical coordinates. Such level surface enjoys a special property, namely it is a Lagrangian sub-manifold which we will define below.

Definition 2.13 Let P be a symplectic manifold of dimension $2n$. A a sub-manifold $G \subset P$ is called a Lagrangian submanifold if $\dim G = n$ and the symplectic form is identically zero on vectors tangent to G, namely

$$\omega(X, Y) = 0, \quad \forall X, Y \in TG.$$

Lemma 2.14 *The manifold M_E defined in (2.19) where $H_1, \ldots, H_n$ are independent and commuting Hamiltonians, is a Lagrangian sub manifold.*

Proof. The gradients

$$\nabla H_j = \left(\frac{\partial H_j}{\partial q^1}, \ldots, \frac{\partial H_j}{\partial q^n}, \frac{\partial H_j}{\partial p_1}, \ldots, \frac{\partial H_j}{\partial p_n} \right)$$

are orthogonal to the surface M_E. Since the vector fields X_{H_j} are orthogonal to ∇H_k because $\{H_j, H_k\} = 0$, it follows that the vector fields X_{H_j} are tangent to the level surface M_E. Furthermore, since the Hamiltonian H_j are linearly independent, it follows that the vector fields X_{H_j}, $j = 1, \ldots, n$ generate all the tangent space TM_E. Therefore the symplectic form is identically zero on the tangent space to M_E, namely $\omega|_{TM_E} \equiv 0$ because

$$\omega(X_{H_j}, X_{H_k}) = -\{H_k, H_j\} = 0.$$

This is equivalent to say that M_E is a Lagrangian submanifold. □

Theorem 2.15 *[Liouville, see e.g. [3]] Consider a completely integrable Hamiltonian system on a non degenerate Poisson manifold P of dimension $2n$ and with canonical coordinates (q, p). Let us suppose that the Hamiltonians $H_1(p, q)$, ..., $H_n(p, q)$ are linearly independent on the level surface M_E (2.19) for a given $E = (E_1, \ldots, E_n)$. The Hamiltonian flows on M_E are integrable by quadratures.*

Proof. By definition the system posses n independent conserved quantities $H_1 = H$, $H_2, \ldots H_n$. Without loosing generality, we assume that (q, p) are canonical coordinates with respect to the symplectic form ω and the Poisson bracket $\{.,.\}$. The idea of the proof is to construct a system of canonical variables that make the evolution trivial. Since the evolution is restricted to the level surface M_E, parametrised by $E = (E_1, \ldots, E_n)$, we need to complete the coordinates E with another set of coordinates $\psi = (\psi_1, \ldots, \psi_n)$ so that the transformation $(q, p) \to (\psi, E)$ is canonical.

For the purpose we also observe that since ∇H_j, $j = 1, \ldots, n$ are linearly independent, it is possible to assume, without loosing in generality that

$$\det \frac{\partial H_j}{\partial p_k} \neq 0.$$

Then by the implicit function theorem we can define

$$p_k = p_k(q, E).$$

Since M_E is a Lagrangian sub-manifold we have

$$0 = \omega|_{TM_E} = \sum_i dp_i(q, E) \wedge dq^i = \sum_{ij} \frac{\partial p_i}{\partial q^j} dq^j \wedge dq^i$$

which implies

$$\frac{\partial p_i}{\partial q^j} - \frac{\partial p_j}{\partial q^i} = 0, \quad i \neq j.$$

The above identity implies that the one form $W = p_i(q, E)dq^i$ is exact on T^*M_E, and therefore there exists a function $S = S(q, E)$ so that $W|_{TM_E} = dS|_{TM_E}$. The function S is the generating function of a canonical transformation Φ which maps the variable $(q, p) \xrightarrow{\Phi} (\psi, E)$ and

$$\Phi^* W = \sum E_i d\psi^i = -\sum \psi^i dE_i$$

and by (2.17)

$$\sum p_i dq^i - \frac{\partial S}{\partial q^i} dq^i - \frac{\partial S}{\partial E_i} dE_i = -\sum \psi^i dE_i$$

so that

$$p_i = \frac{\partial S}{\partial q^i}, \quad \psi_i = \frac{\partial S}{\partial E_i}.$$

In the canonical coordinates (ψ, E) the Hamiltonian flow with respect to the Hamiltonian $H_1 = H$ takes the form

$$\dot{\psi}_i = \{\psi_i, H_1\} = \frac{\partial H_1}{\partial E_i} = \delta_{1i}$$
$$\dot{E}_i = \{E_i, H_1\} = -\frac{\partial H_1}{\partial \psi_i} = 0.$$

So the above equations can be integrated in a trivial way:

$$\psi_1 = t + \psi_1^0, \;\; \psi_i = \psi_i^0, \quad i = 2, \ldots, n \qquad\qquad E_i = E_i^0, \;\; i = 1, \ldots, n$$

where $\psi_i^0$and E_i^0 are constants. Therefore we have shown that the Hamiltonian flow can be integrated by quadratures. Furthermore

$$q = q(t + \psi_1^0, \psi_2^0, \ldots, \psi_n^0, E), \quad p = p(t + \psi_1^0, \psi_2^0, \ldots, \psi_n^0, E).$$

□

We remark that the above theorem is a local theorem, since the existence of the function S relies on a local result, namely Poincare' theorem. In 1968 Arnold observed that if the level surface M_E is compact, Liouville theorem becomes a global theorem and the motion takes place on a torus and is quasi-periodic.

Theorem 2.16 (Arnold) *If the level surface M_{E^0} defined in (2.19) is compact and connected then the level surfaces M_E for $|E - E^0|$ sufficiently small, are diffeomorphic to a torus*

$$M_E \simeq T^n = \{(\phi_1, \ldots, \phi_n) \in \mathbb{R}^n \,|\, \phi_i \sim \phi_i + 2\pi, \, i = 1, \ldots, n\}, \tag{2.20}$$

and the motion on M_E is quasi-periodic, namely

$$\phi_1(t) = \omega_1(E)\, t + \phi_1^0, \ldots, \phi_n(t) = \omega_n(E)\, t + \phi_n^0 \tag{2.21}$$

where $\omega_1(E)$, ..., $\omega_n(E)$ depends on E and the phases ϕ_1^0, ..., ϕ_n^0 are arbitrary.

Proof. To prove the theorem we use a standard lemma (see [3]).

Lemma 2.17 *Let M be a compact connected n-dimensional manifold. If on M there are n linearly independ vector fields $X_1, \ldots, X_n$ such that*

$$[X_i, X_j] = 0, \quad i, j = 1 \ldots, n$$

then $M \simeq T^N$, the n-dimensional torus.

In our case the vector field $X_{H_1}, \ldots, X_{H_n}$ are linearly independent and commuting, so, in the case M_{E^0} is compact and connected, it is also isomorphic to a n-dimensional torus. By continuity, for small values of $|E - E^0|$ the surface M_E is also isomorphic to a torus. The coordinates $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ introduced in the proof of Liouville theorem 2.15 are not angles on the torus. Let us make a change of variable $\phi = \phi(\psi)$ so that the coordinates $\phi = (\phi_1, \ldots, \phi_n)$ are angles on the torus and let $I_1(E), \ldots I_n(E)$ be the canonical variables associated to the angles $(\phi_1, \ldots, \phi_n)$. By definition one has for any Hamiltonian H_m

$$X_{H_m} = \sum_{j=1}^{n} \frac{\partial H_m}{\partial E_j} \frac{\partial}{\partial \psi_j} = \frac{\partial}{\partial \psi_m} = \sum_{j=1}^{n} \frac{\partial H_m}{\partial I_j} \frac{\partial}{\partial \phi_j},$$

since H_m depends only on E and ϕ depends only on ψ. It follows that ϕ_j and ψ_k are related by a linear transformation

$$\phi_j = \sum_m \sigma_{jm} \psi_m, \quad \sigma_{jm} = \sigma_{jm}(E), \quad \det \sigma_{jm} \neq 0.$$

Comparing the above two relations one arrives to

$$\sigma_{jm} = \frac{\partial H_m}{\partial I_j}.$$

Let us verify that (ϕ, I) are indeed canonical variables:

$$\{\phi_j, I_k\} = \{\sum_k \sigma_{jm} \psi_m, I_k\} = \sum_m \sigma_{jm} \{\psi_m, I_k\} = \sum_m \sigma_{jm} \frac{\partial I_k}{\partial E_m} = \sum_m \frac{\partial H_m}{\partial I_j} \frac{\partial I_k}{\partial E_m} = \delta_{jk}.$$

The equation of motions in the variables (ϕ, I) are given by

$$\dot{\phi}_k = \frac{\partial H_1}{\partial I_k} =: \omega_k(E)$$

$$\dot{I}_k = \frac{\partial H_1}{\partial \phi_k} = 0$$

therefore the motion is quasi periodic on the tori. In the variable (p, q), with $p = p(\phi, I)$, $q = q(\phi, I)$, the evolution is given as

$$q = q(\omega_1 t + \phi_1^0, \dots, \omega_n t + \phi_n^0, I)$$

$$p = p(\omega_1 t + \phi_1^0, \dots, \omega_n t + \phi_n^0, I),$$

where $(\phi_1^0, \dots, \phi_n^0)$ are constant phases. $\square$

3 Bi-Hamiltonian geometry and Lax pair

In this subsections we give the basic concepts of bi-Hamiltionian geometry.

Definition 3.1 Two Poisson tensors π_0 and π_1 on a manifold P are called compatible if

$$c_0\pi_0 + c_1\pi_1$$

is a Poisson tensor for any real c_0 and c_1. Such Poisson tensor is also called Poisson pencil.

It follows that the bracket

$$\{f, g\}_\lambda = \{f, g\}_0 + \lambda\{f, g\}_1$$

is a Poisson bracket for any value of λ. Applying the Jacobi identity one obtains that

$$\begin{aligned} &\{f, \{g, h\}_0\}_1 + \{h, \{f, g\}_0\}_1 + \{g, \{h, f\}_0\}_1 + \\ &\{f, \{g, h\}_1\}_0 + \{h, \{f, g\}_1\}_0 + \{g, \{h, f\}_1\}_0 = 0, \end{aligned} \tag{3.1}$$

for any triple of functions $f, g, h \in \mathcal{C}^\infty(P)$. Such identity can be also written in the equivalent form

$$[Y_f, X_g] + [X_f, Y_g] = Y_{\{f,g\}_0} + X_{\{f,g\}_1} \tag{3.2}$$

where $X_f = \{f, .\}_0$ and $Y_f = \{f, .\}_1$.

Definition 3.2 A vector field X on a manifold is called a bi-Hamiltonian system if it is Hamiltonian with respect to two compatible Poisson structures π_1 and π_0

$$X = \{H_1\ .\ ,\}_0 = \{H_0,\ .\ \}_1 \tag{3.3}$$

From now on we assume to have a Poisson manifold P of dimension $2n$ with non degenerate Poisson bracket.

Remark 3.3 Bi-Hamiltonian systems admit large set of first integrals, which make them into integrable Hamiltonian systems. Conversely, a vast majority of known integrable systems turn out to be bi-Hamiltonian. The importance of bi-Hamiltonian systems for the recursive construction of integrals of motion starts with Magri [11] and there is now a very large amount of articles on the subject.

Lemma 3.4 *[11] Let H_0, H_1, ..., be a sequence of functions on Poisson manifold P with compatible Poisson structures π_1 and π_0 satisfying the Lenard-Magri recursion relation*

$$\{\ .\ , H_{p+1}\}_0 = \{\ .\ , H_p\}_1, \quad p = 0, 1, \ldots \tag{3.4}$$

Then

$$\{H_p, H_q\}_1 = \{H_p, H_q\}_0 = 0, \quad p, q = 0, 1, \ldots$$

Proof. Let $p < q$ and $q-p = 2m$ for some $m > 0$. Using the recursion and antisymmetry of the brackets we obtain

$$\{H_p, H_q\}_0 = \{H_p, H_{q-1}\}_1 = -\{H_{q-1}, H_p\}_1 = -\{H_{q-1}, H_{p+1}\}_0 = \{H_{p+1}, H_{q-1}\}_0.$$

Iterating one arrives to

$$\{H_p, H_q\}_0 = \cdots = \{H_{p+m}, H_{q-m}\}_1 = 0$$

since $p + m = q - m$. In a similar way in the case $q - p = 2m + 1$ one obtains

$$\{H_p, H_q\}_0 = \cdots = \{H_n, H_{n+1}\}_0 = \{H_n, H_n\}_1 = 0$$

where $n = p + m = q - m - 1$. $\square$

We remark that this proof uses only (3.4) and the skew symmetry of π_1 an π_0, while it does not uses the assumption of compatibility of the Poisson structures.

However, the assumption that π_1 and π_0 are compatible Poisson structures is essential in order to guarantee the existence of functions H_k fulfilling the Magri recursion relations (3.4) . The question of existence of such functions in the case of an arbitrary bi-Hamiltonian structure is a difficult problem. In the special case π_0 is invertible, one can defined the field $(1, 1)$ tensor $N : TP \to TP$

$$N = \pi_1 \pi_0^{-1} \tag{3.5}$$

which is called the recursion operator or Nijenhuis operator for the bi-Hamiltonian structure. It is called recursion operators, because given a bi-Hamiltonian vector field X the vector fields $N^k X$ will be bi-Hamiltonian. It is called Nijenhuis operator, because it has zero torsion (see below). We recall that for a vector field X one has $(NX)^i = N^i_j X^j$. The lemma 3.4 requires the existence of vector fields $X_{H_{p+1}}$ with respect to π_0 and Y_{H_p} with respect to π_1 such that

$$X_{H_{p+1}} = Y_{H_p}.$$

Then applying the recursive operator N we obtain

$$NX_{H_{p+1}} = Y_{H_{p+1}} = X_{H_{p+2}}.$$

We observe that applying the tensor N to a bi-Hamiltonian vector field $X_{H_{p+1}}$, one obtains a vector field that in general is not a bi-Hamiltonian vector field. The main ingredient of bi-Hamiltonian geometry is the existence of such bi-Hamiltonian vector fields.

Definition 3.5 The torsion of a $(1,1)$ tensor N on a manifold P is the vector valued two-form $T(N)$ defined as

$$T(N)(X,Y) = [NX, NY] - N([NX, Y] + [X, NY]) + N^2[X, Y], \tag{3.6}$$

A $(1,1)$ tensor with vanishing torsion is called Nijenhuis tensor or Nijenhuis operator.

Remark 3.6 The condition (3.6) is equivalent to

$$L_{NX}N - NL_X N = 0 \tag{3.7}$$

for all vector fields on P where L_X is the Lie derivative with respect to X. Indeed

$$T(N)(X,Y) = L_{NX}(NY) - NL_{NX}(Y) - N((L_X(NY) - N(L_X Y)) = (L_{NX}N)Y - N(L_X N)Y.$$

Lemma 3.7 *If (π_1, π_0) are compatible Hamiltonian structures on P and π_0 is invertible, then the recursion operator $N = \pi_1\pi_0^{-1}$ is a Nijenhuis operator.*

Proof. It is enough to show that N vanishes on any pair of vectors of the form $X_f = \pi_0 df$ and $X_g = \pi_0 dg$ with $f, g \in \mathcal{C}^\infty(P)$. We have $Y_f = NX_f$, $Y_g = NX_g$ and

$$\begin{aligned} T(N)(X_f, X_g) &= [Y_f, Y_g] - N([Y_f, X_g] + [X_f, Y_g]) + N^2[X_f, X_g] \\ &= Y_{\{f,g\}_1} - N([Y_f, X_g] + [X_f, Y_g]) + N^2 X_{\{f,g\}_0} \\ &= N(X_{\{f,g\}_1} - [Y_f, X_g] - [X_f, Y_g] + Y_{\{f,g\}_0}) = 0, \end{aligned} \tag{3.8}$$

where we have used in the last identity the relation (3.2). □

Remark 3.8 If X is a bihamiltonian vector field, namely

$$X = \pi_1 dH_1 = \pi_0 dH_0$$

then $L_X\pi_1 = L_X\pi_0 = 0$ (see lemma 2.6). It follows that also $L_X N = 0$. Indeed assuming that π_0 is invertible and using the Leibniz rule of Lie derivative with respect to the product we have

$$L_X(N) = L_X(\pi_1\pi_0^{-1}) = \pi_1 L_X(\pi_0^{-1}) + L_X(\pi_1)\pi_0^{-1} = 0$$

because X is a bi-Hamiltonian vector field.

Lemma 3.9 *IF X is a bi-Hamiltonian vector field with respect to π_0 and π_1 that are invertible Poisson tensors, then $N^k X$, $k \geq 1$ are bi-Poisson vector fields.*

Proof. If π_0 and π_1 are compatible Poisson bracket the relation (3.7) holds. In particular if X is a bi-Hamiltonian vector field we have by Remark 3.8 that $L_X(N) = 0$ so that

$$L_{NX}(N) = 0.$$

Since X is a bi-Hamiltonian vector field, it follows that $X = \pi_0 df = \pi_1 dg$ for some functions f and g. Applying the recursion operator we have

$$NX = N\pi_0 df = \pi_1 df,$$

which means that NX is a Hamiltonian vector field with respect to π_1, and that $L_{NX}(\pi_1) = 0$. It follows that

$$0 = L_{NX}(N) = \pi_1 L_{NX}(\pi_0^{-1}) + L_{NX}(\pi_1)\pi_0^{-1} = \pi_1 L_{NX}(\pi_0^{-1})$$

which is equivalent to say that $L_{NX}(\pi_0) = 0$ because π_1 is invertible. Therefore NX is a bi-Poisson vector field. Repeating the argument $k-1$ times we conclude that $N^k X$ are bi-Poisson vector fields for $k \geq 0$. □

Remark 3.10 A stronger hypothesis namely the assumption that all Poisson vector fields are also Hamiltonian, guarantees that the above recursion relation generates bi-Hamiltonian vector fields. The recursion relation is effective, namely it produces first integrals when, for a given Hamiltonian vector field $X = X(x)$ of the tangent space $T_x P$, the vectors

$$\langle X, NX, \ldots, N^{n-1}X \rangle,$$

span a n-dimensional subspace of $T_x P$, where we assume that $\dim P = 2n$.

We conclude, by drawing some consequences from the equation $L_X(N) = 0$. Let us write it in components,

$$0 = (L_X N)^i_j = \sum_k \left(X^k \frac{\partial}{\partial x_k} N^i_j + \frac{\partial X^k_1}{\partial x_j} N^i_k - \frac{\partial X^i_1}{\partial x_k} N^k_j \right). \tag{3.9}$$

If we interpret N as a matrix with entries N^i_j, the term $X^k \dfrac{\partial}{\partial x_k} N^i_j$ in the r.h.s. of the above relation can be considered as the Lie derivative of N^i_j with respect to the vector field X. We denote by $\mathcal{L}_X N$ the Lie derivative of the components of N with respect to X. Let us define the matrix J with entries

$$J^k_m = \frac{\partial X^k}{\partial x_m}.$$

Then the equation (3.9) can be written in the compact form

$$\mathcal{L}N + JN - NJ = 0,$$

or equivalently, defining as $\frac{d}{dt}$ the flow associated to the vector field X

$$\frac{d}{dt}N = [N, J], \tag{3.10}$$

where now $[N, J] = NJ - JN$ is simply the matrix commutator. Such equation has the so-called Lax form, (see below). From the relation (3.10) it follows that the traces of powers of N are constant of motions.

Lemma 3.11 *If N satisfies equation (3.10), then $Tr(N^k)$ are constants of motions.*

Proof. We need to calculate

$$\frac{d}{dt}\mathrm{Tr}(N^k) = k\mathrm{Tr}\left(N^{k-1}\frac{d}{dt}N\right) = k\mathrm{Tr}\left(N^{k-1}[N, J]\right) = 0$$

because of the ciclicity of the trace. □

With this procedure we have obtained families of constant of motions

Theorem 3.12 *The quantities*

$$H_k = \frac{1}{k}Tr(N^k), \quad k \geq 1, \tag{3.11}$$

satisfies the Magri recursion relation (3.4).

Proof. We have for any vector field X

$$L_{NX}H_k = \mathrm{Tr}\left(N^{k-1}L_{NX}(N)\right), \quad L_X H_{k+1} = \mathrm{Tr}\left(N^{k-1}NL_X(N)\right),$$

so that

$$L_{NX}H_k - L_X H_{k+1} = \mathrm{Tr}\left(N^{k-1}(L_{NX}(N) - NL_X(N)\right) = 0, \quad k \geq 1,$$

where we have used the relation (3.7), which is equivalent to say that N is Nijenhuis operator. We can write the l.h.s. of the above relation in the form

$$L_{NX}H_k - L_X H_{k+1} = \langle NX, dH_k\rangle - \langle X, dH_{k+1}\rangle = \langle X, N^* dH_k\rangle - \langle X, dH_{k+1}\rangle = 0$$

for any vector field X. therefore

$$\pi_0 dH_{k+1} = \pi_1 dH_k, \quad k \geq 1.$$

□

3.1 First integrals associated to a Lax pair

One of the most known method to construct first integrals of a Hamiltonian system is through symmetries of the space P. Another powerful method is due to Lax [**?**] and represents the starting point of the modern theory of integrable systems. Given an ODE

$$\dot{x} = f(x), \quad x = (x^1, \dots, x^N) \tag{3.12}$$

and two $m \times m$ matrices $L = (L_{ij}(x))$, $A = (A_{ij}(x))$, they constitute a *Lax pair* for the dynamical systems if for every solution $x = x(t)$ of (3.12) the matrices $L = (L_{ij}(x(t)))$ and $A = (A_{ij}(x(t)))$ satisfy the equation

$$\dot{L} = [A, L] := LA - AL \tag{3.13}$$

and the validity of (3.13) for $L = L(x)$, $A = A(x)$ implies (3.12).

Theorem 3.13 *Given a Lax pair for the dynamical system (3.12), then the eigenvalues $\lambda_1(x)$, …, $\lambda_m(x)$ of $L(x)$ are integrals of motion for the dynamical system.*

Proof. The coefficients $a_1(x)$, …, $a_m(x)$ of the characteristic polynomial

$$\det(L - \lambda I) = (-1)^m \left[\lambda^m - a_1(x)\lambda^{m-1} + a_2(x)\lambda^{m-2} + \dots + (-1)^m a_m(x)\right] \tag{3.14}$$

of the matrix $L = L(x)$ are polynomials in in tr L, tr L^2, …, tr L^m:

$$a_1 = \mathrm{tr}\ L, \quad a_2 = \frac{1}{2}\left[(\mathrm{tr}\ L)^2 - \mathrm{tr}\ L^2\right], \ a_3 = \dots$$

Next we show that

$$\mathrm{tr}\ L^k, \quad k = 1, 2, \dots \tag{3.15}$$

are first integral of the dynamical system. Indeed for $k = 1$

$$\frac{d}{dt}\mathrm{tr}\ L = \mathrm{tr}\ \dot{L} = \mathrm{tr}\,(A\,L - L\,A) = 0.$$

more generally

$$\frac{d}{dt}\mathrm{tr}\,L^k = k\mathrm{tr}\,([A, L]\,L^{k-1}) = 0. \tag{3.16}$$

Since the coefficients of the characteristic polynomial $L(x)$ are constants of motion it follows that its eigenvalues are constants of motion. □

Another proof of the theorem, close to Lax's original proof, can be obtained observing that the solution of the equation $\dot{L} = [A, L]$ can be represented in the form

$$L(t) = Q(t)L(t_0)Q^{-1}(t) \tag{3.17}$$

where the evolution of $Q = Q(t)$ is determined from the equation

$$\dot{Q} = A(t)Q \tag{3.18}$$

with initial data

$$Q(t_0) = 1.$$

Then the characteristic polynomials of $L(t_0)$ e $Q(t)L(t_0)Q^{-1}(t)$ are the same and consequently the eigenvalues are the same.

Example 3.14 [6] Let us consider in $\mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ the canonical Poisson bracket π_0 and the non degenerate Poisson bracket π_1 given by

$$\pi_1 = \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i<j} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial q_i}.$$

The canonical bracket π_0 and π_1 are compatible brackets. The first traces of the recursion operator $N = \pi_1 \pi_0^{-1}$ are given by

$$H_0 = \frac{1}{2} \mathrm{tr}\, N = \sum_{i=1}^{n} p_i, \quad H_1 = \frac{1}{4} \mathrm{tr}\, N^2 = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

$$H_2 = \frac{1}{6} \mathrm{tr}\, N^3 = \frac{1}{3} \sum_{i=1}^{n} p_i^3 + \sum_{i=1}^{n-1} (p_i + p_{i+1}) e^{q_i - q_{i+1}},$$

and so on. The Hamiltonian H_1 is the Hamiltonian of the open Toda lattice equation with respect to the Poisson bracket π_0. The conserved quantities given by $H_k = \frac{1}{2(k+1)} \mathrm{Tr}\, N^{k+1}, \quad k \geq 0$ are independent and involution with respect to both Poisson brackets π_0 and π_1.

4 The Toda system

Let us consider the system of n points $q_1, q_2, \ldots, q_n$ on the real line interacting with potential

$$U(q_1, \ldots, q_n) = \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

the so called Toda lattice. The Hamiltonian $H(q,p) \in \mathcal{C}^\infty(T^*\mathbb{R}^n)$ takes the form

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \tag{4.1}$$

with Hamilton equations with respect to the canonical Poisson bracket

$$\{q_k, p_j\} = \delta_{kj}, \quad \{q_k, q_j\} = \{p_k, p_j\} = 0, jk = 1, \dots, n \tag{4.2}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = p_k, \quad k = 1, \dots, n$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = \begin{cases} -e^{q_1 - q_2} & \text{if } k = 1 \\ e^{q_{k-1}-q_k} - e^{q_k - q_{k+1}} & \text{if } 2 \le k \le n-1 \\ e^{q_{n-1}-q_n} & \text{if } k = n \end{cases}$$

Since the Hamiltonian is translation invariant, the total momentum is a conserved quantity together with the Hamiltonian.

Flaschka and Manakov separetely showed that the Toda lattice Hamiltonian system is completely integrable. Let us introduce a new set of dependent variables

$$\begin{aligned} a_k &= \frac{1}{2} e^{\frac{q_k - q_{k+1}}{2}}, \quad k = 1, \dots, n-1 \\ b_k &= -\frac{1}{2} p_k, \quad k = 1, \dots, n, \end{aligned} \tag{4.3}$$

with evolution given by the equations

$$\begin{aligned} \dot{a}_k &= a_k(b_{k+1} - b_k), \quad k = 1, \dots, n-1 \\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2), \quad k = 1, \dots, n, \end{aligned} \tag{4.4}$$

where we use the convention that $a_0 = a_n = 0$. Observe that there are only $2n-1$ variables and this is due the translation invariance of the original system. The equations (4.4) have an Hamiltonian form with Hamiltonian

$$H(a,b) = 2\sum_{i=1}^{n} b_i^2 + 4\sum_{i=1}^{n-1} a_i^2$$

with Poisson bracket define on $(\mathbb{R}^*)^{n-1} \times \mathbb{R}^n$ given by

$$\{a_i, b_j\} = -\frac{1}{4}\delta_{ij} a_i + \frac{1}{4}\delta_{i,j-1} a_i, \quad i = 1, \dots, n-1, \;\; j = 1, \dots, n,$$

while all the other entries are equal to zero. We observe that the total momentum is a Casimir of the above Poisson bracket

Next we introduce the tridiagonal $n \times n$ matrices:

$$
\begin{aligned}
L &= \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & 0 \\ a_1 & b_2 & a_2 & & 0 & 0 \\ 0 & a_2 & b_3 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & b_{n-1} & a_{n-1} \\ 0 & & & & a_{n-1} & b_n \end{pmatrix} \\
A &= \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & 0 \\ -a_1 & 0 & a_2 & & 0 & 0 \\ 0 & -a_2 & 0 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & 0 & a_{n-1} \\ 0 & & & & -a_{n-1} & 0 \end{pmatrix}
\end{aligned} \tag{4.5}
$$

where $A = L_+ - L_-$ and we are using the following notation: for a square matrix X we call X_+ the upper triangolar part of X

$$
(X_+)_{ij} = \begin{cases} X_{ij}, & i \leq j \\ 0, & \text{otherwise} \end{cases}
$$

and in a similar way by X_- the lower triangular part of X

$$
(X_-)_{ij} = \begin{cases} X_{ij}, & i \geq j \\ 0, & \text{otherwise.} \end{cases}
$$

A straighforward calculation shows that

Lemma 4.1 *The Toda lattice equations (4.4) are equivalent to*

$$
\frac{dL}{dt} = [A, L] \tag{4.6}
$$

The non periodic Toda lattice equation can sometimes be written in Hessebeg form. Conjugating the matrix L by a diagonal matrix $D = \mathrm{diag}(1, a_1, a_1 a_2, \dots, \prod_{j=1}^{n-1} a_j)$ yelds the matrix $\widehat{L} = DLD^{-1}$

$$\widehat{L} = \begin{pmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ a_1^2 & b_2 & 1 & & 0 & 0 \\ 0 & a_2^2 & b_3 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & b_{n-1} & 1 \\ 0 & & & & a_{n-1}^2 & b_n \end{pmatrix} \tag{4.7}$$

The Toda equations (4.4) take the form

$$\frac{d\widehat{L}}{dt} = -2[\widehat{A}, \widehat{L}] \tag{4.8}$$

where the matrix $\widehat{A} = \widehat{L}_-$ namely

$$\widehat{A} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ a_1^2 & 0 & 0 & & 0 & 0 \\ 0 & a_2^2 & 0 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & 0 & 0 \\ 0 & & & & a_{n-1}^2 & 0 \end{pmatrix} \tag{4.9}$$

It follows from the results of the previous section that the Lax formulation guarantees the existence of conserved quantities, namely the traces

$$H_k = \frac{4}{k+1} \mathrm{tr}\ L^{k+1}, \quad k = 0, \dots, n-1.$$

are conserved quantities. To show the independence of the integrals $H_0, \dots, H_{n-1}$ let us consider the restriction

$$H_i^0(p) := H_i(p, a = 0), \quad i = 0, \dots, n-1,$$

It then follows that the matrix $L = L(b, a = 0)$ is diagonal and the functions $H_i^0(p)$ coincides with i-th symmetric elementary function of the variables $p_1, \dots, p_n$ and so they are linearly independent. To show that the integrals are involution, we will show that the eigenvalues of L are in involution. Before doing that we show that the eigenvalues are all distinct. The following relations hold true.

Lemma 4.2 *(i) The spectrum of L consists of n distinct real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.*

(ii) Let $Lv = \lambda v$ with $v = (v_1, \dots v_n)^t$. Then $v_1 \neq 0$ and $v_n \neq 0$. Furthermore, $v_k = p_k(\lambda)$ where $p_k(\lambda)$ is a polynomial of degree k in λ.

Proof. We will first prove (ii). From the equation $Lv = \lambda v$ one obtains

$$(b_1 - \lambda)v_1 + a_1 v_2 = 0 \tag{4.10}$$

$$a_{k-1}v_{k-1} + (b_k - \lambda)v_k + a_k v_{k+1} = 0, \quad 2 \leq k < n. \tag{4.11}$$

Since $a_1 \neq 0$ clearly $v_1 = 0 \Longrightarrow v_2 = 0$, but then from (4.11) with $k = 2$, since $a_2 \neq 0$, then $v_1 = 0$ and $v_2 = 0$ implies $v_3 = 0$. Hence $v = 0$ if $v_1 = 0$. Therefore $v_1 \neq 0$. In the same way it can be proved that $v_n \neq 0$. From (4.10) and (4.11) it easily follows that v_k is a polynomial of degree k in λ. To prove (i), suppose that v and $\tilde{v}$ are two eigenvalues corresponding to the same eigenvector λ. Then the linear combination $\alpha v + \beta \tilde{v}$, $\alpha, \beta \in \mathbb{R}$ is also an eigenvector of L with eigenvalue λ. But then one can choose $\alpha \neq 0$ and $\beta \neq 0$ so that $\alpha v_1 + \beta \tilde{v}_1 = 0$ and by (ii) it follows that $\alpha v + \beta \tilde{v}$=0 implying that v and $\tilde{v}$ are dependent. □

By the above lemma it follows that

$$L = U \Lambda U^t \tag{4.12}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and U is an orthogonal matrix $UU^t = 1$ with entries $U_{ij} = u_{ij}$ the normalized eigenvectors $u_i = (u_{i1}, \dots u_{in})^t$ of L. From $UU^t = U^t U = 1$ one has

$$(u_i, u_j) = \delta_{ij}, \quad \sum_{k=1}^{n} (u_{kj})^2 = 1, \quad i, j = 1, \dots, n.$$

Proposition 4.3 *The eigenvalues of L commute with respect to the canonical Poisson bracket (4.2).*

Proof. Let λ and μ be two eigenvalues of L with normalized eigenvectors v an w respectively. Then

$$\begin{aligned} \frac{\partial \lambda}{\partial p_i} &= \frac{\partial}{\partial p_i}(v,\, Lv) = \lambda \frac{\partial}{\partial p_i}(v,\, v) + (v,\, \frac{\partial L}{\partial p_i} v) = -\frac{1}{2} v_i^2 \\ \frac{\partial \lambda}{\partial q_i} &= \frac{\partial}{\partial q_i}(v,\, Lv) = (v,\, \frac{\partial L}{\partial q_i} v) = a_i v_i v_{i+1} - a_{i-1} v_i v_{i-1}, \quad i = 1, \dots, n, \end{aligned} \tag{4.13}$$

where we use the fact that $(v, v) = 1$ and we define $a_0 = 0 = a_n$. The same relations hold for the eigenvalue μ. Then one has

$$\begin{aligned}\{\lambda,\, \mu\} &= \sum_{i=1}^{n}\left(\frac{\partial\lambda}{\partial q_i}\frac{\partial\mu}{\partial p_i} - \frac{\partial\lambda}{\partial p_i}\frac{\partial\mu}{\partial q_i}\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}(v_i w_i(a_{i-1}(v_i w_{i+1} - v_{i+1}w_i) + a_{i-2}(w_i v_{i-1} - v_i w_{i-1}))\end{aligned} \tag{4.14}$$

We introduce the quantity $R_i = a_{i-1}(v_i w_{i+1} - v_{i+1}w_i)$, $i = 1, \ldots, n$ with $R_{-1} = R_n = 0$, and we observe that from the equations $Lv = \lambda v$ and $Lw = \mu w$ one obtains $R_i - R_{i-1} = (\mu - \lambda)v_i w_i$. Substituting the above relation in (4.14) one obtains

$$\{\lambda,\, \mu\} = \frac{1}{2(\mu-\lambda)}\sum_{i=1}^{n}(R_i^2 - R_{i-1}^2) = \frac{R_n^2 - R_{-1}^2}{2(\mu-\lambda)} = 0.$$

□

Summarazing, we have proved that the Toda Lattice is a completely integrable system possessing n conserved quantities $H_1, \ldots, H_n$, linearly independent and in involution. It follows that the system can be integrated by quadratures. Let us show how to do this. We know the eigenvalues of $L(t)$, since they are constants of motion. In order to know $L(t)$ at time t we need to know the orthogonal matrix $U = U(t)$, with entries $U_{ij} = u_{ij}$. From (4.6) and (4.12) one has that

$$\dot{U} = AU. \tag{4.15}$$

In particular, the dynamics implied by the above equation of the first row u_{1i}, $i = 1, \ldots, n$ of the matrix U is quite simple.

Lemma 4.4 *The time evolution on the first row of the matrix U, namely the entries u_{1i} $i = 1, \ldots, n$ is given by*

$$u_{1i}(t)^2 = \frac{e^{2\lambda_i t}u_{1i}(0)^2}{\sum_{k=1}^{n} e^{2\lambda_k t}u_{1k}(0)^2}, \quad i = 1, \ldots, n. \tag{4.16}$$

Proof. From (4.15) one has

$$\frac{du_{1i}}{dt} = (AU)_{1i} = a_1 u_{2i}$$

and from the relation $Lv_i = \lambda_i v_i$, with $v_i = (u_{1i}, \ldots, u_{ni})^t$, one reduces the above equation to the form

$$\frac{du_{1i}}{dt} = (\lambda_i - b_1)u_{1i}.$$

The solution is given by

$$u_{1i}(t) = E(t)e^{\lambda_i t}u_{1i}(0), \quad E(t) = exp\left(-\int_0^t b_1(\tau)d\tau\right)$$

Using the normalization conditions

$$1 = \sum_{i=1}^n u_{1i}(t)^2 = E(t)^2 \sum_{i=1}^n e^{2\lambda_i t}u_{1i}(0)^2$$

which implies

$$E(t)^2 = \left(\sum_{i=1}^n e^{2\lambda_i t}u_{1i}(0)^2\right)^{-1} \tag{4.17}$$

one arrives at the statement of the lemma. $\square$

Introducing the notation

$$w_k(t) = u_{1i}(t)^2, \quad k = 1, \ldots, n \tag{4.18}$$

one can see from lemma 4.2 that the orthogonal matrix U can be written in the form

$$U = \begin{pmatrix} \sqrt{w_1(t)}p_0(\lambda_1, t) & \sqrt{w_2(t)}p_0(\lambda_2, t) & \ldots & \sqrt{w_n(t)}p_0(\lambda_n, t) \\ \sqrt{w_1(t)}p_1(\lambda_1, t) & \sqrt{w_2(t)}p_1(\lambda_2, t) & \ldots & \sqrt{w_n(t)}p_1(\lambda_n, t) \\ \vdots & \vdots & & \vdots \\ \sqrt{w_1, t}p_{n-1}(\lambda_1, t) & \sqrt{w_2(t)}p_{n-1}(\lambda_2, t) & \ldots & \sqrt{w_n(t)}p_{n-1}(\lambda_n, t) \end{pmatrix}$$

Since U is an orthogonal matrix, the orthogonality relations on the rows of U take the form

$$\sum_{k=1}^n w_k p_l(\lambda_k)p_j(\lambda_k) = \delta_{lj}. \tag{4.19}$$

In other words, the polynomials $p_j(\lambda)$ are normalized orthogonal polynomials with respect to the discrete weights w_k at the points λ_k. To find the orthogonal polynomials from the weights, is a standard procedure, called the Gram-Schmidt orthogonalization process. Therefore, from the weights $w_1(t), \ldots w_n(t)$ at time t one can get the orthogonal matrix $U(t)$.

4.1 Toda flows and orthogonal polynomials

It is instructive to relate the integration of the Toda flows to orthogonal polynomials. Let $d\mu(\lambda)$ be a positive measure on the real line such that

$$\int_{\mathbb{R}} \lambda^k d\mu(\lambda) < \infty, \quad k \geq 0.$$

Consider the $(n+1)\times(n+1)$ Hankel matrix M_n with entries

$$(M_n)_{ij} = \int_{\mathbb{R}} \lambda^{i+j-2} d\mu(\lambda), \quad i,j = 1,\ldots,n+1.$$

Lemma 4.5 *The matrix M_n is positive definite.*

Proof. It is sufficient to consider the positive integral

$$0 < \int_{\mathbb{R}} (\sum_{k=0}^{n} t_k \lambda^k)^2 d\mu(\lambda) = \int_{\mathbb{R}} \sum_{j,k=0}^{n} t_k t_j \lambda^{k+j} d\mu(\lambda) =< t, M_n t >$$

where $t = (t_0, \ldots, t_n)$. For the arbitrariness of t it follows that M_n is a positive definite matrix. □

We define the determinant

$$D_n = \det M_n \tag{4.20}$$

which is by lemma 4.5 positive. For convenience we are setting $D_{-1} = 1$.

Let us now consider the polynomial of degree n

$$\pi_n(\lambda) = \det \begin{pmatrix} & & M_{n-1} & & \int \lambda^n d\mu(\lambda) \\ & & & & \ldots \\ & & & & \int \lambda^{2n-1} d\mu(\lambda) \\ & & & & \\ \lambda^0 & \lambda^1 & \ldots & \lambda^{n-1} & \lambda^n \end{pmatrix} \tag{4.21}$$

Lemma 4.6 *The polynomials*

$$\begin{aligned} p_0(\lambda) &= \frac{1}{\sqrt{D_0}} \\ p_n(\lambda) &= \frac{\pi_n(\lambda)}{\sqrt{D_n D_{n-1}}} = \sqrt{\frac{D_{n-1}}{D_n}} \left(\lambda^n + O(\lambda^{n-1})\right), \quad n > 0, \end{aligned} \tag{4.22}$$

are orthonormal polynomials with respect to the measure $d\mu(\lambda)$, namely

$$\int_{\mathbb{R}} p_n(\lambda) p_m(\lambda) d\mu(\lambda) = \delta_{nm}. \tag{4.23}$$

Proof. The orthonormality condition (4.23) is equivalent to the conditions $\int_{\mathbb{R}} p_n(\lambda)\lambda^m d\mu(\lambda) = 0$ for $m < n$ and $\int_{\mathbb{R}} p_n(\lambda)^2 d\mu(\lambda) = 1$ Using the fact that the determinant is a multilinear map one has

$$\int_{\mathbb{R}} p_n(\lambda)\lambda^m d\mu(\lambda) = \det \begin{pmatrix} & & M_{n-1} & & \int \lambda^n d\mu(\lambda) \\ & & & & \ldots \\ & & & & \int \lambda^{2n-1} d\mu(\lambda) \\ & & & & \\ \int \lambda^m d\mu(\lambda) & \int \lambda^{m+1} d\mu(\lambda) & \ldots & \int \lambda^{m+n-1} d\mu(\lambda) & \int \lambda^{m+n} d\mu(\lambda) \end{pmatrix} = 0, \quad m < n.$$

The above determinant is equal to zero because the last row of the above matrix is equal to the $(m+1)$th row. Regarding the normalising condition one has

$$\int_{\mathbb{R}} p_n(\lambda)^2 d\mu(\lambda) = \frac{1}{D_n D_{n-1}} \int_{\mathbb{R}} D_{n-1}\lambda^n \pi_n(\lambda) d\mu(\lambda) = 1.$$

□

Lemma 4.7 *The orthogonal polynomials (4.22) satisfy a 3-term recurrence relations*

$$\begin{aligned} \lambda p_0(\lambda) &= a_1 p_1(\lambda) + b_1 p_0(\lambda) \\ \lambda p_n(\lambda) &= a_{n+1} p_{n+1}(\lambda) + b_{n+1} p_n(\lambda) + a_n p_{n-1}(\lambda), \end{aligned} \tag{4.24}$$

with

$$a_{n+1} = \sqrt{\frac{D_{n+1}D_{n-1}}{D_n^2}} \tag{4.25}$$

$$b_{n+1} = \frac{G_n}{D_n} - \frac{G_{n-1}}{D_{n-1}}. \tag{4.26}$$

where G_{n-1} is the determinant of the minor of $D_n(\lambda)$ that is obtained by erasing the $(n+1)$ row and the n column,

Proof. The polynomial $\lambda p_n(\lambda)$ is of degree $n+1$ so one has

$$\lambda p_n(\lambda) = \sum_{k=0}^{n+1} \gamma_k^n p_k(\lambda),$$

for some constants γ_k^n. Multiplying both sides of the above identity by $p_j(\lambda)$, $0 \le j < n-1$ and integrating over $d\mu(\lambda)$ one has, using orthogonality

$$0 = \int_{\mathbb{R}} \lambda p_n(\lambda) p_j(\lambda) d\mu(\lambda) = \gamma_j^n, \quad 0 \le j < n-1.$$

because $\lambda p_j(\lambda)$ is a polynomial of degree at most $j+1$ and $\lambda p_n(\lambda)$ is at most of degree $n+1$. Therefore only $\gamma_{n+1}^n, \gamma_n^n$ and γ_{n-1}^n are different from zero. In order to determine the coefficient γ_{n+1}^n let us observe that

$$p_n(\lambda) = \sqrt{\frac{D_{n-1}}{D_n}}\lambda^n + O(\lambda^{n-1}),$$

and comparing the right and left-handside of (4.24) one has

$$\gamma_{n+1}^n = \sqrt{\frac{D_{n+1}D_{n-1}}{D_n^2}} := a_{n+1} \tag{4.27}$$

Regarding γ_{n-1}^n one has

$$\gamma_{n-1}^n = \int_{\mathbb{R}} \lambda p_n(\lambda) p_{n-1}(\lambda) d\mu(\lambda) = \sqrt{\frac{D_n D_{n-2}}{D_{n-1}^2}}$$

so that $\gamma_{n-1}^n = a_n$. Defining G_{n-1} the determinant of the minor of $D_n(\lambda)$ that is obtained by erasing the $(n+1)$ row and the n column, one has that

$$p_n(\lambda) = \sqrt{\frac{D_{n-1}}{D_n}} \lambda^n - \frac{G_{n-1}}{\sqrt{D_n D_{n-1}}} \lambda^{n-1} + O(\lambda^{n-2})$$

so that comparing the left and righthandside of (4.24) one obtains

$$b_{n+1} = \frac{G_n}{D_n} - \frac{G_{n-1}}{D_{n-1}}. \tag{4.28}$$

□

4.2 Integration of Toda lattice

Now let us consider the measure associated to the Toda lattice

$$d\tilde{\mu}(\lambda) = E^2(t) \sum_{j=1}^n e^{2\lambda_i t} \delta(\lambda - \lambda_i) u_{1,i}(0)^2 d\lambda,$$

with $E(t)$ a function of time as in (4.17). Then it is easy to check that the ratios G_n/D_n in (4.28) are independent from $E(t)$ as well as the ratios $\sqrt{\frac{D_{n+1}D_{n-1}}{D_n^2}}$ in the definition of a_n. Therefore we can set $E(t) = 1$ without loss of generality. It in an easy calculation to derive the identity

$$\frac{\partial D_n}{\partial t} = 2G_n.$$

So using the above identity one can write the coefficient b_{n+1} in the form

$$b_{n+1} = \frac{1}{2} \frac{\partial}{\partial t} \log \frac{D_n}{D_{n-1}}. \tag{4.29}$$

We conclude that the integration of the Toda lattice equation is given by the relation (4.29) and (4.25) with respect to the measure

$$d\mu(\lambda, t) = \sum_{j=1}^n u_{1,i}(0)^2 e^{2\lambda_i t} \delta(\lambda - \lambda_i) d\lambda.$$

We are now interested in determining the evolution of the coefficients a_n and b_n as a function of the parameter t. To operate in a more general setting let us introduce the modified weight

$$d\mu(\lambda) = e^{2\sum_{k=1}^{s}\lambda^k t_k} d\tilde{\mu}(\lambda),$$

with $d\tilde{\mu}(\lambda)$ independent from the times $t_k, k = 1, \ldots, s$ and with $t_1 = t$. Consider the tridiagonal seminfinte matrix L

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \ldots & 0 & 0 & \ldots \\ a_1 & b_2 & a_2 & & 0 & 0 & \ldots \\ 0 & a_2 & b_3 & & & 0 & \ldots \\ & & & & & & \\ \ldots & & & \ldots & & \ldots & \ldots \\ & & & & & & \\ 0 & & & & b_{n-1} & a_{n-1} & \ldots \\ 0 & & & & a_{n-1} & b_n & \ldots \\ \ldots & & & \ldots & & \ldots & \ldots \end{pmatrix} \tag{4.30}$$

and the infinite vector

$$p(\lambda) = \begin{pmatrix} p_0(\lambda) \\ p_1(\lambda) \\ p_2(\lambda) \\ \ldots \\ p_n(\lambda) \\ \ldots. \end{pmatrix}$$

Then the 3-term recurrence relation can be written in the compact form

$$\lambda p(\lambda) = Lp(\lambda). \tag{4.31}$$

Now let us introduce the quasi-polynomials

$$\psi_k(\lambda) = p_k(\lambda) e^{\sum_{k=1}^{s}\lambda^k t_k}.$$

Clearly from the orthonormality of the polynomials $p_k(\lambda)$ it follows that

$$\int_{\mathbb{R}} \psi_k(\lambda)\psi_j(\lambda) d\tilde{\mu}(\lambda) = \delta_{kj}. \tag{4.32}$$

Now we are going to investigate the dependence of ψ_k on the times $t_1, \ldots, t_s$.

Lemma 4.8 *The following relation is satisfied:*

$$\frac{\partial \psi_j(\lambda)}{\partial t_\alpha} = \sum_{m=0}^{\infty} (A_\alpha)_{jm}\psi_m(\lambda), \quad \alpha = 1, \ldots, s, \tag{4.33}$$

with A_α antisymmetric matrix.

Proof. Let us differentiate with respect to t_α the orthonormality relations (4.32)

$$\int_{\mathbb{R}} \frac{\partial \psi_j(\lambda)}{\partial t_\alpha} \psi_k(\lambda) d\tilde{\mu}(\lambda) + \int_{\mathbb{R}} \psi_j(\lambda) \frac{\partial \psi_k(\lambda)}{\partial t_\alpha} d\tilde{\mu}(\lambda) = 0$$

so that

$$\begin{aligned}
&\int_{\mathbb{R}} \sum_m (A_\alpha)_{jm} \psi_m(\lambda) \psi_k(\lambda) d\tilde{\mu}(\lambda) + \int_{\mathbb{R}} \psi_j(\lambda) \sum_m (A_\alpha)_{km} \psi_m(\lambda) d\tilde{\mu}(\lambda) \\
&= (A_\alpha)_{jk} + (A_\alpha)_{kj} = 0
\end{aligned}$$

□

Lemma 4.9 *The following relation is satisfied*

$$A_\alpha = (L^\alpha)_+ - (L^\alpha)_-, \quad \alpha = 1, \ldots, s, \tag{4.34}$$

where $(L^\alpha)_\pm$ is the projection of L^α to the upper/lowe triangular part of L^α.

Proof. We observe that

$$\psi_k(\lambda) = \left(\sqrt{\frac{D_{k-1}}{D_k}} \lambda^k + O(\lambda^{k-1}) \right) e^{\sum_{\beta=1}^s \lambda^\beta t_\beta},$$

so that

$$\frac{\partial \psi_k(\lambda)}{\partial t_\alpha} = \psi_k(\lambda) \frac{\partial}{\partial t_\alpha} \left(\log \sqrt{\frac{D_{k-1}}{D_k}} \right) + \lambda^\alpha \psi_k(\lambda) + O(\lambda^{k-1}) e^{\sum_{\beta=1}^s \lambda^\beta t_\beta},$$

so that for $j > k$

$$\begin{aligned}
A_{kj} &= \int_{\mathbb{R}} \frac{\partial \psi_k(\lambda)}{\partial t_\alpha} \psi_j(\lambda) d\tilde{\mu}(\lambda) = \int_{\mathbb{R}} \lambda^\alpha \psi_k(\lambda) \psi_j(\lambda) d\tilde{\mu}(\lambda) = \int_{\mathbb{R}} \sum_m (L^\alpha)_{km} \psi_m(\lambda) \psi_j(\lambda) d\tilde{\mu} \\
&= (L^\alpha)_{kj}.
\end{aligned}$$

Using the antisymmetry of A_α, (4.34) follows. □

Lemma 4.10 *The semiinfinite matrix L satisfies the Lax equation*

$$\frac{dL}{dt_\alpha} = [A_\alpha, L], \quad \alpha = 1, \ldots, s. \tag{4.35}$$

Proof. We differentiate with respect to t_α the 3-term recurrence relation (4.31) to obtain

$$\frac{dL}{dt_\alpha}\psi + (L-\lambda)\frac{d\psi}{dt_\alpha} = 0 \tag{4.36}$$

where $\psi(\lambda) = p(\lambda)e^{\sum_{k=1}^{s} t_k\lambda^k}$. Using (4.33) one obtains

$$\frac{dL}{dt_\alpha}\psi + (L-\lambda)A_\alpha\psi = \left(\frac{dL}{dt_\alpha} - [A_\alpha, L]\right)\psi = 0$$

so that by the completeness of ψ one has (4.35). $\square$

Remark 4.11 Let $(\lambda_1, \dots, \lambda_n)$ be the zeros of the polynomial $p_n(\lambda)$, then the relation (4.31) takes the form

$$\begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & 0 \\ a_1 & b_2 & a_2 & & 0 & 0 \\ 0 & a_2 & b_3 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & b_{n-1} & a_{n-1} \\ 0 & & & & a_{n-1} & b_n \end{pmatrix} \begin{pmatrix} p_0(\lambda_j) \\ p_1(\lambda_j) \\ p_2(\lambda_j) \\ \\ \dots \\ \\ p_{n-2}(\lambda_j) \\ p_{n-1}(\lambda_j) \end{pmatrix} = \lambda_j \begin{pmatrix} p_0(\lambda_j) \\ p_1(\lambda_j) \\ p_2(\lambda_j) \\ \\ \dots \\ \\ p_{n-2}(\lambda_j) \\ p_{n-1}(\lambda_j) \end{pmatrix}.$$

The above equality says that the zeros of $p_n(\lambda)$ are the eigenvalues of L defined in (4.5) and therefore, by lemma 4.2, its eigenvalues are distinct and real. The eigenvector relative to the eigenvalue λ_j is given by $(p_0(\lambda_j), p_1(\lambda_j), \dots, p_{n-1}(\lambda_j))^t$.

Remark 4.12 From the construction of this section and the relation (4.19), in order to solve the Toda lattice equations, given the Lax matrix $L(0)$ at time $t = 0$, it is sufficient to determine its eigenvalues $\lambda_1, \dots, \lambda_n$ and the first entry of the eigenvectors $u_{1j}(0)$, $j = 1, \dots, n$ and then construct the measure

$$d\mu(\lambda) = \sum_{j=1}^{n} u_{1j}(0)^2 e^{2\lambda_j t}\delta(\lambda - \lambda_j)d\lambda,$$

where $\delta(\lambda)$ is the Dirac delta function. Given the measure $d\mu(\lambda)$ the solution of the Toda lattice equation is obtained from (4.20), (4.29) and (4.27).

Lemma 4.13 *The zeros of the polynomial $p_n(\lambda)$ and $p_{n+1}(\lambda)$ interlace, i.e. between any two zeros of $p_n(\lambda)$ lies exactly one root of $p_{n+1}(\lambda)$.*

Proof. Let us consider the sum

$$
\begin{aligned}
(\mu-\lambda)\sum_{j=0}^{n} p_j(\lambda)p_j(\mu) &= \sum_{j=0}^{n}[(a_{j+1}p_{j+1}(\mu)+b_{j+1}p_j(\mu)+a_jp_{j-1}(\mu))p_j(\lambda) \\
&\quad -(a_{j+1}p_{j+1}(\lambda)+b_{j+1}p_j(\lambda)+a_jp_{j-1}(\lambda))p_j(\mu)] \\
&= \sum_{j=0}^{n}[a_j(p_j(\lambda)p_{j-1}(\mu)-p_{j-1}(\lambda)p_j(\mu)-a_{j+1}(p_{j+1}(\lambda)p_j(\mu)-p_j(\lambda)p_{j+1}(\mu)) \\
&= -a_{n+1}(p_{n+1}(\lambda)p_n(\mu)-p_n(\lambda)p_{n+1}(\mu)),
\end{aligned}
$$

so that

$$
\begin{aligned}
\sum_{j=0}^{n} p_j(\lambda)^2 &= -a_{n+1}\lim_{\mu\to\lambda}\frac{1}{\mu-\lambda}(p_{n+1}(\lambda)p_n(\mu)-p_n(\lambda)p_{n+1}(\mu)) \\
&= -a_{n+1}(p_{n+1}(\lambda)p_n'(\lambda)-p_n(\lambda)p_{n+1}'(\lambda)) > 0,
\end{aligned}
\tag{4.37}
$$

where $'$ means derivative with respect to λ. If λ_j and λ_{j+1} are consecutive zeros of $p_n(\lambda)$ then from the above relation

$$
p_{n+1}(\lambda_j)p_n'(\lambda_j) < 0 \text{ and } p_{n+1}(\lambda_{j+1})p_n'(\lambda_{j+1}) < 0,
$$

because a_{n+1} is positive. Therefore, since $p_n'(\lambda_j)$ and $p_n'(\lambda_{j+1})$ have opposite sign, then $p_{n+1}(\lambda)$ must have a zero between λ_j and λ_{j+1}. □

5 Jacobi Operators

Let us consider the space $l(\mathbb{Z},\mathbb{C})$ of sequences $f=(f_n)_{n\in\mathbb{Z}}$ taking values in the complex numbers $\mathbb{C}$. In the rest we will drop the dependence on $\mathbb{C}$ and keep $\ell(\mathbb{Z})$. One can define a norm

$$
\ell^p(\mathbb{Z}) = \{f\in\ell(\mathbb{Z})|\,|\sum_{n\in\mathbb{Z}}|f_n|^p<\infty\}, \quad 1\le p<\infty
$$
$$
\ell^\infty(\mathbb{Z}) = \{f\in\ell(\mathbb{Z})|\,|\sup_{n\in\mathbb{Z}}|f_n|<\infty\}
$$

Clearly $\ell^2(\mathbb{Z})$ is a Hilbert space. On $\ell(\mathbb{Z})$ we define the endomorphism

$$
\begin{aligned}
\ell(\mathbb{Z}) &\to \ell(\mathbb{Z}) \\
f &\to Lf
\end{aligned}
$$

where L is uniquely determined by its matrix entries $L(m,n)_{m,n\in\mathbb{Z}}$. The order of L is the the smallest nonnegative integer $N=N_++N_-$ such that $L(m,n)=0$ for all m and n with $m-n>N_+$ or $n-m>N_-$.

Let $a, b \in \ell(\mathbb{Z})$ be two real valued sequences satisfying $a_n \in \mathbb{R}^+$ and $b_n \in \mathbb{R}$. We define the second order symmetric operator L as

$$(Lf)_n = a_n f_{n+1} + b_n f_n + a_{n-1} f_{n-1}$$

which is associated with the tridiagonal matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & \\ a_{n-2} & b_{n-1} & a_{n-1} & & & \\ & a_{n-1} & b_n & a_n & & \\ & & a_n & b_{n+1} & a_{n+1} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Let us consider the Jacobi difference equation

$$Lf = zf, \quad z \in \mathbb{Z}, \quad f \in \ell(\mathbb{Z}). \tag{5.1}$$

If $a(n) \neq 0$ the solution is uniquely determined by the two values f_{n_0} and f_{n_0+1}. Therefore one can find exactly two linearly independent solutions to the above equation. We define the discrete Wronkstian

$$W_n(f, g) = a_n(f_n g_{n+1} - f_{n+1} g_n). \tag{5.2}$$

It is easy to check that if f and g satisfy (5.1) then the Wronkstian does not depend on n. Let us introduce a fundamental solutions of (5.1) normalised

$$\begin{aligned} &c(z, n_0, n_0) = 1, \quad s(z, n_0, n_0) = 0 \\ &c(z, n_0 + 1, n_0) = 0, \quad s(z, n_0 + 1, n_0) = 1, \end{aligned} \tag{5.3}$$

where later on, we will omit the dependence on the base point n_0. Since the Wronkstian does not depend on n one has

$$W_n(c(z, n, n_0), s(z, n, n_0)) = a_{n_0}.$$

Any solution of the equation (5.1) can be represented in the form

$$u_n = u_{n_0} c(z, n, n_0) + u_{n_0+1} s(z, n, n_0).$$

Using (5.1) one can write

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{a_{n-1}}{a_n} & \dfrac{z - b_n}{a_n} \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} \tag{5.4}$$

so that we define $U(z, n, n_0)$ as

$$U(z, n, n_0) = \begin{pmatrix} 0 & 1 \\ -\dfrac{a_{n-1}}{a_n} & \dfrac{z - b_n}{a_n} \end{pmatrix}.$$

Then from the above one can define

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \tag{5.5}$$

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