Riemann Surfaces and Integrable Systems

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Chapter 1

Riemann surfaces

1.1 Definition of Riemann surface and basic examples

1.1.1 Complex manifolds. First examples of Riemann surfaces

Bernhard Riemann (1826–1866) introduced the concept of Riemann surface to make sense of multivalued functions like the square root or the logarithm. For the geometric representation of multi-valued functions of a complex variable $w = w(z)$ it is not convenient to regard z as a point of the complex plane. For example, take $w = \sqrt{z}$. On the positive real semiaxis $z \in \mathbb{R}, \quad z > 0$ the two branches $w_1 = +\sqrt{z}$ and $w_2 = -\sqrt{z}$ of this function are well defined by the condition $w_1 > 0$. This is no longer possible on the complex plane. Indeed, the two values $w_{1,2}$ of the square root of $z = r\, e^{i\psi}$

$$w_1 = \sqrt{r}\, e^{i\frac{\psi}{2}}, \quad w_2 = -\sqrt{r}\, e^{i\frac{\psi}{2}} = \sqrt{r} e^{i\frac{\psi+2\pi}{2}}, \tag{1.1.1}$$

interchange when passing along a path

$$z(t) = r\, e^{i\,(\psi+t)}, \quad t \in [0, 2\pi]$$

encircling the point $z = 0$. It is possible to select a branch of the square root as a function of z by restricting the domain of this function for example, by making a cut along the negative real semi-axis. The two functions $w_1(z)$ and $w_2(z)$ defined as in (1.1.1) with $-\pi < \psi < \pi$ are single-valued on the cut plane $\mathbb{C}\backslash(-\infty, 0]$. Riemann's idea was to combine the two branches of the function $\sqrt{z}$ to a single-valued fuction well-defined on a suitable geometric object $\mathcal{S}$. To do this observe that $w_1(z) \to i\sqrt{r}$ and $w_2(z) \to -i\sqrt{r}$ for $z \to -r$ from above the cut $(-\infty, 0]$. In a similar way $w_1(z) \to -i\sqrt{r}$ and $w_2(z) \to i\sqrt{r}$ for $z \to -r$ from below the cut $(-\infty, 0]$. So, the rules to construct the space $\mathcal{S}$ are as follows: one has to take two copies of the complex plane cut along the negative real semi-axis and join the two copies of the complex plane along the cuts glueing the upper side of the cut on one copy with the lower side of the cut on another one. In other words the two sheets have to be glued together in such a way that the branch of the function $\sqrt{z}$ on one sheet joins continuously with the branch defined on the other sheet. The result of this operation is a complex manifold $\mathcal{S}$ of complex dimension one (see below for the precise definition). It can also be treated

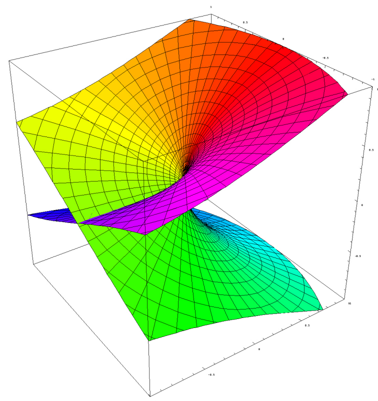


Figure 1.1: The imaginary part of the function $\sqrt{z}$

as a smooth real manifold of dimension two, that is, a surface. The surface shown on figure 1.1 is the imaginary part of $\sqrt{z}$.

A similar procedure of cutting and glueing can be repeated for other multivalued analytic functions. For example the logarithm $\log z$ is a single valued function on $\mathbb{C}\backslash[0,+\infty)$ with infinite number of branches. Each adjacent branch differs by an additive term $2\pi i$. The infinite set of branches attached along the positive real line is shown on the figure 1.2.

T. Cancellerei la parte sotto

The Riemann surface of the multivalued function $\sqrt{z}$ can also be constructed as the zero locus of the polynomial $F(z,w) = w^2 - z$, namely

$$\mathcal{S} = \{(z,w) \in \mathbb{C}^2 \mid F(z,w) = w^2 - z = 0\}$$

(a complex algebraic curve). The function $w : \mathcal{S} \to \mathbb{C}$ is defined as follows: for a point $P = (z,w) \in \mathcal{S}$ we put $w(P) = w$. We leave as an exercise to the reader to verify that the two constructions give the same result. It will also be done below for a general class of algebraic multivalued functions.

In the theory of Riemann surfaces the techniques of working with complex manifolds or with complex algebraic curves both played an important role.

Before doing this we remind that a complex function $f : G \to \mathbb{C}$ where G is a domain in $\mathbb{C}$, can be written in the form $f(z) = u(x,y) + iv(x,y)$, with $z = x+iy$, $x,y \in \mathbb{R}$ and $u(x,y)$ and $v(x,y)$ real functions of (x,y). The function $f(z)$ is holomorphic in G if u and v are real differentiable in G and their derivatives satisfy the Cauchy Riemann equations

$$u_x = v_y, \quad u_y = -v_x, \quad \text{for } z \in G.$$

Alternatively introducing the operators $\partial/\partial z$ and $\partial/\partial\bar{z}$ defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{1.1.2}$$

the Cauchy Riemann equations can be written in the form

$$\frac{\partial}{\partial \bar{z}} f = 0, \quad \text{for } z \in G.$$

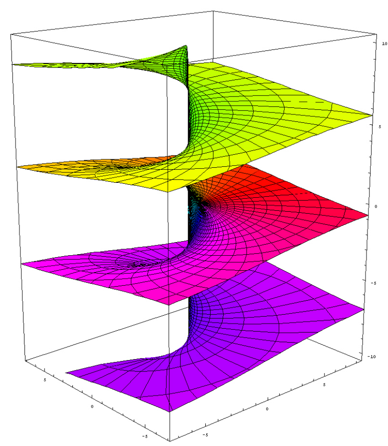


Figure 1.2: The Riemann surface of the function $\log z$

We also recall that a holomorphic function $f : G \to \mathbb{C}$ can be expanded in convergent power series. For this reason it is often called analytic function.

We now introduce some basic properties of complex manifolds.

Definition 1.1.1. *A complex manifold of complex dimension n is a second-countable Hausdorff*[1] *topological space M with a collection of charts* $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ *where* $U_\alpha \subset M$ *is an open subset in M and* $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ *such that*

1. *The sets* U_α *are a covering of M*
$$\bigcup_{\alpha\in\mathcal{A}} U_\alpha = M \tag{1.1.3}$$
2. $\phi_\alpha(U_\alpha)$ *is open in* $\mathbb{C}^n$ *and* $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ *is a homeomorphism onto an open subset in* $\mathbb{C}^n$.
3. *If* $U_{\alpha,\beta} := U_\alpha \cap U_\beta \neq \emptyset$ *then both* $\phi_\alpha(U_{\alpha,\beta})$ *and* $\phi_\beta(U_{\alpha,\beta})$ *are open sets in* $\mathbb{C}^n$ *and*
$$G_{\alpha,\beta} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_{\alpha,\beta}) \to \phi_\beta(U_{\alpha,\beta}) \tag{1.1.4}$$
are holomorphic maps,
$$G_{\alpha,\beta}(z_1, \ldots, z_n) = (w_1(z), \ldots, w_n(z)) \in \phi_\beta(U_{\alpha,\beta}) \subset \mathbb{C}^n, \quad z = (z_1, \ldots, z_n) \in \phi_\alpha(U_{\alpha,\beta}) \subset \mathbb{C}^n$$
$$\frac{\partial w_i}{\partial \bar{z}_j} = 0, \quad i, j = 1, \ldots, n.$$

The collection of charts is called an **atlas** *for the manifold M. The image* $\phi_\alpha(P) = (z_1(P), \ldots, z_n(P)) \in \mathbb{C}^n$ *of a point* $P \in U_\alpha$ *defines* **local coordinates** $z_1(P), \ldots, z_n(P)$ *of the point. The maps* $G_{\alpha,\beta}$ *are called* **transition functions**.

[1] Recall that a Hausdorff topological space is a topological space such that for any pair of distinct points there exist non-intersecting open neighbourhoods. A topogical space X is second-countable if there there exists a countable family $(V_i)_{i\in\mathbb{N}}$ such that any open subset in X can be represented as a union $\bigcup_{i\in I} V_i$ for some $I \subset \mathbb{N}$.

Note that the transition functions $G_{\alpha,\beta}$ are invertible and the inverse maps $G_{\alpha,\beta}^{-1} = G_{\beta,\alpha}$ are holomorphic.

Given two atlases $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in\mathcal{B}}$ on M, we say that they are **equivalent** if their union is still an atlas. An equivalence class of atlas defines a complex analytic structure on M.

The space $\mathbb{C}^n$ is the simplest example of an n-dimensional complex manifold. One can also take an arbitrary open subset $M \subset \mathbb{C}^n$. In these cases it suffices to use atlases consisting just of one chart. Let us consider a less trivial example.

Example 1.1.2. Points of the complex n-dimensional projective space $\mathbb{P}^n$ are defined as equivalence classes of $(n+1)$-dimensional non-zero complex vectors $\in \mathbb{C}^{n+1}\backslash 0$

$$(Z_0, Z_1, \ldots, Z_n) \sim \lambda\,(Z_0, Z_1, \ldots, Z_n)\,, \quad 0 \neq \lambda \in \mathbb{C}.$$

The equivalence class of vectors $(Z_0, Z_1, \ldots, Z_n)$ is denoted by $(Z_0 : Z_1 : \cdots : Z_n)$. The complex numbers Z_α are called homogeneous cordinates of the point.

An atlas consisting of $n+1$ charts $(U_\alpha, \phi_\alpha)_{\alpha=0,1,\ldots,n}$ is defined as follows

$$U_\alpha = \{(Z_0, Z_1, \ldots, Z_n) \in \mathbb{C}^{n+1} \mid Z_\alpha \neq 0\}$$

$$\phi_\alpha\,(Z_0, Z_1, \ldots, Z_n) = \left(\frac{Z_0}{Z_\alpha}, \frac{Z_1}{Z_\alpha}, \ldots, \frac{\widehat{Z_\alpha}}{Z_\alpha}, \ldots, \frac{Z_n}{Z_\alpha}\right)$$

where the hat means that the corresponding term is omitted.

Let us consider the particular cases $n = 1$ and $n = 2$. On $\mathbb{P}^1$ we have two charts U_0 and U_1 with the local coordinates

$$\phi_0(Z_0, Z_1) = \frac{Z_1}{Z_0} := z \quad \text{on} \quad U_0, \quad \phi_1(Z_0, Z_1) = \frac{Z_0}{Z_1} := w \quad \text{on} \quad U_1.$$

On the intersection $U_0 \cap U_1$ we have $z \neq 0$, $w \neq 0$ and the transition functions are

$$w = \frac{1}{z} \quad \text{or} \quad z = \frac{1}{w}.$$

The map ϕ_0 establishes a one-to-one correspondence between U_0 and the complex plane $\mathbb{C}$. The complement $\mathbb{P}^1\backslash U_0$ consists just of one point $(0:1)$. It can be considered as the point at infinity in the complex plane. Indeed, if a point $P \in U_0$ goes to $(0:1)$ then $z(P) \to \infty$. Thus

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}.$$

That means that topologically $\mathbb{P}^1$ is a two-dimensional sphere. For this reason the manifold $\mathbb{P}^1$ is often called Riemann sphere. Another name for $\mathbf{P}^1$ is extended complex plane denoted by $\overline{\mathbb{C}}$.

In a similar way for $\mathbb{P}^2$ the chart U_0 is identified with $\mathbb{C}^2$ and

$$\mathbb{P}^2\backslash U_0 = \{(0, Z_1, Z_2) \neq 0, \mid (0, Z_1, Z_2) \sim \lambda\,(0, Z_1, Z_2) \mid 0 \neq \lambda \in \mathbb{C}\} = \mathbb{P}^1.$$

Therefore

$$\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1.$$

Exercise 1.1.3: Consider the $(2n+1)$-dimensional unit sphere S^{2n+1} defined in the space $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ by the equation

$$|Z_0|^2 + |Z_1|^2 + \cdots + |Z_n|^2 = X_0^2 + Y_0^2 + X_1^2 + Y_1^2 + \cdots + X_n^2 + Y_n^2 = 1$$

where $Z_k = X_k + i\,Y_k$. The group $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ acts on S^{2n+1} by multiplication

$$(Z_0, Z_1, \ldots, Z_n) \sim \lambda\,(Z_0, Z_1, \ldots, Z_n)\,.$$

Prove that the quotient manifold S^{2n+1}/S^1 carries a natural structure of a complex manifold of complex dimension n. Prove that this manifold can be identified with $\mathbb{P}^n$. As a corollary derive that the projective space $\mathbb{P}^n$ is compact for any n.

Exercise 1.1.4: Prove that $\mathbb{P}^1$ is diffeomorphic to the standard unit sphere S^2 in $\mathbb{R}^3$

$$x^2 + y^2 + z^2 = 1.$$

To define a real C^k-smooth n-dimensional manifold, one has to replace $\mathbb{C}^n$ with $\mathbb{R}^n$ and the transition functions are C^k-smooth in their respective variables. An equivalence class of atlases defines a C^k-smooth structure on the manifold. When $k = \infty$ the manifold is simply called smooth manifold or C^∞-smooth manifold.

A complex n-dimensional manifold is also a real C^∞-smooth[2] manifold of dimension $2n$. A natural choice of local coordinates on the real manifold is given by the real and imaginary parts of the complex coordinates

$$x_i = \operatorname{Re} z_i, \quad y_i = \operatorname{Im} z_i, \quad i = 1, \ldots, n.$$

The transition function

$$z = (z_1, \ldots, z_n) \mapsto (w_1(z), \ldots, w_n(z))$$

is a holomorphic change of coordinates. In the new chart define the real coordinates

$$u_i = \operatorname{Re} w_i, \quad v_i = \operatorname{Im} w_i, \quad i = 1, \ldots, n.$$

Further the following identity between real and complex Jacobians holds true

$$\det \begin{pmatrix} \partial u_i/\partial x_j & \partial u_i/\partial y_j \\ \partial v_i/\partial x_j & \partial v_i/\partial y_j \end{pmatrix} = \left| \det \left(\partial w_i/\partial z_j \right) \right|^2. \tag{1.1.5}$$

We leave the proof of this identity as an exercise for the reader.

A real smooth manifold M is *orientable* if there exists an atlas such that all the transition maps $G(x_1, \ldots, x_n) = (G_1(x), \ldots, G_n(x))$ have positive Jacobian determinant $\det \left(\dfrac{\partial G_j(x)}{\partial x_k} \right) > 0$. A choice of such an atlas is called an orientation on M.

From the relation (1.1.5) it follows that a *complex manifold is always orientable.*

We will be concerned with manifolds of complex dimension 1.

[2] It is even a real analytic manifold.

Definition 1.1.5. *A Riemann surface $\mathcal{S}$ is a connected*[3] *one-dimensional complex manifold.*

As it was explained above $\mathcal{S}$ is also a two-dimensional smooth orientable manifold.

Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ define a complex structure on $\mathcal{S}$ and suppose that $P \in U_\alpha \cap U_\beta \neq \emptyset$. Hence the local charts

$$z = \phi_\alpha(P), \quad w = \phi_\beta(P)$$

will be complex-valued functions.

The transition function $\phi_\beta \circ \phi_\alpha^{-1} : z \to w = w(z)$ is *bi-holomorphic*, namely, holomorphic with holomorphic inverse $z = z(w)$

$$\frac{\partial w}{\partial \bar{z}} = 0, \quad \frac{\partial z}{\partial \bar{w}} = 0,$$

where the operators $\partial/\partial w$ and $\partial/\partial \bar{w}$ are defined in a similar way as in (1.1.2). So, in a small neighbourhood of any point $P_0 \in U_\alpha \cap U_\beta$ with $z_0 = \phi_\alpha(P_0)$ and $w_0 = \phi_\beta(P_0)$ we have the power series expansion

$$w(z) = w_0 + \sum_{k>0} a_k (z - z_0)^k, \quad a_1 \neq 0,$$

and

$$z(w) = z_0 + \sum_{k>0} b_k (w - w_0)^k, \quad b_1 \neq 0.$$

Example 1.1.6. Elementary examples of Riemann surfaces

(a) The simplest examples of Riemann surfaces are those defined by one single chart. Any connected open subset of the complex plane is clearly a Riemann surface. Other interesting examples include the complex plane $\mathbb{C}$, the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and the upper half space $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

(b) The projective space $\mathbb{P}^1$, the Riemann sphere or extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ and the sphere $S^2 = \{(x, y, t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 = 1\}$ are Riemann surfaces. In this case the atlas consists of two charts. For the sphere S^2 the two charts are

$$U_1 = S^2 \backslash (0, 0, 1), \phi_1(x, y, t) = \frac{x + iy}{1 - t} \tag{1.1.6}$$

$$U_2 = S^2 \backslash (0, 0, -1), \phi_1(x, y, t) = \frac{x - iy}{1 + t} = \frac{1 - t}{x + iy} \tag{1.1.7}$$

$$\tag{1.1.8}$$

On the intersection $U_1 \cap U_2 \simeq \mathbb{C} \backslash \{0\}$ we have $\phi_2 \circ \phi_1^{-1}(z) = \frac{1}{z}$ where $z = \phi_1(x, y, t)$. It is let as an exercise to show that $\overline{\mathbb{C}}$ and $\mathbb{P}^1$ are Riemann surfaces.

Example 1.1.7. Riemann surface of $\sqrt{z}$.

Consider the complex algebraic curve

$$C = \{(z, w) \in \mathbb{C}^2 \mid w^2 - z = 0\}.$$

[3] In this book we use the word 'connected' for path-connected topological spaces. For manifolds these two notions are equivalent, see e.g. [?].

A chart in a neighbourhood of a point $(z_0, w_0) \in C$ with $z_0 \neq 0$ is defined on the domain $U = \{(z, w = \sqrt{z}) \in C \mid |z - z_0| < \epsilon\}$ with $\epsilon < |z_0|$ where the branch of $\sqrt{z}$ is uniquely defined by the condition $\sqrt{z_0} = w_0$. The coordinate map $U \to \mathbb{C}$ is given by the projection to the z-axis

$$(z, w) \mapsto z.$$

It remains to construct a chart in a neighbourhood of the point $(0, 0) \in C$. Define the domain $V = \{(z = w^2, w) \in C \mid |w| < \epsilon\}$ for some $\epsilon > 0$. The coordinate map $V \to \mathbb{C}$ is given by the projection to the w-axis

$$(z, w) \mapsto w.$$

On the intersection $U \cap V$ we have holomorphic transition functions

$$z(w) = w^2 \quad \text{and} \quad w(z) = \sqrt{z}, \quad w(z_0) = w_0.$$

Example 1.1.8. Complex tori

Let ω, ω' be two complex numbers called half-periods satisfying

$$\operatorname{Im} \frac{\omega'}{\omega} > 0.$$

Define the lattice of points on the complex plane by

$$\Lambda_{\omega,\omega'} = 2\mathbb{Z}\omega + 2\mathbb{Z}\omega' = \{2m\omega + 2n\omega' \mid m, n \in \mathbb{Z}\}. \tag{1.1.9}$$

The half-periods ω, ω' are linearly independent as vectors on the two-dimensional real plane $\mathbb{C} = \mathbb{R}^2$. Therefore two vectors $2m_1\omega + 2n_1\omega'$ and $2m_2\omega + 2n_2\omega'$ of the lattice coincide iff $m_1 = m_2$ and $n_1 = n_2$. In other words the lattice $\Lambda_{\omega,\omega'} \subset \mathbb{C}$ as a subgroup of the additive group of complex numbers is isomorphic to the group $\mathbb{Z} \oplus \mathbb{Z}$.

Consider the quotient

$$T^2_{\omega,\omega'} = \mathbb{C}/\Lambda_{\omega,\omega'} \tag{1.1.10}$$

as the set of equivalence classes of complex numbers, where the equivalence relation is as follows: two complex numbers z and $\tilde{z}$ are equivalent if $\tilde{z} - z \in \Lambda_{\omega,\omega'}$.

The claim is that

- As a real smooth manifold the quotient is diffeomorphic to the two-dimensional torus $T^2_{\omega,\omega'} \simeq S^1 \times S^1$.
- It has a natural structure of compact connected one-dimensional complex manifold namely a compact Riemann surface.

To prove the first statement introduce real coordinates on the complex plane by representing a given complex number z in the form

$$z = 2\omega\, x + 2\omega' y.$$

Such a representation is unique. In these coordinates the quotient becomes equal to

$$\mathbb{C}/\Lambda_{\omega,\omega'} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = S^1 \times S^1.$$

In particular this implies compactness of (1.1.10).

To prove the second statement one needs to construct a complex analytic structure on $T^2_{\omega,\omega'}$. Let $\pi : \mathbb{C} \to T^2_{\omega,\omega'}$ be the projection map. Endow $T^2_{\omega,\omega'}$ with the quotient topology, namely, a set $U \subset T^2_{\omega,\omega'}$ is open if $\pi^{-1}(U)$ is open in $\mathbb{C}$. This definition makes π continuous and since $\mathbb{C}$ is connected so is $T^2_{\omega,\omega'}$. Furthermore, it is easy to check that π is an open map. Indeed, let U be an open set in $\mathbb{C}$. Then by definition the set $\pi(U)$ is open if $\pi^{-1}(\pi(U))$ is. But the latter is certainly open since $\pi^{-1}(\pi(U)) = \bigcup_{m,n\in\mathbb{Z}}(U + 2\omega\, m + 2\omega' n)$ is open.

In order to define a complex chart near a point $p_\alpha \in T^2_{\omega,\omega'}$ choose a representative $z_\alpha \in \pi^{-1}(p_\alpha)$ and consider the parallelogram

$$U(z_\alpha) = \{z_\alpha + 2\omega\, x + 2\omega' y \mid x,\, y \in \mathbb{R}, \quad |x|, |y| < \epsilon\}, \quad 0 < \epsilon \leqslant \frac{1}{2}$$

centered at z_α. The restriction $\pi|_{U(z_\alpha)} : U(z_\alpha) \to \pi(U(z_\alpha))$ is a homeomorphism. So we will use the natural complex coordinate on the parallelogram $U(z_\alpha) \subset \mathbb{C}$ for defining the homeomorphism ϕ_α on $\pi(U(z_\alpha)) \subset T^2_{\omega,\omega'}$. The pair $(\pi(U(z_\alpha)), \phi_\alpha)$ defines a complex chart. For $p \in \pi(U(z_\alpha)) \cap \pi(U(z_\beta))$ let $\phi_\alpha(p) = z$ and $\phi_\beta(p) = \tilde{z}$ so that the transition function $T(z) := \phi_\beta \circ \phi_\alpha^{-1}(z) = \tilde{z}$. Since z and $\tilde{z}$ are the image of the same point p on the torus, it follows that

$$T(z) - z = \Omega(z), \quad \Omega(z) \in \Lambda_{\omega,\omega'}.$$

Since the map T is continuous and $\Lambda_{\omega,\omega'}$ is discrete, it follows that $\Omega(z)$ independent from z. We conclude that the map T is holomorphic. An important remark is to be done. Namely, although the complex tori (1.1.10) are all diffeomorphic as real smooth manifolds they in general define different complex manifolds for different pairs of half-periods.In the next Section more details are given.

1.1.2 Holomorphic maps of Riemann surfaces

We begin this section with the general definition of holomorphic maps between complex manifolds. Let M and N be complex manifolds of complex dimensions m and n respectively. Let $(U_\alpha, \phi_\alpha)_{\alpha\in\mathcal{A}}$

$$\phi_\alpha(P) = (z_1(P), \ldots, z_m(P)) \in \mathbb{C}^m \quad \text{for} \quad P \in U_\alpha \subset M$$

and $(V_\beta, \psi_\beta)_{\beta\in\mathcal{B}}$

$$\psi_\beta(Q) = (w_1(Q), \ldots, w_n(Q)) \in \mathbb{C}^n \quad \text{for} \quad Q \in V_\beta \subset N$$

be atlases on these manifolds.

Definition 1.1.9. *(i) A map $f : M \to N$ is called* <u>*holomorphic*</u> *if for any $P_0 \in U_\alpha$ such that $f(P_0) \in V_\beta$ the superposition*

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : z = (z_1, \ldots, z_m) \to (w_1(z), \ldots, w_n(z))$$

defined on a sufficiently small open neighbourhood of P_0 is a holomorphic map of an open subset in $\mathbb{C}^n$ to $\mathbb{C}^m$.

(ii) Holomorphic maps $f : M \to \mathbb{C}$ are called <u>*holomorphic functions*</u> *on M.*

(iii) *The holomorphic map $f : M \to N$ is called* biholomorphic equivalence *if it is one-to-one and the inverse map $f^{-1} : N \to M$ is also holomorphic. The notation $M \simeq N$ will be used for biholomorphically equivalent complex manifolds.*

We leave as an exercise for the reader to verify that the above definition depends only on the complex analytic structures on the manifolds but not on the choice of atlases.

Example 1.1.10. The projective space $\mathbb{P}^1$, the Riemann sphere $\overline{\mathbb{C}}$ and the sphere $S^2 = \{(x, y, t) \in R^3\, |x^2 + y^2 + t^2 = 1\}$ are biholomorphic equivalent. The biholomorphic equivalence is given by

$$f_1 : \mathbb{P}^1 \to \overline{\mathbb{C}}, \quad [z_1 : z_2] \to \begin{cases} \frac{z_1}{z_2} & \text{if } z_2 \neq 0 \\ \infty & \text{if } z_2 = 0 \end{cases} \qquad f_2 : S^2 \to \overline{\mathbb{C}}, \quad (x, y, t) \to \begin{cases} \frac{x+iy}{1-t} & \text{if } x \neq 0,\ y \neq 0 \\ \infty & \text{if } x = y = 0. \end{cases}$$

Straightforward computations shows that the maps f_1 and f_2 are biholomorphic.

Exercise 1.1.11: Prove that the superposition $g \circ f : M \to L$ of two holomorphic maps $f : M \to N$ and $g : N \to L$ between complex manifolds is holomorphic.

Exercise 1.1.12: Let M be a compact connected one dimensional complex manifold. Prove that any holomorphic function $f : M \to \mathbb{C}$ must be a constant. *Hint:* use the maximum modulus principle.

Example 1.1.13. Let $P(z)$ and $Q(z)$, $z \in \mathbb{C}$, be two polynomials of degrees m and n respectively. Define a holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$ by

$$f((Z_0 : Z_1)) = \left(Z_0^N Q(Z_1/Z_0) : Z_0^N P(Z_1/Z_0)\right), \quad N = \max(m, n).$$

Note that the two homogeneous coordinates of the image, namely

$$\widetilde{Q}(Z_0, Z_1) := Z_0^N Q(Z_1/Z_0) \quad \widetilde{P}(Z_0, Z_1) := Z_0^N P(Z_1/Z_0)$$

are homogeneous polynomials of degree N in the variables Z_0, Z_1. Without loss of generality we can assume that the polynomials $P(z)$ and $Q(z)$ have no common roots. The point $(Z_0 : Z_1) \in U_0 = \mathbb{C}$ with coordinate $z = \frac{Z_1}{Z_0}$ is mapped to

$$f(Z_0 : Z_1) = \begin{cases} (1 : \frac{P(z)}{Q(z)}) & \text{if } Q(z) \neq 0, \\ (0 : 1) & \text{if } Q(z) = 0, \end{cases} \tag{1.1.11}$$

while for $Z_1 \in \mathbb{C}^*$ we have

$$f(0 : 1) = \begin{cases} (0 : 1) & \text{if } \deg P > \deg Q, \\ (1 : 0) & \text{if } \deg P < \deg Q, \\ (\widetilde{Q}(0, 1) : \widetilde{P}(0, 1)) & \text{if } \deg P = \deg Q. \end{cases} \tag{1.1.12}$$

Vice versa, the rational function $\frac{P(z)}{Q(z)}$ can be extended to a holomorphic map from $\mathbb{P}^1$ to $\mathbb{P}^1$ by (1.1.11) and (1.1.12).

The map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is biholomorphic *iff*

$$f(Z_0 : Z_1) = (cZ_1 + dZ_0 : aZ_1 + bZ_0), \quad ad - bc \neq 0.$$

The inverse map

$$f^{-1}(W_0 : W_1) = \frac{1}{ad - bc}(-cW_1 + aW_0 : dW_1 - bW_0).$$

Meromorphic functions on a Riemann surface $\mathcal{S}$ are defined as follows.

Definition 1.1.14. *Let $\mathcal{S}$ be a Riemann surface. Holomorphic maps $f : \mathcal{S} \to \mathbb{P}^1 = \overline{\mathbb{C}}$ are called meromorphic functions on $\mathcal{S}$.*

Denote by z the complex coordinate on the finite part of $\overline{\mathbb{C}}$ and by $\tilde{z} = 1/z$ the complex coordinate near infinity. Take a point $P_0 \in \mathcal{S}$ and choose a local complex coordinate $\phi(P) = \tau$ near this point such that $\phi(P_0) = 0$. Let $f(P_0) = z_0 \in \mathbb{C}$. Then we have a locally defined holomorphic function

$$z = f \circ \phi^{-1}(\tau) = z_0 + \sum_{k \geqslant m} a_k \tau^k, \quad m \geqslant 1, \; a_m \neq 0.$$

If $z_0 = 0$ then the number m is the multiplicity of the zero at P_0 of the meromorphic function f. Consider now the case $f(P_0) = \{\infty\} = \{\tilde{z} = 0\}$. In this case $\tilde{z}$ is a holomorphic function of τ

$$\tilde{z} = \sum_{k \geqslant n} b_k \tau^k, \; n \geqslant 1, \; b_n \neq 0.$$

Then for the function $z = f \circ \phi^{-1}(\tau)$ we obtain an expansion in Laurent series

$$z = f \circ \phi^{-1}(\tau) = \left[\sum_{k \geqslant m} b_k \tau^k \right]^{-1} = \sum_{k \leqslant n} \frac{c_{-k}}{\tau^k}, \quad c_{-n} = \frac{1}{b_n} \neq 0,$$

valid on a punctured disk $0 < |\tau| < \epsilon$, for a sufficiently small ϵ. The point P_0 is called a pole of order n of the meromorphic function f. The multiplicity of a zero and the order of a pole do not depend on the choice of local parameter. An alternative definition of a meromorphic function on a Riemann surface is that the function f is holomorphic in $\mathcal{S}$ outside a discrete subset of points that are poles of this function.

Exercise 1.1.15: Prove that, indeed, the set of poles of a meromorphic function must be discrete. In particular prove that a meromorphic function on a compact connected one-dimensional complex manifold has only a finite number of poles.

Exercise 1.1.16: Prove that any meromorphic function on the Riemann sphere $\overline{\mathbb{C}}$ is a rational function.

Remark 1.1.17. The space of meromorphic functions on a Riemann surface $\mathcal{S}$ is a field. That means that the product fg of two meromorphic functions is meromorphic; the same is true for the ratio f/g provided the function g is not an identical zero. This field will be denoted by $\mathcal{M}(\mathcal{S})$. For example, according to the above Exercise $\mathcal{M}(\overline{\mathbb{C}})$ is isomorphic to the field of rational functions of one variable.

Example 1.1.18. Consider the Riemann surface

$$\mathcal{S} = \{(z, w) \in \mathbb{C}^2 \mid w^2 - z = 0\}.$$

The projections $\pi_z : \mathcal{S} \to \mathbb{C}$ and $\pi_w : \mathcal{S} \to \mathbb{C}$ defined as

$$\pi_z(z, w) = z \quad \text{and} \quad \pi_w(z, w) = w$$

define holomorphic functions on $\mathcal{S}$.

The map π_w establishes a biholomorphic equivalence $\mathcal{S} \simeq \mathbb{C}$. Indeed, the inverse to the map $\pi_w : \mathcal{S} \to \mathbb{C}$ is given by

$$w \mapsto \left(w^2, w\right) \in \mathcal{S}.$$

Example 1.1.19. Let $\mathbb{D} = \{|z| < 1\}$ be the unit disk and $\mathbb{H} = \{\operatorname{Im} w > 0\}$ the upper half-plane. The map

$$w = i\,\frac{1-z}{1+z}$$

establishes a biholomorphic equivalence $\mathbb{D} \simeq \mathbb{H}$ between the unit disk $\mathbb{D}$ and the upper half-plane $\mathbb{H}$. The inverse map is given by

$$z = \frac{i-w}{i+w}.$$

Example 1.1.20. Any holomorphic map from $\mathbb{C} \to \mathbb{D}$ must be a constant, due to the maximum modulus principle. Therefore the complex plane and the unit disk are not biholomorphically equivalent. Nevertheless $\mathbb{C}$ and $\mathbb{D}$ are diffeomorphic to each other by means of the smooth map $\psi : \mathbb{C} \to \mathbb{D}$

$$\psi(z) = \frac{z}{\sqrt{1+|z|^2}} = w$$

with inverse $\psi^{-1}(w) = \dfrac{w}{\sqrt{1-|w|^2}}$.

Remark 1.1.21. Clearly the Riemann sphere is not biholomorphically equivalent either to $\mathbb{C}$ or to $\mathbb{H}$ as it is compact. Indeed combining the results of Examples 1.1.19 and 1.1.20 we conclude that there is no biholomorphic equivalence between $\mathbb{C}$ and $\mathbb{H}$.

The following fundamental result proven in 1907 by Henri Poincaré and Paul Koebe provides a complete classification of simply connected Riemann surfaces.

Uniformization Theorem. *Any simply connected Riemann surface is biholomorphically equivalent to one of these three:*

1. *complex plane* $\mathbb{C}$;
2. *Riemann sphere* $\mathbb{P}^1 = \overline{\mathbb{C}}$;
3. *upper half-plane* $\mathbb{H}$.

For the definition of simply connected topological spaces see below Section 1.3.1. The proof of the Uniformization Theorem can be found in the book [27].

Example 1.1.22. Holomorphic maps of complex tori.

Recall (see Example 1.1.8 above) that a complex torus is a compact Riemann surface $T^2_{\omega,\omega'}$ defined as the quotient of the complex plane over a two-dimensional lattice

$$T^2_{\omega,\omega'} = \mathbb{C}/\{2\omega\, m + 2\omega' n \mid m,\, n \in \mathbb{Z}\}. \tag{1.1.13}$$

Here ω, $\omega' \in \mathbb{C}$ is a pair of half-periods of the lattice. They must satisfy the inequality

$$\operatorname{Im} \frac{\omega'}{\omega} > 0.$$

Vectors $2\omega\, m + 2\omega' n$ of the lattice are called periods. A natural basis in the lattice is given by the periods 2ω, $2\omega'$. All vectors of the lattice are linear combinations with integer coefficients of the basic periods. There are other bases in the lattice that can be obtained in the following way.

Lemma 1.1.23. *Let $2\tilde\omega$, $2\tilde\omega'$ be another basis of the lattice satisfying the inequality $\operatorname{Im} \frac{\tilde\omega'}{\tilde\omega} > 0$. Then*

$$\tilde\omega = d\,\omega + c\,\omega', \quad \tilde\omega' = b\,\omega + a\,\omega' \tag{1.1.14}$$

where the integers a, b, c, d satisfy

$$\det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = 1.$$

Conversely, any matrix from the group $SL(2,\mathbb{Z})$ defines a change of basis in the lattice according to eq. (1.1.14)

Recall that the group $SL(2,\mathbb{Z})$ consists of 2×2 matrices with integer entries and determinant one.

Proof Since the vectors $2\tilde\omega$, $2\tilde\omega'$ belong to the lattice with the basis 2ω, $2\omega'$ they must have the form (1.1.14) with some integer coefficients. Interchanging the roles of the bases we conclude that the inverse of the matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ must also have integer entries hence $\det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \pm 1$. Using the simple identity

$$\operatorname{Im}\frac{\tilde\omega'}{\tilde\omega} = \frac{(ad-bc)|\omega|^2}{|c\,\omega' + d\,\omega|^2}\operatorname{Im}\frac{\omega'}{\omega} \tag{1.1.15}$$

we conclude that the determinant of the matrix must be positive. □

Let us proceed to studying functions on complex tori. First, we already know that any holomorphic function on $T^2_{\omega,\omega'}$ must be a constant, see Exercise 1.1.11 above. It is worthwhile to present the proof of the statement about holomorphic functions on a complex torus in a slightly modified way. Namely, a function $f : \mathbb{C}/\{2\omega\, m + 2\omega' n\} \to \mathbb{C}$ can be considered as a function on $\mathbb{C}$ satisfying

$$f(z+2\omega) = f(z), \quad f(z+2\omega') = f(z) \tag{1.1.16}$$

for any $z \in \mathbb{C}$. Such functions are called doubly periodic. Any doubly periodic holomorphic function will be bounded on the entire complex plane hence, due to Liouville theorem it must be constant.

Definition 1.1.24. *Doubly periodic meromorphic functions on the complex plane are called elliptic functions.*

We conclude that the set of holomorphic maps of the complex torus (1.1.13) to $\mathbb{P}^1$ is the same as the set of elliptic functions on the complex plane. In Section **??** we will construct some important examples of elliptic functions.

Let us now consider holomorphic maps between complex tori. Any such map

$$f : T^2_{\omega,\omega'} \to T^2_{\tilde\omega,\tilde\omega'} \tag{1.1.17}$$

can be considered as a holomorphic function $f(z)$, $z \in \mathbb{C}$ satisfying

$$f(z+2\omega) = f(z) + 2s\,\tilde\omega + 2r\,\tilde\omega', \quad f(z+2\omega') = f(z) + 2q\,\tilde\omega + 2p\,\tilde\omega', \quad p,\, q,\, r,\, s \in \mathbb{Z} \tag{1.1.18}$$

for any $z \in \mathbb{C}$. The derivative $f'(z)$ will be a doubly periodic holomorphic function hence constant. So $f(z) = \lambda z + z_0$ for some $\lambda \neq 0$, $z_0 \in \mathbb{C}$. Thus the holomorphic maps (1.1.17) correspond to pairs $\lambda \neq 0$, $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Mat(2, \mathbb{Z})$. The matrix M must have positive determinant; this can be proven by using the relation (1.1.15). Existence of such a map imposes the following constraint on the periods of the tori

$$\lambda \begin{pmatrix} \omega' \\ \omega \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \tilde{\omega}' \\ \tilde{\omega} \end{pmatrix} \tag{1.1.19}$$

The simplest case is $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then the lattice $\Lambda_{\tilde{\omega},\tilde{\omega}'}$ is obtained from $\Lambda_{\omega,\omega'}$ by rescaling

$$\tilde{\omega} = \lambda\omega, \quad \tilde{\omega}' = \lambda\omega'. \tag{1.1.20}$$

The map

$$f : T^2_{\omega,\omega'} \to T^2_{\lambda\omega,\lambda\omega'}, \quad f(z) = \lambda z \tag{1.1.21}$$

is biholomorphic, $f^{-1}(z) = z/\lambda$. By chosing $\lambda = \frac{1}{2\omega}$ it follows that the tori

$$f : T^2_{\omega,\omega'} \to T^2_{\frac{1}{2},\frac{\tau}{2}} \quad f(z) = \frac{1}{2\omega} z, \quad \tau = \frac{\omega'}{\omega} \tag{1.1.22}$$

are biholomorphic equivalent. For simplicity the torus $T^2_{\frac{1}{2},\frac{\tau}{2}}$ is denoted by T^2_τ. Combining the above observation with lemma 1.1.23 we arrive to the following Theorem.

Theorem 1.1.25. *Let T_τ and $T_{\tau'}$ be two tori defined by the lattices $\{m + n\tau \mid m, n \in \mathbb{N}\}$ and $\{m + n\tau' \mid m, n \in \mathbb{N}\}$ with $\Im(\tau) > 0$ and $\Im(\tau') > 0$. The tori are isomorphic if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{1.1.23}$$

The proof is left as an exercise.

Holomorphic maps between complex tori will be considered up to superpositions with rescalings. This allows to freely choose λ in a suitable way.

One can also use the freedom in the choice of bases in the lattices $\Lambda_{\omega,\omega'}$, $\Lambda_{\tilde{\omega},\tilde{\omega}'}$ in order to reduce the matrix $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ to some canonical form. In this way the matrix M is considered up to transformations of the form

$$M \mapsto AMB, \quad A, B \in SL(2, \mathbb{Z}). \tag{1.1.24}$$

The matrix A corresponds to a change of basis in $\Lambda_{\omega,\omega'}$ and B comes from a change of basis in the lattice $\Lambda_{\tilde{\omega},\tilde{\omega}'}$. The following algebraic statement describes the normal form of the matrix M wrt transformations of the form (1.1.24).

Lemma 1.1.26. *Any 2×2 matrix M with integer entries and* $\det M > 0$ *by a transformation of the form* (1.1.24) *can be reduced to the form*

$$M_{\text{normal}} = \pm \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad d_1 \mid d_2, \tag{1.1.25}$$

where the symbol $d_1 \mid d_2$, stands for d_1 divides d_2. The numbers d_1 and d_2 are determined uniquely.

The proof of Lemma is left as an exercise for the reader.
Summarizing the above arguments we arrive at the following

Proposition 1.1.27. *Any holomorphic map between complex tori modulo biholomorphic rescalings can be reduced to the following standard form*

$$f_n : T^2_{\omega,\omega'} \to T^2_{\omega/n,\omega'}, \quad f_n(z) = z \tag{1.1.26}$$

for some integer $n > 0$.

Holomorphic maps of the form (1.1.26) play an important role in the theory of elliptic functions. For the first nontrivial case $n = 2$ they are related to Landen's transformations that we will explain in Example 1.4.11.

Exercise 1.1.28: Prove that the preimage of any point in the torus wrt the map (1.1.26) consists of n points.

Example 1.1.29. We conclude this section by constructing a meromorphic function on the torus T^2_τ with $\Im(\tau) > 0$.

The Jacobi theta function is defined by the series

$$\theta(z;\tau) = \sum_{-\infty<n<\infty} \exp\left(\pi i \tau n^2 + 2\pi i n z\right). \tag{1.1.27}$$

The function $\vartheta_3(z\,;\,\tau))$ in the standard notation for $\theta(z;\tau)$, see e.g.[4]. Since

$$\left|\exp\left(\pi i \tau n^2 + 2\pi i n z\right)\right| = \exp\left(-\pi \Im\tau n^2 - 2\pi n \Im z\right)$$

the series (1.1.27) converges absolutely and uniformly in the strips $|\Im(z)| \leqslant$ const and defines an entire function of z.

The series (1.1.27) can be rewritten in the form common in the theory of Fourier series:

$$\theta(z) = \sum_{-\infty<n<\infty} \exp(\pi i \tau n^2) e^{2\pi i z n} \tag{1.1.28}$$

The function $\theta(z;\tau)$ has the following periodicity properties:

$$\theta(z+1;\tau) = \theta(z) \tag{1.1.29}$$

$$\theta(z+m\tau;\tau) = \exp(-\pi i m^2 \tau - 2\pi i m z)\theta(z), \quad m \in \mathbb{Z} \tag{1.1.30}$$

The equality (1.1.29) is obvious. The equality (1.1.30) is also easy to prove:

$$\theta(z+m\tau;\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i \tau (n-m)^2 + 2\pi i (n-m)(z+m\tau)\right) = \exp(-\pi i m^2 \tau - 2\pi i m z)\theta(z;\tau).$$

The integer lattice with basis 1 and τ is called the period lattice of the theta function. The remaining Jacobi theta-functions are defined with respect to the lattice $1, \tau$ as

$$\vartheta_1(z\,;\,\tau) := \sum_{-\infty<n<\infty} \exp\left[\pi i \tau \left(n+\frac{1}{2}\right)^2 + 2\pi i \left(z+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right]$$

$$\vartheta_2(z;\tau) := \sum_{-\infty<n<\infty} \exp\left[\pi i\tau\left(n+\frac{1}{2}\right)^2 + 2\pi i z\left(n+\frac{1}{2}\right)\right]$$

$$\vartheta_4(z;\tau) := \sum_{-\infty<n<\infty} \exp\left[\pi i\tau n^2 + 2\pi i\left(z+\frac{1}{2}\right)n\right].$$

The functions $\vartheta_2(z;\tau)$, $\vartheta_3(z;\tau)$ and $\vartheta_4(z;\tau)$ are even functions of z while $\vartheta_1(z;\tau)$ is odd. For simplicity we drop the τ-dependence and write only $\theta(z)$ for $\theta(z;\tau)$.

In the parallelogram Γ defined by the lattice 1 and τ, namely

$$\Gamma := \left\{\frac{1}{2}+\frac{\tau}{2}+x+y\tau \,|\, x,y\in\mathbb{R},\quad |x|\leqslant\frac{1}{2},\ |y|\leqslant\frac{1}{2}\right\}$$

the function $\theta(z)$ has only one zero. Indeed let us consider the integral

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\partial\Gamma}\frac{d}{dz}\log\theta(z)dz = \{\#\text{ of zeros of }\theta(z)\text{ in }\Gamma\}\\ &= \frac{1}{2\pi i}\left(\int_0^1(\log\theta(t))'dt + \int_0^1(\log\theta(1+\tau t))'\tau dt - \int_0^1(\log\theta(\tau+t))'dt + \int_1^0(\log\theta(\tau t))'\tau dt\right).\end{aligned}$$

Using the periodicity properties (1.1.29) and (1.1.30) we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\Gamma}\frac{d}{dz}\log\theta(z)dz &= \frac{1}{2\pi i}\left(\int_0^1(\log\theta(t))'dt - \int_0^1(\log\theta(\tau+t))'dt\right)\\ &= \frac{1}{2\pi i}\left(\int_0^1(\log\theta(t))'dt - \int_0^1[(\log\theta(t))'-2\pi i]dt\right) = 1,\end{aligned}$$

which shows that the number of zeros of $\theta(z)$ in the domain Γ is equal to one. To determine this zero, we use the parity and periodicity property of $\theta(z)$ so that

$$\theta\left(\frac{\tau}{2}+\frac{1}{2}\right) = \theta\left(-\frac{\tau}{2}-\frac{1}{2}\right) = \theta\left(\frac{\tau}{2}+\frac{1}{2}-\tau\right) = e^{-\pi i\tau+2\pi i(\frac{\tau}{2}+\frac{1}{2})}\theta\left(\frac{\tau}{2}+\frac{1}{2}\right) = -\theta\left(\frac{\tau}{2}+\frac{1}{2}\right)$$

which implies that $\frac{\tau}{2}+\frac{1}{2}$ is the only zero for the theta function $\theta(z)$ in the domain Γ. Finally it is left as an exercise to show that for $2m$ complex numbers $v_1,\dots,v_m$ and $c_1,\dots,c_m$ such that $\sum_{j=1}^m v_j = \sum_{j=1}^m c_j$ the function

$$f(z) = \frac{\prod_{j=1}^m\theta(z-v_j)}{\prod_{j=1}^m\theta(z-c_j)}$$

in meromorphic on the torus T_τ^2 with zeros at the points $z = v_j+\frac{1}{2}+\frac{\tau}{2}$ and poles at the points $z = c_j+\frac{1}{2}+\frac{\tau}{2}$, $j=1,\dots,m$ with $v_j\neq c_i$, $i,j=1,\dots,m$.

1.2 Algebraic curves and Riemann surfaces

1.2.1 Algebraic digression: resultant and discriminant

The resultant of two polynomials $f(z)$ and $g(z)$ in one variable is a polynomial in the coefficients of f and g that provides a condition of compatibility of the system

$$\left.\begin{array}{rcl} f(z) & = & 0 \\ g(z) & = & 0 \end{array}\right\}$$

of two algebraic equations. More precisely,

Definition 1.2.1. *Let $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ and $g(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_m$ be two polynomials of degree n and m respectively with $a_i,\ b_j \in \mathbb{C}$ with $a_0 \neq 0$ and $b_0 \neq 0$. The resultant $R(f,g)$ is given by the determinant of the $(n+m)\times(n+m)$ matrix*

$$R(f,g) = \det\begin{pmatrix} a_0 & a_1 & \dots & a_n & 0 & 0 & & & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_n & 0 & 0 & & \dots & 0 \\ \dots & & & \dots & & & & & \dots & \\ 0 & 0 & \dots & \dots & a_0 & a_1 & a_2 & & \dots & a_n \\ b_0 & b_1 & \dots & \dots & b_{m-1} & b_m & 0 & & \dots & 0 \\ 0 & b_0 & b_1 & \dots & \dots & b_{m-1} & b_m & 0 & \dots & 0 \\ \dots & & & \dots & & & & & \dots & \\ 0 & \dots & b_0 & b_1 & \dots & & \dots & & b_{m-1} & b_m \end{pmatrix}. \tag{1.2.1}$$

Lemma 1.2.2. *$R(f,g) = 0$ if and only if f and g have a common zero. The co-rank of the matrix appearing in the determinant is the number of common zeroes.*

Proof. The polynomials $f(z)$ and $g(z)$ have a common root $z = z_0$ if and only if they are divisible by $r(z) = z - z_0$, that is there exist polynomials $\psi(z)$ and $\phi(z)$ such that $f(z) = r(z)\psi(z)$ and $g(z) = r(z)\phi(z)$. Here ψ and ϕ are polynomials of degree at most $n-1$ and $m-1$ respectively. This implies that

$$f(z)\phi(z) = g(z)\psi(z) \tag{1.2.2}$$

where

$$\phi(z) = \alpha_1 z^{m-1} + \cdots + \alpha_{m-1} z + \alpha_m$$

and

$$\psi(z) = \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + \beta_n$$

for some complex coefficients $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_n$.

To write the system in a matrix form we define the spaces $V = \operatorname{span}(z^{m-1}, \dots, 1) \oplus \operatorname{span}(z^{n-1}, \dots, 1)$ and $W = \operatorname{span}(z^{n+m-1}, \dots, 1)$. The space of solutions to the system (1.2.2) coincides with the kernel of the map $\mathcal{M} : V \to W$ given by

$$\mathcal{M}(\phi \oplus \psi) = f\phi - g\psi \in W.$$

The matrix of the linear operator $\mathcal{M}$ in the indicated bases is (up to multiplication of the last n rows by (-1)) precisely the matrix appearing in (1.2.1). Hence the vanishing of the determinant is the necessary and sufficient condition for the solvability of (1.2.2).

Note now that the smallest possible degrees of ψ, ϕ amongst the possible solutions of (1.2.2) are precisely $m-s, n-s$ where s is the number of common roots of the polynomials f and g **(exercise).** Denoting (ϕ_0, ψ_0) such a minimal solution we then observe that we have a s-dimensional freedom of multiplying both sides of the equation $f(z)\phi_0(z) = g(z)\psi_0(z)$ by an arbitrary polynomial of degree $\leqslant s-1$. This means that the kernel of the matrix in (1.2.1) has dimension s.

□

Lemma 1.2.3.
$$R(f,g) = a_0^m b_0^n \prod (x_j - y_k)$$
where x_j and y_k are the roots of the polynomials f and g respectively.

Proof. We have
$$f(z) = a_0 \prod_{i=1}^{n} (z - x_i), \quad g(z) = b_0 \prod_{j=1}^{m} (z - y_i).$$
So
$$a_i = (-1)^i a_0 \times i\text{−th elementary symmetric function of} \quad x_1, \ldots, x_n, \quad i = 1, \ldots, n$$
and a similar representation holds for the coefficients of the polynomial $g(z)$.

The resultant can be considered as a polynomial in the coefficients of f and g,
$$R(f,g) \in \mathbb{C}[a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m]$$
homogeneous of degree m in $a_0, a_1, \ldots, a_n$ and degree n in $b_0, b_1, \ldots, b_m$. Using the elementary symmetric functions we can represent it as an element of the ring of polynomials
$$R(f,g) \in a_0^m b_0^n \, \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]^{S_n \times S_m}$$
symmetric in $x_1, \ldots, x_n$ and in $y_1, \ldots, y_m$. It vanishes if $x_i = y_j$ for some i, j. Therefore it is divisible by $x_i - y_j$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We conclude that $R(f,g)$ is divisible by the polynomial
$$P := a_0^m b_0^n \prod_{i,j} (x_i - y_j). \tag{1.2.3}$$

The polynomial (1.2.3) can be represented in the following way
$$P = a_0^m \prod_{i=1}^{n} g(x_i).$$
Hence it is a homogeneous polynomial of degree n in $b_0, b_1, \ldots, b_m$. Its coefficients are symmetric polynomials in $x_1, \ldots, x_n$ times a_0^m. So they can be represented, in a unique way, as polynomials in $a_0, a_1, \ldots, a_n$. Alternatively P can be written as follows
$$P = (-1)^{mn} b_0^n \prod_{j=1}^{m} f(y_j).$$
Thus P is a homogeneous polynomial of degree m in $a_0, a_1, \ldots, a_n$. We conclude that
$$R(f,g) = \text{const}\, P.$$
In order to prove that const=1 we look at the terms of the highest degree in b_m. It is easy to see that they are equal to $a_0^m b_m^n$ both in R and in P. The lemma is proved. □

Now we address the following question: how to check whether the polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \tag{1.2.4}$$

has multiple roots? It is well known that $z = z_0$ is a mutiple root of $f(z)$ if and only if it satisfies the system

$$\left.\begin{array}{rcl} f(z_0) & = & 0 \\ f'(z_0) & = & 0 \end{array}\right\}.$$

Here $f'(z) = df(z)/dz$. The condition of compatibility of this system is the vanishing of the resultant $R(f, f')$. We arrive at

Definition 1.2.4. *The* <u>*discriminant*</u> *$D(f)$ of the polynomial $f(z)$ in (1.2.4) is equal to*

$$D(f) = \frac{1}{a_0}(-1)^{\frac{n(n-1)}{2}} R(f, f'). \tag{1.2.5}$$

From eq. (1.2.1) we obtain the following expression for the discriminant

$$D(f) = \frac{1}{a_0}(-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & \dots & a_{n-1} & a_n & \dots & 0 \\ \dots & \dots & & \dots & & & \dots & & \dots \\ 0 & 0 & \dots & & \dots & & \dots & a_{n-1} & a_n \\ na_0 & (n-1)a_1 & (n-2)a_2 & \dots & a_{n-1} & 0 & \dots & \dots & 0 \\ 0 & na_0 & (n-1)a_1 & \dots & 2a_{n-2} & a_{n-1} & 0 & \dots & 0 \\ & \dots & & \dots & & & & & \dots \\ 0 & 0 & \dots & \dots & & & \dots & 2a_{n-2} & a_{n-1} \end{pmatrix}. \tag{1.2.6}$$

We put the prefactor $1/a_0$ since the polynomial $R(f, f')$ is divisible by a_0. In this way we can see that $D(f)$ is a homogeneous polynomial in $a_0, a_1, \dots, a_n$ of degree $2n - 2$.

For example, the discriminant of a degree two polynomial $f = a_0 z^2 + a_1 z + a_2$ is equal to $D(f) = a_1^2 - 4a_0 a_2$. For a cubic polynomial $f = a_0 z^3 + a_1 z^2 + a_2 z + a_3$ it is given by the formula

$$D(f) = -\frac{1}{a_0} \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 3a_0 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 \end{pmatrix} = a_1^2 a_2^2 - 4a_0 a_2^3 - 4a_1^3 a_3 + 18 a_0 a_1 a_2 a_3 - 27 a_0^2 a_3^2. \tag{1.2.7}$$

Exercise 1.2.5: Prove that the discriminant as a symmetric polynomial in the roots $z_1, \dots, z_n$ of $f(z)$ can be written in the following form

$$D(f) = a_0^{2n-2} \prod_{i<j} (z_i - z_j)^2.$$

1.2.2 Smooth affine plane curves as Riemann surfaces

Let us consider a polynomial $F(z,w) = \sum_{i=0}^{n} a_i(z) w^{n-i}$ in two complex variables z and w, $a_i(z) \in \mathbb{C}[z]$, $i = 0, 1, \ldots, n$. For simplicity let us assume[4] that $a_0(z) \equiv 1$. Then for any $z \in \mathbb{C}$ the algebraic equation

$$F(z,w) = 0$$

has n roots $w_1(z), \ldots, w_n(z)$ counted with multiplicities. We obtain a n-valued function $w = w(z)$ of complex variable. The basic idea of Riemann surface theory is to replace the domain of the multivalued function $w(z)$ by its graph that is nothing but the complex algebraic curve

$$C := \{(z,w) \in \mathbb{C}^2 \mid F(z,w) = \sum_{i=0}^{n} a_i(z) w^{n-i} = 0\} \tag{1.2.8}$$

and to deal with a single-valued holomorphic function $(z,w) \mapsto w$ on C rather than with a multivalued function on $\mathbb{C}$. We have already considered above the example of the multivalued function $w(z) = \sqrt{z}$. It becomes single-valued on the algebraic curve $w^2 - z = 0$.

In the theory of functions of a complex variable one encounters also more complicated (nonalgebraic) curves, where $F(z,w)$ is not a polynomial. For example, the equation $e^w - z = 0$ determines the Riemann surface of the logarithm or $\sin w - z = 0$ determines the Riemann surface of arcsine. Such surfaces will not be considered here.

From the real point of view the algebraic curve (1.2.8) is a two-dimensional surface in $\mathbb{C}^2 = \mathbb{R}^4$ given by the two equations

$$\left.\begin{array}{l} \Re F(z,w) = 0 \\ \Im F(z,w) = 0 \end{array}\right\}.$$

We will now formulate main conditions that guarantee that this surface is smooth and, moreover, it admits a natural structure of a connected complex manifold of complex dimension one or, according to Definition **??** it is a Riemann surface.

Definition 1.2.6. *An affine plane curve C is a subset in $\mathbb{C}^2$ defined by the equation (1.2.8) where $F(z,w)$ is polynomial in z and w. The curve C is non-singular if for any point $P_0 = (z_0, w_0) \in C$ the complex gradient vector*

$$grad_{\mathbb{C}} F|_{P_0} = \left(\frac{\partial F(z,w)}{\partial z}, \frac{\partial F(z,w)}{\partial w} \right)\Bigg|_{(z=z_0, w=w_0)}$$

does not vanish. If the polynomial $F(z,w)$ is irreducible[5] then the curve C is called irreducible affine plane curve.

In order to define a complex structure on C we need the following complex version of the implicit function theorem.

[4] This can be achieved by a transformation

$$w \mapsto \frac{w}{a_0(z)}, \quad F \mapsto a_0(z)^{n-1} F.$$

[5] A polynomial $F(z,w)$ is called irreducible if it cannot be factorized into a product $F(z,w) = F_1(z,w) F_2(z,w)$ of two nonconstant polynomials.

Lemma 1.2.7. *[Complex implicit function theorem] Let $F(z,w)$ be an analytic function of complex variables z and w in a neighbourhood of the point $P_0 = (z_0, w_0)$ such that $F(z_0, w_0) = 0$ and $\partial_w F(z_0, w_0) \neq 0$. Then there exists a unique function $\phi(z)$ such that $F(z, \phi(z)) = 0$ and $\phi(z_0) = w_0$. This function is analytic in z in some neighbourhood of z_0.*

Proof. Let $z = x + iy$ and $w = u + iv$, $F = f + ig$. Then the equation $F(z,w) = 0$ can be written as the system

$$\begin{cases} f(x,y,u,v) & = & 0 \\ g(x,y,u,v) & = & 0 \end{cases}$$

The conditions of the real implicit function theorem are satisfied for this system: the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}_{(z_0, w_0)}$$

is non-singular because

$$\det \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \left| \frac{\partial F}{\partial w} \right|^2 > 0,$$

(here we use only analyticity in w of the function $F(z,w)$). Thus, in some neighbourhood of (z_0, w_0) there exists a unique smooth function $\phi(z, \bar z) = \phi_1(x,y) + i\phi_2(x,y)$ such that $F(z, \phi(z, \bar z)) = 0$, with $\phi(z_0, \bar z_0) = w_0$. Differentiating the identity $F(z, \phi(z, \bar z)) \equiv 0$ with respect to $\bar z$, we get that

$$0 = \frac{\partial F}{\partial \bar z} + \frac{\partial F}{\partial w}\frac{\partial \phi}{\partial \bar z} = \frac{\partial F}{\partial w}\frac{\partial \phi}{\partial \bar z}$$

due to analyticity of $F(z,w)$. Using $\frac{\partial F}{\partial w} \neq 0$ we conclude that $\frac{\partial \phi}{\partial \bar z} = 0$. That means that $\phi(z)$ is an analytic function of z. $\square$

We arrive to the following main result of this Section.

Theorem 1.2.8. *Let C be the irreducible affine plane curve (1.2.8). If C is non-singular then it has a natural structure of a Riemann surface. Restriction of the coordinates z and w onto the curve defines two holomorphic functions on the Riemann surface.*

Proof. Since $F(z,w)$ is irreducible the curve C is connected, see Theorem 1.3.47 below for the proof. Let us define a complex structure on C. Let $P_0 = (z_0, w_0)$ be a non-singular point of the surface C. Suppose, for example, that the derivative $\frac{\partial F}{\partial w}$ is nonzero at this point. Then by the Lemma 1.2.7, in a neighbourhood U_0 of the point P_0, the points of the curve C admit a parametric representation of the form

$$(z, w(z)) \in U_0 \subset C, \quad w(z_0) = w_0, \tag{1.2.9}$$

where the function $w(z)$ is holomorphic. Therefore, in this case z is a complex local coordinate also called *local parameter* on C in a neighbourhood U_0 of $P_0 = (z_0, w_0) \in C$. For a pair of charts with this type of local coordinate the transition function is the identity.

Similarly, if the derivative $\dfrac{\partial F}{\partial z}$ is nonzero at the point $P_0 = (z_0, w_0)$, then we can take w as a local parameter (an obvious variant of the lemma), and the curve C can be represented in a neighbourhood U_0 of the point P_0 in the parametric form

$$(z(w), w) \in C, \quad z(w_0) = z_0, \tag{1.2.10}$$

where the function $z(w)$ is, of course, holomorphic. Call U_0 the domain of the second type. For a non-singular surface it is possible to use both ways for representing the surface on the intersection of domains of the first and second types, i.e., at points of C where $\dfrac{\partial F}{\partial w} \neq 0$ and $\dfrac{\partial F}{\partial z} \neq 0$ simultaneously. The resulting *transition functions* $w = w(z)$ and, $z = z(w)$ are holomorphic and invertible.

Let us prove that the projections $(z, w) \mapsto z$ and $(z, w) \mapsto w$ are holomorphic on the constructed Riemann surface. Indeed, on a domain of the first kind the first projection is given by the identity function $z \to z$ while the second one is given by the holomorphic function $w(z)$. In a similar way on domains of the second kind we have $z(w)$ and $w \to w$ respectively. □

Remark 1.2.9. If the polynomial $F(z, w) = \sum_{i=0}^{n} a_i(z) w^{n-i}$ is not monic in w then the Riemann surface associated with the algebraic curve $F(z, w) = 0$ can still be constructed but the function w will not be holomorphic but meromorphic on this surface. Poles of this function can be located over zeros of the coefficient $a_0(z)$.

Due to the above Theorem we will denote by $\mathcal{S}$ the Riemann surface corresponding to a non-singular irreducible algebraic curve $C = \{(z, w) \in \mathbb{C}^2 \,|\, F(z, w) = 0\}$. It is equipped with a pair of holomorphic functions z, w that establish a one-to-one correspondence

$$\mathcal{S} \ni P \mapsto (z(P), w(P)) \in C.$$

The Riemann surface $\mathcal{S}$ associated to the curve (1.2.8) is realized as an n-sheeted branched covering of the z-plane. The precise meaning of this is as follows: let $\pi : \mathcal{S} \to \mathbb{C}$ be the projection map from $\mathcal{S}$ to the complex z-plane given by the function z that here will be denoted by

$$\pi(z, w) = z. \tag{1.2.11}$$

Then for almost all z the preimage $\pi^{-1}(z)$ consists of n distinct points

$$(z, w_1(z)), \;\; (z, w_2(z)), \;\ldots\;, (z, w_n(z)) \tag{1.2.12}$$

of the surface $\mathcal{S}$ where $w_1(z), \ldots, w_n(z)$ are the n roots of (1.2.8) for a given value of z. For certain values of z, some of the points of the preimage can merge. This happens at the *ramifications points* (z_0, w_0) of the Riemann surface where the partial derivative $F_w(z, w)$ vanishes (recall that we consider only non-singular curves so far). The point $z_0 \in \mathbb{C}$ is called *branch point* and it is determined by the system of equations

$$\left.\begin{aligned} F(z_0, w) &= 0 \\ F_w(z_0, w) &= 0 \end{aligned}\right\}. \tag{1.2.13}$$

Let $\Delta_F(z) := D(F(z,.))$ be the discriminant of $F(z,w)$ considered as a polynomial in w depending on the parameter z

$$\Delta_F(z) = \frac{1}{a_0(z)}(-1)^{\frac{n(n-1)}{2}} \times \tag{1.2.14}$$

$$\det \begin{pmatrix} a_0(z) & a_1(z) & a_2(z) & \dots & a_{n-1}(z) & a_n(z) & 0 & \dots & 0 \\ 0 & a_0(z) & a_1(z) & \dots & \dots & a_{n-1}(z) & a_n(z) & \dots & 0 \\ \dots & \dots & & \dots & & & \dots & & \dots \\ 0 & 0 & \dots & & \dots & & \dots & a_{n-1}(z) & a_n(z) \\ na_0(z) & (n-1)a_1(z) & (n-2)a_2(z) & \dots & a_{n-1}(z) & 0 & \dots & \dots & 0 \\ 0 & na_0(z) & (n-1)a_1(z) & \dots & 2a_{n-2}(z) & a_{n-1}(z) & 0 & \dots & 0 \\ & \dots & & \dots & & & & & \dots \\ 0 & 0 & \dots & \dots & & & \dots & 2a_{n-2}(z) & a_{n-1}(z) \end{pmatrix}.$$

Proposition 1.2.10. *If $P_0 \in \mathcal{S}$ is a ramification point of the complex algebraic curve (1.2.8) with respect to its projection onto the z-plane then its projection $z_0 = \pi(P_0) \in \mathbb{C}$ satisfies $\Delta_F(z_0) = 0$. If the curve is smooth irreducible then also the converse statement holds true.*

The proof easily follows from the results of the previous section.

It follows that the Riemann surface associated with a smooth irreducible affine algebraic curve has a finite number of ramification points.

The choice of the variables z or w as a local parameter is not always the most convenient. We shall also encounter other ways of choosing a local parameter τ so that near the point (z,w) the curve $\mathcal{S}$ can be represented locally in the form

$$z = z(\tau), \quad w = w(\tau) \tag{1.2.15}$$

where $z(\tau)$ and $w(\tau)$ are holomorphic functions of τ, and

$$\left(\frac{dz}{d\tau}, \frac{dw}{d\tau}\right) \neq (0,0) \tag{1.2.16}$$

on a sufficiently small neighbourhood of the point. We study the structure of the mapping π in (1.2.12) in a neighbourhood of a ramification point $P_0 = (z_0, w_0)$ of $\mathcal{S}$ defined in (1.2.8). Let τ be a local parameter on $\mathcal{S}$ in a neighbourhood of P_0 such that $\tau(P_0) = 0$. Then

$$\begin{aligned} z &= z_0 + a_k\tau^k + O(\tau^{k+1}), \quad a_k \neq 0 \\ w &= w_0 + c_q\tau^q + O(\tau^{q+1}), \quad c_q \neq 0, \end{aligned} \tag{1.2.17}$$

where a_k and c_q are nonzero coefficients. Since w can be taken as the local parameter in a neighbourhood of P_0 it follows that $q = 1$. We get a parametrization of the surface $\mathcal{S}$ in a neighbourhood of a ramification point:

$$\begin{aligned} z &= z_0 + a_k\tau^k + O(\tau^{k+1}), \\ w &= w_0 + b_1\tau + O(\tau^2), \end{aligned} \tag{1.2.18}$$

where $k > 1$. It is easy to check that the number k does not depend on the choice of the local parameter.

Definition 1.2.11. *The number* $\mathrm{mult}_z(P_0) = k$ *is called the* multiplicity *and* $b_z(P_0) = k-1$ *the* ramification index *of the point* $P_0 \in \mathcal{S}$ *wrt the map* $\pi : \mathcal{S} \to \mathbb{C}$, $\pi(z, w) = z$.

So, if P_0 is not a ramification point then $\mathrm{mult}_z(P_0) = 1$ and $b_z(P_0) = 0$.

Exercise 1.2.12: Let $P_0 = (z_0, w_0)$ be a ramification point for the curve (1.2.8) with respect to the projection $(z, w) \to z$. Suppose that the local parameter in the neighbourhood of P_0 is of the form (1.2.18) with $k > 1$. Show that

$$\left.\frac{d^j F(z,w)}{dw^j}\right|_{(z_0,w_0)} = 0, \quad j = 0, \ldots, k-1.$$

Exercise 1.2.13: Prove that the total multiplicity of all the ramification points on $\mathcal{S}$ over $z = z_0$ is equal to the multiplicity of $z = z_0$ as a root of the discriminant of the polynomial $F(z, w)$.

Exercise 1.2.14: Recall that a partition μ of an integer n is a collection of positive integers $\mu = (\mu_1, \ldots, \mu_l)$ such that $\sum_{j=1}^{l} \mu_j = n$. To every smooth algebraic curve C in (1.2.8) of degree n in w and a point $z_0 \in \mathbb{C}$, let $l \leqslant n$ be the number of pre-images $\pi^{-1}(z_0) = P_1 \cup \cdots \cup P_l$, where $\pi : C \to \mathbb{C}$ is the projection $\pi(z, w) = z$. Assign positive integers $(k_1, \ldots, k_l)$ by

$$k_j = \mathrm{mult}_z(P_j), \quad j = 1, \ldots, l.$$

This collection of integers is called the ramification profile of the smooth curve over $z_0 \in \mathbb{C}$. Note that if z_0 is not a branch point then the preimage $\pi^{-1}(z_0)$ consists of n distinct points of multiplicity 1. Show that the ramification profile over any point of the complex plane is a partition of n.

Lemma 1.2.15. *Let* $P_0 = (z_0, w_0)$ *be a ramification point of the Riemann surface* $\mathcal{S}$ *defined in* (1.2.8) *with respect to the projection* $(z, w) \to z$ *and let* $\mathrm{mult}_z(P_0) = k$ *be its multiplicity. Then there are* k *functions* $w_1(z), \ldots, w_k(z)$ *analytic on a sector* $S_{\rho,\phi}$ *of the punctured disc*

$$0 < |z - z_0| < \rho, \quad \arg(z - z_0) < \phi$$

for sufficiently small $\rho > 0$ *and any positive* $\phi < 2\pi$ *such that*

$$F(z, w_j(z)) \equiv 0 \quad \text{for} \quad z \in S_{\rho,\phi}, \quad j = 1, \ldots, k.$$

The functions $w_1(z), \ldots, w_k(z)$ *are continuous in the closure* $\bar{S}_{\rho,\phi}$ *and*

$$w_1(z_0) = \cdots = w_k(z_0) = w_0.$$

Proof. As P_0 is a ramification point we have $F_w(z_0, w_0) = 0$. Therefore, by the non-singularity assumption $F_z(z_0, w_0) \neq 0$. So the complex curve $F(z, w) = 0$ can be locally parametrized in the form $z = z(w)$ where the analytic function $z(w)$ is uniquely determined by the condition $z(w_0) = z_0$. Consider the first nontrivial term of the Taylor expansion of this function

$$z(w) = z_0 + \alpha_k (w - w_0)^k + \alpha_{k+1}(w - w_0)^{k+1} + \ldots, \quad k > 1, \;\; \alpha_k \neq 0,$$

or equivalently

$$z - z_0 = \alpha_k (w - w_0)^k \left(1 + \frac{\alpha_{k+1}}{\alpha_k}(w - w_0) + O((w - w)^2)\right) \quad k > 1, \;\; \alpha_k \neq 0.$$

Introduce an auxiliary function

$$\begin{aligned} f(w) &= \beta(w-w_0)\left[1+\frac{\alpha_{k+1}}{\alpha_k}(w-w_0)+O\left((w-w_0)^2\right)\right]^{\frac{1}{k}} \\ &= \beta(w-w_0)\left[1+\frac{\alpha_{k+1}}{k\,\alpha_k}(w-w_0)+O\left((w-w_0)^2\right)\right], \end{aligned} \tag{1.2.19}$$

where the complex number β is chosen in such a way that $\beta^k = \alpha_k$. The function $f(w)$ is analytic for sufficiently small $|w-w_0|$. Observe that $f'(w_0) = \beta \neq 0$. Therefore the analytic inverse function f^{-1} locally exists. The needed k functions $w_1(z), \ldots, w_k(z)$ can be constructed as follows

$$w_j(z) = f^{-1}\left(e^{\frac{2\pi i\,(j-1)}{k}}(z-z_0)^{1/k}\right), \quad j = 1, \ldots, k, \tag{1.2.20}$$

where we choose an arbitrary branch of the k-th root of $(z-z_0)$ for $z \in S_{\rho,\phi}$. □

The statement of Lemma shows that near a ramification point $P_0 \in \mathcal{S}$ of multiplicity k there are exactly k sheets of the Riemann surface that all merge together at the point P_0.

Example 1.2.16. Elliptic and hyperelliptic Riemann surfaces have the form

$$\mathcal{S} = \{(z,w) \in \mathbb{C}^2 \,|F(z,w) = w^2 - Q_n(z) = 0\}, \tag{1.2.21}$$

where $Q_n(z)$ is a polynomial of degree n with leading coefficient 1. These surfaces are two-sheeted coverings of the z-plane. The non-singularity condition implies that gradient vector $\mathrm{grad}_{\mathbb{C}} F = (-Q_n'(z), 2w) \neq (0,0)$ at any point of $\mathcal{S}$. A point $(z_0, w_0) \in \mathcal{S}$ is singular if

$$w_0 = 0, \quad Q_n'(z_0) = 0. \tag{1.2.22}$$

Together with the condition (1.2.21) for a point (z_0, w_0) to belong to $\mathcal{S}$ we get that

$$Q_n(z_0) = 0, \quad Q_n'(z_0) = 0, \tag{1.2.23}$$

i.e. z_0 is a multiple root of the polynomial $Q_n(z)$. Accordingly, the surface (1.2.21) is non-singular if and only if the polynomial $Q_n(z)$ does not have multiple roots:

$$Q_n(z) = \prod_{i=1}^{n}(z-z_i), \quad z_i \neq z_j, \text{ for } i \neq j. \tag{1.2.24}$$

The surface $\mathcal{S}$ is called elliptic for $n = 3, 4$ and it is called hyperelliptic for $n > 4$. The ramification points of the surface with respect to the map $(z,w) \to z$ are determined by the two equations

$$w^2 = Q_n(z), \;\; w = 0,$$

which gives n ramification points $P_i = (z = z_i, w = 0)$, $i = 1, \ldots, n$. All the ramification points have ramification index equal to one. In a neighbourhood of any point of $\mathcal{S}$ that is not a ramification point, one can take z as a local parameter, and $w = \sqrt{Q_n(z)}$ is a locally defined holomorphic function. In a neighbourhood of a ramification point P_i it is convenient to take

$$\tau = \sqrt{z - z_i}, \tag{1.2.25}$$

as a local parameter. Then near the ramification point P_i, the Riemann surface (1.2.21) has the local parametrization

$$z = z_i + \tau^2, \quad w = \tau \sqrt{\prod_{j \neq i} (\tau^2 + z_i - z_j)} \tag{1.2.26}$$

where $w = w(\tau)$ is a single-valued holomorphic function and $dw/d\tau \neq 0$ for sufficiently small values of τ.

Exercise 1.2.17: Consider the family of n-sheeted Riemann surfaces of the form

$$F(z,w) = \sum_{i+j \leqslant n} a_{ij} z^i w^j \tag{1.2.27}$$

(the so-called planar curves of degree n) for all possible values of the coefficients a_{ij}. Prove that (1) the generic surface of the form (1.2.27) is smooth; (2) there are $n(n-1)$ ramification points on the curve and they all have ramification index 1. In other words, the conditions for the appearance of ramification points of index greater than one are written as a collection of algebraic equations on the coefficients a_{ij}.

We conclude this Section with a brief discussion of Riemann surfaces associated with singular curves. Let C be the algebraic curve defined by an irreducible polynomial equation $F(z,w) = 0$. The goal is to construct a Riemann surface $\mathcal{S}$ along with a map $\rho : \mathcal{S} \to C$ that is biholomorphic away from the singular points of C and their preimages on $\mathcal{S}$. Here we will do it only locally near one singular point and, moreover, only for the simplest case of a nodal singularity. The case of arbitrary singularities will be treated in the next Section.

Let (z_0, w_0) be a singular point of the curve that is,

$$F(z_0, w_0) = 0, \quad F_z(z_0, w_0) = 0, \quad F_w(z_0, w_0) = 0.$$

It is called a node if

$$\det \begin{pmatrix} F_{zz}(z_0, w_0) & F_{zw}(z_0, w_0) \\ F_{zw}(z_0, w_0) & F_{ww}(z_0, w_0) \end{pmatrix} \neq 0.$$

Using Taylor formula rewrite the polynomial F in the form

$$F(z,w) = \frac{1}{2}\left[a(z-z_0)^2 + 2b(z-z_0)(w-w_0) + c(y-y_0)^2\right] + \Delta F(z,w)$$

where

$$a = F_{zz}(z_0, w_0), \quad b = F_{zw}(z_0, w_0), \quad c = F_{ww}(z_0, w_0)$$

and

$$\Delta F(z,w) = \sum_{i+j \geqslant 3} r_{ij} (z-z_0)^i (w-w_0)^j, \quad r_{ij} = \frac{1}{(i+j)!} \frac{\partial^{i+j} F(z_0, w_0)}{\partial z^i \, \partial w^j}.$$

The quadratic term can be factorized into a product of two distinct linear functions. Near the point (z_0, w_0) the term ΔF can be considered as a small perturbation of the leading quadratic term. Therefore, assuming $c \neq 0$ one obtains two solutions of equation $F(z,w) = 0$ in the form of convergent series

$$w_\pm(z) = w_0 - \frac{b \pm \sqrt{b^2 - ac}}{c}(z - z_0) + O\left((z - z_0)^2\right).$$

We are now ready to describe the local structure of the Riemann surface $\mathcal{S}$ and the map $\rho : \mathcal{S} \to C$ near the node $P_0 = (z_0, w_0)$. The surface will consist locally of two small disks D_+ and D_- centred at points $P_\pm$ respectively. The complex coordinates $\tau_\pm$ on the disks can be chosen in such a way that $\tau_\pm(P_\pm) = 0$ and the map $\rho(\tau_\pm) = (z(\tau_\pm), w_\pm(\tau_\pm))$ reads

$$\left.\begin{array}{rcl} z(\tau_\pm) & = & z_0 + \tau_\pm \\ w_\pm(\tau_\pm) & = & w_0 - \frac{b \pm \sqrt{b^2 - ac}}{c}\tau_\pm + O\left(\tau_\pm^2\right) \end{array}\right\}, \quad \tau_\pm \in D_\pm.$$

From the above calculations it follows that the map $\rho : \dot{D}_+ \cup \dot{D}_- \to C \backslash P_0$ of the punctured disks $\dot{D}_\pm = D_\pm \backslash P_\pm$ is locally biholomorphic. But $\rho(P_+) = \rho(P_-) = P_0$.

We did the calculations assuming that $c \neq 0$. If $c = 0$ but $a \neq 0$ then everything goes in a similar way after interchanging the roles of z and w. The picture slightly changes in the case $a = c = 0$. In this case the polynomial $F(z,w)$ takes the form

$$F(z,w) = b(z - z_0)(w - w_0) + \sum_{i+j=3} r_{ij}(z - z_0)^i (w - w_0)^j + \sum_{i+j \geqslant 4} r_{ij}(z - z_0)^i (w - w_0)^j.$$

The map ρ has the form

$$\left.\begin{array}{rcl} z(\tau_+) & = & z_0 + \tau_+ \\ w_+(\tau_+) & = & -\frac{r_{30}}{b}\tau_+^2 + O(\tau_+^3) \end{array}\right\}$$

on D_+ and

$$\left.\begin{array}{rcl} z(\tau_-) & = & z_0 - \frac{r_{03}}{b}\tau_-^2 + O(\tau_-^3) \\ w_-(\tau_-) & = & w_0 + \tau_- \end{array}\right\}$$

on D_-. Observe that in the case $c = 0$ the point $P_- \in \mathcal{S}$ is a ramification point wrt the map $\mathcal{S} \ni P \mapsto z(P) \in \mathbb{C}$.

The above method for constructing the Riemann surface of an algebraic curve near a singular point of the latter is a version of the procedure called <u>resolution of singularities</u>. The constructed Riemann surface is called <u>normalisation</u> of the algebraic curve. The method is based on an efficient algorithm for computing series expansions of all branches of the algebraic function near the singular point. In full generality the algorithm will be explained in the next Section.

1.2.3 Newton polygons and Puiseux series

In this section we explain the use of an algebraic tool for studying the local structure of the Riemann surface $\mathcal{S}$ associated with an algebraic curve C defined by an irreducible polynomial equation

$$F(z,w) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0 \tag{1.2.28}$$

near a singular point $P_0 = (z_0, w_0)$ of the curve. Here the problem will be treated only locally, near one singular point; discussion of the global structure of the Riemann surface $\mathcal{S}$ is postponed till Section 1.3.

Recall (see the previous Section) that locally the Riemann surface $\mathcal{S}$ must consist of a finite number of open disks $\mathcal{S} = D_1 \cup \cdots \cup D_k$ with marked points $P_i \in D_i$, $i = 1, \ldots, k$ and a map

$$\rho : \mathcal{S} \to C \quad \text{satisfying} \quad \rho(P_i) = P_0, \quad i = 1, \ldots, k$$

and establishing a biholomorphic equivalence between $\dot{D}_1 \cup \cdots \cup \dot{D}_k$ and a punctured neighbourhood of the point $P_0 \in C$. Here $\dot{D}_i = D_i \backslash P_i$. On every disk D_i one can choose a local parameter τ such that $\tau(P_i) = 0$ and the restriction of ρ on the disk

$$\rho(\tau) = (z(\tau), w(\tau)), \quad F(z(\tau), w(\tau)) \equiv 0, \quad \rho(0) = P_0$$

is given by a pair of holomorphic functions on D_i of the form[6]

$$\left.\begin{array}{rcl} z(\tau) & = & z_0 + \tau^q \\ w(\tau) & = & w_0 + \alpha_0 \tau^p + \alpha_1 \tau^{p+1} + \alpha_2 \tau^{p+2} \ldots, \quad \alpha_0 \neq 0 \end{array}\right\} \tag{1.2.29}$$

for some integers $p \neq 0$, $q > 0$. The integer p is positive unless z_0 is a root of the leading coefficient $a_0(z)$ of the polynomial $F(z, w)$. It is understood that, in the second line of (1.2.29), the series is convergent for sufficiently small $|\tau|$. In order to show that the map ρ is bi-holomorphic for $\tau \neq 0$, we build the inverse map by first assuming the vanishing of the coefficients $\alpha_1, \alpha_2, \ldots$ in (1.2.29). Let γ and β be integers such that $\gamma p + \beta q = 1$. Then the inverse map is $\rho^{-1}(z, w) = (z - z_0)^\gamma (\frac{w - w_0}{\alpha_0})^\beta = \tau$ that is clearly holomorphic for $w \neq w_0$ and $z \neq z_0$. In the general case let us define the function $h(\tau)$ so that $w(\tau) = w_0 + \tau^p h(\tau)$. Then $g(\tau) = \tau(h(\tau))^\beta$ is holomorphic invertible and $\rho^{-1}(z, w) = g^{-1}\left((z - z_0)^\gamma (w - w_0)^\beta\right) = \tau$.

The Riemann surface $\mathcal{S} = D_1 \cup \cdots \cup D_k$ equipped with the holomorphic function z is a branched covering of a disk $|z - z_0| < \epsilon$ for a sufficiently small ϵ. If, for example, on the disk D_i the expansion of $z(\tau)$, has the form (1.2.29) then the point P_i is ramification point of z of multiplicity q.

If $q = 1$ then the expansion for w can be rewritten in the form of a convergent Taylor series

$$w = w_0 + \alpha_0 (z - z_0) + \alpha_1 (z - z_0)^2 + \ldots.$$

In the general case $q > 1$ eliminating $\tau = (z - z_0)^{\frac{1}{q}}$ one obtains an alternative representation of (1.2.29) as an expansion in fractional powers of $z - z_0$

$$w(z) = w_0 + \alpha_0 (z - z_0)^{\frac{p}{q}} + \alpha_1 (z - z_0)^{\frac{p+1}{q}} + \ldots. \tag{1.2.30}$$

In complex analysis the expansions of the form (1.2.30) are called <u>Puiseux series</u>. For $p > 0$ they can be considered as power series in the variable $(z - z_0)^{\frac{1}{q}}$; if $p < 0$ then they are Laurent series in the same variable. We will present an algorithm of computing Puiseux expansions of all branches of the algebraic function $w(z)$ near a singular point of the curve. Clearly the branches of the algebraic function $w(z)$ obtained one from another by analytic continuation around the point z_0 on the complex plain are identified, namely:

$$w_0 + \sum_{m \geqslant p} \alpha_{m-p} (z - z_0)^{\frac{m}{q}} \sim w_0 + \sum_{m \geqslant p} e^{\frac{2\pi i m j}{q}} \alpha_{m-p} (z - z_0)^{\frac{m}{q}} \quad \text{for } j = 0, 1, \ldots, q - 1.$$

[6] The integers p, q as well as the coefficients α_0, α_1 certainly depend on the label i of the disk. We do not put this dependence explicitly in the formulae in order to avoid too complicated notations.

It is understood that the numbers p, q are chosen in the minimal way i.e., there exists an integer $m \geqslant p$ not divisible by q such that $\alpha_{m-p} \neq 0$.

Let the polynomial (1.2.28) have the form

$$F(z,w) = \sum_{i,j \geqslant 0} a_{ij} z^i w^j. \tag{1.2.31}$$

Without loss of generality we may assume that the singular point in question is the origin,

$$F(0,0) = F_z(0,0) = F_w(0,0) = 0.$$

It will be always assumed that the partial derivative $F_w(z,w)$ does not vanish identically at the points of the curve $F(z,w) = 0$.

Definition 1.2.18. *The Newton polygon of the polynomial (1.2.31) is the convex hull of the set of points (i,j) on the (x,y)-plane defined by*

$$\{(i,j) \in \mathbb{R}^2 \,|\, a_{ij} \neq 0\}.$$

The Newton polygon is a convex set belonging to the first quadrant of the plane. Without loss of generality we may assume that it touches the coordinate axes. In the opposite case we can factor out some powers of z or of w. Actually, for the algorithm only the sides of the polygon

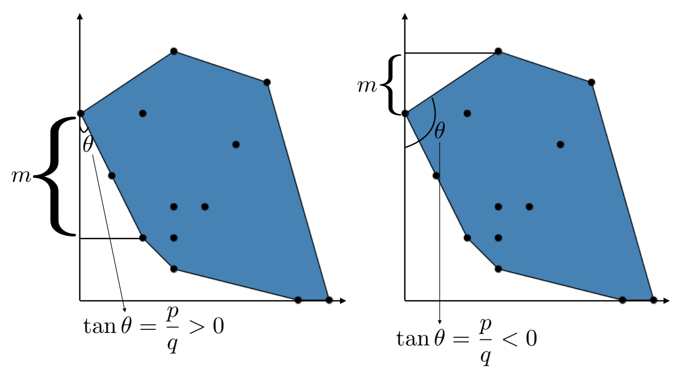


Figure 1.3: Newton polygon

looking towards the y-axis will be relevant, see Fig. 3.1 for an example.

To each side of the Newton polygon looking towards the y axis we associate two numbers, a positive integer m that equals the length of the projection of the side onto the y-axis, and a rational number $\frac{p}{q}$ that is equal to the tangent of the angle between the side and the negative direction of the y-axis. With such a side we will associate m convergent Puiseux expansions of the algebraic function $w(z)$ of the form

$$w = \alpha z^\rho + \alpha' z^{\rho'} + \dots \tag{1.2.32}$$

for rational numbers $\rho < \rho' < \dots$. The exponent of the leading term is equal to the slope of the corresponding side

$$\rho = \frac{p}{q}. \tag{1.2.33}$$

The leading coefficient $\alpha \neq 0$ is determined as a nonzero root of the polynomial

$$P(\omega) = \sum_{(i,j)\in\text{the side}} a_{ij}\omega^j. \tag{1.2.34}$$

Observe that the number of nonzero roots of the polynomial (1.2.34), counted with multiplicities, is equal to m=lenght of the projection of the side onto the y-axis.

Remark 1.2.19. The number of solutions, counted with multiplicities, of the equation $F(z,w) = 0$ written in the form of Puiseux series (1.2.32) is equal to $n = \deg_w(F)$ (the degree of F with respect to the variable w). If the Newton polygon has k sides that faces the y-axis and we denote by m_1, $\dots$, m_k the lengths of their projections onto the y-axis, since the height of the Newton polygon is equal to n we have $m_1 + \cdots + m_k = n$.

Choose a nonzero root $\omega = \alpha$ of (1.2.34). Further inspection shows that the set of nonzero roots of the polynomial $P(\omega)$ is invariant with respect to multiplication by the q-th root of unity (assuming the numbers p, q to be coprime): this follows from the representation

$$P(\omega) = \omega^{j_0} Q\left(\omega^q\right) \tag{1.2.35}$$

for some polynomial Q and a nonnegative integer j_0 (see eq. (1.2.41) below).

In order to determine the next term $w_1 = \alpha' z^{\rho'}$ of the expansion (1.2.32), consider the new polynomial

$$F_1(z_1, w_1) := F\left(z_1^q, \alpha\, z_1^p + w_1\right) \tag{1.2.36}$$

and repeat the above procedure applying it to the side closest to the x-axis. And so on and so forth.

Before explaining the motivations for such an algorithm let us consider an example.

Example 1.2.20. Consider polynomial

$$F(z,w) = 2z^7 - z^8 - z^3 w + (4z^2 + z^3)w^2 + (z^3 - z^4)w^3 - 4zw^4 + 7z^5w^5 + (1 - z^2)w^6 + 5z^6w^7 + z^3w^8. \tag{1.2.37}$$

There are four sides in the Newton polygon of F looking towards the y-axis (see Fig. 1.4); only they will be relevant for determining the Puiseux expansions of various branches of the solutions $w(z)$ near $z = 0$. For the first one with the vertices $(7,0)$ and $(3,1)$ one has $m = 1$, $\frac{p}{q} = 4$. The corresponding part of the polynomial reads $2z^7 - z^3 w$. Solving the equation $2z^7 - z^3 w = 0$ we obtain $w = 2z^4$. This is the leading term of the branch of solution corresponding to the first side of the polygon. In order to compute the first correction let us substitute $w = 2z^4 + w_1$ in $F(z,w)$. Then w_1 is determined from the equation $F_1(z, w_1) := F(z, 2z^4 + w_1) = 0$. In the Newton polygon of F_1 (see Fig. 1.5) take the edge connecting the points $(8,0)$ and $(3,1)$. The corresponding equation $-z^8 - z^3 w_1 = 0$ yields $w_1 = -z^5$. So, the first two terms of expansion of the branch of $w(z)$ associated with the first side of the Newton polygon read $w = 2z^4 - z^5 + O(z^6)$. Higher order

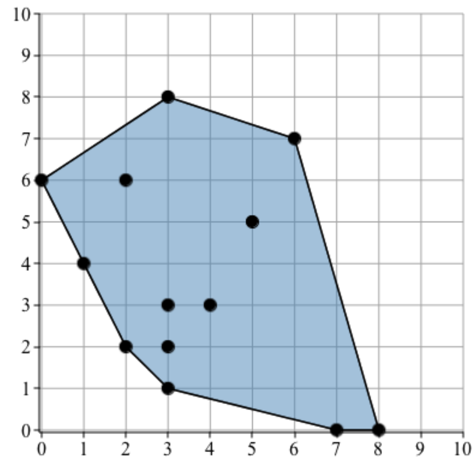


Figure 1.4: Newton polygon of the polynomial (1.2.37).

terms can be obtained by iterating the above procedure. This is an ordinary point of the Riemann surface with respect to the map $z : \mathcal{S} \to \mathbb{C}$.

In a similar way to the second side $(3,1)$–$(2,2)$ of the Newton polygon in Figure 1.4, with $m = 1$, $\frac{p}{q} = 1$, one associates the leading term $w = \frac{1}{4}z$. From the side $(5,0)$–$(3,1)$ of the Newton polygon of $F_1(z, w_1) := F\left(z, \frac{1}{4}z + w_1\right)$ (see Fig. 1.6) one finds the next correction etc. This gives the second branch of $w(z)$ near another ordinary point of the Riemann surface $w = \frac{1}{4}z - \frac{3}{64}z^2 + O(z^3)$.

For the third side of the polygon in Figure 1.4, one has $m = 4$, $\frac{p}{q} = \frac{1}{2}$. It corresponds the equation $4z^2w^2 - 4zw^4 + w^6 = w^2(w^2 - 2z)^2 = 0$. So, at the leading order one has two pairs of double roots $w^{(1)} = w^{(2)} = \sqrt{2}\, z^{\frac{1}{2}}$ and $w^{(3)} = w^{(4)} = -\sqrt{2}\, z^{\frac{1}{2}}$. We will see now that these double roots split at the next approximation. Indeed, in order to treat the pair $w^{(1)}$ and $w^{(2)}$ we have to substitute $w = \sqrt{2}\, z^{\frac{1}{2}} + w_1$ and obtain a new polynomial in $z_1 = z^{\frac{1}{2}}$ and w_1. For the side $(7,0)$–$(4,2)$ of the Newton polygon (see Fig. 1.7) of such a new polynomial it corresponds the equation $-\sqrt{2}z_1^7 + 16z_1^4 w_1^2 = 0$ that yields $w_1 = \pm\frac{2^{\frac{1}{4}}}{4} z_1^{\frac{3}{2}}$. One obtains the following pair of distinct expansions

$$w^{(1)} = \sqrt{2}\, z^{\frac{1}{2}} + \frac{2^{\frac{1}{4}}}{4} z^{\frac{3}{4}} + \ldots \qquad\qquad w^{(2)} = \sqrt{2}\, z^{\frac{1}{2}} - \frac{2^{\frac{1}{4}}}{4} z^{\frac{3}{4}} + \ldots.$$

Similarly for $w^{(3)}$ and $w^{(4)}$ one has

$$w^{(3)} = -\sqrt{2}\, z^{\frac{1}{2}} + i\frac{2^{\frac{1}{4}}}{4} z^{\frac{3}{4}} + \ldots \qquad\qquad w^{(4)} = -\sqrt{2}\, z^{\frac{1}{2}} - i\frac{2^{\frac{1}{4}}}{4} z^{\frac{3}{4}} + \ldots.$$

It can be shown that the higher order terms will contain integer powers of $z^{1/4}$; all four expansions $z^{(1)}, \ldots, z^{(4)}$ are four branches of the same algebraic function. These branches merge at $z = 0$. One obtains one ramification point of multiplicity 4 of the Riemann surface $(\mathcal{S}, z)$.

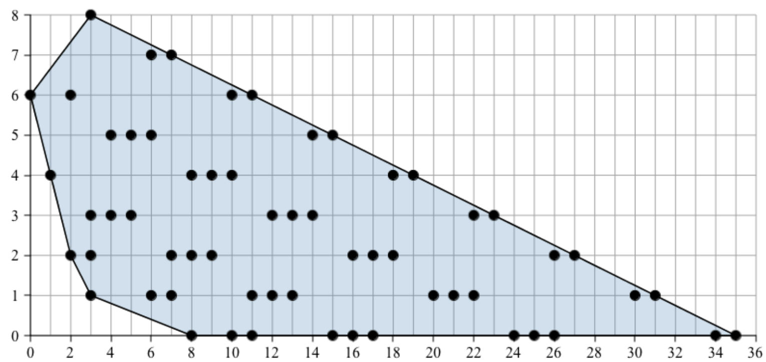


Figure 1.5: Newton polygon of the polynomial $F(z, 2z^4 + w_1)$.

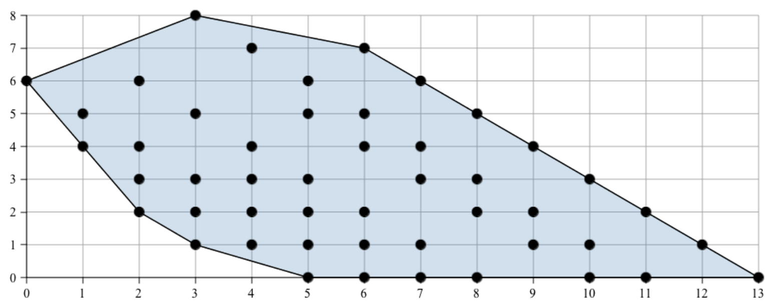


Figure 1.6: Newton polygon of the polynomial $F(z, \frac{1}{4}z + w_1)$.

The fourth side of the Newton polygon in Figure 1.4, with $m = 2$, $\frac{p}{q} = -\frac{3}{2}$ yields the equation $w^6 + z^3 w^8 = 0$ that is, $w^{(1,2)} = \pm i z^{-\frac{3}{2}}$. At the next order one has to analyze the equation $F\left(z_1^2, i z_1^{-3} + w_1\right) = 0$ (here, like above we denote $z_1 = z^{1/2}$). To obtain a polynomial equation one has to multiply the result by z_1^{15}, see the corresponding Newton polygon on Fig. 1.8. For the first correction one obtains the equation $z_1 - 2iw_1 = 0$. This gives a ramification point of multiplicity 2:

$$w^{(1)} = i z^{-\frac{3}{2}} - \frac{i}{2} z^{\frac{1}{2}} + \ldots \qquad w^{(2)} = -i z^{-\frac{3}{2}} + \frac{i}{2} z^{\frac{1}{2}} + \ldots .$$

Actually, when z goes to zero these two branches tend to infinity. So, the last point is an infinite point of the Riemann surface $(\mathcal{S}, z)$. Note that the leading term $a_0(z) = z^3$ vanishes at the singular point.

Remark 1.2.21. To compute the branches of $w(z)$ at $z \to \infty$ one can use the above algorithm applied at the right-looking sides of the Newton polygon. For the example (1.2.37) of an algebraic curve we obtain two expansions, namely, a Laurent series in $1/z$

$$w(z) = -5z^3 - \frac{12}{25} z^{-4} + \frac{1}{5} z^{-6} + \ldots$$

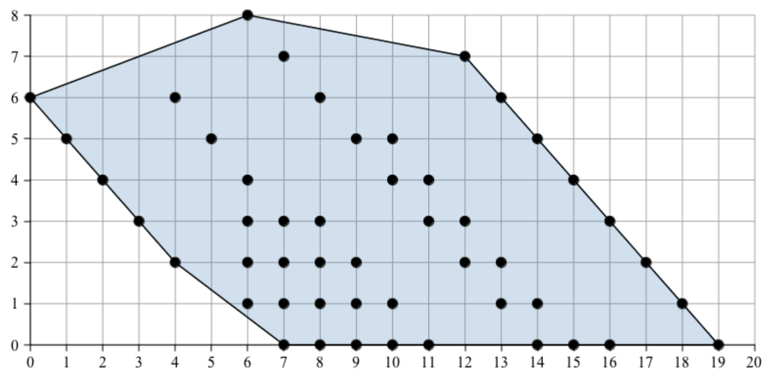


Figure 1.7: Newton polygon of the polynomial $F(z_1^2, \sqrt{2}z_1 + w_1)$.

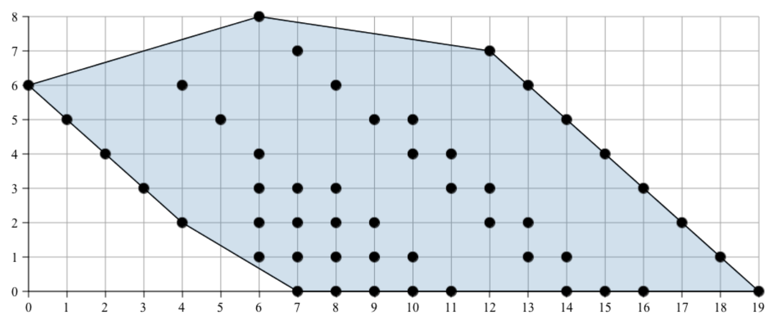


Figure 1.8: Newton polygon of the polynomial $z_1^{15}F(z_1^2, i\, z_1^{-3} + w_1)$.

for the side $(3,8)$–$(6,7)$ and a Laurent–Puiseux series in $z^{-1/7}$

$$w(z) = \frac{z^{2/7}}{5^{1/7}} - \frac{2}{7}\frac{1}{5^{1/7} z^{5/7}} + \dots$$

along with 6 other branches $w_k(z) = w\left(z\, e^{2\pi i k}\right)$, $k = 1, \dots, 6$ for the side $(6,7)$–$(8,0)$. So we have two infinity points P_1, P_2 on the Riemann surface. The function z has a simple pole at P_1 and a pole of order 7 at P_2. In other words, P_1 is an ordinary point of the Riemann surface with respect to its projection onto the extended z-plane $\mathbb{C}$. The point P_2 is a ramification point of multiplicity 7. The function w has poles of order 3 and 2 at the points P_1, P_2 respectively.

In order to justify the above algorithm let us first make a digression about zeroes of families of analytic functions. Let $f(x)$ be an analytic function in x with a simple zero at $x = x_0$,

$$f(x_0) = 0, \quad f'(x_0) \neq 0.$$

Consider the perturbed equation

$$f(x) = \epsilon \tag{1.2.38}$$

where ϵ is a small parameter. The claim is that the solution to (1.2.38) remains close to x_0. Moreover, such a solution is an analytic function in ϵ for sufficiently small $|\epsilon|$. Indeed, the inverse function f^{-1} such that $f^{-1}(0) = x_0$ is well defined due to the assumption $f'(x_0) \neq 0$ and it is analytic on a neighborhood of 0. Then

$$x = f^{-1}(\epsilon) = x_0 + \frac{\epsilon}{f'(x_0)} + O(\epsilon^2).$$

All terms of the expansion in powers of ϵ are uniquely determined from eq. (1.2.38).

Consider now the case of a multiple zero. Let x_0 be a root of $f(x)$ of multiplicity k,

$$f(x_0) = f'(x_0) = \cdots = f^{(k-1)}(x_0) = 0, \quad f^{(k)}(x_0) \neq 0.$$

Then, after adding of a small perturbation the multiple root splits in k different roots that are analytic functions in $\epsilon^{\frac{1}{k}}$. Indeed, the Taylor expansion of $f(x)$ at $x = x_0$ starts from a term of degree k

$$f(x) = c_k(x - x_0)^k + c_{k+1}(x - x_0)^{k+1} + \ldots, \quad c_k = \frac{f^{(k)}(x_0)}{k!} \neq 0.$$

Denote $\tilde{f}$ the k-th root of $f(x)$

$$\tilde{f}(x) = b_k(x - x_0)\left[1 + \frac{c_{k+1}}{c_k}(x - x_0) + \ldots\right]^{\frac{1}{k}}$$

for some choice of $b_k = c_k^{\frac{1}{k}}$. This function is analytic and invertible on a neighborhood of x_0. Thus the equation

$$f(x) = \left[\tilde{f}(x)\right]^k = \epsilon$$

can be solved by a convergent Puiseux series

$$x = \tilde{f}^{-1}\left(\epsilon^{\frac{1}{k}}\right) = x_0 + b_k^{-1}\epsilon^{\frac{1}{k}} + O\left(\epsilon^{\frac{2}{k}}\right).$$

Choosing various branches of $\epsilon^{\frac{1}{k}}$ one obtains expansions of all k distinct roots of the perturbed equation.

For more complicated perturbations the splitting of the multiple root of the equation $f(x) = 0$ may not take place. Consider, for example, a more general perturbation of the form

$$f(x) = g(\epsilon)$$

where

$$g(\epsilon) = d_l\epsilon^l + d_{l+1}\epsilon^{l+1} + \ldots, \quad d_l \neq 0$$

is analytic near $\epsilon = 0$. Then the deformation of a k-multiple root x_0 of the unperturbed equation $f(x) = 0$ will be determined from an equivalent equation

$$\tilde{f}(x) = [g(\epsilon)]^{\frac{1}{k}} = d_l^{\frac{1}{k}}\epsilon^{\frac{l}{k}}\left[1 + \frac{d_{l+1}}{d_l}\epsilon + \ldots\right]^{\frac{1}{k}} = d_l^{\frac{1}{k}}\epsilon^{\frac{l}{k}} + \frac{d_{l+1}}{k\, d_l^{\frac{k-1}{k}}}\epsilon^{\frac{l}{k}+1} + \ldots.$$

If $\frac{l}{k} = \frac{l_1}{k_1}$ where k_1 and l_1 are coprime integers then solutions to the equation $f(x) = g(\epsilon)$ are represented by convergent Puiseux series in $\epsilon^{\frac{1}{k_1}}$. In the case $k_1 < k$ we conclude that the function $x(\epsilon)$ lives on a Riemann surface with k/k_1 ramification points of order k_1.

The above considerations can also be applied to the more general equations of the form

$$f(x) = \epsilon\, g(x,\epsilon)$$

where the function $g(x,\epsilon)$ is analytic near the point $(x_0, 0)$. We leave the details as an exercise for the reader.

Let us apply the above ideas to the derivation of the Newton polygon algorithm. Let us fix a side of the Newton polygon facing the y-axis with the lowest vertex (i_0, j_0) and the slope $\frac{p}{q}$. For simplicity we will only consider the sides with positive slope p/q. Any point on the side can be written in the form

$$\begin{aligned} i &= i_0 - p\,l \\ j &= j_0 + q\,l \end{aligned} \tag{1.2.39}$$

for some integer $l = 0, 1, \ldots$. So, the terms of the polynomial F corresponding to the vertices on the side can be written as follows

$$\sum_{(i,j)\in\text{the side}} a_{ij} z^i w^j = z^{i_0 + \frac{q}{p} j_0} \sum_{(i,j)\in\text{the side}} a_{ij} \left(\frac{w}{z^{\frac{p}{q}}}\right)^j = z^{\frac{q i_0 + p j_0}{q}} P\,(\omega) \tag{1.2.40}$$

where we put

$$\omega = \frac{w}{z^{\frac{p}{q}}}$$

and the polynomial $P(\omega)$ was defined in (1.2.34). Observe that the polynomial Q in (1.2.35) is equal to

$$Q(x) = \sum_{l\geqslant 0} a_{i_0 - pl, j_0 + ql} x^l. \tag{1.2.41}$$

We will now rewrite other terms of the polynomial $F(z,w)$ in the variables z, ω. In this way it will become clear that the sum of other monomials in $F(z,w)$ can be considered as a perturbation of the leading term (1.2.40). The small parameter of the perturbation will be some fractional power of z.

Consider a monomial $a_{IJ} z^I w^J$ in F for a point (I, J) sitting inside the Newton polygon. The points on the side of the Newton polygon satisfy the equation

$$\frac{i - i_0}{p} + \frac{j - j_0}{q} = 0.$$

Hence the coordinates (I, J) satisfy

$$\frac{I - i_0}{p} + \frac{J - j_0}{q} = r > 0,$$

for some rational number $r = r(I, J)$. Thus

$$z^I w^J = z^{i_0 + \frac{p}{q} j_0 + p\, r} \left(\frac{w}{z^{\frac{p}{q}}} \right)^J = z^{i_0 + \frac{p}{q} j_0 + p\, r} \omega^J.$$

We arrive at the following representation of the polynomial $F(z, w)$

$$F(z, w) = z^{\frac{q i_0 + p j_0}{q}} \left[P(\omega) + \sum_{I,J} a_{IJ} (z^p)^{r(I,J)} \omega^J \right]$$

where the sum is taken for (I, J) inside the Newton polygon and the exponents $r(I, J) > 0$ for all terms in the sum. As there is a finite number of terms one can choose an integer t such that the numbers

$$s(I, J) := r(I, J)\, t\, p$$

are all integers. Introducing new variable $\epsilon = z^{\frac{1}{t}}$ we apply the above perturbative procedure to solve the equation

$$P(\omega) + \sum_{I,J} a_{IJ} \epsilon^{s(I,J)} \omega^J = 0$$

in the form of a Puiseux series of the form

$$\omega = \alpha + \alpha' \epsilon^\sigma + \dots, \quad \sigma > 0$$

for every root $\omega = \alpha$ of a multiplicity k of the polynomial $P(\omega)$. This gives a branch of the algebraic function $w(z)$

$$w = \alpha\, z^{\frac{p}{q}} + \alpha' z^{\frac{p}{q} + \frac{\sigma}{t}} + \dots .$$

Summarizing the above considerations we arrive to the following.

Theorem 1.2.22. *Let us consider the algebraic curve C described by the zero locus of the polynomial*

$$F(z, w) = w^n + a_1(z) w^{n-1} + \dots + a_n(z) = 0, \tag{1.2.42}$$

where the coefficients $a_1(z), \dots a_n(z)$ are polynomials in z. Let us suppose that $(z_0, w_0) \in C$ is a singular point such that $\pi^{-1}(z_0) = (z_0, w_0)$ with π the projection to the z- plane. Then there exist positive integers $m_1, \dots, m_k$ satisfying

$$m_1 + \dots + m_k = n$$

and k functions $f_1, \dots f_k$ analytic on a neighborhood of z_0 such that all solutions $w(z)$ to (1.2.42) for sufficiently small $|z - z_0|$ can be written in the form

$$w(z) = w_0 + f_j \left((z - z_0)^{\frac{1}{m_j}} \right), \quad j = 1, \dots, k. \tag{1.2.43}$$

Observe that, for $m_j > 1$ the formula (1.2.43) defines m_j series due to ambiguity up to conjugation

$$w(z) \sim \tilde{w}(z) = w_0 + f_j \left(e^{\frac{2\pi i \ell}{m_j}} (z - z_0)^{\frac{1}{m_j}} \right), \quad \ell = 0, 1, \dots, m_j - 1.$$

Remark 1.2.23. It follows from the theorem that near the singular point (z_0, w_0), the polynomial equation (1.2.42) can be written in the form

$$F(z,w) = \prod_{j=1}^{k} \prod_{\ell=1}^{m_j} \left(w - w_0 - f_j \left(e^{\frac{2\pi i \ell}{m_j}} (z - z_0)^{\frac{1}{m_j}} \right) \right),$$

for $|z - z_0|$ sufficiently small.

Remark 1.2.24. Let $(z_0, w_0) \in C$ be a singular point for the curve C. In the situation described in the theorem 1.2.22 we obtain k points $P_1, \ldots, P_k$ on the Riemann surface $\mathcal{S}$ of the algebraic curve C. The holomorphic map $\rho : \mathcal{S} \to C$ with local structure near the point P_j given by (1.2.29) is obtained from the Puiseux expansion $f_j\left((z - z_0)^{\frac{1}{m_j}}\right)$.

We conclude this Section with an elegant algebraic statement. Consider the space

$$\mathbb{C}\langle\langle z \rangle\rangle = \bigcup_{q=1}^{\infty} \mathbb{C}((z^{\frac{1}{q}}))$$

of Puiseux Laurent series with arbitrary fractional exponents. It is easy to see that this is a field.

Theorem 1.2.25 (Puiseux)**.** *The field* $\mathbb{C}\langle\langle z \rangle\rangle$ *is algebraically closed.*

That is, all solutions of a polynomial equation with coefficients in the field $\mathbb{C}\langle\langle z \rangle\rangle$ belong to the same field. The theorem of Puiseux is a generalization of the fundamental theorem of algebra. The constructive proof is obtained by extending the Newton-Puiseux method developed in this section to the case when the coefficients $a_j(z)$ are not polynomials in z but Puiseux Laurent series with arbitrary fractional exponents. Details of the proof can be found in [28].

1.2.4 Smooth projective curves as compact Riemann surfaces

In this subsection we define Riemann surfaces as algebraic curves in $\mathbb{P}^2$.

Definition 1.2.26. *Let $Q(X, Y, Z)$ be a homogeneous non-zero polynomial of degree d in the variables X, Y and Z. The locus*

$$C = \{(X : Y : Z) \in \mathbb{P}^2 \mid Q(X, Y, Z) = 0\} \tag{1.2.44}$$

is the projective curve defined by the polynomial Q.

Remark 1.2.27. Observe that the curve C is well defined since the condition $Q(X, Y, Z) = 0$ is independent from the choice of homogeneous coordinates due to $Q(\lambda X, \lambda Y, \lambda Z) = \lambda^d Q(X, Y, Z)$. Furthermore C is a closed subset of $\mathbf{P}^2$ and therefore it is compact.

Recall that the space $\mathbb{P}^2$ can be covered with three open subsets homeomorphic to $\mathbb{C}^2$:

$$U_0 = \{(X : Y : Z) \in \mathbb{P}^2 \mid X \neq 0\}$$

$$U_1 = \{(X : Y : Z) \in \mathbb{P}^2 \mid Y \neq 0\}$$

$$U_2 = \{(X : Y : Z) \in \mathbb{P}^2 \mid Z \neq 0\}.$$

The homeomorphism on U_0 is given by the map $(X : Y : Z) \to (Y/X, Z/X) \in \mathbb{C}^2$ and similarly for the other open subsets U_1 and U_2.

The intersection of C with any of the U_i is an affine plane curve. For example

$$C_0 = C \cap U_0 = \{(u, v) \in \mathbb{C}^2 \mid Q(1, u, v) = 0\}.$$

Now we show that under non-singularity assumptions, C is a compact Riemann surface.

Definition 1.2.28. *The curve (1.2.44) is non-singular if there are no non-zero solutions to the following system of four equations*

$$Q = \frac{\partial Q}{\partial X} = \frac{\partial Q}{\partial Y} = \frac{\partial Q}{\partial Z} = 0.$$

Exercise 1.2.29: Show that the projective curve C defined in (1.2.44) is non-singular if and only if its intersections $C_i = C \cap U_i$, $i = 1, 2, 3$ with the charts U_i are all non-singular. *Hint*: use Euler identity for homogeneous functions of degree d

$$XQ_X + YQ_Y + ZQ_Z = Qd. \tag{1.2.45}$$

Suppose that C is a smooth projective curve. In order to define a complex manifold structure on C let us recall that each C_i is a smooth affine plane curve and hence a Riemann surface. The coordinate charts are given by the projections onto coordinate axes. For example for the curve C_0 the coordinate charts are Y/X or Z/X and the transition functions are the same as those obtained for smooth affine plane curves. One needs to check that the complex structures given on each C_i are compatible.

Proposition 1.2.30. *Suppose that the projective curve C in (1.2.44) is non-singular. Then C is a compact Riemann surface.*

Proof. We will show that the complex structures given on each C_i are compatible. Let $P \in C_0 \cap C_1$ where $P = (X : Y : Z)$ and $X \neq 0$ and $Y \neq 0$. Since each smooth affine plane curve is non-singular (see exercise 1.2.29), we can assume without loss of generality that Q_X and Q_Z are non-zero on C. Let $\phi_0 : C_0 \to \mathbb{C}$ with $\phi_0(P) = Y/X$ and with locally defined inverse $\phi_0^{-1}(Y/X) = [1 : Y/X : h(Y/X)]$ where h is a holomorphic function on some open domain in $\mathbb{C}$. Let $\phi_1 : C_1 \to \mathbb{C}$ with $\phi_1(P) = Z/Y$ with locally defined inverse $\phi_1^{-1} = [g(\frac{Z}{Y}), 1, \frac{Z}{Y}]$ where $g(\frac{Z}{Y})$ is holomorphic for $Y \neq 0$ and non-zero since we assume $X \neq 0$. Then $\phi_1 \circ \phi_0^{-1}(Y/X) = Xh(Y/X)/Y$ which is holomorphic because $Y \neq 0$, $X \neq 0$ and $h(Y/X)$ is holomorphic. In the same way $\phi_0 \circ \phi_1^{-1}(Z/Y) = \frac{1}{g(Z/Y)}$ which is holomorphic because $Y \neq 0$ and g is nonzero. Similar checks can be done with the other coordinate charts. □

Lemma 1.2.31. *Let $Q(X, Y, Z)$ and $F(X, Y, Z)$ be two homogeneous polynomials of degree d and m respectively. Suppose that $Q(0, 0, Z) \neq 0$ and $F(0, 0, Z) \neq 0$. Then the resultant*

$$R(Q, F)(X, Y)$$

is a homogeneous polynomial in X and Y of degree dm.

Proof. According to the assumptions, $Q(X, Y, Z) = q_0 Z^d + q_1(X, Y)Z^{d-1} + \cdots + q_d(X, Y)$ where $q_j(X, Y)$ are homogeneous polynomials of degree j in X and Y, $j = 0, \ldots, d$ and $F(X, Y, Z) =$

$f_0Z^m + f_1(X,Y)Z^{m-1} + \cdots + f_m(X,Y)$ where $f_j(X,Y)$ are homogeneous polynomials of degree j, $j = 0, \ldots, m$.

Then according to the definition of resultant in (1.2.1)

$$R(Q,F)(X,Y) = det \begin{pmatrix} q_0 & q_1 & q_2 & \ldots & q_d & 0 & 0 & & & \ldots \\ 0 & q_0 & q_1 & \ldots & & q_d & 0 & & \ldots & 0 \\ \ldots & & & \ldots & & & & & \ldots & \\ 0 & \ldots & \ldots & 0 & q_0 & q_1 & q_2 & & \ldots & q_d \\ f_0 & f_1 & f_2 & \ldots & \ldots & f_{m-1} & f_m & 0 & & \ldots \\ 0 & f_0 & f_1 & \ldots & \ldots & \ldots & f_{m-1} & f_m & 0 & \ldots \\ \ldots & & & \ldots & & & & & \ldots & \ldots \\ 0 & \ldots & f_0 & f_1 & \ldots & & \ldots & & f_{m-1} & f_m \end{pmatrix}. \qquad (1.2.46)$$

We multiply the second row by $\lambda \neq 0$, the third row by λ^2 and so on till the m-th row that is multiplied by λ^{m-1}. Then we multiply the $(m+2)$-th row by λ, the $(m+3)$-th by λ^2 and so on till the $(m+d)$-th that is multiplied by λ^{d-1} one has

$$\begin{aligned} R(Q,F)(\lambda X, \lambda Y) &= \frac{1}{\lambda^{\frac{1}{2}(d-1)d}\lambda^{\frac{1}{2}m(m-1)}} \\ &\times \det \begin{pmatrix} q_0 & \lambda q_1 & \ldots & \lambda^d q_d & 0 & 0 & & \ldots & 0 \\ 0 & \lambda q_0 & \lambda^2 q_1 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & & & \ldots & & & & \ldots & \\ 0 & 0 & \ldots & \ldots & \lambda^{m-1} q_0 & \lambda^m q_1 & \ldots & \ldots & \lambda^{d+m-1} q_d \\ f_0 & \lambda f_1 & \ldots & \ldots & \lambda^{m-1} f_{m-1} & \lambda^m f_m & 0 & \ldots & 0 \\ 0 & \lambda f_0 & \lambda^2 f_1 & \ldots & \ldots & \lambda^m f_{m-1} & \lambda^{m+1} f_m & \ldots & 0 \\ \ldots & & & \ldots & & & & \ldots & \\ 0 & \ldots & \lambda^{d-1} f_0 & \lambda^d f_1 & \ldots & & \ldots & \lambda^{m+d-2} f_{m-1} & \lambda^{m+d-1} f_m \end{pmatrix} \\ &= \lambda^{md} R(Q,F)(X,Y), \end{aligned}$$

where we use the fact that and $q_j(\lambda X, \lambda Y) = \lambda^j q_j(X,Y)$ and $f_j(\lambda X, \lambda Y) = \lambda^j f_j(X,Y)$. The above relation shows that the resultant $R(Q,F)(X,Y)$ is a homogeneous polynomial in X and Y of degree md. □

Theorem 1.2.32 (Bézout's theorem)**.** *Let C and $\mathcal{D}$ be two projective curves defined by the homogenous polynomials $Q(X,Y,Z)$ and $F(X,Y,Z)$ of degree d and m respectively. If C and $\mathcal{D}$ do not have common components then they intersect in dm points counted with multiplicity.*

Proof. By Lemma 1.2.3, C and $\mathcal{D}$ have a common component if and only if their resultant is identically zero. Consider the case in which C and $\mathcal{D}$ do not have common components. Without loss of generality we assume that $[0:0:1]$ does not belong to both curves. With this assumption $Q(X,Y,Z) = q_0(X,Y)Z^d + q_1(X,Y)Z^{d-1} + \cdots + q_d(X,Y)$ where $q_j(X,Y)$ are homogeneous polynomials of degree j in X and Y, $j = 0, \ldots, d$ and $q_0(0,0) \neq 0$. In the same way $F(X,Y,Z) = f_0(X,Y)Z^m + f_1(X,Y)Z^{m-1} + \cdots + f_m(X,Y)$ where $f_j(X,Y)$ are homogeneous polynomials of degree j, $j = 0, \ldots, m$ and $f_0(0,0) \neq 0$. Therefore the resultant is a homogeneous polynomial of degree md by lemma 1.2.31 and it has md zeros counting their multiplicity. □

Lemma 1.2.33. *If the projective curve C defined in (1.2.44) is non-singular, then the polynomial $Q(X,Y,Z)$ is irreducible. If C is irreducible, then it has at most a finite number of singular points.*

Proof. Let us suppose that the polynomial is reducible, namely $Q = Q_1 Q_2$ where Q_1 and Q_2 are homogeneous polynomials in X, Y and Z of degree d_1 and $d - d_1$. The condition of C being singular takes the form

$$Q_2 Q_1 = 0, \quad Q_2 \partial_X Q_1 + Q_1 \partial_X Q_2 = 0, \quad Q_2 \partial_Y Q_1 + Q_1 \partial_Y Q_2 = 0, \quad Q_2 \partial_Z Q_1 + Q_1 \partial_Z Q_2 = 0.$$

Such system of equations has always a solution as long as there is a point P in the intersection of the curves defined by $Q_1 = 0$ and $Q_2 = 0$. But this is always the case. Indeed let us consider the resultant $R(Q_1, Q_2)(X, Y)$ of the polynomials $Q_1(X, Y, Z)$ and $Q_2(X, Y, Z)$ with respect to Z. Assuming that $Q_1(0, 0, 1) \neq 0$ and $Q_2(0, 0, 1) \neq 0$ the resultant $R(Q_1, Q_2)(X, Y)$ is a homogeneous polynomial of degree $d_1(d - d_1)$. Therefore the curves defined by the equations $Q_1(X, Y, Z) = 0$ and $Q_2(X, Y, Z) = 0$ intersect by Bézout's theorem in $d_1(d - d_1)$ points counted with multiplicity. We conclude that if Q is reducible, then C is singular. Suppose that C is irreducible and defined by a polynomial Q of degree n. Then Q and Q_Z do not have a common component so that the resultant $R(Q, Q_Z)(X, Y)$ is a homogeneous polynomial of degree $n(n - 1)$ not identically zero. Since the singular points of C are contained among the zeros of the resultant, their number is finite. □

Example 1.2.34. The simplest example of a projective curve is a projective line $\mathbb{P}^1 \subset \mathbb{P}^2$ given by a linear equation

$$\alpha X + \beta Y + \gamma Z = 0$$

where $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Every such line is uniquely specified by the homogeneous coordinates $(\alpha : \beta : \gamma)$. We obtain an isomorphism

$$\{\text{lines in } \mathbb{P}^2\} \simeq \mathbb{P}^2.$$

A line in $\mathbb{P}^2$ is uniquely specified by a pair of points on it assuming the points to be in general position. In this case "general position" simply means that the points are distinct. In the multidimensional case we say that the points $P_1, \ldots, P_n$ in $\mathbb{P}^n$ are not in general position if there exists a subspace $\mathbb{P}^{n-2} \subset \mathbb{P}^n$ containing all these points.

Exercise 1.2.35: Prove that equation of the tangent line to a projective curve C defined by a homogeneous polynomial $Q(X, Y, Z)$ at a non-singular point (X_0, Y_0, Z_0) can be written in the form

$$X\, Q_X(X_0, Y_0, Z_0) + Y\, Q_Y(X_0, Y_0, Z_0) + Z\, Q_Z(X_0, Y_0, Z_0) = 0.$$

Example 1.2.36. The next example is a conic defined by a homogeneous equation of degree 2

$$C_A := \left\{ (X : Y : Z) \in \mathbb{P}^2 \mid |Q_A(X, Y, Z) = \begin{pmatrix} X & Y & X \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0 \right\}$$

where the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

In order to spell out the condition of smoothness it suffices to observe that the three partial derivatives Q_X, Q_Y, Q_Z are equal to $2A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ so that the condition of smoothness is

$$\det A \neq 0.$$

Now let O be a nonsingular 3×3 matrix. Then the matrix $B = O^t AO$ is a non singular matrix that defines the conic C_B as the zero locus of the polynomial equation

$$Q_B(X,Y,Z) = \begin{pmatrix} X & Y & X \end{pmatrix} O^t AO \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0.$$

Clearly the conic C_A and C_B are isomorphic, the isomorphism is the linear map $(X,Y,Z) \to (X,Y,Z)O^T$.

Exercise 1.2.37: Show that any conic is determined by five points belonging to it. Further show that five points in $\mathbb{P}^2$ uniquely determine a conic if their images w.r.t the Veronese map

$$\mathbb{P}^2 \to \mathbb{P}^5, \quad (X : Y : Z) \mapsto (X^2 : Y^2 : Z^2 : XY : YZ : ZX).$$

are in general position (see exercise 1.2.34).

Exercise 1.2.38: Prove that the tangent line to a smooth conic intersects with it only at the tangency point.

Exercise 1.2.39: Let $Q(X,Y,Z)$ be an irreducible homogeneous polynomial of degree d defining a smooth projective curve C. Suppose that the equation $Q(X,Y,1) = 0$ locally defines Y as a holomorphic function of X.

(1) Show that
$$\frac{d^2Y(X)}{dX^2} = \frac{1}{Q_Y^3} \det \begin{pmatrix} Q_{XX} & Q_{XY} & Q_X \\ Q_{YX} & Q_{YY} & Q_Y \\ Q_X & Q_Y & 0 \end{pmatrix}. \tag{1.2.47}$$

(2) A point $(X_0 : Y_0 : 1)$ is an inflection point for the curve C if and only if $\frac{d^2Y(X)}{dX^2}$ vanishes at X_0. Calculate the number of inflection points of the cubic defined by the homogeneous polynomial $Q(X,Y,Z) = Y^2Z - (X-Z)(X-aZ)X$ with $a \neq 0, 1$.

(3) Prove that a smooth point P of a projective curve is an inflection point iff it has multiplicity at least three as the intersection point of the curve with its tangent line at the point P.

(4) Prove that the tangent line at a smooth inflection point of a cubic has no other intersections with the curve but the tangency point.

(5) Prove that inflection points on the projective curve $Q(X,Y,Z) = 0$ can be determined by the hessian equation
$$\det \begin{pmatrix} Q_{XX} & Q_{XY} & Q_{XZ} \\ Q_{YX} & Q_{YY} & Q_{YZ} \\ Q_{ZX} & Q_{ZY} & Q_{ZZ} \end{pmatrix} = 0. \tag{1.2.48}$$
Derive that on any smooth plane cubic there are 9 distinct inflection points.

(6) Prove that the inflection points of the projectivization of the smooth elliptic curve $w^2 = 4z^3 - g_2 z - g_3$ are at the infinite point and at the points $(z_i, \pm w_i)$, $i = 1, \ldots, 4$ where z_i are the roots of the equation

$$48z^4 - 24g_2 z^2 - 48g_3 z - g_2^2 = 0.$$

Exercise 1.2.40: Let C be a smooth plane cubic and $P_0 \in C$ a point on it. (1) Prove that there exists a unique structure of an abelian group on C such that

- $P + Q + R = 0$ for any triple of points in the intersection of C with a line.
- $P_0 + P = P$ for any $P \in C$.

(2) Let $P \in C$ be such that the line tangent to C at the point P passes via P_0. Prove that P is an element of order 2 in the group.

(3) Prove that the inflection points of the curve have order 3 in the group.

Compactification of an affine plane curve

At the beginning of this Section we have seen that the intersection of a projective curve C in $\mathbf{P}^2$ with any of the open charts $U_i \simeq \mathbb{C}^2$ is an affine plane curve. For example

$$C_2 = C \cap U_2 = \{(z, w) \in \mathbb{C}^2 \mid Q(z, w, 1) = 0\}.$$

Clearly we can proceed also in the opposite direction. Namely given an affine plane curve C_2 in $\mathbb{C}^2$ defined by the polynomial equation $F(z, w) = 0$,

$$C_2 = \{(z, w) \in \mathbb{C}^2 \,|\, F(z, w) = 0\},$$

we can compactify such a curve in the projective space $\mathbf{P}^2$ in the following way. Let

$$F(z, w) = \sum_{i+j \leqslant k} a_{ij} z^i w^j.$$

Define the homogeneous polynomial of degree k by

$$Q(X, Y, Z) = Z^k F\left(\frac{X}{Z}, \frac{Y}{Z}\right). \tag{1.2.49}$$

A complex compact curve C is given in $\mathbf{P}^2$ by the homogeneous equation

$$C := \{(X : Y : Z) \in \mathbf{P}^2 \;\mid\; Q(X, Y, Z) = 0\}. \tag{1.2.50}$$

The affine part of the curve $C \cap U_2$ (where $Z \neq 0$) coincides with C_2. The projective curve C is compact and thus we have compactified the affine plane curve C_2 by adding the points at infinity given by the equation

$$Q(X, Y, 0) = 0. \tag{1.2.51}$$

Remark 1.2.41. Even if the curve C_2 is non-singular, the projective curve C might be singular.

Example 1.2.42. $C_2 = \{(z,w) \in \mathbb{C}^2 \mid w^2 = z\}$. A local parameter at the branch point $(z = 0, w = 0)$ is given by $\tau = \sqrt{z}$, i.e. $z = \tau^2$, $w = \tau$. The compactification C has the form $C = \{(X : Y : Z) \in \mathbf{P}^2 \mid Y^2 = XZ\}$. The point at infinity is given by solving the equation (1.2.51), that gives $P^\infty = [1 : 0 : 0]$. We determine the local coordinates near the point P^∞. For $X \neq 0$ we introduce the coordinates u, v

$$u = \frac{Y}{X} = \frac{w}{z}, \quad v = \frac{Z}{X} = \frac{1}{z}, \tag{1.2.52}$$

which define the affine curve $u^2 = v$. The point at infinity is given by $(v = 0, u = 0)$ which is clearly a ramification point for the curve defined by the equation $u^2 = v$ and $\sqrt{v}$ is a local parameter near this point. Therefore a parametrization of the C in a neighbourhood of P^∞ takes the form

$$z = \frac{1}{u^2}, \quad w = \frac{1}{u}.$$

Example 1.2.43. $C_2 = \{w^2 = z^2 - a^2\}$, $a \neq 0$. The branch points are $(z = \pm a, w = 0)$ and the corresponding local parameters are $\tau_\pm = \sqrt{z \pm a}$. The compactification is the conic $C = \{Y^2 = X^2 - a^2 Z^2\}$. The points at infinity are given by solving the equation (1.2.51), that gives $P^\infty_\pm = [1 : \pm 1 : 0]$. Making the substitution (1.2.52) we get the form of the curve C in a neighbourhood of the ideal line: $u^2 = 1 - a^2 v^2$. For $v = 0$ we get that $u = \pm 1$. We can take $v = 1/z$ as a local parameter in a neighbourhood of each of these points. The form of the surface C in a neighbourhood of these points $P_\pm$ is as follows:

$$z = \frac{1}{v}, \quad w = \pm\frac{1}{v}\sqrt{1 - a^2 v^2}, \quad v \to 0 \tag{1.2.53}$$

where $\sqrt{1 - a^2 v^2}$ is, for small v, a single-valued holomorphic function, and the branch of the square root is chosen to have value 1 at $v = 0$.

Example 1.2.44. Let us consider the class of hyperelliptic Riemann surfaces

$$C_2 = \{(z,w) \in \mathbb{C}^2 \mid F(z,w) = w^2 - P_N(z) = 0\}, \tag{1.2.54}$$

where $P_N(z) = \prod_{j=1}^N (z - a_j)$, and $a_i \neq a_j$ for $i \neq j$.

If we consider the projective curve defined by the zeros of homogeneous polynomial

$$Q(X, Y, Z) = Y^2 Z^{N-2} - Z^N P_N(X/Z) = 0$$

one can check that the curve is singular at the point $[0 : 1 : 0]$ if $N \geqslant 4$. Therefore, for $N \geqslant 4$, the embedding of C_2 in $\mathbf{P}^2$ results in a singular surface. For $N = 3$ the projective curve

$$Y^2 Z = (X - a_1 Z)(X - a_2 Z)(X - a_3 Z)$$

is a compact smooth elliptic curve. By a projective transformation such curve can be reduced to the form

$$Y^2 Z = X(X - Z)(X - \lambda Z), \quad \lambda \in C \backslash \{0, 1\}.$$

The point at infinity is given by $P^\infty = [0 : 1 : 0]$. For $Y \neq 0$ the substitution $u = X/Y$ and $v = Z/Y$ gives the curve

$$Q(u, 1, v) = v - u(u - v)(u - \lambda v) = 0$$

The point $(0,0)$ is a branch point for the above curve. Indeed for $(u,v) \neq 0$ the projection $\pi : (u,v) \to v$ is a local coordinate. The preimage $\pi^{-1}(v)$ consists of three points. At the point $(0,0)$ one has $Q_u(0,1,0) = 0$ and $Q_{uu}(0,1,0) = 0$ so that the preimage of $\pi^{-1}(0)$ consists of a single point. Therefore a local coordinate near the point $(0,0)$ takes the form

$$u = \tau(1 + o(\tau)), \quad v = \tau^3(1 + o(\tau)).$$

We look for the holomorphic tail of the above expansions in the form

$$u = \tau g(\tau), \quad v = \tau^3 g(\tau)$$

with $g(\tau)$ analytic and invertible in a neighbourhood of $\tau = 0$. Plugging the above ansatz in the equation $Q(u,1,v) = v - u(u-v)(u-\lambda v) = 0$ one obtains that

$$g(\tau) = \frac{1}{\sqrt{(1-\tau^2)(1-\lambda\tau^2)}}.$$

Since

$$z = \frac{X}{Z} = \frac{u}{v}, \quad w = \frac{Y}{Z} = \frac{1}{v}$$

one has that a local coordinate near the point at infinity for the curve C is given by

$$z = \frac{1}{\tau^2}, \quad w = \frac{1}{\tau^3}\sqrt{(1-\tau^2)(1-\lambda\tau^2)}.$$

The above examples show that a smooth affine plane curve can sometimes be made into a compact Riemann surface by embedding the affine curve into the projective space. In general such embedding produces a singular projective curve that can still be turned into a compact Riemann surface once the problems with the singular points have been fixed. In the next Section we will show how to do it by using simple topological arguments about covering spaces.

1.3 Compact Riemann surfaces: a topological viewpoint

1.3.1 Topological digression: coverings, fundamental group and monodromy

Let X, Y be two topological spaces and $p : X \to Y$ a surjective continuous map. We additionally assume the space Y to be connected[7].

Definition 1.3.1. *The triple (X, Y, p) is called a covering if for any point $P \in Y$ there exists an open neighbourhood $U_P \ni P$ such that the preimage $p^{-1}(U_P)$ is a disjoint union of open subsets $U_\alpha \subset X$, $\alpha \in F$ such that the restriction of p onto U_α is a homeomorphism $p : U_\alpha \to U_P$ for any $\alpha \in F$. Here F is an at most countable discrete set.*

X is called the covering space, Y the base of the covering, p the covering map. The set F can be naturally identified with the preimage $p^{-1}(P)$ of the point P. It is called the fiber over P.

[7] Recall that in this book 'connected' means 'path-connected', cf. footnote 3.

Our first claim is that the fiber over P does not depend on P. Indeed, let Q be another point in the base. If the intersection of U_P with U_Q is not empty then there is an obvious one-to-one correspondence between the fibers over P and over Q. In general we connect the points by a path

$$\gamma : [0,1] \to Y, \quad \gamma(0) = P, \ \gamma(1) = Q.$$

Due to compactness of the path there exists a finite sequence $t_1 = 0 < t_2 < \cdots < t_N = 1$ such that the open domains $U_{\gamma(t_i)}$, $i = 1, \ldots, N$ cover the path $\gamma([0,1])$. Then passing from $U_{\gamma(t_i)}$ to $U_{\gamma(t_{i+1})}$ step by step we obtain a one-to-one correspondence between the fibers over the end points.

If the fiber is a finite set of n points then we say that the covering is of degree n or also n-sheeted covering.

Example 1.3.2. The Cartesian product $X = Y \times F$ for an arbitrary discreet set F with the covering map $p(P, \alpha) = P$ is an example of trivial covering.

Definition of covering

Definition 1.3.3. *Two coverings (X, Y, p) and (X', Y, p') are called equivalent if there exists a homeomorphism $f : X \to X'$ such that $p' \circ f = p$. A covering equivalent to the trivial one will itself be called trivial.*

Exercise 1.3.4: Let (X, Y, p) be a covering of degree $n \neq 1$ with connected covering space X. Prove that it is not trivial.

Example 1.3.5. Define a map of the punctured disk $\dot{D} = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ to itself by

$$p(z) = z^n.$$

This is a covering of degree n.

Example 1.3.6. The map

$$p : \mathbb{C} \to \mathbb{C}^*, \quad p(z) = e^z$$

is a covering. The fiber can be identified with the set of integers since $e^{2\pi i n} = 1$ for any $n \in \mathbb{Z}$.

Before we proceed to further constructions from the theory of coverings we need to recall the notion of homotopy. It formalizes the idea of deformations of continuous maps between topological spaces.

Definition 1.3.7. *Let X, Y be two topological spaces and f_0, $f_1 : X \to Y$ two continuous maps. These maps are homotopic if there exists a continuous map $F : X \times [0,1] \to Y$ called homotopy between f_0 and f_1 such that*

$$F(P, 0) = f_0(P), \quad F(P, 1) = f_1(P) \quad \forall\, P \in X.$$

We will use notation $f_0 \sim f_1$ for homotopic maps. Clearly it is an equivalence relation.

Example 1.3.8. A path on a topological space is a continuous map of the segment $[0,1]$ to this space. Let $\gamma_{0,1} : [0,1] \to Y$ be two paths on the topogical space Y. A homotopy between these paths is a continuous map of the square $\Gamma : [0,1] \times [0,1] \to Y$ such that

$$\Gamma(t, 0) = \gamma_0(t), \quad \Gamma(t, 1) = \gamma_1(t) \quad \forall\, 0 \leqslant t \leqslant 1.$$

In the particular case where the end points of the two paths coincide

$$\gamma_0(0) = \gamma_1(0) = P, \quad \gamma_0(1) = \gamma_1(1) = Q$$

for $P, Q \in Y$ it is convenient to consider homotopies with fixed end points imposing the following boundary conditions

$$\Gamma(0,s) = P, \quad \Gamma(1,s) = Q \quad \forall\, 0 \leqslant s \leqslant 1.$$

In the more specific case $P = Q$ we are dealing with loops on the space Y with the base point P. In this case Γ is a homotopy between the two loops with fixed base point.

We now return to coverings.

Lemma 1.3.9. *Let (X, Y, p) be a covering and $\gamma : [0,1] \to Y$ be a path on the base of the covering.*
1. Then for any $\hat{P} \in p^{-1}(\gamma(0))$ there exists a unique path $\hat{\gamma} : [0,1] \to X$ on the covering space with prescribed initial point $\hat{\gamma}(0) = \hat{P}$ such that $p(\hat{\gamma}(t)) = \gamma(t)$ for all $t \in [0,1]$. The path $\hat{\gamma}$ is called the lift of γ with prescribed initial point.

2. Let $\gamma_0 : [0,1] \to Y$ and $\gamma_1 : [0,1] \to Y$ be two homotopic paths on the base with the same initial and end points. Denote $\hat{\gamma}_0 : [0,1] \to X$, $\hat{\gamma}_0 : [0,1] \to X$ their lifts with the same initial point $\hat{\gamma}_0(0) = \hat{\gamma}_1(0)$.Then

$$\hat{\gamma}_0(1) = \hat{\gamma}_1(1).$$

Proof Let us first assume that the entire path γ belongs to the open domain $U_P \subset Y$ from the definition of a covering where $P = \gamma(0)$. Denote $\hat{U}_P \subset X$ the component of the preimage $p^{-1}(U_P)$ containing $\hat{P}$. Then the lift is obtained by

$$\hat{\gamma}(t) = \left(p\,|_{\hat{U}_P}\right)^{-1}(\gamma(t)). \tag{1.3.1}$$

In the general case we split $[0,1]$ in small segments $[t_{i-1}, t_i]$, $i = 1, \dots, N$, $t_0 = 0$, $t_N = 1$ such that $\gamma_i := \gamma|_{[t_{i-1},t_i]} \subset U_{\gamma(t_{i-1})}$. Such spltting always exists due to compactness of the segment $[0,1]$. Then, following the above procedure we construct the lift of γ_1 with the initial point $\hat{P}$, the lift of γ_2 with the initial point = the end point of γ_1 etc.

Let us now consider a homotopy $\Gamma(t,s)$ between the paths γ_0 and γ_1 with fixed end points

$$\Gamma(t,0) = \gamma_0(t), \quad \Gamma(t,1) = \gamma_1(t), \quad \Gamma(0,s) \equiv P, \quad \Gamma(1,s) \equiv Q.$$

We represent it as a family of curves depending on the parameter $s \in [0,1]$

$$\gamma_s(t) = \Gamma(t,s), \quad t \in [0,1].$$

All these curves have their initial point at P and the end point at Q. Denote $\hat{\gamma}_s : [0,1] \to X$ the lift of the path γ_s with the initial point $\hat{P}$ and define a map $\hat{\Gamma} : [0,1] \times [0,1] \to X$ by

$$\hat{\Gamma}(t,s) = \hat{\gamma}_s(t).$$

By definition it satisfies

$$p \circ \hat{\Gamma} = \Gamma$$

Let us prove continuity of this map. First, it is continuous for sufficiently small t. Indeed, since $\Gamma(0,s) \equiv P$ there exists $\epsilon > 0$ such that $\Gamma(t,s) \in U_P$ for $0 \leqslant t < \epsilon$ and any $s \in [0,1]$. So the lift of the curves $\gamma_s(t)$ for $0 \leqslant t < \epsilon$ can be obtained by

$$\hat{\gamma}_s(t) = \left(p\,|_{\hat{U}_P}\right)^{-1}(\gamma_s(t))$$

(cf. eq. (1.3.1)) hence the continuity on $[0,\epsilon) \times [0,1]$.

Suppose that the set of points $(t,s) \in [0,1] \times [0,1]$ where $\hat{\Gamma}$ fails to be continuous is non-empty. Denote t_0 the lower bound of those values of t for which $\hat{\Gamma}$ is not continuous for some $s = s_0$. We already know that $t_0 \geqslant \epsilon > 0$. Denote $R = \Gamma(t_0, s_0) = \gamma_{s_0}(t_0)$, $\hat{R} = \hat{\Gamma}(t_0, s_0) = \hat{\gamma}_{s_0}(t_0)$. Let $U_R \subset Y$ and $\hat{U}_R \subset X$ be open neighbourhoods of the points R and $\hat{R}$ respectively such that the map $p : \hat{U}_R \to U_R$ is a homeomorphism. Choose $\epsilon' > 0$ such that $\Gamma(t,s) \in U_R$ for $|t - t_0| < \epsilon'$, $|s - s_0| < \epsilon'$. As the curve $\hat{\gamma}_{s_0}(t)$ passes through the point $\hat{R}$ it must have the form

$$\hat{\gamma}_{s_0}(t) = \left(p\,|_{\hat{U}_R}\right)^{-1}(\gamma_{s_0}(t))$$

for $|t - t_0| < \epsilon'$. Take $t_0 - \epsilon' < t_1 < t_0$ so that $\hat{\gamma}_{s_0}(t_1) \in \hat{U}_R$. The map $\hat{\Gamma}$ is continuous at the point (t_1, s_0). So there exists $\delta > 0$ such that $\hat{\gamma}_s(t_1) = \hat{\Gamma}(t_1, s) \in \hat{U}_R$ for $|s - s_0| < \delta$. Assume additionally that $\delta \leqslant \epsilon'$. Then

$$\hat{\Gamma}(t,s) = \left(p\,|_{\hat{U}_R}\right)^{-1}(\Gamma(t,s))$$

for $|t - t_0| < \epsilon'$, $|s - s_0| < \delta$ hence it is continuous in this region. This contradicts the assumption about (t_0, s_0). Thus $\hat{\Gamma}$ is continuous everywhere on $[0,1] \times [0,1]$.

It remains to prove that $\hat{\Gamma}(1,s) \equiv \hat{Q}$ for $s \in [0,1]$. Indeed, $\hat{\Gamma}(1,s)$ is a continuous path but it must belong to $p^{-1}(\Gamma(1,s)) = p^{-1}(Q)$. The latter set is discrete hence $\hat{\Gamma}(1,s) \equiv \hat{\gamma}_0(1) = \hat{Q}$ that implies that $\hat{\Gamma}(1,1) = \hat{\gamma}_1(1) = \hat{\gamma}_0(1)$. □

Remark 1.3.10. The above Lemma is a particular case of the Covering Homotopy Theorem. Namely, given a covering (X, Y, p) and two continuous maps $f : Z \to Y$ and $\hat{f} : Z \to X$ of a topological space Z in a suitable class satisfying $p \circ \hat{f} = f$ and, moreover, a homotopy

$$F : Z \times [0,1] \to Y, \quad F\,|_{Z\times\{0\}} = f$$

then there exists a unique covering homotopy

$$\hat{F} : Z \times [0,1] \to X \quad \text{satisfying} \quad \hat{F}\,|_{Z\times\{0\}} = \hat{f}.$$

For the proof see e.g. [26].

Using the operation of lifting paths from the base of a covering to the covering space we now define monodromy transformations acting on the fiber over a given point in the base.

Definition 1.3.11. *Let (X, Y, p) be a covering and $P_0 \in Y$ a point on the base. Monodromy transformations are bijections σ_γ of the fiber $F = p^{-1}(P_0)$ defined for any loop $\gamma : [0,1] \to Y$ with $\gamma(0) = \gamma(1) = P_0$. Namely, for any given point $Q \in p^{-1}(P_0)$ we put $\sigma_\gamma(Q) = Q' \in p^{-1}(P_0)$ if $Q' = \hat{\gamma}(1)$ is the end point of the lift $\hat{\gamma}$ of the loop γ with the initial point $\hat{\gamma}(0) = Q$.*

Due to Lemma the monodromy transformation σ_γ depends only on the homotopy class of the loop γ with the base point P_0. To put this observation into a proper algebraic setting we need to recall the definition of fundamental group of a topological space.

Elements of the fundamental group $\pi_1(Y, P_0)$ are equivalence classes of loops $\gamma : [0,1] \to Y$, $\gamma(0) = \gamma(1) = P_0$ wrt homotopies with fixed base point P_0. The product of two loops γ_1, γ_2 is the loop that we denote $\gamma_1\gamma_2$ defined in the following way

$$(\gamma_1\gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}.$$

The inverse of a loop γ is the same loop run in the opposite direction

$$\gamma^{-1}(t) := \gamma(1-t), \quad t \in [0,1].$$

The unit of the group is the homotopy class of the constant loop $\gamma(t) \equiv P_0$.

According to the definition the fundamental group depends on the choice of the base point. But for a connected space Y the fundamental groups $\pi_1(Y, P_0)$ and $\pi_1(Y, Q_0)$ are isomorphic for any pair of points P_0, Q_0. An isomorphism $\pi_1(Y, Q_0) \to \pi_1(Y, P_0)$ is established by choosing a path from P_0 to Q_0. It depends only on the homotopy class of the path with fixed end points.

Example 1.3.12. The fundamental group of the unit disk $\mathbb{D} = \{|z| < 1\}$ is trivial, $\pi_1(\mathbb{D}, \{0\}) = 1$. A homotopy of a loop with the base point 0 to the trivial one can be obtained by using the contraction $z \mapsto t\,z$, $t \in [0,1]$ of the unit disk to the central point. In a similar way the complex plane can be contracted to one point so $\pi_1(\mathbb{C}, \{0\}) = 1$.

Contractions appear as the simplest example of homotopy equivalence between topological spaces.

Definition 1.3.13. *Two topological spaces X and Y are homotopy equivalent if there are two continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.*

Homeomorphic spaces are homotopy equivalent but not vice versa. The simplest example is the unit disk an the space consisting of one point. The map $f : \mathbb{D} \to \mathrm{pt}$ maps the disk to the point and $g : \mathrm{pt} \to \{0\} \in \mathbb{D}$ is an embedding of the point in the disk. The superposition $f \circ g$ is the identity map of the point to itself. The homotopy $F : \mathbb{D} \times [0,1] \to \mathbb{D}$ between $g \circ f$ and the identity map $\mathbb{D} \to \mathbb{D}$ can be constructed as $F(z,t) = t\,z$.

Exercise 1.3.14: Prove that the punctured plane $\mathbb{C}^*$ is homotopy equivalent to the unit circle $S^1 = \{|z| = 1\}$.

Exercise 1.3.15: Let $f : X \to Y$ be a continuous map of topological spaces. Choose a point $P_0 \in X$ and let $Q_0 = f(P_0)$. Define a map

$$f_* : \pi_1(X, P_0) \to \pi_1(Y, Q_0), \quad f_*\gamma(t) = f(\gamma(t)).$$

Prove that the map f_* is well defined and it is a group homomorphism.

Exercise 1.3.16: Let (X, Y, p) be a covering. Choose a point $P_0 \in Y$ in the base and let $Q_0 \in p^{-1}(P_0)$. Prove that

$$p_* : \pi_1(X, Q_0) \to \pi_1(Y, P_0)$$

is a monomorphism.

Exercise 1.3.17: Let the maps $f : X \to Y$ and $g : Y \to X$ establish a homotopy equivalence between the spaces X and Y. Prove that the fundamental groups $\pi_1(X, P_0)$ and $\pi_1(Y, Q_0)$ where $Q_0 = f(P_0)$ are isomorphic.

Definition 1.3.18. *A connected topological space is called simply connected if its fundamental group consists only of the unit element.*

In other words, every loop on a simply connected space is homotopic to the constant one.

The Riemann surfaces $\mathbb{C}$, $\mathbb{D}$, $\mathbb{H}$ are all simply connected. More generally, if a space is homotopy equivalent to a point then it is simply connected. The Riemann sphere $\mathbb{P}^1$ is an example of simply connected space that is not homotopy equivalent to a point, see [26] for the proof.

Example 1.3.19. The simplest example of a non-simply connected space is the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Take the loop

$$\gamma : [0,1] \to S^1, \quad \gamma(t) = e^{2\pi i t}, \quad \gamma(0) = \gamma(1) = 1.$$

The homotopy classes of this loop and of its powers

$$\gamma^n(t) = e^{2\pi i n t}, \quad n \in \mathbb{Z}$$

are pairwise distinct and any other loop on S^1 is homotopic to one of these. Thus

$$\pi_1(S^1, \{1\}) \simeq \mathbb{Z}.$$

Proofs of the above statements using universal coverings will be given below, see Example 1.3.29.

Let us return to monodromy transformations. Recall that for any loop γ on the base Y of the covering (X, Y, p) with the base point P_0 we have constructed a bijection of the fiber $F = p^{-1}(P_0)$ onto itself. Denote it $\mu(\gamma) \in Aut(F)$. The monodromy transformation $\mu(\gamma)$ depends only on the homotopy class of the loop γ with fixed base point P_0. We obtain a map

$$\mu : \pi_1(Y, P_0) \to Aut\,(F)\,. \tag{1.3.2}$$

Here and below $Aut(F)$ is the set of bijections $F \to F$. It has a natural group structure defined by superposition of bijections.

Proposition 1.3.20. *The map (1.3.2) is an anti-homomorphism of the groups that is, for any $a,\ b \in \pi_1(Y, P_0)$ we have*

$$\mu(ab) = \mu(b)\mu(a). \tag{1.3.3}$$

Proof easily follows from the above definitions. □

We will often omit "anti" if there is no confusion.

Definition 1.3.21. *The (anti)homomorphism (1.3.2) is called the monodromy of the covering (X, Y, p).*

Example 1.3.22. To compute the monodromy of the covering $p : \dot{D} \to \dot{D}$ of the punctured disk $\dot{D} = \{0 < |z| < 1\}$ to itself, $p(z) = z^n$ (see Example 1.3.5 above) we have to study the behaviour of branches of the algebraic function $p^{-1}(z) = \sqrt[n]{z}$ under analytic continuation along the loop $\gamma(t) = r\,e^{2\pi i t}$, $0 \leqslant t \leqslant 1$ for some $0 < r < 1$ that is a generator of the fundamental group $\pi_1(\dot{D}, \{r\}) = \mathbb{Z}$. Choose the first branch in such a way that

$$\left(p^{-1}(\gamma(t))\right)_1 = r^{1/n} e^{\frac{2\pi i t}{n}}.$$

Passing from $t = 0$ to $t = 1$ we obtain the second branch

$$\left(p^{-1}(\gamma(t))\right)_2 = r^{1/n} e^{\frac{2\pi i (t+1)}{n}}$$

and so on up to the n-th branch

$$\left(p^{-1}(\gamma(t))\right)_n = r^{1/n} e^{\frac{2\pi i (t+n-1)}{n}}.$$

One more step in the analytic continuation takes us back to the first branch. We conclude that, with the chosen labelling of the branches the monodromy $\mu(\gamma)$ is the cyclic permutation $1 \mapsto 2$, $2 \mapsto 3, \ldots, n-1 \mapsto n$, $n \mapsto 1$. In the theory of symmetric group S_n such a permutation is called a cycle of length n. It is denoted by $(12\ldots n) \in S_n$.

Example 1.3.23. For the covering $p : \mathbb{C} \to \mathbb{C}^*$, $p(z) = e^z$ over the punctured complex plane (see Example 1.3.6 above) we have $p^{-1}(z) = \log z$. Using the well known formula $\log(ze^{2\pi i}) = \log z + 2\pi i$ we conclude that monodromy $\mu(\gamma)$ along the generator $\gamma(t) = e^{2\pi i t}$ of the fundamental group $\pi_1(\dot{D}, \{1\}) = \mathbb{Z}$ acts on the fiber $F = \mathbb{Z}$ by shifts $n \mapsto n+1 \ \forall\, n \in \mathbb{Z}$.

Definition 1.3.24. *The monodromy representation (1.3.2) is called reducible if there exists a nonempty subset in the fiber $p^{-1}(P_0)$ different from the fiber itself and invariant wrt to the image of μ. Otherwise it is called irreducible.*

It is easy to see that reducibility/irreducibility of monodromy does not depend on the choice of the point P_0 in the base of the covering.

Remark 1.3.25. Irreducibility of the monodromy representation μ implies that the bijections from the image of μ act transitively on the fiber, and vice versa. Recall that action of a group on a set is called transitive if for any pair of points x, y in the set there exists an element of the group that maps x to y.

Let (X, Y, p) be a covering of finite degree n and μ its monodromy representation.

Lemma 1.3.26. *The covering space X is connected if and only if the monodromy (1.3.3) is irreducible.*

Proof. Assuming irreducibility of the monodromy let us prove the connectivity of X. It suffices to prove that any point $Q_0 \in p^{-1}(P_0)$ can be connected with any other point $Q \in p^{-1}(P)$ for an arbitrary $P \in Y$. To this end let us choose a path $\gamma \subset Y$ in the base from P to P_0. Denote $\hat{\gamma}$ the lift of γ to the covering space X with the initial point Q. Denote $\tilde{Q} \in p^{-1}(P_0)$ the final point of $\hat{\gamma}$. Due to transitivity of the monodromy there exists a loop $\delta \subset Y$ starting and ending at P_0 such that its lift $\hat{\delta}$ that begins from $\tilde{Q}$ has Q_0 as its end point. The composition $\hat{\gamma}\,\hat{\delta}$ connects Q with Q_0.

Conversely, assume that X is connected. Choose a pair of points Q_0, $Q_0' \in p^{-1}(P_0)$. Let $\sigma \subset X$ be a path connecting Q_0 with Q_0'. The projection $p(\sigma)$ is a loop on Y such that the monodromy $\mu\,(p(\sigma))$ interchange Q_0 with Q_0'. $\square$

Definition 1.3.27. *A covering (X, Y, p) is called universal if the covering space X is connected and simply connected.*

Let (X, Y, p) be a universal covering. We will now establish a one-to-one correspodence between points of its fiber and elements of the fundamental group of the base.

Proposition 1.3.28. *Choose a point $P_0 \in Y$ on the base of the universal covering and another point $Q_0 \in X$ satisfying $p(Q_0) = P_0$. For any loop $\gamma \in \pi_1(Y, P_0)$ define a point $Q_\gamma \in p^{-1}(P_0)$ by the monodromy action on Q_0*

$$Q_\gamma = \mu(\gamma)(Q_0).$$

The map

$$\pi_1(Y, P_0) \to p^{-1}(P_0), \quad \gamma \mapsto Q_\gamma \tag{1.3.4}$$

is one-to-one. It satisfies

$$\mu(\gamma_1)(Q_{\gamma_2}) = Q_{\gamma_2\gamma_1}. \tag{1.3.5}$$

Proof Let $Q \in p^{-1}(P_0)$ be any point. It can be connected with Q_0 by a path $\hat{\gamma}$, $\hat{\gamma}(0) = Q_0$, $\hat{\gamma}(1) = Q$. The projection $\gamma = p(\hat{\gamma})$ is a loop on the base with the base point P_0. Then $Q = Q_\gamma$. So the map (1.3.4) is surjective.

Let us now prove injectivity of the map (1.3.4). Suppose $Q_\gamma = Q_0$ for some $\gamma \in \pi_1(Y, P_0)$. That means that the lift $\hat\gamma \subset X$ of the loop γ with the initial point Q_0 returns to Q_0, i.e. it is a loop. As the covering space X is simply connected the loop $\hat\gamma$ is homotopic to the constant one. Projecting this homotopy to Y we obtain a homotopy between γ and the constant loop. That is, $\gamma \sim e$.

The last point is about eq. (1.3.5). It easily follows from the property (1.3.3) of the monodromy representation. □

Example 1.3.29. The covering

$$p : \mathbb{R} \to S^1, \quad x \mapsto e^{2\pi i x}$$

of the real line over the unit circle $\{|z| = 1\} \subset \mathbb{C}$ is universal. The fiber over the point $z = 1$ consists of all integers $\mathbb{Z} \subset \mathbb{R}$. Use this point as a marked point P_0 on the base and choose the point $x = 0$ as the marked point Q_0 in the fiber over P_0. For the loop $\gamma(t) = e^{2\pi i t}$, $t \in [0, 1]$ we have $p^{-1}(\gamma(t)) = t$ as the lift starting at Q_0. Hence $Q_\gamma = 1$. In a similar way for the n-th power of the loop γ

$$\gamma^n(t) = e^{2\pi i n t}, \quad n \in \mathbb{Z}$$

we have $Q_{\gamma^n} = n$. This gives a one-to-one correspondence between the infinite cyclic group generated by the loop γ and the fiber over the point $z = 1$. According to Proposition this implies that the fundamental group of the circle coincides with this infinite cyclic group i.e., $\pi_1(S^1, \{1\}) \simeq \mathbb{Z}$.

Exercise 1.3.30: Compute the fundamental group of the n-dimensional torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n times).

Example 1.3.31. The covering $p : \mathbb{C} \to \mathbb{C}^*$, $p(z) = e^z$ over the punctured complex plane (see Example 1.3.6) is universal. Repeating the arguments from the previous Example we conclude that $\pi_1(\mathbb{C}^*, \{1\}) \simeq \mathbb{Z}$. This is not a big surprise as the punctured complex plane is homotopy equivalent to the unit circle; we leave as an exercise for the reader to construct explicitly such a homotopy equivalence.

The story becomes much more involved for the complex plane with two or more punctures. The fundamental group of the complex plane with K punctures is the free group with K generators. We will denote it by $\mathcal{F}_K$. Elements of the group are words made of symbols $a_1, \dots, a_K$ and a_1^{-1}, $\dots$, a_K^{-1}. The product of two words is defined by concatenation: we write the first word on the left then continue with the second one on the right. One rule is to be imposed: having in a word two neighboring symbols $a_i a_i^{-1}$ or $a_i^{-1} a_i$ we just erase them. For example,

$$a_1 a_2^{-1} a_1 \times a_1^{-1} a_2 a_1 = a_1 a_1.$$

The unit is the empty word and the inverse to the word $a_{i_1}^{\pm} \dots a_{i_s}^{\pm}$ is $a_{i_s}^{\mp} \dots a_{i_1}^{\mp}$. For $K = 1$ one obtains the infinite cyclic group $\mathcal{F}_1 \simeq \mathbb{Z}$. For $K \geqslant 2$ the group $\mathcal{F}_K$ is non-abelian.

So, the claim is that

$$\pi_1\left(\mathbb{C} \setminus (\{z_1\} \cup \dots \cup \{z_K\}), z_*\right) \simeq \mathcal{F}_K \tag{1.3.6}$$

where $z_i \neq z_j$ for $i \neq j$, $z_* \neq z_i$ for any $i = 1, \dots, K$. To establish the isomorphism (1.3.6) we choose loops $\alpha_1, \dots, \alpha_K$ on the punctured plane as follows. The loops must have no pairwise intersections either self-intersections except for the common point z_*; the loop α_i has inside only one puncture

z_i; it goes around it in the anticlockwise direction. The homotopy classes of these loops correspond to the generators of the free group

$$\pi_1\left(\mathbb{C}\backslash(\{z_1\}\cup\cdots\cup\{z_K\}),z_*\right)\ni\alpha_i\leftrightarrow a_i\in\mathcal{F}_K.$$

To justify the above statement for $K\geqslant 2$ we will construct a universal covering over the complex plane with K punctures and describe the action of the fundamental group on the fiber. We will do like it was done above for the case of complex plane with one puncture, namely, we replace the complex plane with K punctures by a homotopy equivalent space that is bouquet of K circles, see Figure 1.9 for the case $K=2$. Construction of the homotopy equivalence is left as an exercise to the reader. The fundamental groups of the punctured plane and of the bouquet are isomorphic so we will be computing the latter.

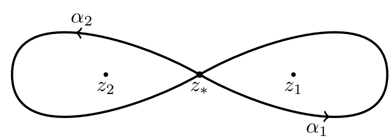


Figure 1.9: Generators of the fundamental group of complex plane with two punctures

The bouquet is not a manifold for $K\geqslant 2$. Nevertheless Proposition 1.3.28 remains valid also in this case. The universal cover of the bouquet of K circles is an infinite graph with no cycles (such graphs are called trees) with all vertices of valency $2K$, see Figure[8] 1.10 for $K=2$. The edges of the graph are oriented and labelled by symbols $a_1,\ldots,a_K$ (on Figures 1.9 and 1.10 for $K=2$). At every vertex there are K incoming edges labelled by $a_1,\ldots,a_K$ and K outgoing edges with the same labels. The covering map from the graph to the bouquet of oriented circles acts as follows: the vertices of the graph go to the common point z_*, any edge labelled by a_i goes to the i-th circle according to the orientation. Choose a vertex of the graph. Then the lift of a product $\alpha_{i_1}^{\pm}\ldots\alpha_{i_s}^{\pm}$ of s loops with the initial point at the marked vertex will be a walk of length s on the graph that starts from the marked vertex and goes successively along the edges $a_{i_1}^{\pm},\ldots,a_{i_s}^{\pm}$ in positive or negative directions according to the signs $\pm$. The isomorphism (1.3.6) readily follows from this description of the lift.

Exercise 1.3.32: Compute the fundamental group of the Riemann sphere with K punctures.

Remark 1.3.33. From Theorem 1.3.34 it follows that the universal covering of the complex plane with K punctures is a simply connected Riemann surface. So, according to the Uniformization Theorem (see Section 1.1.2 above) it must be biholomorphically equivalent to the one of three: $\mathbb{P}^1$, $\mathbb{C}$ or the upper half plane $\mathbb{H}$. For $K=1$ we already know that the universal covering of the punctured complex plane is $\mathbb{C}$. It turns out that the universal covering of the complex plane with $K\geqslant 2$ punctures is $\mathbb{H}$. For more details see below Section **??**.

[8] This graph is perhaps the most known example of Cayley graphs. There are many ways to draw this graph; we have chosen the one found in the book by W.Fulton [26] as the most appropriate one to illustrate ideas of topology of coverings.

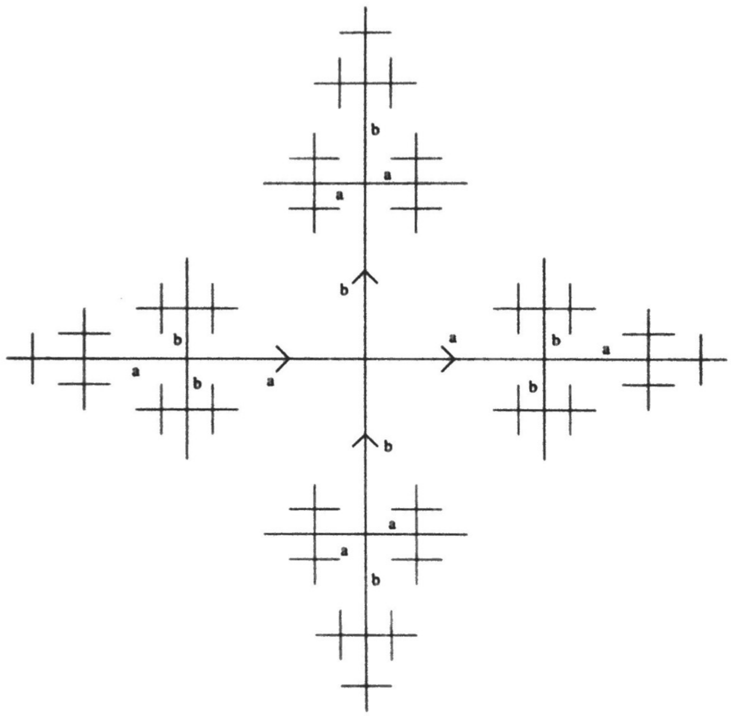


Figure 1.10: Universal covering of figure-eight

Theorem 1.3.34. *Let M be a smooth connected manifold. Then there exists a smooth manifold $\hat{M}$ and a smooth locally diffeomorphic map $p : \hat{M} \to M$ such that the triple $(\hat{M}, M, p)$ is a universal covering. Such a covering is unique up to an equivalence in the sense of Definition 1.3.3. A similar statement holds true for connected complex manifolds M. Then the covering space $\hat{M}$ is a complex manifold as well and the covering map p is holomorphic and locally biholomorphic.*

Proof Let $P_0 \in M$ be an arbitrary point. We define $\hat{M}$ as the set of equivalence classes of paths $\gamma : [0, 1] \to M$ such that $\gamma(0) = P_0$. Two paths γ and $\tilde{\gamma}$ are called equivalent if $\tilde{\gamma}(1) = \gamma(1)$ and they are homotopic with fixed end points. Denote $[\gamma]$ the equivalence class of a path γ. The map $p : \hat{M} \to M$ is defined as follows

$$p([\gamma]) = \gamma(1).$$

Observe that the preimage $p^{-1}(P_0)$ of the point P_0 can be naturally identified with the fundamental group $\pi_1(M, P_0)$. Therefore it is at most countable as it follows from the following Lemma.

Lemma. *The fundamental group of any manifold is at most countable.*

For the proof see e.g. [21].

We now continue the proof of Theorem 1.3.34 by introducing a topology on the set $\hat{M}$. For any point $P \in M$ we define a family of admissible pairs (U, ϵ) where U is a chart of an atlas on M with local coordinates $x_1, \ldots, x_n$ such that $P \in U$ and a positive number ϵ satisfies the following condition: the ball $B_\epsilon(P)$ of radius ϵ centered at P

$$B_\epsilon(P) := \{\sum_{i=1}^{n}(x_i - x_i^0)^2 < \epsilon^2\}, \quad x_i^0 = x_i(P), \quad i = 1, \ldots, n$$

is entirely contained in U. Here n is the dimension of the manifold M. Now, let γ be a path on M with $\gamma(0) = P_0$ and let (U, ϵ) be an admissible pair for the point $\gamma(1)$. Define a subset $V_{(U,\epsilon)}(\gamma) \subset \hat{M}$ consisting of the equivalence classes of the paths of the form $\gamma\rho$ where ρ is a radial path inside the ϵ-ball centered at $\gamma(1)$ with the initial point $\gamma(1)$. Here the product of the paths γ and ρ is defined like it was done above for the product of loops, namely, we first go along γ from $\gamma(0) = P_0$ to $\gamma(1) = \rho(0)$ then proceed along ρ till $\rho(1)$. Clearly the subset $V_{(U,\epsilon)}(\gamma)$ depends only on the equivalence class of the path γ. For any pair (U, ϵ) admissible for $\gamma(1)$ the subset $V_{(U,\epsilon)}(\gamma)$ will be considered as an open neighbourhood of the point $[\gamma]$. We leave as an exercise for the reader to verify that this collection of open subsets defines a base[9] of topology on $\hat{M}$.

For any pair (U, ϵ) admissible for a point $P \in M$ the full preimage of the ϵ-ball centered at P is equal to

$$p^{-1}(B_\epsilon(P)) = \bigcup_{[\gamma]\in p^{-1}(P)} V_{(U,\epsilon)}(\gamma) \tag{1.3.7}$$

where by definition $p^{-1}(P) = \{[\gamma] \mid \gamma(0) = P_0,\ \gamma(1) = P\}$. It is easy to see that

$$V_{(U,\epsilon)}(\gamma_1) \cap V_{(U,\epsilon)}(\gamma_2) = \varnothing \quad \text{if} \quad [\gamma_1] \neq [\gamma_2].$$

Thus the full preimage (1.3.7) is a disjoint union of open subsets $V_{(U,\epsilon)}(\gamma)$ with $\gamma \in p^{-1}(P)$. Finally we observe that the map

$$p : V_{(U,\epsilon)}(\gamma) \to B_\epsilon(\gamma(0))$$

is one-to-one and, therefore it is a homeomorphism. We conclude that $(\hat{M}, M, p)$ is a covering indeed.

Let us now prove that the space $\hat{M}$ is connected. We have to show that any pair of points $[\gamma_1]$, $[\gamma_2]$ can be connected by a path. It suffices to prove it for the particular case of constant path $\gamma_1 = \gamma_{\text{id}}$ where $\gamma_{\text{id}}(t) \equiv P_0$. Then the needed path $\Gamma : [0,1] \to \hat{M}$ has the form $\Gamma(s) = [\gamma_2(st)]$.

It remains to prove that $\hat{M}$ is simply connected. A loop with the base point at the constant path γ_{id} can be considered as a map of the square to M

$$\Gamma(t, s) \in M, \quad (t, s) \in [0,1] \times [0,1] \quad \text{satisfying} \quad \Gamma(t, 0) = \Gamma(t, 1) \equiv P_0 \quad \text{and} \quad \Gamma(0, s) \equiv P_0.$$

Take the map of the unit cube with the coordinates (t, s, r) given by

$$\mathbf{\Gamma}(t, s, r) = \Gamma(rt, s), \quad r \in [0,1].$$

[9] A base of topology on a set X is a collection of subsets $V_\alpha \subset X$ covering X such that for any pair V_α, V_β with non-empty intersection and any point $x \in V_\alpha \cap V_\beta$ there exists V_γ such that $x \in V_\gamma \subset V_\alpha \cap V_\beta$. Using a base one can introduce topology on the set X defining the open subspaces as unions of arbitrary families of elements of the base. We refer the reader to the book [**?**] for further details.

It provides a homotopy between the original loop (for $r = 1$) and the constant loop $\mathbf{\Gamma}(t, s, 0) \equiv P_0$. Thus $\pi_1(\hat{M}, [\gamma_{\mathrm{id}}]) = 1$.

Uniqueness of the universal covering follows from

Lemma 1.3.35. *Let $(\hat{M}_1, M, p_1)$ and $(\hat{M}_2, M, p_2)$ be two universal coverings. Fix a point P_0 in the base and choose points $P_1 \in \hat{M}_1$ and $P_2 \in \hat{M}_2$ such that $p_1(P_1) = p_2(P_2) = P_0$. Then there exists a unique map $f : \hat{M}_1 \to \hat{M}_2$ satisfying $p_2 \circ f = p_1$ such that $f(P_1) = P_2$. Moreover, this map is a homeomorphism.*

Proof Let $P \in \hat{M}_1$ be an arbitrary point. Choose a path $\gamma_1 \subset \hat{M}_1$ connecting P_1 with P and let $\gamma = p_1(\gamma_1) \subset M$ be its projection to the base. Its initial point is $\gamma(0) = P_0$. Denote $\gamma_2 \subset \hat{M}_2$ the lift of γ to $\hat{M}_2$ with the initial point $\gamma_2(0) = P_2$. Put $f(P) := \gamma_2(1)$. The choice of the lift γ_2 is unique due to the condition $f(P_1) = P_2$.

Due to connectedness and simply-connectedness of $\hat{M}_1$ the construction of f works for any point $P \in \hat{M}_1$ and it does not depend on the choice of the path γ_1. Observe that $p_1(P) = p_2(f(P))$. That is the map $f : \hat{M}_1 \to \hat{M}_2$ satisfies the condition $p_2 \circ f = p_1$ from Definition 1.3.3 of equivalence of coverings. The inverse map $f^{-1} : \hat{M}_2 \to \hat{M}_1$ can be constructed in a similar way. Therefore the map f is one-to-one. Let us now prove that it is continuous.

Let $U_2 \subset \hat{M}_2$ be an open neighbourhood of $f(P)$. For a sufficiently small $\epsilon > 0$ we can find an open ϵ-ball $B_\epsilon(Q) \subset M$ centered at the point $Q := p_1(P) = p_2(f(P))$ and two open neighbourhoods $V_1 \subset \hat{M}_1$ and $V_2 \subset \hat{M}_2$ containing the points P and $f(P)$ respectively such that the projections

$$p_1 : V_1 \to B_\epsilon(Q) \quad \text{and} \quad p_2 : V_2 \to B_\epsilon(Q)$$

are homeomorphisms. Put

$$U_1 := V_1 \cap p_1^{-1}(p_2(U_2)) \subset \hat{M}_1.$$

It is an open subset in $\hat{M}_1$. Obviously it contains the point P. We will now prove that $f(U_1) \subset U_2$.

Let $P' \in U_1$ be an arbitrary point. In order to compute $f(P')$ we choose a path $\gamma_1' \subset \hat{M}_1$ connecting P_1 with P' in the following way

$$\gamma_1' = \gamma_1 \rho_1$$

where $\rho_1 \subset V_1$ is the lift of the radial path ρ in $B_\epsilon(Q)$ from $Q = p_1(P)$ to $p_1(P')$ with the initial point P. Recall that $p_1(P') \in p_2(U_2)$. Now we have to lift the path $p_1(\gamma_1') = \gamma\rho$ to $\hat{M}_2$ with the initial point P_2. The resulting lift has the form

$$\gamma_2' = \gamma_2 \rho_2 \quad \text{where} \quad \rho_2 = p_2^{-1}(\rho) \cap V_2.$$

Hence the end point $\rho_2(1) = f(P')$ belongs to $V_2 \cap U_2$.

The continuity of the inverse map f^{-1} can be proved in a similar way. This completes the proof of Lemma. $\square$

We have completed the construction of the topological space $\hat{M}$ of the universal covering as well as of the covering map $p : \hat{M} \to M$ that is a local homeomorphism. We have now to prove that the universal covering space over a smooth manifold is a smooth manifold itself. Similarly if the base is a complex manifold then so is the universal covering space. This follows from

Lemma 1.3.36. *A covering space X over a complex*[10] *manifold Y inherits a structure of complex manifold. With respect to the constructed complex structure the covering map $p : X \to Y$ becomes locally biholomorphic.*

Proof Let $(V_\beta, \phi_\beta)_{\beta\in B}$ be a complex atlas on Y. For any point $P \in Y$ and any $\beta \in B$ such that $P \in V_\beta$ denote $U_{P,\beta} = U_P \cap V_\beta$. We obtain a new complex atlas $\left(U_{P,\beta}, \phi_\beta|_{U_{P,\beta}}\right)_{P\in Y,\,\beta\in B}$ on Y. The components of the preimages $p^{-1}\left(U_{P,\beta}\right)$ with the coordinate maps $Q \mapsto \phi_\beta|_{U_{P,\beta}}(p(Q))$ provide a complex atlas on X. This structure is second-countable since the fiber of the covering is at most countable. The Lemma and, therefore the Theorem 1.3.34 is proved. □

Exercise 1.3.37: Let $G \subset \pi_1(M, P_0)$ be a subgroup of the fundamental group of a connected manifold M. Prove that there exists a covering $(\hat{M}_G, M, p)$ such that $\pi_1(\hat{M}_G, Q_0) \simeq G$ where $Q_0 \in p^{-1}(P_0)$.

We will now define an action of the fundamental group of the base on the universal covering space. Let us first recall some basics about group actions.

Let G be a group and X a topological space.

Definition 1.3.38. *1. We say that the group G acts on the space X if for any $g \in G$ there is a homeomorphism*

$$T_g : X \to X \quad \text{satisfying} \quad T_{g_1} \circ T_{g_2} = T_{g_1 g_2} \quad \forall\, g_1,\, g_2 \in G. \tag{1.3.8}$$

In particular T_e=id. Here e is the unit of the group.

2. A point $x \in X$ is fixed for the map T_g if $T_g(x) = x$. The action (1.3.8) is called fixed points free if T_g has no fixed points for $g \neq e$.

3. The group G acts discontinuously on the space X if for any $x \in X$ there exists an open neighbourhood $V_x \ni x$ such that $T_g(V_x) \cap V_x = \emptyset$ for any $g \neq e$.

Exercise 1.3.39: Let the group G act discontinuously and fixed points free on the space X. Define the quotient space X/G in the following way. Points of X/G are orbits

$$\mathcal{O}_x = \bigcup_{g\in G} T_g(x).$$

To introduce a base of topology on X/G define subsets

$$\mathcal{V} = \bigcup_{y\in V_x} \mathcal{O}_y \subset X/G$$

for any $x \in X$. Here the open neighbourhood V_x of the point x is as in the part 3 of the above Definition. Prove that the triple $(X, X/G, p)$ where the map $p : X \to X/G$ is given by

$$p(x) = \mathcal{O}_x$$

is a covering.

[10]Needless to say that the construction works for smooth real manifolds as well.

Example 1.3.40. Define an action of the group of integers on the real line by

$$\mathbb{R} \ni x \mapsto x + n, \quad n \in \mathbb{Z}. \tag{1.3.9}$$

Clearly this group action is fixed point free. For any interval I of the length less than 1 and any nonzero integer n we have $I \cap I + n = \emptyset$. So the group $\mathbb{Z}$ acts on $\mathbb{R}$ discontinuously. The quotient of the real line over this action coincides with the quotient $\mathbb{R}/\mathbb{Z}$ of the additive group of real numbers over the subgroup of integers. As a real one-dimensional manifold it can be identified with the unit circle $|z| = 1$ on the complex z-plane by the map

$$z = e^{2\pi i x}.$$

So the factorization map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ coincides with the covering of Example **??**.

Another way for the identification of the quotient $\mathbb{R}/\mathbb{Z}$ with the circle is the following one. Consider the segment $[0, 1]$. In the equivalence class of any non-integer real number there is a unique representative belonging to the inner part of the segment. Integers have two equivalent representatives at the end points of the segment. Thus to obtain the set of equivalence classes wrt the action (1.3.9) one has to identify the end points of the segment resulting in a circle.

The above construction is the simplest example of a fundamental domain for a group action on a topological space X. Roughly speaking a fundamental domain is a subset $D \subset X$ of unique representatives of all orbits of the group action. For the group action (1.3.9) the choice $D = [0, 1)$ is fine. In many cases however it is more convenient to slightly modify the definition assuming that D is a closed subset containing representatives of all orbits containing no equivalent pairs of points in the internal part of D but some repetitions allowed on the boundary. The segment $D = [0, 1]$ fits into this modified definition. In sequel all examples of fundamental domains will also be treated according to the modified version of the definition.

Example 1.3.41. An action of the group $\mathbb{Z} \oplus \mathbb{Z}$ on the real plane $\mathbb{R}^2$ will be defined by

$$(x, y) \mapsto (x + m, y + n), \quad m, n \in \mathbb{Z}. \tag{1.3.10}$$

The quotient $\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$ can be identified with the two-dimensional torus $T^2 = S^1 \times S^1$ by

$$(x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y}).$$

For the fundamental domain (see the previous Example) one can choose the unit square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$. The points on the opposite sides of the square must be identified as

$$(x, 0) \sim (x, 1), \quad 0 \leqslant x \leqslant 1, \quad (0, y) \sim (1, y), \quad 0 \leqslant y \leqslant 1$$

in order to obtain the set of all orbits of the action (1.3.10). After gluing together the opposite sides of the square we again obtain a torus.

The above construction can be easily generalised to multidimensional tori.

We will now explain an important construction of an action of the fundamental group of a manifold on its universal covering space.

Theorem 1.3.42. *Let M be a connected smooth manifold and $(\hat{M}, M, p)$ its universal covering. Then the fundamental group of M acts on $\hat{M}$ by diffeomorphisms*

$$T_\gamma : \hat{M} \to \hat{M} \quad \forall \gamma \in \pi_1(M, P_0)$$

discontinuously and fixed points free. Here $P_0 \in M$ is an arbitrary point. If M is a complex manifold then the maps T_γ are biholomorphic.

Proof Choose a point $Q_0 \in \hat{M}$ such that $p(Q_0) = P_0$. Connect Q_0 with a given point $Q \in \hat{M}$ by a path $\hat{\gamma}_Q$. Let $\gamma_Q = p(\hat{\gamma}_Q)$ be its projection to M. For any loop $\gamma \in \pi_1(M, P_0)$, define a new path

$$\gamma'_Q = \gamma\,\gamma_Q.$$

Let $\hat{\gamma}'_Q$ be the lift of γ'_Q with the initial point Q_0. Denote Q' the end point of $\hat{\gamma}'_Q$ and put

$$T_\gamma(Q) = Q'.$$

As the space $\hat{M}$ is simply connected the resulting point $T_\gamma(Q)$ does not depend on the choice of the path $\hat{\gamma}_Q$. It depends only on the homotopy class of the loop γ. The superposition $T_{\gamma_1}(T_{\gamma_2}(Q))$ for two loops γ_1, γ_2 in $\pi_1(M, P_0)$ can be obtained by lifting the path $\gamma_1\gamma_2\gamma_Q$. Therefore

$$T_{\gamma_1} \circ T_{\gamma_2} = T_{\gamma_1\gamma_2}.$$

In particular $T_{\gamma^{-1}} = (T_\gamma)^{-1}$. Thus for any $\gamma \in \pi_1(M, P_0)$ the map $T_\gamma : \hat{M} \to \hat{M}$ is a bijection. Contiinuity of this map as well as of the inverse map can be proved in the way similar to the proof of continuity of the map f in Lemma 1.3.35. If Q is a fixed point of T_γ then the paths γ_Q and $\gamma\,\gamma_Q$ are homotopic with fixed end points. Hence γ is homotopic to the constant loop. This implies that the action of the fundamental group $\pi_1(M, P_0)$ on the universal covering space $\hat{M}$ is fixed points free.

It remains to prove that it acts discontinuously. To this end for a given point $Q \in \hat{M}$ we choose an open neighbourhood $V_{p(Q)}$ of its projection $p(Q)$ such that the full preimage $p^{-1}\left(V_{p(Q)}\right)$ is homeomorphic to $V_{p(Q)} \times F$. Points of the fiber F of the universal covering can be identified, via the monodromy action, with elements of the fundamental group. Let U_Q be the component of the preimage $p^{-1}\left(V_{p(Q)}\right)$ containing the point Q. Then the images $T_\gamma(U_Q)$ for $\gamma \in \pi_1(M, P_0)$ will have no intersections. □

We will return to these constructions in Section **??** considering universal coverings of Riemann surfaces.

1.3.2 Riemann surface of an algebraic function: the general case

Let us return to the study of Riemann surfaces of algebraic functions. For an irreducible monic polynomial

$$F(z, w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) \tag{1.3.11}$$

of degree n in w introduce a finite set of <u>critical points</u> $Crit \subset \mathbb{C}$ taking zeros of the discriminant of F

$$Crit = \{z \in \mathbb{C} \mid \Delta_F(z) = 0\}.$$

Denote

$$\dot{C} = C \backslash \pi^{-1}(Crit) \tag{1.3.12}$$

where C is the complex algebraic curve

$$C = \{(z, w) \in \mathbb{C}^2 \mid F(z, w) = 0\} \tag{1.3.13}$$

and

$$\pi : C \to \mathbb{C}, \quad \pi(z,w) = z \tag{1.3.14}$$

is the projection. The punctured curve $\dot{C}$ has a natural structure of a one-dimensional complex manifold.

Lemma 1.3.43. *$\dot{C}$ is a n-sheeted covering space of $\mathbb{C}\backslash Crit$ with respect to the projection*

$$\pi : \dot{C} \to \mathbb{C}\backslash Crit. \tag{1.3.15}$$

The map π is defined in (1.3.14).

Proof Let z_0 be a point in $\mathbb{C}\backslash Crit$. Then for every point $P \in \pi^{-1}(z_0)$ one can use z as a local coordinate. In other words, there exists a positive number ϵ_P and a neighbourhood U_P of P such that the map

$$\pi : U_P \to \{|z - z_0| < \epsilon_P\}$$

is biholomorphic. Put

$$\epsilon = \min_{P \in \pi^{-1}(z_0)} \epsilon_P$$

and denote $U = \{|z - z_0| < \epsilon\}$. Order the points $P_1, \dots, P_n$ in $\pi^{-1}(z_0)$. Then the preimage $\pi^{-1}(U)$ has the form

$$\pi^{-1}(U) = \bigcup_{i=1}^{n} U_i, \quad U_i = \bigcup_{z \in U} \{(z, w_i(z))\}$$

where $w_i(z)$ is the branch of the algebraic function $w(z)$ near P_i. The biholomorphic map

$$\varphi : \pi^{-1}(U) \to U \times \{1, 2, \dots, n\}, \quad (z, w_i(z)) \mapsto (z, i)$$

is the needed homeomorphism. □

Choose a complex number $z_* \in \mathbb{C}\backslash Crit$.

Definition 1.3.44. *The monodromy (anti)homomorphism*

$$\mu : \pi_1 \left(\mathbb{C}\backslash Crit, z_*\right) \to Aut\left(\pi^{-1}(z_*)\right) \tag{1.3.16}$$

of the covering (1.3.15) (see the Definition 1.3.21) is called the monodromy of the algebraic function $w(z)$ defined by the polynomial equation $F(z,w) = 0$.

The preimage $\pi^{-1}(z_*)$ consists of n distinct points. Ordering them in an arbitrary way

$$(z_*, w_1(z_*)), \dots, (z_*, w_n(z_*))$$

we can rewrite (1.3.16) as a homomorphism into symmetric group

$$\mu : \pi_1 \left(\mathbb{C}\backslash Crit, z_*\right) \to S_n. \tag{1.3.17}$$

Recall that a change of the base point z_* gives rise to an equivalent representation.

Example 1.3.45. For the hyperelliptic curve

$$C = \{(z,w) \in \mathbb{C}^2 \mid w^2 = \prod_{i=1}^{k}(z-a_i)\}, \quad a_i \neq a_j \quad \text{for} \quad i \neq j$$

the set *Crit* consists of the branch points

$$Crit = \bigcup_{i=1}^{k}\{a_i\}.$$

The punctured curve

$$\dot{C} = C \backslash \bigcup_{i=1}^{k}\{(a_i, 0)\}$$

is a two-sheet covering of $\mathbb{C}\backslash Crit$.

For a loop $\gamma \subset \mathbb{C}\backslash Crit$ encircling just one branch point the monodromy along γ changes the sign of $w(z) = \sqrt{\prod_{i=1}^{k}(z-a_i)}$. Thus

$$\mu(\gamma) = (12) \in S_2$$

is the permutation between 1 and 2. For a loop encircling two branch points the monodromy is trivial. More generally, for a loop γ encircling m branch points

$$\mu(\gamma) = \begin{cases} (12) \in S_2, & m = \text{odd} \\ \text{id} \in S_2, & m = \text{even}. \end{cases}$$

Let $f = f(z,w)$ be a function on $\dot{C}$. Restricting it at the points of the preimage $\pi^{-1}(z_*)$ ordered in some way we obtain n numbers $f(z_*, w_1(z_*)), \ldots, f(z_*, w_n(z_*))$. We say that f is monodromy invariant over z_* if, for any loop $\gamma \in \pi_1(\mathbb{C}\backslash Crit, z_*)$ we have

$$f(z_*, w_{\mu(\gamma)(i)}(z_*)) = f(z_*, w_i(z_*)) \quad \forall\, i \in \{1, 2, \ldots, n\}.$$

For example the symmetric functions

$$w_1(z_*)^k + \cdots + w_n(z_*)^k$$

for any integer k are always monodromy invariant. If $f(z,w)$ is locally holomorphic near the points in $\pi^{-1}(z_*)$ then the above invariance holds true over z in some neighbourhood of z_*. Finally, if $f(z,w)$ is meromorphic on $\dot{C}$ then, applying analytic continuation we obtain monodromy invariance over any point $z \in \mathbb{C}\backslash Crit$. In this case we will simply say that the meromorphic function f is monodromy invariant.

Proposition 1.3.46. *Assume irreducibility of the monodromy (1.3.16). Let $f = f(z,w)$ be a meromorphic function on $\dot{C}$ growing at most polynomially at the punctures in $\pi^{-1}(Crit)$ as well as at infinity. Suppose f is invariant with respect to the monodromy representation. Then $f = f(z)$ and this is a rational function of the complex variable z.*

Proof Because of the monodromy invariance and transitivity of the monodromy, the function $f(z,w)$ depends only on z. Therefore it is a meromorphic function on $\overline{\mathbb{C}}\backslash(Crit \cup \{\infty\})$. Due to the assumptions about the polynomial growth, it has removable singularities (See e.g. []) at the points of the set $Crit \cup \{\infty\}$. Hence it can be extended to a meromorphic function $\overline{\mathbb{C}}$. So it must be a rational function. □

We will now prove connectedness of $\dot{C}$.

Theorem 1.3.47. *For an irreducible polynomial $F(z,w)$ the manifold $\dot{C}$ is connected.*

Proof Suppose $\dot{C}$ is not connected. According to Lemma 1.3.26 it implies that the monodromy action is not transitive. That means that there exists a partition

$$\{1,2,\ldots,n\} = I \sqcup J$$

into two nonempty sets $I = \{i_1,\ldots,i_p\}$ and $J = \{j_1,\ldots,j_q\}$, $p+q=n$ such that, for a given point $z \in \mathbb{C}\backslash Crit$ after a suitable ordering of the points in the preimage $\pi^{-1}(z)$ the subsets

$$\{(z,w_{i_1}(z)),\ldots,(z,w_{i_p}(z))\} \quad \text{and} \quad \{(z,w_{j_1}(z)),\ldots,(z,w_{j_q}(z))\}$$

are both invariant with respect to the monodromy

$$\pi_1\left(\mathbb{C}\backslash Crit, z\right) \to Aut\left(\pi^{-1}(z)\right).$$

Let us assume the action of the monodromy on both subsets I and J to be irreducible. Consider two polynomials

$$F_I = (w - w_{i_1}(z))\ldots(w - w_{i_p}(z)) \quad \text{and} \quad F_J = (w - w_{j_1}(z))\ldots(w - w_{j_q}(z)).$$

They are locally well defined and monodromy-invariant. We will now extend them onto $\mathbb{C}\backslash Crit$. Let $z' \notin Crit$ be another point. Connect it by a path $\gamma \subset \mathbb{C}\backslash Crit$ with z. The n lifts of γ establish a one-to-one correspondence between the sets $\pi^{-1}(z)$ and $\pi^{-1}(z')$. Denote I' and J' the images of the subsets I and J with respect to this correspondence. They have the same cardinalities p and q respectively. These subsets are monodromy-invariant with respect to the representation

$$\pi_1\left(\mathbb{C}\backslash Crit, z'\right) \to Aut\left(\pi^{-1}(z')\right).$$

Analytically continuing the roots $w_{i_1}(z),\ldots,w_{i_p}(z)$ of the polynomial F_I along the lifts of γ that start at the points of I we obtain the needed extension of F_I to the point z'. In a similar way we extend the polynomial F_J. Due to the above arguments about monodromy-invariance the resulting extensions do not depend on the choice of the path γ. So, according to Proposition 1.3.46 the coefficients of these polynomials are rational functions in z. Clearly $F = a_0(z)F_IF_J$. This contradicts irreducibility. In the more general case where the action of monodromy on I and/or J is reducible we split them into smaller subsets such that the monodromy acts irreducibly on each of them. Repeating the above arguments we arrive at a factorization of $F(z,w)$ into a product of more than two factors. □

We now pass to the main point of this Section: to the construction of Riemann surface of an algebraic function. We do it in the following way. Start from the open manifold $\dot{C}$ as in eq. (1.3.12).

Then add to it a finite number of points and introduce local coordinates on neighbourhoods of these points. The last step is to compactify the resulting Riemann surface. To this end we add a finite number of points at infinity. Remarkably all prescriptions for this construction are encoded in the monodromy of the covering $\dot{C} \to \mathbb{C} \backslash Crit$.

Theorem 1.3.48. *Let C be the complex algebraic curve $F(z, w) = 0$ defined by an irreducible monic polynomial of degree n in w. Then there exists a compact Riemann surface $\mathcal{S}$ and two holomorphic maps $\hat{z} : \mathcal{S} \to \overline{\mathbb{C}}_z$ and $\hat{w} : \mathcal{S} \to \overline{\mathbb{C}}_w$ onto the extended complex z- and w-plane respectively such that*

(i)

$$F(\hat{z}(P), \hat{w}(P)) = 0 \quad \forall P \in \mathcal{S};$$

(ii) the map

$$\rho : \mathcal{S} \backslash \hat{z}^{-1}(Crit \cup \{\infty\}) \to \dot{C}, \quad P \mapsto (z = \hat{z}(P), w = \hat{w}(P))$$

is biholomorphic;

(iii) if the algebraic curve C is smooth then $\mathcal{S} \backslash \hat{z}^{-1}(\{\infty\}) = C$.

Let us begin with constructing the finite part $\mathcal{S}_{\text{finite}}$ of the Riemann surface; the infinite points will be added later. We will follow the notations introduced in the beginning of this section. Let z_0 be a zero of the discriminant $\Delta_F(z)$. Choose a point z_1 close to z_0 but away from $Crit$. Order in an arbitrary way the points in the preimage $\pi^{-1}(z_1)$. Consider the monodromy transformation $\sigma \in S_n$

$$\sigma : \pi^{-1}(z_1) \to \pi^{-1}(z_1)$$

generated by lifting the anticlockwise loop around z_0. Decompose the permutation σ into product of cycles

$$\sigma = (i_1, \ldots, i_p)(j_1, \ldots, j_q) \cdots (l_1, \ldots, l_s)$$

of the lengths $p, q, \ldots, s$,

$$p + q + \cdots + s = n$$

corresponding to a partition of the set $\{1, 2, \ldots, n\}$ into disjoint union of subsets $\{i_1, \ldots, i_p\}$, $\{j_1, \ldots, j_q\}$, $\ldots$, $\{l_1, \ldots, l_s\}$. Such a decomposition always exists and is unique [3]. For every such cycle we add to $\dot{C}$ a point. It will be a ramification point with respect to $\hat{z}$ of the new Riemann surface $\mathcal{S}_{\text{finite}}$ of the ramification index = length of the cycle - 1. We will explain the construction for the first cycle $(i_1, \ldots, i_p)$.

In a sufficiently small neighbourhood of z_1 we have p branches $w_{i_1}(z), \ldots, w_{i_p}(z)$. Anticlockwise analytic continuation around z_0 permutes them cyclically

$$\begin{array}{rcl} w_{i_1}\left(z_0 + (z - z_0)e^{2\pi i}\right) & = & w_{i_2}(z) \\ w_{i_2}\left(z_0 + (z - z_0)e^{2\pi i}\right) & = & w_{i_3}(z) \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ w_{i_{p-1}}\left(z_0 + (z - z_0)e^{2\pi i}\right) & = & w_{i_p}(z) \\ w_{i_p}\left(z_0 + (z - z_0)e^{2\pi i}\right) & = & w_{i_1}(z) \end{array}$$

Take a punctured disk

$$\dot{D} = \{\tau \in \mathbb{C} \mid 0 < |\tau| < \epsilon\}$$

for a sufficiently small ϵ and consider the function

$$\tilde{w}(\tau) = w_{i_1}\left(z_0 + \tau^p\right)$$

on $\dot{D}$. Since $w_{i_1}\left(z_0 + (z - z_0)e^{2p\,\pi i}\right) = w_{i_1}(z)$ we conclude that $\tilde{w}(\tau)$ is a single valued holomorphic function on $\dot{D}$. We obtain a holomorphic map

$$\dot{D} \to \dot{C}, \quad \tau \mapsto \left(z_0 + \tau^p, \tilde{w}(\tau)\right). \tag{1.3.18}$$

~~Note that the projection of the image of $\dot{D}$ to z-plane is again a punctured disk~~

$$\dot{D}_0 = \{z \in \mathbb{C} \mid 0 < |z - z_0| < \epsilon^p\}$$

~~and the degree of the through map~~

$$\dot{D} \to \dot{D}_0$$

~~equals p.~~ It remains to observe that $w_{i_k}(z) \to w_0$ when $z \to z_0$ along radial directions, for any $k = 1, \ldots, p$. Here $w = w_0$ is a root of the equation $F(z_0, w) = 0$. Therefore $\tau = 0$ is a removable singularity for the function $\tilde{w}(\tau)$. We define it as the new point $P_1 \in \mathcal{S}_{\text{finite}}$ added to $\dot{C}$. The map

$$D = \{\tau \in \mathbb{C} \mid |\tau| < \epsilon\} \to \mathcal{S}, \quad \tau \mapsto \begin{cases} \left(z_0 + \tau^p, \tilde{w}(\tau)\right), & \tau \neq 0 \\ P_1, & \tau = 0 \end{cases}$$

provides a chart on a neighbourhood of P_1. Put

$$\begin{aligned} &\hat{z}(P) = z, \quad P = (z, w) \in \dot{C} \\ &\hat{z}\left(z_0 + \tau^p, \tilde{w}(\tau)\right) = z_0 + \tau^p, \quad \hat{z}(P_1) = z_0 \end{aligned}$$

and

$$\begin{aligned} &\hat{w}(P) = w, \quad P = (z, w) \in \dot{C} \\ &\hat{w}\left(z_0 + \tau^p, \tilde{w}(\tau)\right) = \tilde{w}(\tau), \quad \hat{w}(P_1) = w_0. \end{aligned}$$

It is easy to see that all properties of these maps formulated in the Theorem hold true. This completes the construction for the first cycle in the decomposition of the monodromy σ . For other cycles the construction is identical, so we obtain a new point $P_2 \in \mathcal{S}_{\text{finite}}$ for the second cycle etc. Then we proceed to other zeros of the discriminant.

The last step is in compactification of the Riemann surface $\mathcal{S}_{\text{finite}}$. Denote $\zeta = 1/z$ the local coordinate near the infinite point of $\mathbb{C}_z$. Let us rewrite the algebraic equation $F(z, w) = 0$ in the variables ζ, w taking

$$\tilde{F}(\zeta, w) = \zeta^N F\left(\frac{1}{\zeta}, w\right)$$

where $N = \max\{\deg a_0(z), \ldots, \deg a_n(z)\}$. This polynomial is not monic in w so we do one more substitution (cf. footnote 4 above) by introducing a new variable

$$\omega = w\,\zeta^N = \frac{w}{z^N}$$

and define a new polynomial

$$F_\infty(\zeta,\omega) = \zeta^{N(n-1)}\tilde{F}\left(\zeta, \frac{\omega}{\zeta^N}\right) = \zeta^{nN} F\left(\frac{1}{\zeta}, \frac{\omega}{\zeta^N}\right) = \sum_{j=0}^{n} a_j(\frac{1}{\zeta})\zeta^{jN}\omega^{n-j}$$

monic of degree n in ω. We can now proceed with analysis of the local monodromy $\sigma_\infty \in S_n$ interchanging the branches of the algebraic function $\omega(\zeta)$ as the result of the anticlockwise analytic continuation along a small loop around the point $\zeta = 0$. Factorizing σ_∞ into a product of cycles we obtain the prescription for adding to $\mathcal{S}_{\text{finite}}$ some points over the infinite point $\zeta = 0$ of the extended complex plane $\overline{\mathbb{C}}_z$. One important observation about the local monodromy σ_∞ has to be taken into account. Namely, to the small loop $|\zeta| = \epsilon$ around infinity running in the anticlockwise direction it corresponds the big loop $|z| = 1/\epsilon$ running in the clockwise direction. Hence the monodromy σ_∞ describes the clockwise analytic continuation of the branches of the algebraic function $w(z)$ along such a big loop. Let ϵ be so small that the circle $|z| = 1/\epsilon$ contains inside all zeros $z_1, \ldots,$ z_K of the discriminant. Choose a point z_* on the circle such that the segments connecting z_* with $z_1, \ldots, z_K$ do not have common internal points. Order the zeros of the discriminant in such a way that the segments follow in the anticlockwise direction, looking from their common point z_* . Running along the i-th segment from z_* to a point close to z_i then along a small loop around z_i in the anticlockwise direction and finally returning back to z_* along the same segment we obtain a loop $\gamma_i \in \pi_1(\mathbb{C}\backslash Crit, z_*)$. Denote $\sigma_i = \mu(\gamma_i)$ the monodromy along the loop γ_i. Our claim is that

$$\sigma_\infty = [\gamma_1\gamma_2\ldots\gamma_K]^{-1}. \tag{1.3.19}$$

Indeed, the loop given by the circle $|z| = 1/\epsilon$ run in the anticlockwise direction is homotopic to the product $\gamma_1\gamma_2\ldots\gamma_K$. Inverting the direction we obtain the inverse of this element in the fundamental group $\pi_1(\mathbb{C}\backslash Crit, z_*)$. This proves eq. (1.3.19).

We are now to prove that the constructed one-dimensional complex manifold $\mathcal{S}$ is connected and compact. Connectednes immediately follows from Theorem 1.3.47. Let us prove compactness.

Let $Q_1, Q_2, \ldots$ be an infinite sequence of points in $\mathcal{S}$. Due to compactness of the Riemann sphere there exists a subsequence $Q_{i_1}, Q_{i_2}, \ldots$ such that $\hat{z}(Q_{i_s})$ converges to some point $z_0 \in \overline{\mathbb{C}}$ for $s \to \infty$ (it may happen that $z_0 = \infty$. Let us first consider the case where z_0 is not a branch point wrt the map $\hat{z}$ that is at all points $P_1, P_2, \ldots, P_n$ of the preimage $\hat{z}^{-1}(z_0)$ the derivative of $\hat{z}$ wrt the local parameter does not vanish. Then $\hat{z}$ is locally biholomorphic near every point $P_1, \ldots, P_n$. There exists at least one point P_j such that its neighbourhood contains an infinite number of points Q_{i_s}. This subsequence of subsequence converges to P_j.

Consider now the case where z_0 is a branch point. Then the preimage consists of $m < n$ points $P_1, \ldots, P_m$ of multiplicities $k_1, \ldots, k_m$ respectively. Therefore for every $j = 1, \ldots, m$ there exists a neighbourhood U_j of P_j such that the map

$$\hat{z} : U_j \backslash P_j \to \{0 < |z - z_0| < \epsilon\}$$

for some $\epsilon > 0$ is a covering of degree k_j. Repeating the above arguments we obtain a subsequence of subsequence of points $Q_i \in \mathcal{S}$ convergent to P_j for some $j = 1, \ldots, m$. This completes the proof of compactness of $\mathcal{S}$.

It remains to consider the case of smooth algebraic curves C. Let $(z_0, w_0) \in C$ be a ramification point of the ramification index $p - 1$. One can use the w-coordinate as a local parameter near this point. Another coordinate is a holomorphic function of the local parameter on some neighbourhood of w_0

$$z = z_0 + \sum_{k\geqslant 1} c_k (w - w_0)^k.$$

Due to our assumptions the first nonzero coefficient is c_p. Introduce a holomorphic function

$$\tau(w) = c_p^{1/p}(w - w_0)\left[1 + \sum_{k\geqslant 1} \frac{c_{k+p}}{c_p}(w - w_0)^k\right]^{\frac{1}{p}}.$$

The inverse function is also holomorphic for sufficiently small $|\tau|$; denote it $\tilde{w}(\tau)$. We have

$$z = z_0 + \tau^p, \quad w = w_0 + \tilde{w}(\tau). \tag{1.3.20}$$

So in this case we do not need to add new points as the function $\tilde{w}(\tau)$ is holomorphic at $\tau = 0$ and $\tilde{w}(0) = 0$. As the projection $(z, w) \to z$ has degree p near (z_0, w_0) the local monodromy around z_0 of the function $w(z)$ defined by (1.3.20) is a cycle of length p. The Theorem is proved. □

Example 1.3.49. Consider the algebraic function $w(z)$ defined by equation

$$w^2 = z^2(z + 1).$$

The discriminant is equal to $4z^2(z+1)$, so $Crit = \{0\} \cup \{-1\}$. The point $(z = -1, w = 0)$ is a smooth point of the corresponding algebraic curve C; it is a ramification point of the ramification index 1. Another point $(z = 0, w = 0)$ is a singular point of C. Near $z = 0$ the function $w(z)$ has two branches $w_{1,2}(z) = \pm z\sqrt{z + 1}$. The analytic continuation along the circle $|z| = r$, $r < 1$ does not interchange these two branches. Therefore the corresponding monodromy is the identity

$$\mathrm{id} = (1)(2) \in S_2.$$

So we have to add two points P_1, P_2 to the punctured curve

$$\mathcal{S}_{\text{finite}} = (C \backslash \{(0, 0)\}) \cup \{P_1\} \cup \{P_2\}$$

$$\hat{\pi}\,(P_{1,2}) = 0 \in \mathbb{C}, \quad \rho\,(P_{1,2}) = (0, 0) \in C$$

and these points are not ramification points of $\mathcal{S}$ with respect to $\hat{\pi} : \mathcal{S} \to \mathbb{C}$.

As we have only one branch point on $\mathbb{C}$ then, due to (1.3.19) there is also a branch point at infinity. The monodromy around infinity interchanges the two branches of the algebraic function $\omega(\zeta)$ defined by equation

$$\omega^2 = \zeta^3 + \zeta^4.$$

Here

$$z = \frac{1}{\zeta}, \quad w = \frac{\omega}{\zeta^3}.$$

Observe that this curve has a cuspidal singularity at $(\zeta = 0, \omega = 0)$. According to the constructions of the Theorem we have to add one point P_∞ to $\mathcal{S}_{\text{finite}}$ and introduce a local parameter τ near this point by

$$\zeta = \tau^2.$$

We obtain a function

$$\tilde{\omega}(\tau) = \tau^3 \sqrt{1+\tau^2}$$

holomorphic for $|\tau| < 1$. In the original coordinates we have

$$z = \frac{1}{\tau^2}$$

and the function w has a pole of order 3 at P_∞

$$w = \frac{1}{\tau^3} \sqrt{1+\tau^2}.$$

It is easy to find a realization of the Riemann surface $\mathcal{S}_{\text{finite}}$ as a smooth algebraic curve. To this end consider the curve defined by the equation

$$\tilde{w}^2 = z + 1.$$

Obviously it is smooth. It has two points $P_{1,2} = (z = 0, \tilde{w} = \pm 1)$ above $z = 0$. Define a map

$$\rho : \mathcal{S}_{\text{finite}} \to C, \quad \rho(z, \tilde{w}) = (z, z\tilde{w}).$$

It is biholomorphic for $z \neq 0, \infty$ and it maps both the points P_1 and P_2 to $(0, 0)$. Adding, like above a point P_∞ to $\mathcal{S}_{\text{finite}}$ we obtain a realization of $\mathcal{S}$.

Example 1.3.50. Consider the hyperelliptic curve

$$w^2 = z^{2n+1} + a_1 z^{2n} + \cdots + a_{2n+1} = \prod_{i=1}^{2n+1} (z - z_i), \quad z_i \neq z_j \quad \text{for} \quad i \neq j.$$

It has $2n + 1$ branch points $z = z_1, \ldots, z = z_{2n+1}$. The monodromy around every of these points is the permutation $(12) \in S_2$. From eq. (1.3.19) using the obvious identity $(12)^2 = \text{id}$ it follows that the monodromy around infinity is the same permutation (12). Therefore the Riemann surface of the algebraic function $w(z)$ has one infinite point P_∞ and it is a ramification point of the ramification index 1. Like in the previous example we introduce the local parameter near P_∞ by

$$z = \frac{1}{\tau^2}.$$

The function z has a pole of order 2 at P_∞ and the function

$$w = \frac{1}{\tau^{2n+1}} \sqrt{1 + a_1\tau^2 + a_{2n+1}\tau^{4n+2}}$$

has a pole of order $2n + 1$.

Example 1.3.51. Consider now a hyperelliptic curve with even number of branch points

$$w^2 = z^{2n+2} + a_1 z^{2n+1} + \cdots + a_{2n+2} = \prod_{i=1}^{2n+2} (z - z_i), \quad z_i \neq z_j \quad \text{for} \quad i \neq j.$$

Applying again eq. (1.3.19) we conclude that the monodromy around infinity is trivial. Therefore the Riemann surface of the algebraic function $w(z)$ in this case has two infinite points $P_\infty^\pm$ and these are not ramification points. That means that the local parameter τ near this points coincides with $\zeta = 1/z$ or

$$z = \frac{1}{\tau}.$$

Thus the function z has two simple poles at the infinite points $P_\infty^\pm$ and w has two poles of order $n+1$

$$w = \pm \frac{1}{\tau^{n+1}} \sqrt{1 + a_1\tau + \cdots + a_{2n+2}\tau^{2n+2}}, \quad (z, w) \to P_\infty^\pm.$$

We conclude this section with three remarks.

Remark 1.3.52. The constructions we used in the proof of Theorem 1.3.48 are close to the Riemann's original approach to the idea of Riemann surface. Taking n copies of complex plane with cuts between the critical points he glues the copies along the cuts where the rules of glueing are prescribed by the action of monodromy. The simplest example of this procedure was already considered above in Section 1.1.1 in the construction of Riemann surface of $\sqrt{z}$. Further examples will be considered below in Section **??**.

On this way Riemann arrived at the following important result.

Riemann Existence Theorem. *Let $z_1, \ldots, z_K$ be distinct points of complex plain and*

$$\mu : \mathcal{F}_K \to S_n$$

an (anti)homomorphism of the free group with K generators to the symmetric group S_n such that the image acts transitively on the set $\{1, 2, \ldots, n\}$. Then there exists a n-sheeted Riemann surface with branch points at $z_1, \ldots, z_K$ and, possibly, at infinity (see eq. (1.3.19) *above) with the monodromy μ.*

Exercise 1.3.53: Prove Riemann Existence Theorem for $n = 2$ and arbitrary K.

Exercise 1.3.54: Prove Riemann Existence Theorem for $K = 1$ and arbitrary n.

Remark 1.3.55. In this Section we have started from an irreducible polynomial equation $F(z, w) = 0$ to construct what was called compact Riemann surface of the algebraic function $w(z)$ defined by this equation. It turns that any compact Riemann surface can be obtained in this way. The precise statement is given by the following theorem.

Theorem. *Let $\mathcal{S}$ be a compact Riemann surface. Then there exist two meromorphic functions $z, w : \mathcal{S} \to \overline{\mathbb{C}}$ satisfying the identity*

$$F(z(P), w(P)) = 0 \quad \forall\, P \in \mathcal{S}$$

for some irreducible polynomial $F(z, w)$. $\mathcal{S}$ coincides with the Riemann surface of the algebraic function $w(z)$ defined by the equation $F(z, w) = 0$.

The Theorem will be proven in Section 3.1.5 below.

Remark 1.3.56. The most powerful tool for computing the local monodromy of an algebraic function around a critical point uses Newton polygons to obtain expansions of the branches of this function in Puiseux series. Let us illustrate this procedure using the results of Example 1.4 above. Namely, we will describe the local structure of the Riemann surface associated with the algebraic curve (1.2.37) near the branch point $z = 0$. There were obtained 8 different Puiseux expansions

$$w_1(z) = 2z^4 + \dots,\ w_2(z) = \frac{1}{4}z + \dots,\ w_3(z) = \sqrt{2}\,z^{\frac{1}{2}} + \frac{2^{\frac{1}{4}}}{4}z^{\frac{3}{4}} + \dots,\ w_4(z) = \sqrt{2}\,z^{\frac{1}{2}} - \frac{2^{\frac{1}{4}}}{4}z^{\frac{3}{4}} + \dots,$$

$$w_5(z) = -\sqrt{2}\,z^{\frac{1}{2}} + i\,\frac{2^{\frac{1}{4}}}{4}z^{\frac{3}{4}} + \dots,\ w_6(z) = -\sqrt{2}\,z^{\frac{1}{2}} - i\,\frac{2^{\frac{1}{4}}}{4}z^{\frac{3}{4}} + \dots,$$

$$w_7(z) = i\,z^{-\frac{3}{2}} - \frac{i}{2}z^{\frac{1}{2}} + \dots,\ w_8(z) = -i\,z^{-\frac{3}{2}} + \frac{i}{2}z^{\frac{1}{2}} + \dots$$

of solutions $w(z)$ to the equation $F(z, w) = 0$ of degree 8 in w. They correspond to 8 sheets of the Riemann surface. Label the branches of $w(z)$ according to the order they were written above. The local monodromy $z \mapsto z\,e^{2\pi i}$ around $z = 0$ is given by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 6 & 5 & 3 & 4 & 8 & 7 \end{pmatrix}.$$

It factorizes into product of four cycles

$$(1)(2)(3645)(78).$$

Thus there are four points on the Riemann surface over $z = 0$, two of them regular i.e. of multiplicity 1, one point of multiplicity 4 and one of multiplicity 2.

Exercise 1.3.57: Prove that the monodromy group of the Riemann surface of the algebraic function defined by a generic polynomial equation of the form (1.2.27) coincides with the complete symmetric group S_n . *Hint*. Show that the branch points of such a surface can be labeled by pairs of distinct numbers $i \neq j$, $(i, j = 1, ..., n)$ in such a way that a circuit about the images of the points P_{ij} and P_{ji} gives rise to a transposition of the i-th and j-th points of the fiber (when these points are suitably numbered).

1.3.3 Meromorphic functions on compact Riemann surfaces and branched coverings of $\mathbb{P}^1$

Recall that a meromorphic function on a Riemann surface $\mathcal{S}$ is nothing but a holomorphic map

$$f : \mathcal{S} \to \mathbb{P}^1$$

of the surface to the Riemann sphere. The points in the preimage of the infinite point $\{\infty\} \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ are called poles of f, other points on $\mathcal{S}$ will be called ordinary points. If $P \in \mathcal{S}$ is a pole of f then the function can be expanded in a Laurent series

$$f = \sum_{i \geqslant -m} c_i \tau^i, \quad m > 0, \quad c_{-m} \neq 0$$

convergent on some punctured neighbourhood of the point P. Here τ is a local parameter near $P \in \mathcal{S}$ such that $\tau(P) = 0$. The positive number m is called the order of the pole. Near an ordinary point the function can be expanded in a convergent power series

$$f = f(P) + \sum_{i \geqslant 1} c_i \tau^i.$$

The number

$$m = \min_{i \geqslant 1} \{ i \mid c_i \neq 0 \}$$

is called the multiplicity $\mathrm{mult}_f(P)$ of the ordinary point P wrt the map $f : \mathcal{S} \to \mathbb{P}^1$, cf. Definition 1.2.11. If P is a pole of f of order m then we put $\mathrm{mult}_f(P) = m$. It is easy to see that the multiplicity of a point is independent of the choice of local parameter. Moreover, one can always choose a local parameter τ near a point $P \in \mathcal{S}$, either an ordinary one or a pole, and a local parameter ζ near $f(P) \in \overline{\mathbb{C}}$ such that the map f is locally written as

$$\zeta = \tau^m, \quad m = \mathrm{mult}_f(P). \tag{1.3.21}$$

The points of multiplicity one will be called regular points of the meromorphic function. All other points in $\mathcal{S}$ will be called ramification points of f.

Example 1.3.58. Let $\mathcal{S}$ be the compact Riemann surface of an algebraic function $w(z)$ defined by an irreducible polynomial equation $F(z, w) = w^n + a_1(z) w^{n-1} + \cdots + a_n(z) = 0$. From the construction of Theorem 1.3.48 we have two holomorphic maps

$$z : \mathcal{S} \to \overline{\mathbb{C}}, \quad w : \mathcal{S} \to \overline{\mathbb{C}}$$

(we now omit hats over z and w used in the Theorem) satisfying the identity

$$F\left(z(P), w(P)\right) = 0 \quad \forall\, P \in \mathcal{S}.$$

Let us look at regular points and ramification points on $\mathcal{S}$ wrt the map $f = z$. First, let $Crit \subset \overline{\mathbb{C}}$ be the finite subset in the Riemann sphere consisting of all zeros of the discriminant $\Delta_F(z)$ plus the infinite point. Denote $\dot{\mathcal{S}} = \mathcal{S} \backslash z^{-1}(Crit)$. Then any point in $\dot{\mathcal{S}}$ is a regular point wrt the map z. Moreover, the holomorphic map $z : \dot{\mathcal{S}} \to \overline{\mathbb{C}} \backslash Crit$ is a covering of degree n.

So, the ramification points can be found only in the finite set $z^{-1}(Crit)$. Let z_0 be a point in $Crit$. We associate with it a partition of n

$$z_0 \in Crit \quad \Rightarrow \quad \text{a partition} \quad (m_1, \ldots, m_l), \quad m_i > 0, \quad m_1 + \cdots + m_l = n \tag{1.3.22}$$

called the ramification profile of $\mathcal{S}$ over $z_0 \in Crit$. Namely, choose a point z_* close to z_0 and order the n points in the preimage $z^{-1}(z_*)$. Denote $\mu_0 \in S_n$ the permutation generated by a small anticlockwise loop around z_0 wrt the monodromy representation

$$\mu : \pi_1(\overline{\mathbb{C}} \backslash Crit, z_*) \to S_n$$

of the covering $\left(\dot{\mathcal{S}}, \overline{\mathbb{C}} \backslash Crit, z\right)$. The permutation μ_0 can be factorized, in a unique way, into a product of l cycles of the lengths $m_1, \ldots, m_l$. This is the partition in question. Now we are ready to describe the preimage $z^{-1}(z_0) \in \mathcal{S}$ of the point $z_0 \in Crit$. It consists of l points $P_1, \ldots, P_l$ of multiplicities $m_1, \ldots, m_l$ respectively. Since the preimage of any point away from $Crit$ consists of n regular points in $\mathcal{S}$ we assign $(1, 1, \ldots, 1)$ (n times) as the ramification profile over $z_0 \notin Crit$.

Exercise 1.3.59: Describe the ramification points of the holomorphic map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by a polynomial of degree n

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad a_0 \neq 0, \quad \text{for} \quad z \in \mathbb{C}.$$

The above Example is a brief summary of the constructions and results of Section 1.3.2. Our nearest goal is to extend them to arbitrary non-constant meromorphic functions on arbitrary compact Riemann surfaces.

Proposition 1.3.60. *Let $f : \mathcal{S} \to \overline{\mathbb{C}}$ be a non-constant holomorphic map of a compact Riemann surface $\mathcal{S}$. Then*

- *The map f is surjective.*
- *The preimage of any point in $\overline{\mathbb{C}}$ is a finite subset of $\mathcal{S}$.*
- *The number of ramification points on $\mathcal{S}$ is finite.*

Proof For any open subset $U \subset \mathcal{S}$ its image $f(U) \subset \overline{\mathbb{C}}$ is open. This can be easily proven by using (1.3.21). So $f(\mathcal{S})$ is an open subset in $\overline{\mathbb{C}}$. Since $\mathcal{S}$ is compact its image is also a closed subset. Therefore $f(\mathcal{S}) = \overline{\mathbb{C}}$ as $\overline{\mathbb{C}}$ is a connected Hausdorff topological space.

Let us now consider the preimage $f^{-1}(z_0) \subset \mathcal{S}$ of a given point $z_0 \in \overline{\mathbb{C}}$. Suppose it consists of an infinite set of points $P_1, P_2, \ldots$. By definition $f(P_i) = z_0$ for any i. Due to compactness of the Riemann surface one can choose a convergent subsequence $P_{i_k} \to P_0 \in \mathcal{S}$ (the so-called accumulation point of the infinite set). Using the following uniqueness statement from complex analysis

Lemma 1.3.61. *Let f_1, f_2 be two functions holomorphic on an open connected domain $U \subset \mathbb{C}$ taking equal values at the points of an infinite subset with an accumulation point in U. Then $f_1 \equiv f_2$.*

along with connnectedness of $\mathcal{S}$ we conclude that $f \equiv z_0$. Such a contradiction proves the second part of Proposition.

Proof of the third statement of Proposition is quite similar. Namely, if $P_i \in \mathcal{S}$ is a ramification point then $df(P_i)/d\tau = 0$ where τ is a local parameter near P_i. If the set of such points is infinite then, using again the above Lemma and connectedness of $\mathcal{S}$ we conclude that f is a constant map. □

Definition 1.3.62. *Let $f : \mathcal{S} \to \overline{\mathbb{C}}$ be a non-constant holomorphic map of a compact Riemann surface. A point $z_0 \in \overline{\mathbb{C}}$ is called a branch point wrt this map if $z_0 = f(P_0)$ for some ramification point $P_0 \in \mathcal{S}$. The finite set of all branch points will be denoted $Branch \subset \overline{\mathbb{C}}$.*

Remark 1.3.63. If $\mathcal{S}$ is the compact Riemann surface of an algebraic function $w(z)$ then the set of all branch points wrt the map $f(z, w) = z$ belongs to the set *Crit* (see above) but not necessarily coincides with it.

Theorem 1.3.64. *1. Let $f : \mathcal{S} \to \overline{\mathbb{C}}$ be a non-constant holomorphic map of a compact Riemann surface. Denote $\dot{\mathbb{C}} = f^{-1}\left(\overline{\mathbb{C}} \backslash Branch\right)$. Then the triple*

$$\left(\dot{\mathcal{S}}, \overline{\mathbb{C}} \backslash Branch, f\,|_{\dot{\mathcal{S}}}\right) \tag{1.3.23}$$

is a covering of a finite degree n for some $n \geqslant 1$.

2. Let $z_0 \in \overline{\mathbb{C}}$ be a branch point and $\{P_1\} \cup \cdots \cup \{P_l\} = f^{-1}(z_0)$. Denote

$$m_i = mult_f P_i, \quad i = 1, \ldots, l.$$

Then the ramification profile over z_0 wrt the covering (1.3.23) equals $(m_1, \ldots, m_l)$. In particular

$$m_1 + \cdots + m_l = n.$$

Proof Let $\{P_1\} \cup \cdots \cup \{P_n\} = f^{-1}(z_0) \subset \mathcal{S}$ be the full preimage of a point $z_0 \in \overline{\mathbb{C}} \backslash Branch$. All the points $P_1, \ldots, P_n$ are regular. So for every $i = 1, \ldots, n$ there is an open neighbourhood $P_i \in U_i \subset \mathcal{S}$ such that the restriction

$$f : U_i \to V_i \quad \text{for some open neighbourhood} \quad V_i \subset \overline{\mathbb{C}} \backslash Branch \quad \text{of} \quad z_0$$

is biholomorphic. Put $V = \bigcap_{i=1}^n V_i$. Then $f^{-1}(V)$ is biholomorphically equivalent to $V \times \{1, 2, \ldots, n\}$. This proves the first part of Theorem since $\overline{\mathbb{C}} \backslash Branch$ is a connected complex manifold.

Let us proceed to the second part. Choose local parameters $\tau_1, \ldots, \tau_l$ near the points $P_1, \ldots, P_l$ respectively and a local parameter ζ near the branch point z_0 in such a way that $\tau_k(P_k) = 0$ and the map f near P_k has the form

$$\zeta = \tau_k^{m_k}.$$

Note that the local parameter ζ is chosen independently of k; this always can be done. Near P_k the points in the preimage $f^{-1}(\zeta_*)$ for small $|\zeta_*|$ have the form

$$Q_1^{(k)} = \zeta_*^{\frac{1}{m_k}}, \quad Q_2^{(k)} = \omega\, \zeta_*^{\frac{1}{m_k}}, \quad \ldots, \quad Q_{m_k}^{(k)} = \omega^{m_k - 1} \zeta_*^{\frac{1}{m_k}} \quad \text{where} \quad \omega = e^{\frac{2\pi i}{m_k}}.$$

Replacing $\zeta_* \mapsto e^{2\pi i} \zeta_*$ we obtain the action of the monodromy around the branch point z_0 of the covering (1.3.23)

$$\left(Q_1^{(k)}, Q_2^{(k)}, \ldots, Q_{m_k}^{(k)}\right) \mapsto \left(Q_2^{(k)}, \ldots, Q_{m_k}^{(k)}, Q_1^{(k)}\right).$$

This is a cycle of the length m_k, $k = 1, \ldots, l$. □

The following corollary of the Theorem has a particular importance.

Corollary 1.3.65. *Let f be a holomorphic map of a compact Riemann surface to the Riemann sphere. Then*

$$\sum_{\{P \in \mathcal{S} \,|\, f(P) = z_0\}} mult_f(P)$$

does not depend on $z_0 \in \overline{\mathbb{C}}$.

Definition 1.3.66. *The number of sheets of the covering (1.3.23) is called the degree of a meromorphic function f on a compact Riemann surface. It will be denoted by $\deg f$.*

According to Corollary 1.3.65 the degree of a meromorphic function is equal to the number of points, counted with multiplicities, in the preimage of any point in $\overline{\mathbb{C}}$.

Example 1.3.67. The meromorphic function f on $\overline{\mathbb{C}}$ defined by a polynomial of degree n has a single pole at infinity of order n. Thus $\deg f = n$. Applying Corollary 1.3.65 we arrive at the Main Theorem of Algebra saying that the number of roots, counted with multiplicities of a polynomial of degree n is equal to n.

Exercise 1.3.68: Let f be a meromorphic function on a compact Riemann surface $\mathcal{S}$ having only one pole of order 1. Prove that $f : \mathcal{S} \to \overline{\mathbb{C}}$ is a biholomorphic equivalence.

Definition 1.3.69. *A compact Riemann surface is called rational if it is biholomorphically equivalent to the Riemann sphere.*

Exercise 1.3.70: Prove that a compact Riemann surface is rational if and only if there exists a meromorphic function of degree 1 on it.

Exercise 1.3.71: Prove that the field of meromorphic functions (see Remark 1.1.17) on a rational Riemann surface is isomorphic to the field $\mathbb{C}(z)$ of rational functions of one variable.

To conclude this section we briefly discuss more general holomorphic maps between Riemann surfaces. Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a non-constant holomorphic map between compact Riemann surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. The following general propertes of such maps can be established in a way similar to the particular case $\mathcal{S}_2 = \mathbb{P}^1$ considered above. Namely,

- the number of ramification points in $\mathcal{S}_1$ is finite;
- the number of branch points in $\mathcal{S}_2$ is finite;
- the number of points in the preimage $f^{-1}(Q) \subset \mathcal{S}_1$ counted with multiplicities does not depend on the choice of the point $Q \subset \mathcal{S}_2$. This number is called degree of the map f and denoted $\deg f$.

We leave as an exercise to the reader to formulate the precise definitions of a ramification point, branch point, multiplicity of a point.

Example 1.3.72. The holomorphic map (1.1.26) between complex tori has degree n. The sets of ramification points and branch points both are empty.

Exercise 1.3.73: Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a non constant holomorphic map of Riemann surfaces. Prove that if $\mathcal{S}_1$ is compact then so is $\mathcal{S}_2$.

Given a holomorphic map $f : \mathcal{S}_1 \to \mathcal{S}_2$ of compact Riemann surfaces and a meromorphic function φ_2 on $\mathcal{S}_2$ one can construct a meromorphic function φ_1 by using the pullback $\varphi_1 = f^*\varphi_2$

$$f^*\varphi_2(P) = \varphi_2(f(P)).$$

One obtains a homomorphism

$$f^* : \mathcal{M}(\mathcal{S}_2) \to \mathcal{M}(\mathcal{S}_1)$$

of the fields of meromorphic functions. It is an isomorphism iff f is a biholomorphic equivalence.

Proposition 1.3.74. *A holomorphic map $f : \mathcal{S}_1 \to \mathcal{S}_2$ of compact Riemann surfaces is biholomorphic iff* $\deg f = 1$.

Proof of Proposition is left as an exercise for the reader.

Exercise 1.3.75: Consider the compact Riemann surface $\mathcal{S}$ of the algebraic function defined by equation $w^n = P_m(z)$ where $P_m(z)$ is a polynomial of degree m in z with distinct roots. Consider the group of automorphisms of $\mathcal{S}$ of the form

$$J: (z,w) \to (z, e^{2\pi i\, j/n} w), \quad j = 0, 1, \dots, n-1$$

and define the equivalence relation $(z_1, w_1) \simeq (z_2, w_2)$ if $z_1 = z_2$ and $w_1 = e^{2\pi i\, j/n} w_2$ for some j. Show that the quotient surface $\mathcal{S}/J$ is well defined and it is biholomorphic to $\mathbb{P}^1$. Determine the ramification points of the projection map

$$\pi : \mathcal{S} \to \mathcal{S}/J.$$

Example 1.3.76. Consider the hyperelliptic Riemann surface $\mathcal{S}$ of $w^2 = P_{2g+2}(z)$. We show that any such surface is biholomorphically equivalent to some surface $\tilde{\mathcal{S}}$ of the form $\tilde{w}^2 = \tilde{P}_{2g+1}(\tilde{z})$. Let z_0 be one of the zeros of the polynomial $P_{2g+2}(z)$, and let

$$\tilde{z} = \frac{1}{z - z_0}, \quad \tilde{w} = \frac{w}{(z - z_0)^{g+1}}.$$

The point $(z_0, 0) \in \mathcal{S}$ goes to the infinite point of $\tilde{\mathcal{S}}$. The two infinite points $P_\pm \in \mathcal{S}$ where $z \to \infty$ and $w/z^{g+1} \to \pm 1$ go to $(0, \pm 1) \in \tilde{\mathcal{S}}$. The inverse mapping has the form

$$z = z_0 + \frac{1}{\tilde{z}}, \quad w = \frac{\tilde{w}}{\tilde{z}^{g+1}}.$$

If $P_{2g+2}(z) = (z - z_0) \prod_{i=1}^{2g+1} (z - z_i)$, then $\tilde{P}_{2g+1}(\tilde{z}) = \prod_{i=1}^{2g+1} (1 + (z_0 - z_i)\tilde{z})$.

1.3.4 Rational versus meromorphic functions on compact Riemann surfaces

Let $\mathcal{S}$ be the compact Riemann surface of an algebraic function $w(z)$ defined by an irreducible polynomial equation $F(z,w) = 0$. How can we construct meromorphic functions $f : \mathcal{S} \to \mathbb{P}^1$ on it? We already have two meromorphic functions denoted by the same symbols z and w satisfying the identity $F(z(p), w(p)) = 0$ for any $p \in \mathcal{S}$. More generally we can take a rational function of two variables

$$R(z,w) = \frac{P(z,w)}{Q(z,w)}, \quad P(z,w),\ Q(z,w) \in \mathbb{C}[z,w] \tag{1.3.24}$$

and restrict it on $\mathcal{S}$, i.e., define

$$f(p) = R(z(p), w(p)), \quad p \in \mathcal{S}. \tag{1.3.25}$$

The following simple statement says that, under a natural assumption about the denominator the above construction produces a meromorphic function on the Riemann surface.

Proposition 1.3.77. *Assume that the restriction on $\mathcal{S}$ of the polynomial $Q(z,w)$ does not vanish identically. Then the rational function (1.3.24), (1.3.25) is meromorphic on $\mathcal{S}$.*

Proof Any algebraic combination of the meromorphic functions z, w is a meromorphic function on $\mathcal{S}$. So the functions $P(z(p), w(p))$, $Q(z(p), w(p))$ are meromorphic and the latter one is not an identical zero. The ratio of these meromorphic functions is also meromorphic. $\square$

We will now prove the converse statement.

Theorem 1.3.78. *Let $\mathcal{S}$ be the compact Riemann surface of the algebraic function $w(z)$ defined by an irreducible equation*

$$F(z, w) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0 \tag{1.3.26}$$

Let f be a meromorphic function on $\mathcal{S}$. Then f can be represented as a rational function of z and w.

Proof Within this proof it will be convenient to redenote by $\pi : \mathcal{S} \to \overline{\mathbb{C}}$ the function $p \mapsto z(p)$. Take a generic point z on the complex plane such that its preimage $\pi^{-1}(z) \subset \mathcal{S}$ consists of n distinct points. Ordering them in an arbitrary way we obtain two n-tuples of locally well defined functions $w_1(z), \ldots, w_n(z)$ and $f_1(z), \ldots, f_n(z)$. Consider the following combinations

$$\begin{array}{rcc} b_1(z) & = & f_1(z) + \cdots + f_n(z) \\ b_2(z) & = & w_1(z)f_1(z) + \cdots + w_n(z)f_n(z) \\ & & \cdots \\ b_n(z) & = & w_1(z)^{n-1}f_1(z) + \cdots + w_n(z)^{n-1}f_n(z). \end{array} \tag{1.3.27}$$

They do not depend on the choice of the order of points in the preimage hence they are monodromy invariant. Due to the monodromy invariance of $b_1(z), \ldots, b_n(z)$ they are rational functions in z. We now look at (1.3.27) as at a system of linear equations for $f_1(z), \ldots, f_n(z)$. The determinant $D(w_1(z), \ldots, w_n(z))$ of the matrix of coefficients of the system is nothing but the Vandermonde determinant

$$D(w_1(z), \ldots, w_n(z)) = \prod_{i>j} \left(w_i(z) - w_j(z)\right).$$

According to Exercise 1.2.5 it is equal to

$$D(w_1(z), \ldots, w_n(z)) = \frac{\sqrt{\Delta_F(z)}}{a_0(z)^{n-1}}$$

where $\Delta_F(z)$ is the discriminant of the polynomial $F(z, w)$. So it is not an identical zero.

Using Kramer rule write an explicit formula for the solution of the linear system. For $f_1(z)$ we have

$$f_1(z) = \frac{D(b(z), w_2(z), \ldots, w_n(z))}{D(w_1(z), w_2(z), \ldots, w_n(z))}$$

where $D(b(z), w_2(z), \ldots, w_n(z))$ is obtained from the Vandermonde determinant by replacing the first column $\left(1, w_1(z), \ldots, w_1(z)^{n-1}\right)$ by $(b_1(z), \ldots, b_n(z))$. Multiplying both the numerator and denominator by the Vandermonde

$$f_1(z) = \frac{a_0(z)^{2n-2}}{\Delta_F(z)} D(b(z), w_2(z), \ldots, w_n(z)) D(w_1(z), w_2(z), \ldots, w_n(z))$$

we obtain a polynomial in $w_1(z)$ whose coefficients are rational functions in z combined with symmetric polynomials in $w_2(z), \dots, w_n(z)$. These symmetric polynomials can be expressed via the coefficients of

$$(w - w_2(z))\dots(w - w_n(z)) = \frac{1}{a_0(z)}\frac{F(z,w)}{w - w_1(z)} = \frac{1}{a_0(z)}\left[a_0(z)w^{n-1} + (a_0(z)w_1(z) + a_1(z))\,w^{n-2} + \dots\right].$$

The coefficients of this polynomial are rational functions in z and $w_1(z)$. We finally arrive at an expression of the form

$$f_1(z) = R(z, w_1(z))$$

where $R(z,w)$ is some rational function in two variables. For other functions $f_k(z)$ we obtain similar expressions

$$f_k(z) = R(z, w_k(z)), \quad k = 2, \dots, n$$

with the same $R(z,w)$. Therefore $f = R(z,w)$. $\square$

Example 1.3.79. Let $\mathcal{S}$ be the hyperelliptic Riemann surface

$$w^2 = P_{2n+1}(z), \quad P_{2n+1}(z) = \prod_{i=1}^{2n+1}(z - a_i), \quad a_i \neq a_j \quad \text{for} \quad i \neq j.$$

The functions z and w are holomorphic in the finite part of $\mathcal{S}$. These functions have poles at the infinite point of $\mathcal{S}$, namely, z has a double pole and w has a pole of order $2n+1$. The function z has on $\mathcal{S}$ two simple zeros at the points $z = 0$, $w = \pm\sqrt{P_{2n+1}(0)}$ that merge into a single double zero if $P_{2n+1}(0) = 0$. The function w has $2n+1$ simple zeros on $\mathcal{S}$ at the branch points. The function $1/(z - a_i)$ has a unique second order pole at the i-th ramification point on $\mathcal{S}$ and a double zero at infinity. More general rational functions on $\mathcal{S}$ have the form

$$R(z,w) = \frac{P_0(z) + P_1(z)\,w}{Q_0(z) + Q_1(z)\,w}$$

for some polynomials $P_{0,1}(z)$, $Q_{0,1}(z)$. Multiplying both the numerator and the denominator by $Q_0(z) - Q_1(z)\,w$ we can rewrite the function in the form

$$R(z,w) = R_0(z) + R_1(z)\,w$$

where $R_{0,1}(z)$ are rational functions of z.

Exercise 1.3.80: Describe poles and zeros of the meromorphic function

$$R(z,w) = \frac{w}{\prod_{i=1}^{n}(z - a_i)}$$

on the hyperelliptic Riemann surface of the above example.

Exercise 1.3.81: On the same hyperelliptic surface, consider n points $p_1 = (z_1, w_1), \dots, p_n = (z_n, w_n)$ in the finite part of $\mathcal{S}$ satisfying $z_i \neq z_j$ for $i \neq j$ and $w_1 \cdots w_n \neq 0$. Construct a meromorphic function f on $\mathcal{S}$ with simple poles at $p_1, \dots, p_n$ and at infinity. Prove that such a function is unique up to a transformation $f \mapsto af + b$, $a \neq 0$, $b \in \mathbb{C}$,

Exercise 1.3.82: Prove that any meromorphic function on the Riemann surface of the algebraic function $w(z)$ defined by eq. (1.3.26) can be represented in the form

$$f = R_0(z) + R_1(z)w + \cdots + R_{n-1}(z)w^{n-1}$$

where $R_0(z), \ldots, R_{n-1}(z)$ are rational functions of z.

Exercise 1.3.83: Let $\mathcal{S}$ be a compact Riemann surface represented as a smooth projective curve in $\mathbb{P}^2$. Prove that any meromorphic function f on $\mathcal{S}$ can be represented in the form

$$f(X:Y:Z) = \frac{P(X,Y,Z)}{Q(X,Y,Z)}.$$

Here $(X : Y : Z)$ are homogeneous coordinates of a point on $\mathcal{S}$, P and Q are homogeneous polynomials of the same degree such that Q does not vanish identically on the curve.

Exercise 1.3.84: Let $\mathcal{S}$ be a compact Riemann surface and f a degree n meromorphic function on it. Let g be another meromorphic function on $\mathcal{S}$. Prove that these functions are algebraically dependent that is, there exists a polynomial $F(z,w)$ of degree n in w such that

$$F\left(f(p), g(p)\right) = 0 \quad \forall\, p \in \mathcal{S}.$$

Example 1.3.85. Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be two compact Riemann surfaces realized as smooth projective curves in $\mathbb{P}^2$ defined by homogeneous polynomial equations $Q(X,Y,Z) = 0$ and $\tilde{Q}(X,Y,Z) = 0$ respectively. A map $f : \mathcal{S} \to \tilde{\mathcal{S}}$ is called rational if it can be represented in the form

$$f(X:Y:Z) = (A(X,Y,Z) : B(X,Y,Z) : C(X,Y,Z))$$

where A, B, C are three homogeneous polynomials of the same degree such that none of them vanishes identically on $\mathcal{S}$ and satisfying

$$(A(X,Y,Z), B(X,Y,Z), C(X,Y,Z)) \neq (0,0,0) \quad \forall\, (X:Y:Z) \in \mathcal{S}$$

and

$$\tilde{Q}\left(A(X,Y,Z), B(X,Y,Z), C(X,Y,Z)\right) = 0 \quad \forall\, (X:Y:Z) \in \mathcal{S}.$$

Let us prove that the map f is holomorphic. Consider a point $p \in \mathcal{S}$ belonging to the chart $U_Z = \{(X : Y : Z) \,|\, Z \neq 0\}$ on the projective plane and assume that $\tilde{p} = f(p) \in \tilde{\mathcal{S}}$ belongs to the same chart. Then locally, near p the map f in the coordinates $x = X/Z$, $y = Y/Z$ is given by a pair of rational functions

$$f : (x,y) \mapsto (\tilde{x}, \tilde{y}) = \left(\frac{A(x,y,1)}{C(x,y,1)}, \frac{B(x,y,1)}{C(x,y,1)}\right).$$

Due to smoothness of the curve $\mathcal{S}$ one of the coordinates x or y can be used as a local parameter near p; let it be $x =: \tau$ then $y = y(\tau)$ is a locally defined holomorphic function. In a similar way near $\tilde{p}$ assume that, say, $\tilde{y} =: \tilde{\tau}$ works as a local parameter on $\tilde{\mathcal{S}}$. Then the map f is locally given by the holomorphic function

$$\tilde{\tau} = \frac{B(\tau, y(\tau), 1)}{C(\tau, y(\tau), 1)}.$$

In a similar way one can consider other combinations of charts on $\mathbb{P}^2$ and other choices of local parameters on the curves.

Let us now prove the converse statement saying that any holomorphic map between smooth projective curves is rational. To this end take two meromorphic functions $\tilde{x} = X/Z$, $\tilde{y} = Y/Z$ on $\tilde{\mathcal{S}}$. Their pullbacks $f^*\tilde{x}$ and $f^*\tilde{y}$ are meromorphic functions on $\mathcal{S}$. Hence, according to Theorem 1.3.78 they are rational functions on the curve $\mathcal{S}$

$$\tilde{x} = \frac{a(x,y)}{c(x,y)}, \quad \tilde{y} = \frac{b(x,y)}{c(x,y)}$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$ are polynomials; we have reduced the two fractions to a common denominator $c(x,y)$. Let p, q, r be non-negative integers such that

$$Z^p a(X/Z, Y/Z) = A(X,Y,Z), \quad Z^q b(X/Z, Y/Z) = B(X,Y,Z), \quad Z^r c(X/Z, Y/Z) = C(X,Y,Z)$$

with some homogeneous polynomials A, B, C. Denote $m = \max(p,q,r)$. Then f coincides with the rational map

$$f(X:Y:Z) = \left(Z^{m-p}A(X,Y,Z) : Z^{m-q}B(X,Y,Z) : Z^{m-r}C(X,Y,Z)\right).$$

Exercise 1.3.86: Let $\mathcal{S}$ be a non-singular projective curve defined as $\mathcal{S} := \{(X:Y:Z) \in \mathbf{P}^2 \,|\, Q(X,Y,Z) = 0\}$ where Q is an irreducible homogeneos polynomial of degree $n \geqslant 2$. Show that the map

$$(X:Y:Z) \to (Q_X : Q_Y : Q_Z)$$

from $\mathcal{S}$ to $\mathbf{P}^2$ is well defined. The image of such a map is called the dual curve $\hat{\mathcal{S}}$ to $\mathcal{S}$. Find the dual curves for a conic and for the Fermat cubic $x^3 + y^3 + z^3 = 0$. Show that the map is holomorphic but it does not have a holomorphic inverse if $n \geqslant 3$.

Example 1.3.87. Let C be the algebraic curve defined by an irreducible polynomial equation $F(z,w) = 0$. Denote by $\mathcal{S}$ the compact Riemann surface of an algebraic function $w(z)$ defined by the same equation. The surface is equipped with a pair of meromorphic functions $\hat{z}$, $\hat{w}$ that define a map

$$\rho : \mathcal{S} \to C, \quad \rho(P) = (\hat{z}(P), \hat{w}(P))$$

biholomorphic outside a finite number of points, see Theorem 1.3.48 above. We want to compare rational functions on C and on $\mathcal{S}$ especially for the case when the curve has singularities. More precisely, we have a natural pullback map

$$\rho^* : \{\text{rational functions globally defined on } C\} \to \{\text{meromorphic functions on } \mathcal{S}\} \tag{1.3.28}$$

$$\rho^*(f)(P) = f(\rho(P)), \quad P \in \mathcal{S}, \quad \text{for a rational function } f \text{ on } C.$$

What is the image of this map?

Let us begin with a simple example of the curve

$$C : \; w^2 = z^3 + z^2.$$

The Riemann surface $\mathcal{S}$ is rational. It can be described by the equation $\tilde{w}^2 = \tilde{z} + 1$, see Example **??** above. The map ρ has the form[11]

$$\rho(\tilde{z}, \tilde{w}) = (z, w) \quad \text{where} \quad z = \tilde{z},\ w = \tilde{z}\,\tilde{w}.$$

The two points $P_\pm = (\tilde{z} = 0, \tilde{w} = \pm 1) \in \mathcal{S}$ go to the same point $\rho(P_\pm) = (0, 0)$ on C. Thus the pullback ρ^* of any rational function globally defined on the curve C consists of meromorphic functions on $\mathcal{S}$ taking equal values at the points $P_\pm$. It remains to observe that, due to the rationality of $\mathcal{S}$ the space of meromorphic functions on it is isomorphic to the space of rational functions of the variable $\tilde{w}$. Therefore the image of the map (1.3.28) consists of rational functions $f(\tilde{w})$ satisfying $f(1) = f(-1)$.

In a similar way one can deal with rational functions globally defined on an irreducible algebraic curve with n nodal singularities assuming rationality of the corresponding compact Riemann surface (as an example one can take the curve $w^2 = z \prod_{i=1}^n (z - z_i)^2$). Then the image of the pullback map (1.3.28) consists of rational functions of one variable satisfying

$$f(a_i) = f(b_i), \quad i = 1, \ldots, n$$

for some pairwise distinct complex numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$.

1.4 Example: complex tori and elliptic functions

Let $T^2 = T^2_{\omega,\omega'}$ be a complex torus

$$T^2 = \mathbb{C}/\Lambda_{\omega,\omega'} \tag{1.4.1}$$

where

$$\Lambda_{\omega,\omega'} = \{2m\omega + 2n\omega' \mid m, n \in \mathbb{Z}\} \tag{1.4.2}$$

be the period lattice defined by a pair of complex numbers ω, ω' satisfying

$$\Im(\omega'/\omega) > 0.$$

We already know that there are no non-constant holomorphic functions on the torus and any meromorphic functions on T^2 can be considered as doubly periodic meromorphic function on the complex plane

$$f(z + 2\omega) = f(z), \quad f(z + 2\omega') = f(z) \quad \forall\, z \in \mathbb{C}.$$

Such functions will be called elliptic for the reasons that will be explained later.

Values of an elliptic function at any point of the complex plane are uniquely determined by its restriction onto the fundamental parallelogram consisting of complex numbers z of the form

$$z = z_0 + 2x\omega + 2y\omega', \quad 0 \leqslant x,\ y \leqslant 1 \tag{1.4.3}$$

[11] One can formally invert ρ in the class of rational maps

$$\rho^{-1}(z, w) = (z, w/z).$$

That means that $\mathcal{S}$ and C are birationally equivalent. Observe that the function w/z is not defined at the singular point $(0, 0)$ of the curve C.

We leave as an exercise to the reader to prove birational equivalence between an arbitrary irreducible algebraic curve and the corresponding compact Riemann surface constructed in Theorem 1.3.48.

for a given $z_0 \in \mathbb{C}$. There is only finite number of poles of an elliptic function inside the parallelogram or on its boundary. Choosing appropriately the vertex z_0 we can free the boundary of (1.4.3) of the poles of the function.

Proposition 1.4.1. *Let $z_1, \ldots, z_k$ be the poles of an elliptic function f inside a fundamental parallelogram (1.4.3). Assume that there are no poles on the boundary of the parallelogram. Then*

$$\sum_{i=1}^{k} \operatorname*{Res}_{z=z_i} f(z)\, dz = 0.$$

Proof According to Cauchy theorem

$$\sum_{i=1}^{k} \operatorname*{Res}_{z=z_i} f(z)\, dz = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

where C is the boundary of the parallelogram oriented in the anti-clockwise direction. On the opposite sides of the boundary the function takes equal values. So the contour integral in the above equation vanishes. □

Corollary 1.4.2. *There is no elliptic functions with only one simple pole in the fundamental parallelogram.*

Remark 1.4.3. According to Exercise 1.3.25 the above Corollary implies that the complex torus is not biholomorphically equivalent to the Riemann sphere. In Section 2.1 below we give another proof of this statement based on simple topological arguments.

Exercise 1.4.4: For a given elliptic function $f(z)$ of degree n choose a fundamental parallelogram containing neither zeros nor poles of f on its boundary. Denote $a_1, \ldots, a_n$ the zeros and $b_1, \ldots, b_n$ the poles of f inside the parallelogram repeated according to their multiplicities. Prove that

$$\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \in \Lambda_{\omega,\omega'}.$$

Hint: consider the integral

$$\frac{1}{2\pi i} \oint z \frac{f'(z)}{f(z)}\, dz$$

over the boundary of the parallelogram.

We now construct the first example of an elliptic function with one double pole in the parallelogram. The Weierstrass elliptic function, $\wp(z)$ is defined by

$$\wp(z) = \wp(z \,|\, \omega, \omega') = \frac{1}{z^2} + \sum_{m^2+n^2 \neq 0} \left[\frac{1}{(z - w_{mn})^2} - \frac{1}{w_{mn}^2} \right]. \tag{1.4.4}$$

Here and below we use the notation

$$w_{mn} = 2m\omega + 2n\omega', \quad m, n \in \mathbb{Z} \tag{1.4.5}$$

for the points of the lattice. It is not difficult to verify that the (1.4.4) converges absolutely and uniformly on compact sets not containing points of the period lattice. Therefore, it defines a meromorphic function of z having double poles at the lattice nodes. This function is obviously doubly periodic: $\wp(z + 2k\omega + 2l\omega') = \wp(z)$, $k, l \in \mathbb{Z}$. It is an even function $\wp(-z) = \wp(z)$.

Exercise 1.4.5: Let f be a meromorphic function on the complex torus (1.4.1) having only one pole of order two at $z = 0$. Prove that

$$f(z) = a\wp(z) + b, \quad a, b \in \mathbb{C}.$$

The Laurent expansions of the functions $\wp(z)$ and $\wp'(z)$ have the following forms as $z \to 0$

$$\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{20} + \frac{g_3 z^4}{28} + \dots, \tag{1.4.6}$$

$$\wp'(z) = -\frac{2}{z^3} + \frac{g_2 z}{10} + \frac{g_3 z^3}{7} + \dots, \tag{1.4.7}$$

where

$$\begin{aligned} g_2 &= 60 \sum_{m^2+n^2\neq 0} w_{mn}^{-4} \\ g_3 &= 140 \sum_{m^2+n^2\neq 0} w_{mn}^{-6}, \end{aligned} \tag{1.4.8}$$

(verify!). This implies that the Laurent expansion of the function $(\wp')^2 - 4\wp^3 + g_2\wp + g_3$ has the form $O(z)$ as $z \to 0$. Hence, this doubly periodic function is constant, and thus equals zero. Conclusion: the Weierstrass function $\wp(z)$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3. \tag{1.4.9}$$

Let us now map the torus $T^2(\omega, \omega')$ to the elliptic curve C, where

$$C: \; Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3 \tag{1.4.10}$$

by setting $f : T^2(\omega, \omega') \to C$ with

$$f(z) = \begin{cases} (\wp(z), \wp'(z), 1), & z \neq 0 \\ (0, 1, 0), & z = 0 \end{cases} \tag{1.4.11}$$

Theorem 1.4.6. *1. The elliptic curve (1.4.10) is non-singular.*

2. *The map (1.4.11) of the complex torus (1.4.1) to the Riemann surface (1.4.10) is a biholomorphic isomorphism.*

3. *Any non-singular elliptic curve of the form (1.4.10) is biholomorphically equivalent to a complex torus of the form (1.4.1)*

Proof As the Weierstrass function has on the torus only one pole of order two, the degree of the holomorphic map $\wp : T^2 \to \overline{\mathbb{C}}$ is equal to two. That means that for a given $u \in \mathbb{C}$ the equation $\wp(z) = u$ has two solutions counted with multiplicities. If z is a solution then so is $-z$ since the function is even. These two solutions are distinct iff $2z \notin \Lambda_{\omega,\omega'}$. Therefore the ramification points of the holomorphic map coincide with the half-periods of the lattice. All of them have multiplicity two.

Modulo the lattice there are four half-periods: $\omega_0 = 0$ and

$$\omega_1 = \omega, \quad \omega_2 = -\omega - \omega', \quad \omega_3 = \omega'. \tag{1.4.12}$$

The point ω_0 makes the preimage of the infinite point in $\overline{\mathbb{C}}$. Denote

$$e_i = \wp(\omega_i), \quad i = 1, 2, 3. \tag{1.4.13}$$

Lemma 1.4.7. *1. The complex numbers e_1, e_2, e_3 are pairwise distinct.*

2. They are roots of the cubic equation $4u^3 - g_2 u - g_3 = 0$ where g_2, g_3 are defined by eqs. (1.4.8).

Proof Suppose, for example that $e_1 = e_2$. Then the full preimage $\wp^{-1}(e_1)$ consists of two points ω_1 and ω_2 of the total multiplicity four – a contradiction.

To prove the second part of Lemma we observe that $\wp'(z)$ is an odd function. So

$$\wp'(\omega_i) = -\wp'(-\omega_i) = -\wp'(-\omega_i + 2\omega_i) = -\wp'(\omega_i) \quad \Rightarrow \quad \wp'(\omega_i) = 0, \quad i = 1, 2, 3.$$

Substituting $z = \omega_i$ in eq. (1.4.9) we obtain

$$0 = 4e_i^3 - g_2 e_i - g_3, \quad i = 1, 2, 3.$$

□

The first statement of Theorem readily follows from Lemma. To prove the second statement it suffices to prove that the degree of the map (1.4.11) is equal to one. That is, for a given point (X, Y, Z), $Z \neq 0$, of the curve (1.4.10) we have to solve the system of equations

$$\begin{cases} \wp(z) &= u, \quad u = \frac{X}{Z}, \quad v = \frac{Y}{Z} \\ \wp'(z) &= v \end{cases}$$

If $v \neq 0$ then the first equation has two distinct solutions z and $-z$. The second equation selects only one of them since $\wp'(z) \neq \wp'(-z)$ in this case. Let us now consider the case $v = 0$. Then, we have $u = e_1$, e_2 or e_3. We already know that the equation $\wp'(z) = 0$ has three distinct solutions $z = \omega_1, \omega_2$ and ω_3. Since $\wp'$ is a degree three meromorphic function on the torus there are no other solutions. So we have uniqueness of the solution to the system also in this case. Finally for the point at infinity $(0, 1, 0)$ of (1.4.10) the unique point in the preimage is $z = 0$.

The proof of the third part of Theorem follows from the following lemma.

Lemma 1.4.8. *Consider the affine curve $v^2 = 4u^3 - g_2 u - g_3$ and defined the integrals*

$$\omega_i = \int_\infty^{e_i} \frac{du}{\sqrt{4u^3 - g_2 u - g_3}}, \quad i = 1, 2, 3$$

for a suitable choice of the square root. The solution to the differential equation

$$\left(\frac{du}{dz}\right)^2 = 4u^3 - g_2 u - g_3 \tag{1.4.14}$$

satisfying

$$u(z) = \frac{1}{z^2} + O\left(\frac{1}{z}\right), \quad z \to 0$$

has the form

$$u(z) = \wp(z \,|\, \omega, \omega')$$

where $\omega = \omega_1, \quad \omega' = \omega_3.$

Proof The differential equation (1.4.14) can be solved by quadratures. Indeed we can write it in the form

$$\frac{du}{\sqrt{4u^3 - g_2 u - g_3}} = dz$$

so that

$$z(u) = \int_\infty^u \frac{du'}{\sqrt{4(u')^3 - g_2 u' - g_3}}, \quad z(u) = \pm \frac{1}{\sqrt{u}} + O(u^{-\frac{5}{2}})$$

It is more convenient to use the notation

$$z(P) = \int_\infty^P \frac{du'}{v'}, \quad P = (u, v).$$

For $P \to P + \gamma$ where γ is a loop in C we have

$$z(P) \to z(P) + \int_\gamma \frac{du'}{v'}$$

The inverse map

$$z \mapsto P(z) = (u(z), v(z))$$

satisfies

$$u\left(z + \oint_\gamma \frac{du'}{v'}\right) = u(z), \quad v\left(z + \oint_\gamma \frac{du'}{v'}\right) = v(z)$$

Choose γ_i the path from ∞ to e_i on the first sheet and back to the second sheet so that

$$\int_{\gamma_i} \frac{du'}{v'} = 2\omega_i$$

so

$$u(z + 2\omega_i) = u(z), \quad v(z + 2\omega_i) = v(z), \;\; i = 1, 2, 3.$$

One has $\omega_1 + \omega_2 + \omega_3 = 0$. So we choose $2\omega := 2\omega_1$ and $2\omega' := 2\omega_3$. Then $u(z)$ and $v(z)$ are elliptic functions on the torus $T^2_{(\omega,\omega')}$. Further

$$u(z) + \frac{1}{z^2} + O(z^{-4}), \quad v(z) = -\frac{2}{z^3} + O(z) \quad z \to 0$$

We conclude that $u(z) = \wp(z)$ and $v(z) = \wp'(z)$.

Namely, the inverse function to the solution in question can be written as elliptic integral

$$z(Z) = \int_\infty^Z \frac{dZ}{\sqrt{4Z^3 - g_2 Z - g_3}}.$$

For sufficiently large $|Z|$ the function

$$z(Z) = \pm \frac{1}{\sqrt{Z}} + O\left(Z^{-5/2}\right) \tag{1.4.15}$$

is well defined up to a sign. We can extend it to a (multivalued) function $z(P)$, $P = (Z, W)$ on the elliptic Riemann surface (1.4.10) by the integral

$$z(P) = \int_\infty^P \frac{dZ}{W}$$

along some path from the infinite point of the surface to the point P. For a given P it depends only on the homotopy class of the path with fixed endpoints. A change of the homotopy class changes the integral as

$$z(P) \to z(P) + \oint_\gamma \frac{dZ}{W}$$

for a loop γ on the Riemann surface. Therefore the inverse map

$$z \mapsto P(z) = (Z(z), W(z))$$

satisfies

$$Z\left(z + \oint_\gamma \frac{dZ}{W}\right) = Z(z), \quad W\left(z + \oint_\gamma \frac{dZ}{W}\right) = W(z)$$

for any loop γ. Take the following particular loops γ_i, $i = 1, 2, 3$ as follows: choose a path from infinity to e_i on one sheet of the Riemann surface then return back along the same path on another sheet. Then

$$\oint_{\gamma_i} \frac{dZ}{W} = 2\omega_i, \quad i = 1, 2, 3,$$

see eq. (**??**). We obtain

$$Z(z + 2\omega_i) = Z(z), \quad W(z + 2\omega_i) = W(z), \quad i = 1, 2, 3.$$

It is easy to see that, under a suitable choice of orientations on the loops one has $\omega_1 + \omega_2 + \omega_3 = 0$. So we choose $2\omega := 2\omega_1$ and $2\omega' := 2\omega_3$ as two independent periods. It remains to prove that $\mathrm{Im}(\omega'/\omega) > 0$. To this end consider the following integral

$$\frac{i}{2}\iint \frac{dZ}{W} \wedge \overline{\frac{dZ}{W}} = \frac{i}{2}\iint \frac{dZ \wedge d\overline{Z}}{|W|^2} > 0$$

over the Riemann surface. Applying Stokes theorem rewrite it as a contour integral

$$\frac{i}{2}\oint z(P)\,\overline{\frac{dZ}{W}}$$

over two sides of the loops γ_1 and γ_3. The latter is equal (cf. the proof of Lemma 3.1.16 below) to $2i(\omega\,\overline{\omega'} - \omega'\overline{\omega}) = 4|\omega|^2 \mathrm{Im}\,\frac{\omega'}{\omega}$.

We conclude that $Z(z)$ and $W(z)$ are elliptic functions on the complex torus $T^2_{\omega,\omega'}$. They have poles only at $z = 0$. From (1.4.15) it follows that

$$Z(z) = \frac{1}{z^2} + O(z^2), \quad W(z) = -\frac{2}{z^3} + O(z) \quad \text{for} \quad z \to 0.$$

Hence

$$Z(z) = \wp(z\,|\,\omega,\omega'), \quad W(z) = \wp'(z\,|\,\omega,\omega').$$

This completes the proof of Lemma and, therefore of Theorem. □

Exercise 1.4.9: Prove that any elliptic function $f(z)$ with period lattice $\{2m\omega + 2n\omega'\}$ can be represented in the form

$$f(z) = P\,[\wp(z)] + Q\,[\wp(z)]\,\wp'(z)$$

where P and Q are rational functions.

Exercise 1.4.10: Prove the following addition theorem for the Weierstrass function

$$\det\begin{pmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(u+v) & -\wp'(u+v) \end{pmatrix} = 0 \quad \forall\, u,\, v. \tag{1.4.16}$$

Derive that the map (1.4.11) is an isomorphism of the group of points on the torus $T^2 = \mathbb{C}/\{2\omega\mathbb{Z}\oplus 2\omega'\mathbb{Z}\}$ to the group of points on the cubic (1.4.10) with the marked point at infinity, see Exercise 1.2.40 above.

Example 1.4.11. Let us briefly consider behaviour of elliptic functions under holomorphic maps between complex tori. Take the first nontrivial case of the degree two map

$$f_2 : T^2_{\omega,\omega'} \to T^2_{\frac{\omega}{2},\omega'},$$

see eq. (1.1.26) above. To compute the pullback of the Weierstrass function $\wp(z\,|\,\frac{\omega}{2},\omega')$ on the torus $T^2_{\omega/2,\omega'}$ we have to express it via $\wp(z) = \wp(z\,|\,\omega,\omega')$. Proof of the resulting expression

$$\wp\left(z\,|\,\frac{\omega}{2},\omega'\right) = \wp(z) + \wp(z-\omega) - e_1$$

(the so-called Landen transformation for Weierstrass functions) is left as an exercise to the reader.

Exercise 1.4.12: Prove that

$$\left(\wp\left(\frac{2}{3}m\omega + \frac{2}{3}n\omega'\right), \wp'\left(\frac{2}{3}m\omega + \frac{2}{3}n\omega'\right)\right), \quad 0 \leqslant m,\, n \leqslant 2$$

are the inflection points of the cubic (1.4.10), see Exercise 1.2.39 above.

Exercise 1.4.13: Let $\wp(z)$ be the Weierstrass function with a rectangular lattice of periods

$$\omega \in \mathbb{R}_{>0}, \quad \omega' \in i\,\mathbb{R}_{>0}.$$

(1) Prove that $\wp(z)$ takes real values on the lines of four types

$$\operatorname{Re} z = 2m\omega, \quad \text{or} \quad i \operatorname{Im} z = 2n\omega'$$

and

$$\operatorname{Re} z = (2m+1)\omega, \quad \text{or} \quad i \operatorname{Im} z = (2n+1)\omega'$$

with $m, n \in \mathbb{Z}$.

(2) Prove that the coefficients g_2, g_3 given by eqs. (1.4.8) are real.

(3) Prove that the roots (1.4.13) of the cubic polynomial $4Z^3 - g_2 Z - g_3$ are real and satisfy the inequalities

$$e_1 > e_2 > e_3.$$

Observe that $e_1 > 0$ and $e_3 < 0$.

(4) Prove that $\wp(z)$ restricted onto the line $i \operatorname{Im} z = (2n+1)\omega'$, $n \in \mathbb{Z}$ satisfies

$$e_3 \leqslant \wp(z) \leqslant e_2$$

and its restriction onto the line $\operatorname{Re} z = (2m+1)\omega$, $m \in \mathbb{Z}$ satisfies

$$e_2 \leqslant \wp(z) \leqslant e_1.$$

(5) Prove that any elliptic Riemann surface (1.4.10) with real branch points is biholomorphically equivalent to a complex torus with a rectangular lattice of periods.

Define the Weierstrass ζ- and σ-functions useful in the theory of elliptic functions by quadratures

$$\zeta'(z) = -\wp(z), \quad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z) \tag{1.4.17}$$

assuming that the integration constants are chosen in such a way that, for $z \to 0$

$$\zeta(z) = \frac{1}{z} + O\left(z^3\right), \quad \sigma(z) = z + O\left(z^5\right). \tag{1.4.18}$$

They are given by the following expansion

$$\zeta(z) = \zeta(z\,|\,\omega, \omega') = \frac{1}{z} + \sum_{m^2+n^2 \neq 0} \left[\frac{1}{z - w_{mn}} + \frac{1}{w_{mn}} + \frac{z}{w_{mn}^2}\right]. \tag{1.4.19}$$

and infinite product

$$\sigma(z) = \sigma(z\,|\,\omega, \omega') = z \prod_{m^2+n^2 \neq 0} \left\{\left(1 - \frac{z}{w_{mn}}\right) \exp\left[\frac{z}{w_{mn}} + \frac{z^2}{2w_{mn}^2}\right]\right\}. \tag{1.4.20}$$

The Weierstrass ζ-function has simple poles at the points of the period lattice. The function $\sigma(z)$ is an entire function on the complex plane. It has simple zeros at the points of the period lattice. These functions satisfy

$$\zeta'(z) = -\wp(z), \quad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z). \tag{1.4.21}$$

The functions $\zeta(z)$ and $\sigma(z)$ are not elliptic; under a translation of the argument by a vector of the period lattice they transform according to

$$\zeta(z+2\omega) = \zeta(z) + 2\eta, \quad \zeta(z+2\omega') = \zeta(z) + 2\eta' \tag{1.4.22}$$

$$\sigma(z+2\omega) = -\sigma(z)\exp[2\eta(z+\omega)], \;\; \sigma(z+2\omega') = -\sigma(z)\exp[2\eta'(z+\omega')] \tag{1.4.23}$$

where η and η' are constants depending on the period lattice.

Exercise 1.4.14: Prove that

$$\eta = \zeta(\omega), \quad \eta' = \zeta(\omega'). \tag{1.4.24}$$

Exercise 1.4.15: Prove the transformation law (1.4.23).

Exercise 1.4.16: Integrating $\zeta(z)$ over the fundamental parallelogram centered at the origin, prove Legendre relation

$$\eta\,\omega' - \eta'\omega = \frac{\pi i}{2}. \tag{1.4.25}$$

Exercise 1.4.17: Prove that the sum

$$\sum_{k=1}^{n} c_k \zeta(z - z_k) + c_0 \tag{1.4.26}$$

is an elliptic function in z iff the coefficients $c_1, \ldots, c_n$ satisfy

$$c_1 + \cdots + c_n = 0.$$

Prove that any elliptic function with only simple poles can be represented in the form (1.4.26).

Exercise 1.4.18: Derive the following expression for the elliptic function $\zeta(u+v) - \zeta(u) - \zeta(v)$

$$\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}. \tag{1.4.27}$$

Exercise 1.4.19: Prove that the function

$$\prod_{k=1}^{n} \frac{\sigma(z - a_k)}{\sigma(z - b_k)} \tag{1.4.28}$$

is an elliptic function in z iff

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k.$$

Prove that any elliptic function can be represented in the form (1.4.28), up to a constant factor.

Exercise 1.4.20: Prove the following identity:

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} = \wp(v) - \wp(u). \tag{1.4.29}$$

Exercise 1.4.21: Prove the following generalization of the previous identity

$$\det \begin{pmatrix} 1 & \wp(u_0) & \wp'(u_0) & \dots & \wp^{(n-1)}(u_0) \\ 1 & \wp(u_1) & \wp'(u_1) & \dots & \wp^{(n-1)}(u_1) \\ . & \dots & \dots & \dots & \dots \\ 1 & \wp(u_n) & \wp'(u_n) & \dots & \wp^{(n-1)}(u_n) \end{pmatrix} = (-1)^{\frac{n(n-1)}{2}} 1!\,2!\dots n! \frac{\sigma(u_0+u_1+\dots+u_n)\prod_{i<j}\sigma(u_i-u_j)}{\sigma^{n+1}(u_0)\sigma^{n+1}(u_1)\dots\sigma^{n+1}(u_n)}$$

for any $n \geqslant 1$ and arbitrary $u_0, u_1, \dots, u_n$.

Exercise 1.4.22: Show that for an arbitrary $\lambda \neq 0$

$$\begin{aligned} \wp(\lambda z \,|\, \lambda\omega, \lambda\omega') &= \lambda^{-2}\wp(z \,|\, \omega, \omega') \\ \zeta(\lambda z \,|\, \lambda\omega, \lambda\omega') &= \lambda^{-1}\zeta(z \,|\, \omega, \omega') \\ \sigma(\lambda z \,|\, \lambda\omega, \lambda\omega') &= \lambda\, \sigma(z \,|\, \omega, \omega') \end{aligned} \tag{1.4.30}$$

Exercise 1.4.23: Consider the Korteweg–de Vries (KdV) equation

$$\dot{u} = 6uu' - u''' \tag{1.4.31}$$

(here $u = u(x,t)$, the dot stands for the derivative with respect to t, and the prime stands for the derivative with respect to x). Show that any (complex) periodic solution of KdV in the form of a traveling wave $u = u(x - ct)$ has the form

$$u(x,t) = 2\wp(x - ct - x_0) - \frac{c}{6}, \tag{1.4.32}$$

where the Weierstrass function $\wp$ corresponds to some elliptic curve (1.4.10), and the velocity c and the phase x_0 are arbitrary.

Exercise 1.4.24: (see [8]). Look for a solution of the KdV equation in the form

$$u(x,t) = 2\wp(x - x_1(t)) + 2\wp(x - x_2(t)) + 2\wp(x - x_3(t)). \tag{1.4.33}$$

Derive for the functions $x_j(t)$ the system of differential equations

$$\ddot{x}_j = 12 \sum_{k \neq j} \wp(x_j - x_k), \quad j = 1, 2, 3, \tag{1.4.34}$$

(a particular case of Calogero–Moser system and its integrals

$$\sum_{k \neq j} \wp'(x_j - x_k) = 0, \quad j = 1, 2, 3. \tag{1.4.35}$$

Integrate this system by quadratures.

Exercise 1.4.25: (see [**?**]). For the elliptic curve (1.4.10) construct a new elliptic curve $w^2 = 4\tilde{P}_3(z)$ with the third-degree polynomial

$$\tilde{P}_3(z) = (z^2 - 3g_2)\left(z + 9\frac{g_3}{g_2}\right). \tag{1.4.36}$$

Denote by $\tilde{\wp}$ the corresponding Weierstrass function. Let $\xi_{ij} = \wp(x_i(t) - x_j(t))$, $i \neq j$, where the quantities $x_i(t)$ are defined in the previous Exercise. Show that the functions $\xi_{12}(t)$, $\xi_{23}(t)$, and $\xi_{13}(t)$ are the roots of the cubic equation

$$4\xi^3 - g_2\xi - \frac{1}{3}g_3 + \frac{1}{2}g_2\tilde{\wp}(6i\sqrt{3g_2}t) = 0 \tag{1.4.37}$$

Other properties of the functions, $\wp$, ζ and σ and of other elliptic functions as well, can be found, for example, in the texts [2] and [**?**], or in the handbook [4].

Chapter 2

Topological properties of Riemann surfaces

2.1 Genus of a compact Riemann surface

An arbitrary Riemann surface is also a real smooth oriented two-dimensional manifold. What can be said about the topology of this manifold? From the topological point of view, Riemann surfaces are quite simple as the following theorem shows.

Theorem 2.1.1. *[18] Any compact connected orientable smooth two-dimensional manifold (= surface) is homeomorphic to a sphere with $g \geqslant 0$ handles. The number of handles is called the genus of the surface. Surfaces of different genera are not homeomorphic.*

Each surface of genus g can be obtained from a genus $g-1$ surface by removing two discs and connecting the resulting holes with a cylinder. The surface of genus 0 is the usual sphere. See Figure 2.1 for examples of surfaces of positive genus.

Let us compute the genus of the surfaces in the examples 1.2.42-1.2.44. We begin with example 1.2.43 namely the curve $C = \{(z,w) \in \mathbb{C}^2 \mid w^2 = z^2 - a^2\}$, $a \neq 0$. Let $\mathcal{S}$ be the compactification of C obtained by adding two points $\infty^\pm$ at infinity. We want to show that the genus of $\mathcal{S}$ is equal to zero. For the purpose let us consider $\mathcal{S}$ as a two sheeted branched covering of the Riemann sphere $\pi : \mathcal{S} \to \bar{\mathbb{C}}$, $\pi(z,w) = z$. Delete the segment $[-a,a]$ with endpoints at the branch points from the z-plane $\bar{\mathbb{C}}$. Off this segment it is possible to distinguish the two branches $w_\pm = \pm\sqrt{z^2 - a^2}$ of the two-valued function $w(z) = \sqrt{z^2 - a^2}$. The preimage $\pi^{-1}(\bar{\mathbb{C}} \backslash [-a,a])$ on $\mathcal{S}$ splits into two pieces, with the mapping π an isomorphism on each of them. The branches $w_+(z)$ and $w_-(z)$ are interchanged in passing from one edge of the cut $[-a,a]$ to the other. Therefore, the surface is glued together from two identical copies of spheres with cuts according to the rule indicated in the figure 2.2

After the gluing we again obtain a sphere, i.e., the genus g is equal to zero. Example 1.2.42 is analogous to Example 1.2.43, but the cut must be made between the points 0 and ∞, i.e. the point at infinity must be considered as a branch point. Again the genus is equal to zero.

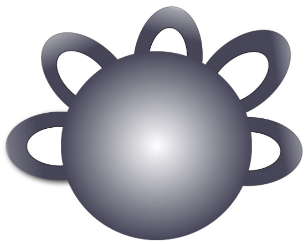


Figure 2.1: A sphere with five handles

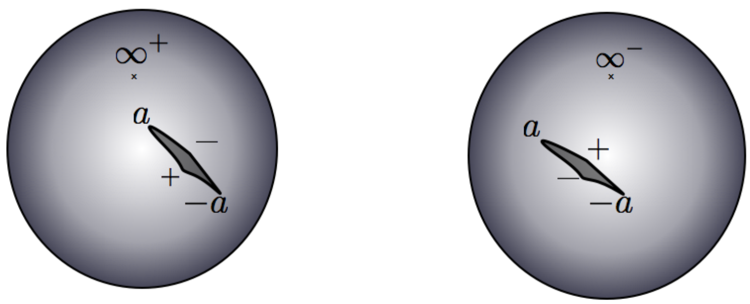


Figure 2.2: The cuts of the algebraic function $\sqrt{z^2 - a^2}$

Remark 2.1.2. It is not difficult to prove that the compact Riemann surface $\mathcal{S}$ of the algebraic function $w(z) = \sqrt{z^2 - a^2}$ is biholomorphically equivalent to the Riemann sphere. Indeed, consider a family of parallel lines

$$w = z - a\,s$$

depending on a complex parameter s. For $s \neq 0$ every such line intersects the curve C in a unique point with the coordinates

$$z(s) = a\frac{1 + s^2}{2s}, \quad w(s) = a\frac{1 - s^2}{2s}.$$

We obtain a one-to-one map

$$\mathbb{C}\backslash\{0\} \to \mathcal{S}, \quad s \mapsto (z(s), w(s)).$$

For $s \to 0$ both $z(s)$ and $w(s)$ go to infinity but the ratio $w(s)/z(s) \to 1$. That means that the image of the point $s = 0$ coincides with the point $\infty^+ \in \mathcal{S}$. In a similar way for $s \to \infty$ both $z(s)$ and $w(s)$ go to infinity but the ratio $w(s)/z(s) \to -1$. That means that the image of the point $s = 0$ coincides with the point $\infty^- \in \mathcal{S}$. So we have a one-to-one holomorphic map from $\overline{\mathbb{C}}$ to $\mathcal{S}$. The inverse map is given by

$$s(z,w) = \frac{a}{z+w}.$$

In Example 1.2.44 for the curve described by the equation $w^2 = \prod_{j=1}^{n}(z-z_j)$ it is necessary to split up the branch points arbitrarily into pairs and make cuts (arcs) in $\overline{\mathbb{C}}$ joining the paired branch points. If n is odd one of the branch points is at ∞. The surface $\mathcal{S}$ is glued together from two identical copies of a sphere with such cuts, with the edges of the corresponding cuts glued together in "cross-wise" fashion (see figure 2.4 for $n = 4$).

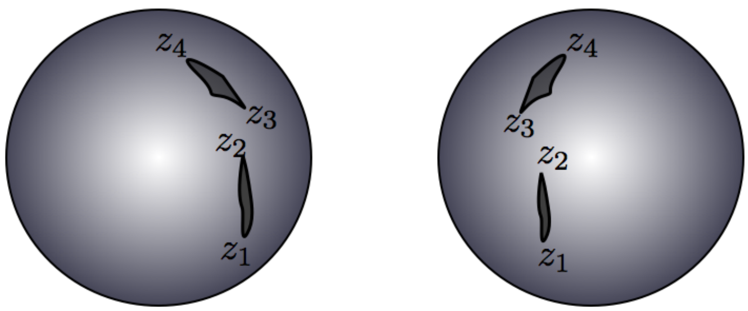


Figure 2.3: Opening of the cuts of the two branches of the function $\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

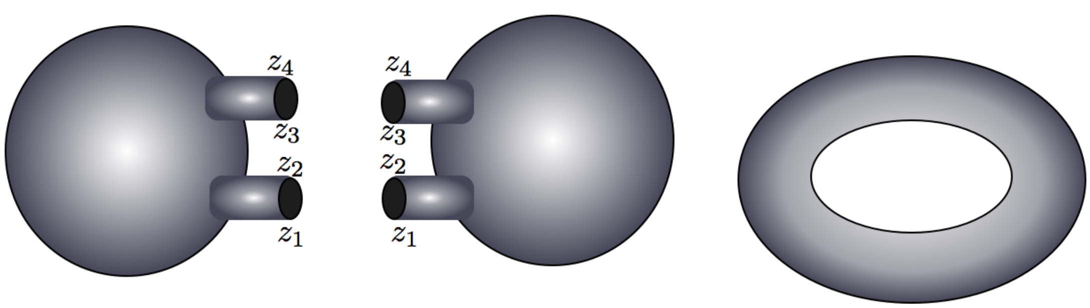


Figure 2.4: The Riemann surface of $w^2 = (z-a_1)(z-a_2)(z-a_3)(z-a_4)$ is glued from two copies of the extended complex plane cut along the intervals $[z_1,z_2]$ and $[z_3,z_4]$. The resulting surface topologically is a torus.

It is not hard to see that in the case $n = 4$ one obtains a sphere with one handle, and, in the general case one obtains a sphere with $n/2 - 1$ handles for n even and $(n-1)/2$ for n odd.

2.1.1 Genus of a Riemann surface and the Riemann-Hurwitz formula

We derive a formula for the computation of the genus of a compact Riemann surface by computing first the Euler characteristic of the surface.

A triangulation of a two-dimensional compact surface M is a decomposition of M into closed subsets homeomorphic to triangles such that each pair fits in one of the following three types

- disjoint
- meet at a vertex
- meet at an edge.

We state the following theorem.

Theorem 2.1.3. *[18] Every compact connected orientable 2-dimensional manifold M can be triangulated.*

Given a 2-dimensional compact manifold M (possibly with boundary) and a triangulation of the manifold with

- $e =$ # of edges;
- $v =$ # of vertices;
- $t =$ # of triangles,

we can associate to such triangulation the *Euler characteristic.*

Definition 2.1.4. *The quantity*

$$E(M) = v - e + t \tag{2.1.1}$$

is called the Euler characteristic of the manifold M with respect to the given triangulation.

Proposition 2.1.5. *The Euler number is independent from the choice of the triangulation. For a compact Riemann surface $\mathcal{S}$ of genus g the Euler number is*

$$E(\mathcal{S}) = 2 - 2g. \tag{2.1.2}$$

Proof. We consider compact surfaces with no boundaries. Given a triangulation, one can *refine* the triangulation by adding a vertex inside a triangle and three edges. This operation replaces one triangle with three triangles an it is easy to check that the Euler number remains unchanged. Another way to refine the triangulation is to add a point on an edge, so that two triangles are replaced by four triangles. Also in this case the Euler number remains unchanged. These operations define elementary refinements. A general refinement is obtained by making a sequence of elementary refinements. Therefore a given triangulation and any of its refinement have the same Euler number. Now the main point is to show that two triangulations have a common refinement. It is sufficient to superimpose two triangulations and add the necessary number for points to make the union of these two triangulations a triangulation. Then the triangulation obtained in this way is a refinement of both the triangulations. This is enough to show that the

Euler number does not depend on the triangulation. Now let us make the computation of the Euler number for a compact Riemann surface of genus g. We use an inductive argument. For the sphere $\mathcal{S}_0$, choosing a triangulation as shown in the figure 2.1.1, with 4 vertices, 4 triangles and 6 edges, one obtains that the Euler number is equal to 2. For the disc $\bar{D} = \{z \in \mathbb{C} \,|\, |z| \leqslant 1\}$, the Euler number is equal to $E(\bar{D}) = 1$ and for the cylinder $C_{cylinder}$ of finite length the Euler number $E(C_{cylinder}) = 0$, (see figure 2.5).

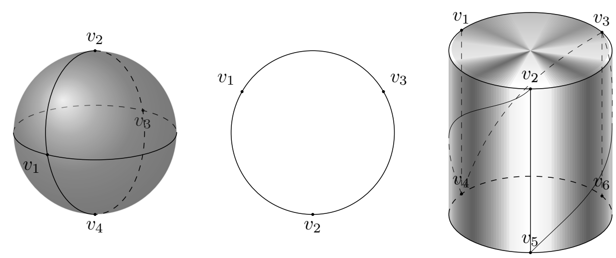


Figure 2.5: Triangulation of the sphere with 4 vertices, 6 edges and 4 triangles. Triangulation of the disc with 3 vertices, 3 edges and one triangle.Triangulation of the cylinder with 6 vertices, 12 edges and 6 triangles.

The torus can be obtained from the sphere by removing two discs and connecting them with a cylinder. It is simple to check that the Euler number of the torus $\mathcal{S}_1$ can be obtained as

$$E(\mathcal{S}_1) = E(\mathcal{S}_0) - 2E(\bar{D}) + E(C_{cylinder}) = 2 - 2 + 0 = 0. \tag{2.1.3}$$

Indeed removing two disks from a genus zero surface, the Euler number decreases by two, because it is just sufficient to subtract from the Euler formula the two discs that are homeomorphic to two triangles. Next we add a cylinder to connect the two discs. In order to compute the Euler number of the resulting surface, it is sufficient to add the contribution of the cylinder (8 edges and 6 triangles for a triangulation like in figure 2.1.1). The resulting Euler characteristics then can be written as in (2.1.3).

This procedure can be iterated. Indeed the surface $\mathcal{S}_g$ of genus g can be obtained from the surface of genus $\mathcal{S}_{g-1}$ by removing two discs and connecting them with a cylinder. Therefore one has

$$E(\mathcal{S}_g) = E(\mathcal{S}_{g-1}) - 2E(\bar{D}) + E(C_{cylinder})$$

which implies

$$E(\mathcal{S}_g) = 2 - 2g.$$

□

We apply this result to calculate the genus of a branched covering over the Riemann sphere.

Proposition 2.1.6. *Let $\mathcal{S}$ be a compact Riemann surface and $f : \mathcal{S} \to \overline{\mathbb{C}}$ a non-constant holomorphic map of degree n. Let $P_1, \ldots, P_k \in \mathcal{S}$ be the ramification points with respect to the map f with multiplicities $m_1, \ldots, m_k$ respectively. Denote $b_i = m_i - 1$, $i = 1, \ldots, k$ the ramification indices of these points and let*

$$b = \sum_{j=1}^{k} b_j$$

be the total ramification index. Then the genus of $\mathcal{S}$ is equal to

$$g = \frac{b}{2} - n + 1. \tag{2.1.4}$$

Proof. Consider a triangulation of $\bar{\mathbb{C}}$ such that the set of vertices of the triangulation contains the points $f(P_1), \ldots, f(P_k)$. Suppose that, for each triangle T on $\bar{\mathbb{C}}$ the restriction of f onto every connected component of the preimage of the interior part of T is a homeomorphism onto the interior of T. In this way the triangulation of $\bar{\mathbb{C}}$ can be lifted to a triangulation of $\mathcal{S}$. Let the triangulation of $\overline{\mathbb{C}}$ have v vertices, t triangles and e edges. Then the triangulation of $\mathcal{S}$ has

- $\tilde{t} = nt$ triangles
- $\tilde{e} = ne$ edges
- $\tilde{v} = nv - b$ vertices.

So the Euler characteristic of the surface $\mathcal{S}$ equals

$$2 - 2g = nv - b - ne + nt = n(v - e + t) - b = 2n - b.$$

The Proposition is proved. □

The equation (2.1.4) is the celebrated Riemann–Hurwitz formula. A generalization of it to holomorphic maps between compact Riemann surfaces will be given below.

As an application of the proposition 2.1.6 we calculate the genus of a smooth projective curve

$$\mathcal{S} = \{(X : Y : Z) \in \mathbf{P}^2 \,|\, Q(X, Y, Z) = 0\}$$

where Q is a homogeneous polynomial of degree n. Suppose that $(0 : 0 : 1) \notin \mathcal{S}$ so that $Q(0, 0, Z) = c\,Z^n \neq 0$ with $c \neq 0$. Then the map

$$\phi : \mathcal{S} \to \mathbf{P}^1, \quad \phi(X, Y, Z) = (X : Y)$$

realises $\mathcal{S}$ as a n-sheeted covering of $\mathbf{P}^1$. Let us calculate the total ramification number of this map. The ramification points are obtained by solving the equations

$$Q(X, Y, Z) = 0, \quad Q_Z(X, Y, Z) = 0.$$

The solution of the above two equations are given by the zeros of the resultant $R(Q, Q_Z)$ with respect to Z. Since $R(Q, Q_Z)$ is a homogeneous polynomial of degree $n(n-1)$ in X and Y, the total number of ramification points counting their multiplicity is $n(n-1)$.

Recall that the ramification number of a ramification point $P_0 = (X_0 : Y_0 : Z_0)$ indicated as $b_\phi(P_0)$ is the order of the zero of $Q(X_0, Y_0, Z)$ at $Z = Z_0$ minus one. We can write

$$Q(X_0, Y_0, Z) = \prod_{0 \leqslant j \leqslant s} (Z - Z_j)^{m_j}$$

where $\sum_j m_j = n$ and $Z_0, \dots, Z_s$ are distinct complex numbers, $Z_j = Z_j(X_0, Z_0)$. With the above notation the branching number of each branch point $P_j = (X_0 : Y_0 : Z_j)$ is $b_\phi(P_j) = m_j - 1$. So a regular point is simple zero of $Q(X_0, Y_0, Z)$ a ramification point with ramification number one is a double zero, and in general a ramification point with ramification number $m-1$ is a zero of order m of $Q(X_0, Y_0, Z)$. So if the number of distinct roots of the discriminant is $n(n-1)$ it means that the curve has $n(n-1)$ branch points with multiplicity one, so that the total ramification number is $n(n-1)$. If the discriminant has for example $n(n-1)-k$ distinct roots, $k > 0$, it means that some of the branch points have branching number bigger than one. However the total ramification number remains equal to $n(n-1)$. Then we can apply formula 2.1.4 to obtain

$$g = \frac{1}{2}(n-1)n - n + 1.$$

We summarise the above discussion with the following Lemma.

Lemma 2.1.7. *The genus of a smooth projective curve of degree n is given by*

$$g = \frac{1}{2}(n-2)(n-1). \tag{2.1.5}$$

Exercise 2.1.8: Calculate the genus of the normalisations of the following curves

- $w^3 = (z-1)(z-2)(z-3)(z-4)$,
- $w^n = z^n + a^n, \quad a \neq 0.$

Exercise 2.1.9: Let us consider the reducible curve

$$C_0 = \{(z,w) \in \mathbb{C}^2 \mid (w - p_1(z)(w - p_2(z))(w - p_3(z)) = 0\}$$

with

$$p_i(z) = a_i z + b_i, \quad i = 1,2,3$$

and a_i and b_i $i = 1,2,3$ complex constants such $a_i b_j - a_j b_i \neq 0$ for $i \neq j$. Furthermore let us assume that the polynomials $p_i(z)$ satisfy the relation

$$p_1(z) + p_2(z) + p_3(z) = 0.$$

Consider the curve

$$C := \{(z,w) \in \mathbb{C}^2 \mid w^3 + w[p_1(z)p_2(z) + p_1(z)p_3(z) + p_2(z)p_3(z))] - p_1(z)p_2(z)p_3(z)(1+h) = 0\} \tag{2.1.6}$$

where h is a small complex constant. Let $\mathcal{S}$ be the normalisation of C. Determine

- how many points have been added to C to obtain $\mathcal{S}$;
- the genus of $\mathcal{S}$;
- the branch points (only in the form of the expansion in h, namely $z_i(h) = z_i(0) + hz_i'(0) + \dots$);
- the monodromy of $\mathcal{S}$ considered as a degree 3 branched covering of the z-plane.

Exercise 2.1.10: Let us consider the curve

$$C := \{(z,w) \in \mathbb{C}^2 \mid (w - z^2)(z - w^2) + hzw = 0\},$$

where h is a small non-zero constant. Determine

- the normalisation $\mathcal{S}$ of C and the genus of $\mathcal{S}$;
- the monodromy of $\mathcal{S}$ with respect to the projection to the z plane.

Exercise 2.1.11: Calculate the genus of the normalization of the singular curves

1. $w^3 = (z - a_1)^2(z - a_2)(z - a_3)^2(z - a_4)$,
2. $w^3 = z^3(z - a_3)^2(z - a_4)$.

For each singular point calculate the number of points in the preimage of the map ϕ defined in theorem **??**.

Exercise 2.1.12: For which value of λ the following curves are non-singular?

1. $X^3 + Y^3 + Z^3 + 3\lambda XYZ = 0$,
2. $X^3 + Y^3 + Z^3 + \lambda(X + Y + Z)^3 = 0$.

Describe the singularities when they exist and calculate the genus of the corresponding Riemann surface.

Exercise 2.1.13 (Plücker's formula)**:** . Let C be a projective curve of degree n with k nodes and no other singularities. Show that the genus of the Riemann surface S obtained by resolving singularities on the curve is equal to

$$g = \frac{1}{2}(n-1)(n-2) - k.$$

2.2 Homology

In this section we define the homology of a compact Riemann surface $\mathcal{S}$. Given a triangulation of the Riemann surface $\mathcal{S}$, we define the verteces as 0-*simplex*, the edges as 1-*simplex* and the triangles as 2-*simplex*. The orientation on the manifold induces an orientation on the triangles that can be used to orient the edges bounding each triangle.

Definition 2.2.1. *A (simplicial)* $0, 1, 2$-**chain** *is a formal sum of vertices P_j, edges $\mathcal{S}_j$ or triangles T_j*

$$c_0 = \sum n_j P_j \qquad c_1 = \sum m_j \mathcal{S}_j \qquad c_2 = \sum k_j T_j, \quad n_j, m_j, k_j \in \mathbb{Z}.$$

The element $-c_1$ is the edge with opposite orientation and $-t$ is the triangle with opposite orientation. The vertices P_1, P_2, $P_3, \ldots$ can be used to identify edges and triangles. For example $\langle P_1 P_2 \rangle$ is the oriented edge from P_1 to P_2 and $\langle P_1, P_2, P_3 \rangle$ is the oriented triangle with sides the oriented edges $\langle P_1 P_2 \rangle$, $\langle P_2 P_3 \rangle$ and $\langle P_3 P_1 \rangle$. The sets of p–chains C_p have the (natural) structure of free abelian groups (just by formal sums). A closed curve $\tilde{\gamma}$ can be homotopically deformed to a chain of edges in the triangulation $\mathcal{T}$ thus defining a cycle (**Exercise**: prove that it is a cycle!); this can be called a **simple cycle**.

With this notation we define the boundary operator δ.

Definition 2.2.2. *The boundary operator $\delta : C_n \to C_{n-1}$ with $n = 0, 1, 2$ is defined as follows:*

$$\delta c_0 = 0, \quad c_0 \in C_0$$

$$\delta \langle P_1 P_2 \rangle = P_2 - P_1$$

$$\delta \langle P_1, P_2, P_3 \rangle = \langle P_1 P_2 \rangle + \langle P_2 P_3 \rangle + \langle P_3 P_1 \rangle.$$

The above relation defines δ on 1 and 2-simplex and it can be extend to 1 and 2-chain by linearity.

The **fundamental property** is that $\delta^2 \equiv 0$: indeed (we need to check this only for C_2)

$$\delta\delta(T) = \delta\left(\langle P_1 P_2 \rangle + \langle P_2 P_3 \rangle + \langle P_3 P_1 \rangle\right) = P_2 - P_1 + P_3 - P_2 + P_1 - P_3 = 0\,. \tag{2.2.1}$$

Definition 2.2.3. *A p–chain c_p such that $\delta c_p = 0 \in C_0$ is called a p–**cycle**. A chain which is the boundary of another chain is called a p–**boundary**. Clearly any p-boundary is a p-cycle, but not viceversa.*

In our case, being the manifold of real dimension 2, all the interesting information is contained in C_1; the 1–cycles and 1–boundaries are the following subgroups of C_1:

$$\mathcal{Z}_n = \{c_n \in C_n \,|\, \delta c_n = 0\}, \quad \mathcal{B}_n = \{c_n \in C_n \,|\, \exists c_{n+1} \in C_{n+1}, \ \ c_n = \delta c_{n+1}\}.$$

From the above definition it is clear that

$$\mathcal{B}_n \subseteq \mathcal{Z}_n \subseteq C_n\,.$$

Definition 2.2.4. *The* **first homology group** *of $\mathcal{S}$ is denoted by $H_1(\mathcal{S}, \mathbb{Z})$ and is*

$$H_1(\mathcal{S}, \mathbb{Z}) := \frac{\mathcal{Z}_1(\mathcal{S})}{\mathcal{B}_1(\mathcal{S})}\,. \tag{2.2.2}$$

This homology group can be shown to be independent of the choice of triangulation $\mathcal{T}$ (more precisely the homology groups corresponding to two triangulations are isomorphic).

Remark 2.2.5. The other homology groups are defined similarly: in particular $H_0(\mathcal{S}, \mathbb{Z})$ is made of the classes of points that cannot be joined by cycles. It is simple to show that $H_0(\mathcal{S}, \mathbb{Z}) = \mathbb{Z}^k$ where k is the number of connected components of $\mathcal{S}$ (hence for connected Riemann surfaces $k = 1$). The generator is the class of any vertex. Regarding $H_2(\mathcal{S}, \mathbb{Z})$ we have that if $\mathcal{S}$ is compact, then C_2 consists of one 2-chain, namely the chain that covers all the surface and $\mathcal{B}_2 = \emptyset$. Therefore $H_2(\mathcal{S}, \mathbb{Z}) = \mathbb{Z}$.

Therefore the only nontrivial group is $H_1(\mathcal{S}, \mathbb{Z})$. One has

Proposition 2.2.6. *Let $\mathcal{S}$ be a connected compact Riemann surface of genus g. The first homology group $H_1(\mathcal{S}, \mathbb{Z})$ is isomorphic to the Abelianization of the first homotopy group, namely*

$$H_1(\mathcal{S}, \mathbb{Z}) \simeq \frac{\pi_1(\mathcal{S})}{[\pi_1(\mathcal{S}), \pi_1(\mathcal{S})]}\,, \tag{2.2.3}$$

where $[\,.,\,.]$ is the standard commutator. The group $H_1(\mathcal{S}, \mathbb{Z})$ is a **free Abelian group** *with $2g$ generators and hence it is isomorphic to $\mathbb{Z}^{2g}$. These generators can be chosen as (classes of) simple cycles.*
Any cycle can be written as sum of simple cycles (with coefficients in $\mathbb{Z}$).

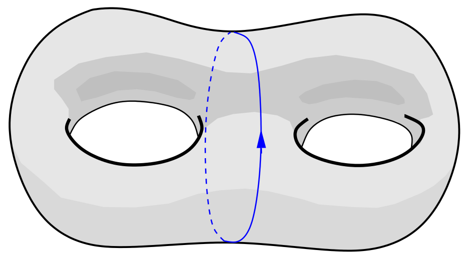


Figure 2.6: The blue contour is not homotopic to the trivial loop but it is homologous to zero because it separates the surface.

Let $\mathcal{S}$ be a compact Riemann surface of genus g and let $[\gamma_1], \dots, [\gamma_{2g}]$ be the set of generators of $\pi_1(\mathcal{S})$. Then any element $[\gamma] \in \pi_1(\mathcal{S})$ can be uniquely written as

$$[\gamma]_{\pi_1} = [\gamma_{k_1}]_{\pi_1}^{j_1} \circ [\gamma_{k_2}]_{\pi_1}^{j_2} \circ \dots [\gamma_{k_n}]_{\pi_1}^{j_n}, \quad k_1, \dots k_n \in \{1, 2, \dots, 2g\}$$

with $j_1, \dots, j_n \in \mathbb{Z}$ and we use the subscript π_1 to denote the elements of the homotopy group. Then the corresponding element $[\gamma]_{H_1}$ in the homology class is obtained as

$$[\gamma]_{H_1} = j_1[\gamma_{k_1}]_{H_1} + j_2[\gamma_{k_2}]_{H_1} + \dots + j_n[\gamma_{k_n}]_{H_1}, \quad k_1, \dots k_n \in \{1, 2, \dots, 2g\}.$$

This in particular also shows that the homology is independent from the triangulation.

Remark 2.2.7. A cycle may be **Homologous** to the trivial cycle but **not homotopic** to a point, for example the one in Fig. 2.6.

In the rest of this section we simply denote as γ an element in the homology. Let $a_1, \dots, a_g, b_1, \dots, b_g$ be a basis in $H_1(\mathcal{S}, \mathbb{Z})$. Then any cycle γ is homologous to a linear combination of the basis with integer coefficients:

$$\gamma \simeq \sum_{i=1}^{g} m_i a_i + \sum_{i=1}^{g} n_i b_i, \quad m_i,\ n_i \in \mathbb{Z}.$$

Intersection number

The notion of intersection number is more general than the one given here as it applies to any two submanifolds of complementary dimensions. In our case of complex one-dimensional manifold (i.e. real surface) two submanifolds of complementary dimension must have both dimension 1 (i.e. they must be curves) or 0 and 2 (points and domains). The latter case is rather degenerate (although not meaningless) and we focus only on the first case.

Given two simple cycles γ and η we represent them as smooth closed curves and we consider their intersection: again, possibly by a small deformation of one or both contours we can reduce to the situation that

(a) the intersection is finite and

(b) all intersections occur **transversally**, i.e. the tangents to γ and η at the point of intersection are not parallel.

Given $p \in \gamma \cap \eta$ one such point of intersection, we associate a number $\nu(p) \in \{+1, -1\}$ as follows. Let z be a local coordinate at p: the two (arcs) of γ and η now are arcs in a neighbourhood of $z(p) = 0$ crossing each other transversally. We denote by $\dot{\gamma}_0$ and $\dot{\eta}_0$ the two tangent vectors at $z(p) = 0$; if the determinant of their components is positive we set $\nu(p) = 1$, if it is negative we set $\nu(p) = -1$. In other words the number $\nu(p)$ indicates the orientation of the axis spanned by $\dot{\gamma}_0$ and $\dot{\eta}_0$ (in this order!) relative to the orientation of the standard $\Re(z)$, $\Im(z)$ axes.

Definition 2.2.8. *The intersection number between γ and η is then defined by*

$$\gamma * \eta := \sum_{p \in \gamma \cap \eta} \nu(p) \ . \tag{2.2.4}$$

It follows immediately from the definition that $\gamma * \eta = -\eta * \gamma$ and the intersection number is an integer. One can also prove that:

Proposition 2.2.9. *The intersection number is invariant under smooth homotopy deformations of γ and η.*

Therefore the intersection number depends only on the homotopy classes of γ and η, which we then denote by $[\gamma] * [\eta]$.

In particular it makes sense to compute the **self-intersection** of a cycle

$$[\gamma] * [\gamma] = 0 \ . \tag{2.2.5}$$

This makes sense because in the actual computation one chooses two different representatives in the same class of γ which intersect transversally: the fact that the result is zero then follows from the antisymmetry.

Note also that the intersection number depends on the orientation of the contours: if we reverse one contour the intersection number changes sign

$$[\gamma] * [\eta] = -[\gamma]^{-1} * [\eta] \ . \tag{2.2.6}$$

Moreover:

Lemma 2.2.10. *The intersection number of any boundary β with any cycle γ vanishes $\gamma * \beta = 0$.*

Proof. A boundary β is a collection of simple cycles that bound a domain. if γ is a symple cycle it must traverse the boundary of this domain an even number of times, and two consecutive crossing count with opposite sign, hence cancel out. ◻

This lemma implies that the intersection number is well defined as a pairing on the first homology group. More in fact is true

Theorem 2.2.11. *The intersection pairing*

$$* : H_1(\mathcal{S}, \mathbb{Z}) \times H_1(\mathcal{S}, \mathbb{Z}) \to \mathbb{Z} \tag{2.2.7}$$

is a bilinear skew–symmetric map. If $\mathcal{S}$ is a compact Riemann surface then it is **nondegenerate***.*

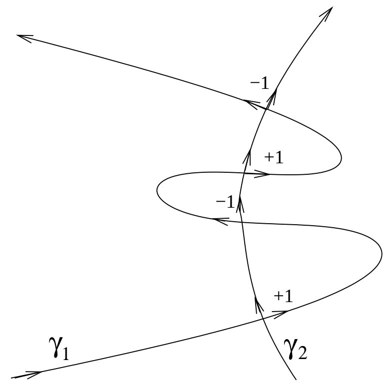


Figure 2.7: Intersection of γ_1 and γ_2.

2.2.1 Homology of a compact Riemann surface of genus g

We have said that $H_1(\mathcal{S}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2g}$ and that the intersection pairing is antisymmetric and nondegenerate. It can be shown that there are simple cycles

$$\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\} \tag{2.2.8}$$

that generate $H_1(\mathcal{S}, \mathbb{Z})$ and such that

$$\alpha_i * \alpha_j = 0, \quad \beta_i * \beta_j = 0\,, \quad \alpha_i * \beta_j = \delta_{ij}\,. \tag{2.2.9}$$

Definition 2.2.12. *A basis of $H_1(\mathcal{S}, \mathbb{Z})$ satisfying (2.2.9) is called a* **canonical basis***.*

A canonical basis exists but it is not unique. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)^t$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_g)^t$ denote the column vectors of the $2g$ generators and let us suppose we make a transformation

$$\begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \tag{2.2.10}$$

where the $2g \times 2g$ matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is integer valued and non-singular. The basis $\boldsymbol{\alpha}', \boldsymbol{\beta}'$ will be a set of generators provided that S^{-1} is also integer–valued and hence the determinant of S must be ± 1.

Moreover if we want that the new basis is also canonical this forces

$$J := \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix} * (\boldsymbol{\alpha}' \boldsymbol{\beta}') = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} * (\boldsymbol{\alpha}, \boldsymbol{\beta}) \tag{2.2.11}$$

so that

$$J = SJS^t \tag{2.2.12}$$

Matrices of dimension $2g \times 2g$ satisfying (2.2.12) form a group, the symplectic group, denoted by $Sp(g, \mathbb{Z})$.

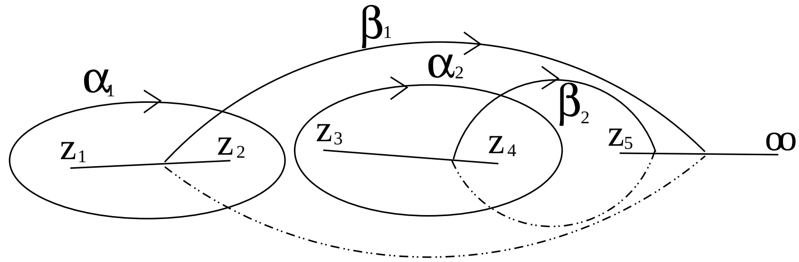


Figure 2.8: Homology basis.

Example 2.2.13. Let us construct a canonical basis of cycles on the hyperelliptic surface $w^2 = \prod_{i=1}^{2g+1}(z - z_i)$, $g \geqslant 1$. We represent this surface in the form of two copies of $\mathbb{C}$ (sheets) with cuts along the segments $[z_1, z_2]$, $[z_3, z_4]$, $\ldots$, $[z_{2g+1}, \infty]$. A canonical basis of cycles can be chosen as indicated in the Figure 2.8 for $g = 2$ (the dashed lines represent the parts of a_1 and a_2 lying on the lower sheet).

2.2.2 Canonical dissection of a compact Riemann–surface and Poincare polygon

We take a basepoint P_0 and consider the homotopy group $\pi_1(\mathcal{S}, P_0)$ of loops based at P_0. Amongst these there are $2g$ generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ whose homology classes form a canonical basis. Although these loops are only identified by their homotopy classes, we will think of them as concrete choices of (smooth) closed curves on the surface with basepoint P_0.

Definition 2.2.14. *The canonical dissection of $\mathcal{S}$, called the Poincare' polygon of $\mathcal{S}$, is the simply connected domain $\widetilde{\mathcal{S}}$ obtained by removing the $2g$ generators identified above.*

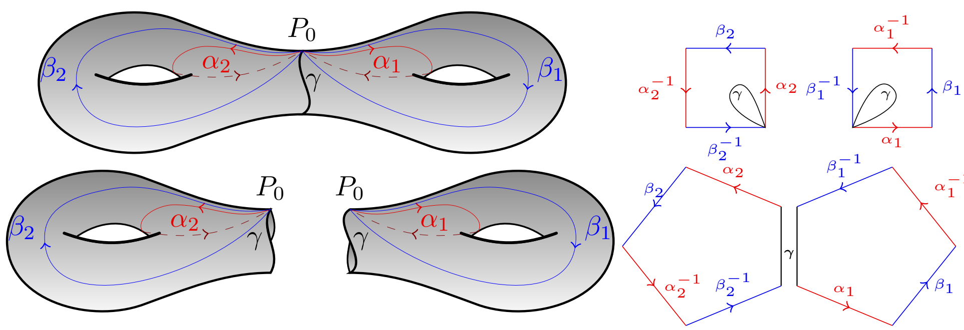


Figure 2.9: Dissection of a surface of genus one and two.

The boundary $\partial\widetilde{\mathcal{S}}$ of this domain consists of **both sides** of each generator and hence consists of $4g$ arcs. We show inductively how to get the domain $\widetilde{\mathcal{S}}$ from the surface $\mathcal{S}$. In figure (2.9) each torus is cut along its cycles so that the simply connected domain $\widetilde{S}$ is the rectangle. One can repeat this operation inductively in the following way. The surface of genus 2 is cut along the line γ which decomposes the surface is two tori with boundary. Then each torus is dissected along its canonical basis of cycles and the polygons obtained are identified along the side γ so that $\widetilde{S}$ coincides with the 8-gone (see Figure 2.9 and 2.10). In the general case one can repeat the dissection by cutting out of a sphere g disks bounded by curves $\gamma_1, \ldots, \gamma_g$. By flattening the resulting surface, one obtains a polygon with g sides with symbol $\gamma_1, \ldots, \gamma_g$. We then attach to each side γ_j the handle $\alpha_j\beta_j\alpha_j^{-1}\beta_j^{-1}\gamma_j$ for $j = 1, \ldots, g$, thus obtaining the normal form of genus g (see Figure 2.10) for the case of genus one and two).

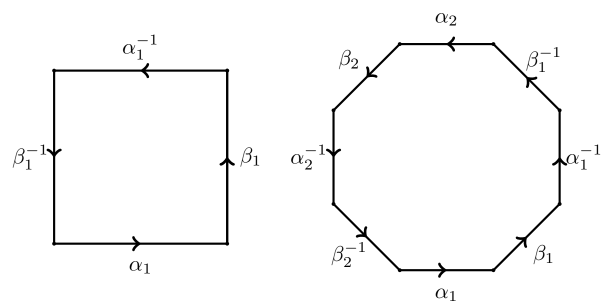


Figure 2.10: Poincaré polygon for surfaces of genus one and two.

Chapter 3

Differentials on a Riemann surface.

3.1 Holomorphic differentials

We consider a complex-one dimensional manifold M with with an atlas of charts $\{U_\alpha, \phi_\alpha\}$ with

$$\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}$$

and $\phi_\alpha(P) = z_\alpha \in V_\alpha$ and $P \in U_\alpha$. Here we are identifying $\mathbb{C}$ with $\mathbb{R}^2$ by writing $z_\alpha = x_\alpha + iy_\alpha$ with x_α and y_α standard coordinates on $\mathbb{R}^2$.

Definition 3.1.1. *A smooth one 1-form (also called differential) ω on M is an assignment of a collection of two smooth functions $u_\alpha(x_\alpha, y_\alpha)$ and $v_\alpha(x_\alpha, y_\alpha)$ to each local coordinate $z_\alpha = x_\alpha + iy_\alpha$ in U_α such that*

$$\omega = u_\alpha(x_\alpha, y_\alpha)dx_\alpha + v_\alpha(x_\alpha, y_\alpha)dy_\alpha \tag{3.1.1}$$

transform under change of coordinates as a (1,0)-tensor. Namely if $z_\beta = x_\beta + iy_\beta$ is another local coordinate such that $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\begin{pmatrix} u_\beta(x_\beta, y_\beta) \\ v_\beta(x_\beta, y_\beta) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_\alpha}{\partial x_\beta} & \dfrac{\partial y_\alpha}{\partial x_\beta} \\ \dfrac{\partial x_\alpha}{\partial y_\beta} & \dfrac{\partial y_\alpha}{\partial y_\beta} \end{pmatrix} \begin{pmatrix} u_\alpha(x_\alpha, y_\alpha) \\ v_\alpha(x_\alpha, y_\alpha) \end{pmatrix}$$

with $x_\alpha = x_\alpha(x_\beta, y_\beta)$ and $y_\alpha = y_\alpha(x_\beta, y_\beta)$.

Using the basis $dz_\alpha = dx_\alpha + idy_\alpha$, $d\bar{z}_\alpha = dx_\alpha - idy_\alpha$, we can rewrite ω in the form

$$\omega = h_\alpha(z_\alpha, \bar{z}_\alpha)\, dz_\alpha + g_\alpha(z_\alpha, \bar{z}_\alpha)\, d\bar{z}_\alpha, \tag{3.1.2}$$

where

$$h_\alpha = \frac{1}{2}(u_\alpha - iv_\alpha), \quad g_\alpha = \frac{1}{2}(u_\alpha + iv_\alpha).$$

The two parts $h(z_\alpha, \bar{z}_\alpha)\, dz_\alpha$ and $g(z_\alpha, \bar{z}_\alpha)\, d\bar{z}_\alpha$ of the expression (3.1.2) will be called $(1,0)$- and $(0,1)$-forms respectively. The above expression shows that the decomposition of ω in $(1,0)$ and

$(0,1)$ form is invariant under local change of coordinates, if and only if the change of coordinates is holomorphic, namely

$$\frac{\partial \bar{z}_\alpha}{\partial z_\beta} = 0, \quad \frac{\partial z_\alpha}{\partial \bar{z}_\beta} = 0.$$

The above conditions in real coordinates are equivalent to the Cauchy-Riemann equation. For a one-complex dimensional manifold M that has a complex structure (namely a Riemann surface), the decomposition of a one form in $(1,0)$ and $(0,1)$ form is invariant under local change of coordinates. From now on we will consider only holomorphic change of coordinates.

Definition 3.1.2. *A one form ω is called holomorphic is the functions $h_\alpha(z_\alpha, \bar{z}_\alpha)$ in (3.1.2) are all holomorphic functions and $g_\alpha \equiv 0$, namely*

$$\omega = h(z_\alpha) dz_\alpha.$$

A one form ω is called antiholomorphic if

$$\omega = g(\bar{z}_\alpha) d\bar{z}\alpha.$$

In a similar way to one form we can define two-forms.

Definition 3.1.3. *A smooth two form η on M is an assignment of a smooth function $f_\alpha(z_\alpha, \bar{z}_\alpha)$ such that*

$$\eta = f_\alpha(z_\alpha, \bar{z}_\alpha) dz_\alpha \wedge d\bar{z}_\alpha$$

is invariant under coordinate change.

The exterior multiplication satisfies the conditions

$$dz_\alpha \wedge dz_\alpha = 0, \quad d\bar{z}_\alpha \wedge d\bar{z}_\alpha = 0, \quad dz_\alpha \wedge d\bar{z}_\alpha = -d\bar{z}_\alpha \wedge dz_\alpha.$$

Under holomorphic change of coordinates $z_\beta = z_\beta(z_\alpha)$, $\bar{z}_\beta = \bar{z}_\beta(\bar{z}_\alpha)$ one has

$$\eta = f_\beta(z_\beta, \bar{z}_\beta) dz_\beta \wedge d\bar{z}_\beta = f_\alpha(z_\alpha, \bar{z}_\alpha) dz_\alpha \wedge d\bar{z}_\alpha$$

where

$$f_\beta(z_\beta, \bar{z}_\beta) = f_\alpha(z_\alpha, \bar{z}_\alpha) \left| \frac{dz_\alpha}{dz_\beta} \right|^2.$$

We define Ω^k for $k = 0, 1, 2$ as the set of smooth functions, smooth one forms and smooth two-forms on M respectively. We define the exterior derivative

$$d : \Omega^k \to \Omega^{k+1}, \quad k = 0, 1, 2$$

as follows. For $f \in \Omega^0$,

$$df(z, \bar{z}) = f_z dz + f_{\bar{z}} d\bar{z},$$

For one forms $\omega \in \Omega^1$, with $\omega = h(z, \bar{z})dz + g(z, \bar{z})d\bar{z}$ in a given coordinate chart, the exterior derivative takes the form

$$d\omega = dh \wedge dz + dg \wedge d\bar{z}$$

and for two forms, $\eta \in \Omega^2(M)$

$$d\eta = 0.$$

Clearly the fundamental property of the exterior differentiation is

$$d^2 = 0.$$

We can decompose the exterior derivative operator d according to the decomposition of 1-form in $(0,1)$ and $(1,0)$ forms

$$d = \partial + \bar{\partial}$$

so that for $h \in \Omega^{0,0} := \Omega^0$ in a local chart

$$\partial : \Omega^0 \to \Omega^{1,0}, \quad \partial h(z,\bar{z}) = h_z dz,$$

and

$$\bar{\partial} : \Omega^0 \to \Omega^{0,1}, \quad \bar{\partial} h(z,\bar{z}) = h_{\bar{z}} d\bar{z}.$$

In general we get the diagram

$$\begin{tikzcd} \Omega^{0,1} \arrow[r, "\partial"] & \Omega^2 \\ \Omega^0 \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"'] & \Omega^{1,0} \arrow[u, "\bar{\partial}"'] \end{tikzcd}$$

where $\Omega^2 = \Omega^{1,1}$. Also in this case $\partial^2 = 0$ and $\bar{\partial}^2 = 0$.

Definition 3.1.4. *A one form ω is called exact if there is a function $f \in \Omega^0$ such that $df = \omega$. A one form $\omega \in \Omega^1$ is called closed if $d\omega = 0$.*

Lemma 3.1.5. *A $(1,0)$-form $\omega = h(z,\bar{z})\, dz$ is closed if and only if the function $h(z,\bar{z})$ is holomorphic.*

It follows that all the *holomorphic differentials*, locally can be written in the form $\omega = h(z)dz$ where $h(z)$ is a holomorphic function. Holomorphic differentials are closed differentials.

Definition 3.1.6. *The first de Rham cohomology group is defined as*

$$H^1_{deRham}(\mathcal{S}) = \frac{\textit{Closed 1-forms}}{\textit{Exact 1-forms}} = \frac{\ker(d : \Omega^1 \to \Omega^2)}{Im(d : \Omega^0 \to \Omega^1)}.$$

A similar definition can be obtained for the Dolbeault cohomology groups $H^{1,0}(\mathcal{S})$ and $H^{0,1}(\mathcal{S})$ with respect to the operator $\bar{\partial}$:

$$H^{1,0}(\mathcal{S}) := \frac{\ker(\bar{\partial} : \Omega^{1,0} \to \Omega^2)}{(\bar{\partial} : \Omega^0 \to \Omega^{1,0})} = \ker(\bar{\partial} : \Omega^{1,0} \to \Omega^2),$$

$$H^{0,1}(\mathcal{S}) := \frac{\ker(\bar{\partial} : \Omega^{0,1} \to \Omega^2)}{(\bar{\partial} : \Omega^0 \to \Omega^{0,1})} = \frac{\Omega^{0,1}}{\text{Image}(\bar{\partial} : \Omega^0 \to \Omega^{0,1})}.$$

A non trivial result shows that there are isomorphisms among the above three groups [17]. By denoting $\overline{H^{0,1}(\mathcal{S})}$ the complex conjugate of the group $H^{0,1}(\mathcal{S})$, one has the following theorem.

Theorem 3.1.7. *The Dolbeault cohomology groups $H^{1,0}(\mathcal{S})$ and $\overline{H^{0,1}(\mathcal{S})}$ are isomorphic*

$$H^{1,0}(\mathcal{S}) \simeq \overline{H^{0,1}(\mathcal{S})} \tag{3.1.3}$$

and the first de-Rham cohomology group is isomorphic to

$$H^1_{deRham}(\mathcal{S}) \simeq H^{1,0}(\mathcal{S}) \oplus H^{0,1}(\mathcal{S}). \tag{3.1.4}$$

The relation (3.1.3) shows that the complex vector spaces $H^{1,0}(\mathcal{S})$ and $H^{0,1}(\mathcal{S})$ have the same dimension. The relation (3.1.4) shows that the dimension of the complex vector space $H^{1,0}(\mathcal{S})$ and $H^{0,1}(\mathcal{S})$ is half the dimension of the complex vector space $H^1_{deRham}(\mathcal{S})$.

3.1.1 Integration

We can integrate one forms on curves of the Riemann surface $\mathcal{S}$, two-forms on domains of $\mathcal{S}$ and 0-forms on zero dimensional domains of $\mathcal{S}$, namely points. Let c_0 be a 0-chain,

$$c_0 = \sum_i n_i P_i, \quad P_i \in \mathcal{S}$$

then for $f \in \Omega^0(\mathcal{S})$ the integral of f over a 0-chain c_0 is

$$\int_{c_0} f = \sum_i n_i f(P_i)$$

A one form ω can be integrated over a one-chain c. If the piece-wise differentiable path $c : [0,1] \to \mathcal{S}$ is contained in a single coordinate disc with coordinates $z = x + iy$, then the integral of ω over the one-chain c takes the form

$$\int_c \omega = \int_0^1 h(z(t), \bar{z}(t)) \frac{dz}{dt} dt + \int_0^1 g(z(t), \bar{z}(t)) \frac{d\bar{z}(t)}{dt} dt$$

By the transition formula for ω the above integral is independent from the choice of the coordinate chart z. In a similar way a two-form η can be integrated over two chains D. Again restricting to a single coordinate chart one has

$$\int\int_D \eta = \int\int_D f(z, \bar{z}) dz d\bar{z}.$$

The integral is well defined and extends in a obvious way to an arbitrary two-chain.

Theorem 3.1.8 (Stokes theorem). *Let D be a domain of $\mathcal{S}$ with a piece-wise smooth boundary ∂D and let ω be a smooth one-form. Then*

$$\int_D d\omega = \int_{\partial D} \omega. \tag{3.1.5}$$

As a consequence of Stokes theorem, the integral of closed forms ω on any closed oriented contour (cycle) γ on $\mathcal{S}$ does not depend on the homology class of γ. Recall that two cycles γ_1 and γ_2 are said to be homologous if their difference $\gamma_1 - \gamma_2 = \gamma_1 \cup (-\gamma_2)$ (where $(-\gamma_2)$ is the cycle with the opposite orientation) is the oriented boundary of some domain D on $\mathcal{S}$ with $\partial D = \gamma_1 - \gamma_2$. Then for a close differential ω and from Stokes theorem we obtain

$$0 = \int_D d\omega = \int_{\partial D} \omega = \int_{\gamma_1 - \gamma_2} \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega.$$

In addition, the integral of a close differential ω on a close cycle γ is independent from the cohomology class. Let $\omega' = \omega + df$ for some smooth function f, then

$$\int_\gamma \omega = \int_\gamma (\omega' - df) = \int_\gamma \omega'.$$

We summarise the above discussion with the following proposition.

Proposition 3.1.9. *The integration is a paring between the first homology group $H_1(\mathcal{S}, \mathbb{Z})$ and the first cohomology group $H^1_{deRham}(\mathcal{S}, \mathbb{C})$*

$$\int : H_1(\mathcal{S}, \mathbb{Z}) \times H^1_{deRham}(\mathcal{S}, \mathbb{C}) \to \mathbb{C}$$

The pairing is non-degenerate.

Proof. We need to prove that the pairing is non-degenerate. Consider a smooth one-form ω such that

$$\int_\gamma \omega = 0$$

for all $\gamma \in H_1(\mathcal{S}, \mathbb{Z})$. It follows that the function

$$f(P) = \int_{P_0}^{P} \omega$$

is well defined and it does not depend on the path of integration between P_0 and P. Therefore $df = \omega$, namely the equivalent class of ω in the de-Rham cohomology is zero, $[\omega] = 0$ in $H^1_{deRham}(\mathcal{S}, \mathbb{C})$. □

As a consequence of the above proposition we have the following lemma.

Lemma 3.1.10. *The dimension of the space $H^1_{deRham}(\mathcal{S}, \mathbb{C})$ is less then or equal to $2g$ where g is the genus of the compact Riemann surface $\mathcal{S}$.*

Proof. Suppose by contradiction, that there are $\omega_1, \ldots, \omega_s$, $s > 2g$ independent closed differentials in $H^1_{deRham}(\mathcal{S}, \mathbb{C})$. Then let us consider a basis of the homology $\mathcal{S}_j$, $j = 1 \ldots, 2g$ and construct the matrix with entries

$$c_{jk} = \int_{\mathcal{S}_j} \omega_k, \quad j = 1, \ldots 2g, \;\; k = 1, \ldots s.$$

Such matrix has rank at most equal to $2g$, and therefore one can find nonzero constants $a_1, \ldots, a_s$ such that the differential $\omega = \sum_{k=1}^{s} a_k \omega_s$ has all its periods equal to zero, namely

$$\int_{\mathcal{S}_j} \omega = 0, \quad j = 1, \ldots 2g.$$

By proposition 3.1.9 it follows that $[\omega] = 0$ and we arrive to a contradiction. □

As a consequence of the above lemma we have the following corollary for the dimension of the space of holomorphic differentials.

Corollary 3.1.11. *The space of holomorphic differentials on a Riemann surface of genus g is no more than g-dimensional.*

Actually the number of independent holomorphic differentials is indeed equal to g.

Theorem 3.1.12. *The space of holomorphic differentials on a Riemann surface $\mathcal{S}$ of genus g has dimension g.*

We do not give a proof of the above theorem that is constructive (see [18] or [17]). However for a Riemann surface given as the zeros of a polynomial equation one can determine explicitly the holomorphic differentials.

Example 3.1.13. Let us consider holomorphic differentials on a hyperelliptic Riemann surface

$$\mathcal{S} = \{w^2 = P_{2g+1}(z)\}, \quad P_{2g+1}(z) = \prod_{k=1}^{2g+1}(z - z_k)$$

of genus $g \geqslant 1$. Let us check that the differentials

$$\eta_k = \frac{z^{k-1}dz}{w} = \frac{z^{k-1}dz}{\sqrt{P_{2g+1}(z)}}, \quad k = 1, \dots, g \tag{3.1.6}$$

are holomorphic. Indeed, holomorphicity at any finite point but branch point is obvious as the denominator does not vanish. We verify holomorphicity in a neighbourhood of the i-th branch point $P_i = \{z = z_i, \quad w = 0\}$. Choosing the local parameter τ in a neighbourhood of P_i in the form $\tau = \sqrt{z - z_i}$, we get from (1.2.26) that $\eta_k = \psi_k(\tau)d\tau$, where the function

$$\psi_k(\tau) = \frac{2(z_i + \tau^2)^{k-1}}{\sqrt{\prod_{j \neq i}(\tau^2 + z_i - z_j)}}$$

is holomorphic for small τ.

At the point at infinity the differentials η_k can be written in terms of the local parameter $\tau = z^{-\frac{1}{2}}$ in the form $\eta_k = \phi_k(\tau)d\tau$, where the functions

$$\phi_k(\tau) = -2\tau^{2(g-k)}\left[\prod_{i=1}^{2g+1}(1 - z_i\tau)\right]^{-\frac{1}{2}}, \quad k = 1, \dots, g$$

are holomorphic for small τ.

In the same way it can be verified that the differentials $\eta_k = z^{k-1}dz/w$, $k = 1, \dots, g$ are holomorphic on the Riemann surface of the curve $w^2 = P_{2g+2}(z)$ with $P_{2g+2}(z)$ an even polynomial with $2g + 2$ distinct roots.

Newton polygon and holomorphic differentials

In general let us consider the non-singular irreducible affine plane curve $C := \{(z,w) \in \mathbb{C}^2, | F(z,w) = \sum_{j=0}^{n} a_j(z) w^{n-j}\}$, where $a_j(z)$ are polynomials in z. Let $\mathcal{S}$ be the Riemann surface of the curve C. The one form

$$\omega = \frac{z^{i-1} w^{j-1} dz}{F_w(z,w)}, \quad i,j \geqslant 1, \tag{3.1.7}$$

is clearly holomorphic for all values where z and w are holomorphic. Indeed the only other possible points where such differential might have poles are the zeros of F_w, namely the branch points with respect to the projection $\pi_z : \mathcal{S} \to \mathbb{C}$, $\pi_z(z,w) = z$. At these branch points, one needs to take w as local coordinate. Since $F_z dz + F_w dw = 0$ one has

$$\frac{dz}{F_w} = -\frac{dw}{F_z}.$$

Therefore at the branch points where $F_w = 0$ one can write the differential ω in the form $\omega = -\dfrac{z^{j-1} w^{k-1} dw}{F_z}$. Since we assume that the curve C is non-singular, $F_z \neq 0$ at the points where $F_w = 0$.

In order to determine for which coefficients (i,j) the differential ω in (3.1.7) remains holomorphic when z and w go to infinity, we exploit again the Newton polygon.

We recall that the Newton polygon $\mathcal{N}$ of the polynomial $F(z,w) = \sum_{i,j \geqslant 0} a_{ij} z^i w^j$ is the convex hull of the set of points (i,j) of the (x,y)-plane defined by

$$\mathcal{N} = \text{Convex Hull}\{(i,j) \in \mathbb{Z}^2 \,|\, a_{ij} \neq 0\}.$$

We define $\widehat{\mathcal{N}} = \mathcal{N} \backslash \partial \mathcal{N}$ where $\partial\mathcal{N}$ is the boundary of $\mathcal{N}$. We have the following theorem

Theorem 3.1.14. *Suppose that the affine plane curve* $C = \{(z,w) \in \mathbb{C}^2 \,| F(z,w) = \sum_{j=0}^{n} \sum_{i=0}^{M} a_{ij} z^i w^j = 0\}$ *is connected and non singular and let* $\mathcal{S}$ *be the compact Riemann surface of the curve* C. *Then the basis of holomorphic differentials of* $\mathcal{S}$ *is*

$$\omega = \frac{z^{i-1} w^{j-1}}{F_w} dz, \quad (i,j) \in \widehat{\mathcal{N}}. \tag{3.1.8}$$

Proof. The Riemann surface $\mathcal{S}$ has two meromorphic function z and w. Therefore we need to show that ω in (3.1.8) remains holomorphic at the poles of z and w when $(i,j) \in \widehat{\mathcal{N}}$. We assume, without loss of generality, that $a_{0n} \neq 0$ and $a_{00} \neq 0$. Further let

$$m_n = \max_{i=0,\dots,M} \{i \,|\, a_{in} \neq 0\}, \quad m_0 = \max_{i=0,\dots,M} \{i \,|\, a_{i0} \neq 0\}$$

In this way the Newton polygon takes the form depicted in the figure. Suppose that the total number of edges of the Newton polygon is ℓ. We divide the edges of the Newton polygon in two subsets:

- the edges that face the y axis, including the horizontal edges.
- the remaining edges.

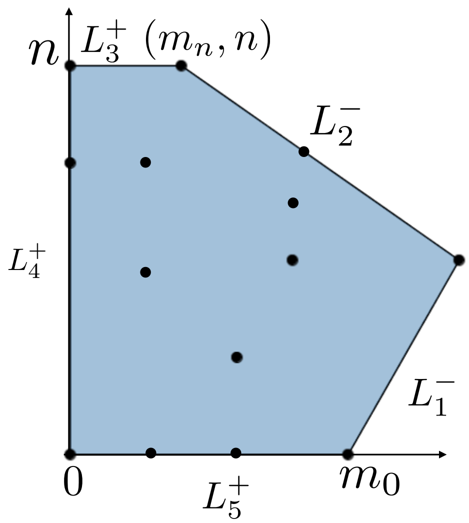


Figure 3.1: Example of Newton polygon

We number each edge starting from the rightmost edge that does not face the y axis and we proceed numbering the edges anti-clockwise as in Figure 3.1. For each edge that does not face the y axis we associate the line that contains it

$$L_s^-(x,y) = x\,q_s + yp_s - m_s = 0, \quad x, y \in \mathbb{R}, \;\; s = 1, 2, \ldots, \ell_1,$$

while for the remaining edges, including the horizontal edges we associate the lines

$$L_s^+(x,y) = x\,q_s + yp_s - m_s = 0, \quad x, y \in \mathbb{R}, \;\; s = \ell_1, \ell_1 + 1, \ldots \ell,$$

where we assume in both cases that $q_s, m_s \in \mathbb{N} \cup \{0\}$, and $p_s \in \mathbb{Z}$. We define the set of integer lattice points

$$\begin{aligned} D_s &= \{(i,j) \in \mathbb{Z}^2 | L_s^-(i,j) < 0\}, \quad s = 1, 2, \ldots, \ell_1, \\ D_s &= \{(i,j) \in \mathbb{Z}^2 | L_s^+(i,j) > 0\}, \quad s = \ell_1, \ell_1 + 1, \ldots \ell. \end{aligned} \tag{3.1.9}$$

Then clearly the interior of the Newton polygon is given by

$$\hat{\mathcal{N}} = \cap_{s=1}^{\ell} D_s.$$

Since the function z has degree n the number of its poles counting multiplicity is equal to n. The local coordinate of the function z at each of its poles is obtained from the slope of each line L_s^-

$s = 1, \dots \ell_1$. Indeed to each line L_s^- we associate the expansion in the local coordinate t

$$z = \frac{1}{t^{q_s}}, \quad w \simeq \frac{c_{0s}}{t^{p_s}}, \quad s = 1, \dots, \ell_1, \tag{3.1.10}$$

where we assume that $(p_s, q_s) \neq (0,0)$. We substitute the above expansion into the equation of the curve to obtain

$$F(z(t), w(t)) = \sum_{(i,j)\in\mathcal{N}} a_{ij} c_{0s}^j t^{-iq_s - jp_s} = t^{-m_s} \Big(\sum_{(i,j)\in\mathcal{N}\cap L_s^-} a_{ij} c_{0s}^j + O(t) \Big) .$$

The coefficient c_{0s} is obtained from

$$\sum_{(i,j)\in\mathcal{N}\cap L_s^-} a_{ij} c_{0s}^j = 0.$$

The number of distinct solutions of the above equation corresponds to the length of the projection of the corresponding edge of the Newton polygon on the y axis. In order to study the behaviour of the differential (3.1.8) near the poles of the function z we first consider the expansion of each term using the local coordinate (3.1.10):

$$dz = -q_s \frac{dt}{t^{q_s - 1}} \tag{3.1.11}$$

$$F_w(z(t), w(t)) = \sum_{i,j\in\mathcal{N}} j a_{ij} c_{0s}^{j-1} t^{-iq_s - jp_s + p_s} = t^{p_s - m_s} \Big(\sum_{i,j\in\mathcal{N}\cap L_s^-} a_{ij} c_{0s}^{j-1} + O(t) \Big) \tag{3.1.12}$$

so that the differential ω in (3.1.8) takes the form

$$\begin{aligned} \omega = \frac{z^{i-1} w^{j-1}}{F_w} dz &= -q_s \frac{t^{-(i-1)q_s - (j-1)p_s} t^{-q_s+1} dt}{t^{p_s - m_s} (\sum_{i,j\in\mathcal{N}\cap L_s^-} a_{ij} c_{0s}^{j-1} + O(t))} \\ &= \text{cons } t^{-iq_s - jp_s + m_s + 1} (1 + O(t)) dt, \end{aligned}$$

where the constant factor is not important for our purposes. In view of (3.1.15), for $(i,j) \in \widehat{\mathcal{N}}$ one has $iq_s + jp_s - m_s < 0$ for $s = 1, \dots, \ell_1$ so we conclude from the above expansion that ω is holomorphic near the poles of z.

Finally, we need to study the behaviour of ω near the set of poles of w. The local behaviour of the function w near its poles is described by the slope of the edges that are not facing the positive x-axis. For the example in the Figure 3.1 this corresponds to the edges L_2^- and L_3^+. In the general case the edges L_s^-, for $s = r \dots, \ell_1$ correspond to poles of both the functions z and w and the computation has already been performed above. The edges L_s^+, $s = \ell_1 + 1, \dots \ell_1 + k$ with $k \geqslant 0$ correspond only to poles of the function w while the function z assumes finite values that are the zeros of the polynomial $\sum_{(i,n)\in\mathcal{N}} a_{in} z^i$. In this case the local coordinate near such points is described by

$$z = c_{0s} t^{q_s}, \quad w \simeq t^{p_s}, \quad s = \ell_1 + 1, \dots \ell_1 + k. \tag{3.1.13}$$

Plugging the above local coordinate in $F(z(t), w(t))$ one can determine the constant c_{0s} and the number of solutions corresponds to the length of the projection of the corresponding segment onto the x axis. Then plugging the local coordinate (3.1.13) in ω one obtains

$$\omega = \text{cons } t^{iq_s + jp_s - m_s - 1} (1 + O(t)) dt, \quad s = \ell_1 + 1, \dots \ell_1 + k.$$

In view of (3.1.15), for $(i,j) \in \widehat{\mathcal{N}}$ one has $iq_s + jp_s - m_s \geqslant 1$ for $s = \ell_1 + 1, \dots \ell_1 + k$, so we conclude from the above expansion that ω is holomorphic near the poles of w.

□

Example 3.1.15. Consider the algebraic curve $C := \{(z,w) \in C^2 \,|\, F(z,w) = w^3 + zw^4 + z^5 w + zw^2 + z^2 w^2 + 1 = 0\}$. Using Maple, it is possible to verify that the curve is non singular since the system of equations $F = 0$, $F_w = 0$ and $F_z = 0$ does not have solutions. The Newton polygon is given in the Figure 3.2. The edges L_1^- and L_2^- describe the poles of the function z, with total multiplicity

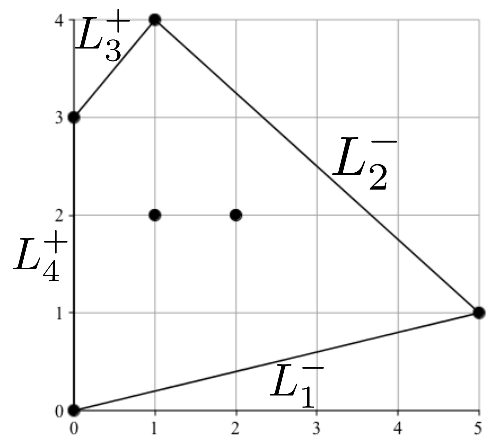


Figure 3.2: Newton polygon

equal to 4 which is equal to the length of the projection of the edges onto the y-axis. The edges L_2^- and L_3^+ describe the poles of the function w with total multiplicity equal to 5 which is equal to the length of the projection of these edges onto the x-axis. The edges L_3^+ and L_4^+ facing the y axis describe the behaviour of the function z near $z = 0$ while the side L_1^- that is facing the x axis describes the behaviour of the function w near $w = 0$. The corresponding lines are

$$\begin{aligned} &L_1^-(x,y) = x - 5y = 0, \quad L_2^-(x,y) = 3x + 4y - 19 = 0 \\ &L_3^+(x,y) = x - y + 3 = 0 \quad L_4^+(x,y) = x = 0\,. \end{aligned} \tag{3.1.14}$$

- Edge L_1^-; it corresponds to the local parameter of the form

$$z = \frac{1}{\tau}, \quad w = \tau^5 (c_0 + \sum_{k \geqslant 1} c_k \tau^k)$$

Plugging the above ansatz into the equation of the curve one obtains $F(z(t), w(t)) = c_0 + 1 + O(t) = 0$ which implies $c_0 = -1$. We denote this point as $P^{\infty,0} = (\infty, 0)$. It is a first order pole for z while it is a zero of order five for the function w.

- Edge L_2^-; it correspond to a local parameter of the form

$$z = \frac{1}{\tau^3}, \quad w = \frac{1}{\tau^4}(c_0 + \sum_{k\geqslant 1} c_k \tau^k)$$

  Plugging the ansatz into the equation of the curve we obtain $F(z(t), w(t)) = \frac{1}{\tau^{19}}(c_0(c_0^3 + 1) + O(\tau)) = 0$ so that $c_0 = e^{\pi i j/3}$ for $j = 1,2,3$. Since locally the function $w \sim -z^{\frac{4}{3}}$, the corresponding point P^∞ that needs to be added to make C a compact Riemann surface is a branch point of multiplicity 3 with respect to the projection $\pi_z(z,w) = z$ and of multiplicity 4 with respect to the projection $\pi_w(z,w) = w$. The point P^∞ is a pole of multiplicity 3 for the function z and it is a pole of multiplicity four for the function w.

- Edge L_3^+; it corresponds to a local parameter of the form

$$z = \tau, \quad w = \frac{1}{\tau}(c_0 + \sum_k c_k \tau^k)$$

  Plugging the ansatz into the equation of the curve we obtain $F(z(\tau), w(\tau)) = \dfrac{1}{\tau^3}(c_0^3(c_0+1) + O(\tau)) = 0$ which implies $c_0 = -1$. We denote this point as $P^{0,\infty}$. It is a simple zero for z and a first order pole for w.

- Edge L_4^+; it correspond to a local parameter of the form

$$z = \tau, \quad w = (c_0 + \sum_k c_k \tau^k)$$

  Plugging the above local coordinate into the equation of the curve one obtains $F(z(\tau), w(\tau)) = c_0^3 + 1 + o(\tau) = 0$, so that $c_0 = e^{\pi i j/3}$ with $j = 1,2,3$. Namely the meromorphic function z has three simple zeros at the points $P_j^0 = (0, e^{\pi i j/3})$, $j = 1,2,3$.

Let us define the domains

$$D_s = \{(i,j) \in \mathbb{Z}^2 | L_s^-(i,j) < 0\}, \quad s = 1,2,$$
$$D_s = \{(i,j) \in \mathbb{Z}^2 | L_s^+(i,j) > 0\}, \quad s = 3,4.$$

Then the interior of the Newton polygon $\widehat{N}$ is equal to

$$\widehat{\mathcal{N}} = \bigcap_{s=1}^{4} D_s.$$

Let us check that the differential $\omega = \dfrac{z^{i-1}w^{j-1}}{F_w}dz$ is holomorphic for $(i,j) \in \widehat{\mathcal{N}}$. For example let us consider the the differential ω in the local coordinate the point $P^{\infty,0}$. We have

$$F_w(z(\tau), w(\tau)) = \frac{1}{\tau^4}(1 + O(\tau)), \quad dz = -\frac{1}{\tau^2}d\tau$$

so that

$$\omega = -\tau^{-i+1}\tau^{4j-4}\tau^2(1+O(\tau))d\tau = -\tau^{-(i-4j+1)}(1+O(\tau))d\tau$$

which is holomorphic for small τ when $(i,j) \in D_4$. In a similar way writing the differential ω in local coordinates near the other points, one concludes that $\omega = \dfrac{z^{i-1}w^{j-1}}{F_w}dz$ is a holomorphic differential for $(i,j) \in \bigcap_{s=1}^4 D_s = \widehat{\mathcal{N}}$.

3.1.2 Riemann bilinear relations

In this section we prove several technical assertions regarding the periods of close differential and holomorphic differentials. Such relations are known as Riemann bilinear relations

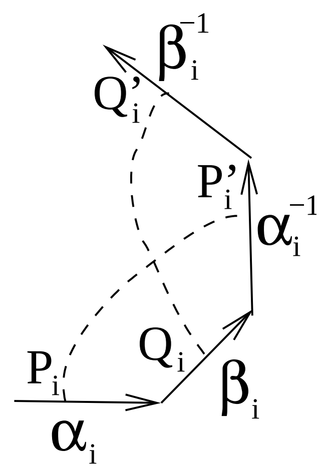
Lemma 3.1.16. *Let ω_1 and ω_2 be two closed differentials on a surface $\mathcal{S}$ of genus $g \geqslant 1$. Denote their periods with respect to a canonical basis of cycles $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, by A_i, B_i and A'_i, B'_i:*

$$A_i = \int_{\alpha_i} \omega, \quad B_i = \int_{\beta_i} \omega, \quad A'_i = \int_{\alpha_i} \omega', \quad B'_i = \int_{\beta_i} \omega'. \tag{3.1.15}$$

Denote by $f = \int \omega$ the primitive of ω, then

$$\int\int_{\mathcal{S}} \omega \wedge \omega' = \oint_{\partial\tilde{\mathcal{S}}} f\omega' = \sum_{i=1}^{g}(A_iB'_i - A'_iB_i). \tag{3.1.16}$$

Proof. The first of the equalities in (3.1.16) follows from Stokes' formula, since $d(f\omega') = \omega \wedge \omega'$. Let us prove the second. We have that



$$\oint_{\partial\tilde{\mathcal{S}}} f\omega' = \sum_{i=1}^{g}\left(\int_{\alpha_i} + \int_{\alpha_i^{-1}}\right) f\omega' + \sum_{i=1}^{g}\left(\int_{\beta_i} + \int_{\beta_i^{-1}}\right) f\omega'.$$

To compute the i-th term in the first sum we use the fact that $f(P) = \int_{P_0}^{P} \omega$ where P_0 is a point in the interior of $\tilde{\mathcal{S}}$:

$$f(P_i) - f(P_i') = \int_{P_0}^{P_i} \omega - \int_{P_0}^{P_i'} \omega = \int_{P_i'}^{P_i} \omega = -B_i \tag{3.1.17}$$

since the cycle $P_i'P_i$, which is closed on $\mathcal{S}$, is homologous to the cycle β_i (see the figure; a fragment of the boundary $\partial\tilde{\mathcal{S}}$ is pictured). Similarly, the jump of the function f in crossing the cut β_i has the form

$$f(Q_i) - f(Q_i') = \int_{Q_i'}^{Q_i} \omega = A_i \tag{3.1.18}$$

since the cycle $Q_i'Q_i$ on $\mathcal{S}$ is homologous to the cycle a_i. Moreover, $\omega'(P_i') = \omega'(P_i)$ and $\omega'(Q_i') = \omega'(Q_i)$ because the differential ω' is single-valued on $\mathcal{S}$. We have that

$$\begin{aligned}\int_{\alpha_i} f(P_i)\omega'(P_i) + \int_{\alpha_i^{-1}} f(P_i')\omega'(P_i') &= \int_{\alpha_i} f(P_i)\omega'(P_i) - \int_{\alpha_i} (f(P_i) + B_i)\omega'(P_i) \\ &= -B_i \int_{\alpha_i} \omega'(P_i) = -B_i A_i'\end{aligned}$$

where the minus sign appears because the edge a_i^{-1} occurs in $\partial\tilde{\mathcal{S}}$ with a minus sign. Similarly,

$$\left(\int_{\beta_i} + \int_{\beta_i^{-1}} \right) f\omega' = A_i B_i'.$$

Summing these equalities, we get (3.1.16). The lemma is proved. □

We derive some important consequences for periods of holomorphic differentials from the lemma 3.1.16. Everywhere we denote by $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ the canonical basis of cycles on $\mathcal{S}$.

Corollary 3.1.17. *Let ω be a nonzero holomorphic differential on $\mathcal{S}$, and $A_1, \ldots, A_g, B_1, \ldots, B_g$ its corresponding periods with respect to the canonical homology basis $\alpha_1 \ldots, \alpha_g$ and $\beta_1 \ldots, \beta_g$, then*

$$\Im \left(\sum_{i=1}^{g} A_k \bar{B}_k \right) < 0. \tag{3.1.19}$$

Proof. Take $\omega' = \overline{\omega}$ in the lemma 3.1.16. Then $A_i' = \bar{A}_i$ and $B_i' = \bar{B}_i$ for $i = 1, \ldots, g$. We have that

$$\frac{i}{2} \int\int_{\mathcal{S}} \omega \wedge \omega' = \frac{i}{2} \int\int |f|^2 dz \wedge d\bar{z} = \int\int_{\mathcal{S}} |f|^2 dx \wedge dy > 0.$$

Here $z = x + iy$ is a local parameter, and $\omega = f(z)dz$. In view of (3.1.16) this integral is equal to

$$\frac{i}{2} \sum_{k=1}^{g} A_k \bar{B}_k - \bar{A}_k B_k = -\Im \left(\sum_{k=1}^{g} A_k \bar{B}_k \right).$$

The corollary is proved. □

Corollary 3.1.18. *If all the α-periods of a holomorphic differential are zero, then $\omega = 0$.*

This follows immediately from Corollary 3.1.17.

Corollary 3.1.19. *On a surface $\mathcal{S}$ of genus g there exists a basis $\omega_1, \ldots, \omega_g$ of holomorphic differentials such that*

$$\oint_{\alpha_j} \omega_k = \delta_{jk}, \quad j, k = 1, \ldots, g. \tag{3.1.20}$$

Proof. Let $\eta_1, \ldots, \eta_g$ be an arbitrary basis of holomorphic differentials on $\mathcal{S}$. The matrix

$$A_{jk} = \oint_{\alpha_j} \eta_k \tag{3.1.21}$$

is non-singular. Indeed, otherwise there are constants $c_l, \ldots, c_g$ such that $\sum_k A_{jk} c_k = 0$. But then $\sum_k c_k \eta_k = 0$, since this differential has zero a-periods. This contradicts the independence of the differentials $\eta_1, \ldots, \eta_g$. Consider

$$\omega_j = \sum_{k=1}^{g} \tilde{A}_{kj} \eta_k, \quad j = 1, \ldots, g, \tag{3.1.22}$$

where the matrix $(\tilde{A}_{kj})$ is the inverse of the matrix (A_{jk}), $\sum_k \tilde{A}_{ik} A_{kj} = \delta_{ij}$. Then the differentials ω_j define the desired basis. □

A basis $\omega_1, \ldots, \omega_g$ satisfying the conditions (3.1.20) will be called a normal basis of holomorphic differentials (with respect to a canonical basis of cycles $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$) .

Corollary 3.1.20. *Let $\omega_1, \ldots \omega_g$ be a normalized basis of holomorphic differentials, and let*

$$B_{jk} = \oint_{\beta_j} \omega_k, \quad j, k = 1, \ldots, g. \tag{3.1.23}$$

Then the matrix (B_{jk}) is symmetric and has positive-definite imaginary part.

Proof. Let us apply the lemma 3.1.16 to the pair $\omega = \omega_j$ and $\omega' = \omega_k$. By (3.1.16) we have that

$$0 = \sum_i (\delta_{ij} B_{ik} - \delta_{ik} B_{ij}) = (B_{jk} - B_{kj}).$$

The symmetry is proved. Next, we apply Corollary 3.1.17 to the differential $\sum_{j=1}^g x_j \omega_j$ where all the coefficients $x_1, \ldots, x_g$ are real. We have that $A_k = x_k$, $B_k = \sum_j x_j B_{kj}$ which implies

$$\Im(\sum_k x_k \sum_j x_j \bar{B}_{kj}) = \sum_{k,j} \Im(\bar{B}_{kj}) x_k x_j < 0.$$

The lemma is proved. □

Definition 3.1.21. *The matrix (B_{jk}) is called a period matrix of the Riemann surface $\mathcal{S}$.*

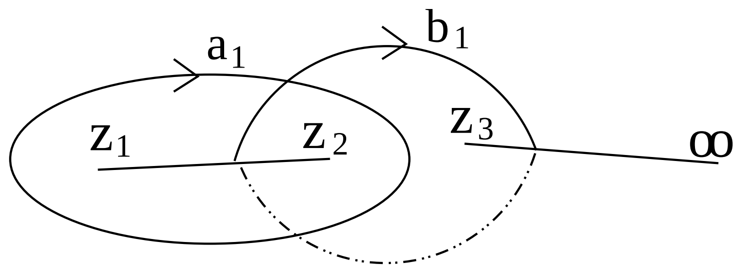


Figure 3.3: Homology basis.

Example 3.1.22. We consider a surface $\mathcal{S}$ of the form $w^2 = P_3(z)$ of genus $g = 1$ (an elliptic Riemann surface). Let $P_3(z) = (z - z_1)(z - z_2)(z - z_3)$ and choose a basis of cycles as shown in the figure 2.7. We have that

$$\omega_1 = \omega = \frac{adz}{\sqrt{P_3(z)}}, \quad a = \left(\oint_{\alpha_1} \frac{dz}{\sqrt{P_3(z)}} \right)^{-1}.$$

Note that

$$\oint_{\alpha_1} \frac{dz}{\sqrt{P_3(z)}} = 2 \int_{z_1}^{z_2} \frac{dz}{\sqrt{P_3(z)}}.$$

The period matrix is the single number

$$B = \oint_{\beta_1} \frac{adz}{\sqrt{P_3(z)}} = \frac{\int_{z_2}^{z_3} \frac{dz}{\sqrt{P_3(z)}}}{\int_{z_1}^{z_2} \frac{dz}{\sqrt{P_3(z)}}}, \quad \Im(B) > 0. \tag{3.1.24}$$

Example 3.1.23. . Consider a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z) = \prod_{i=1}^{2g+1}(z - z_i)$ for genus $g \geqslant 2$, and choose a basis of cycles as indicated in the figure 3.4 (there $g = 2$). A normal basis of holomorphic differentials has the form

$$\omega_j = \frac{\sum_{k=1}^{g} c_{jk} z^{k-1} dz}{\sqrt{P_{2g+1}(z)}}, \quad j = 1, \ldots, g. \tag{3.1.25}$$

Here (c_{jk}) is the matrix inverse to the matrix (A_{jk}) where

$$A_{jk} = 2 \int_{z_{2j-1}}^{z_{2j}} \frac{z^{k-1} dz}{\sqrt{P_{2g+1}(z)}}, \quad j, k = 1, \ldots, g. \tag{3.1.26}$$

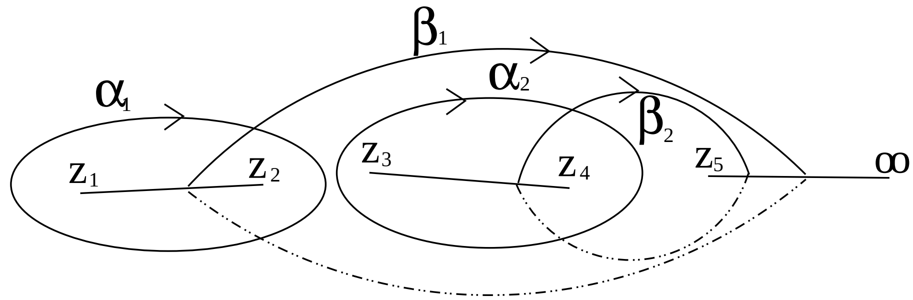


Figure 3.4: Homology basis.

3.1.3 Meromorphic differentials, their residues and periods

Meromorphic (Abelian) differentials on a Riemann surface differ from holomorphic differentials by the possible presence of singularities of pole type. If a surface is given in the form $F(z,w) = 0$, then the Abelian differentials have the form $\omega = R(z,w)dz$ or, equivalently, $\omega = R_1(z,w)dw$, where $R(z,w)$ and $R_1(z,w)$ are rational functions. For example, on a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z)$ the differential $w^{-1}z^{k-1}dz$ has for $k > g$ a unique pole at infinity of multiplicity $2(k-g)$ (see Example 3.1.13). Suppose that the differential ω has a pole of multiplicity k at the point P_0 i.e., can be written in terms of a local parameter z, $z(P_0) = 0$, in the form

$$\omega = \left(\frac{c_{-k}}{z^k} + \cdots + \frac{c_{-1}}{z} + O(1)\right) dz \tag{3.1.27}$$

(the multiplicity of the pole does not depend on the choice of the local parameter z).

Definition 3.1.24. *The residue* $\mathrm{Res}_{P=P_0}\,\omega(P)$ *of the differential* ω *at a point* P_0 *is defined to be the coefficient* c_{-1}.

Lemma 3.1.25. *The residue* $\mathrm{Res}_{P=P_0}\,\omega(P)$ *does not depend on the choice of the local parameter* z.

Proof. This residue is equal to

$$c_{-1} = \frac{1}{2\pi i}\oint_C \omega$$

where C is an arbitrary small contour encircling P_0. The independence of this integral on the choice of the local parameter is obvious. The lemma is proved. □

Theorem 3.1.26 (The Residue Theorem)**.** *. The sum of the residues of a meromorphic differential ω on a Riemann surface, taken over all poles of this differential, is equal to zero.*

Proof. Let $P_1, \ldots, P_N$ be the poles of ω. We encircle them by small contours $C_1, ..., C_N$ such that

$$\operatorname*{Res}_{P_i} \omega = \frac{1}{2\pi i}\oint_{C_j} \omega, \quad j = 1, \ldots, N,$$

(the contours C_i run in the positive direction), and cut out the domains bounded by $C_1, \dots, C_N$ from the surface $\mathcal{S}$. This gives a domain $\mathcal{S}'$ with oriented boundary of the form $\partial\mathcal{S}' = -C_1 - \cdots - C_N$ (the sign means reversal of orientation). The differential ω is holomorphic on $\mathcal{S}'$. By Stokes' formula,

$$\sum_{j=1}^{N} \operatorname*{Res}_{P_j} \omega = \frac{1}{2\pi i} \sum_{j=1}^{N} \oint_{C_j} \omega = -\frac{1}{2\pi i} \oint_{\partial\mathcal{S}'} \omega = -\frac{1}{2\pi i} \int\int_{\mathcal{S}'} d\omega = 0,$$

since $d\omega = 0$. The theorem is proved. □

We present the simplest example of the use of the residue theorem: we prove that the number of zeros of a meromorphic function is equal to its number of poles (counting multiplicity). Let $P_1, \dots, P_k$, be the zeros of the meromorphic function f, with multiplicities $m_1, \dots, m_k$ a nd let $Q_1, ..., Q_l$ be the poles of this function, with multiplicities $n_1, \dots, n_k$. Consider the logarithmic differential $d(ln f)$. This is a meromorphic differential on $\mathcal{S}$ with simple poles at $P_1, \dots, P_k$ with residues $m_1, \dots, m_k$ and at the points $Q_1, \dots, Q_l$ with residues $-n_1, \dots, -n_l$. By the residue theorem: $m_1 + \cdots + m_k - n_1 - \cdots - n_k = 0$, which means that the assertion to be proved is valid. One more example. For any elliptic function $f(z)$ on the torus $T^2 = \mathbb{C}/\{2m\omega + 2n\omega'\}$ the residues at the poles are defined with respect to the complex coordinate z (in $\mathbb{C}$). These are the residues of the meromorphic differential $f(z)dz$, since dz is holomorphic everywhere. Conclusion: the sum of the residues of any elliptic function (over all poles in a lattice parallelogram) is equal to zero. We formulate an existence theorem for meromorphic differentials on a Riemann surface $\mathcal{S}$ (see [**?**] for a proof).

Theorem 3.1.27. *Suppose that $P_1, \dots, P_N$ are points of a Riemann surface $\mathcal{S}$ and $z_1, \dots, z_N$ are local parameters centered at these points, $z_i(P_i) = 0$, and the collection of principal parts is*

$$\left(\frac{c^{(i)}_{-k_i}}{z_i^{k_i}} + \cdots + \frac{c^{(i)}_{-1}}{z_i} \right) dz_i, \quad i = 1, \dots, N. \tag{3.1.28}$$

Assume the condition

$$\sum_{i=1}^{N} c^i_{-1} = 0. \tag{3.1.29}$$

Then there exists on $\mathcal{S}$ a meromorphic differential with poles at the points $P_1, \dots, P_N$, and principal parts (3.1.28).

Any meromorphic differential can be represented as the sum of a holomorphic differential and the following elementary meromorphic differentials.

1. Abelian differential of the second kind Ω^n_P has a unique pole of multiplicity $n+1$ at P and a principal part of the form

$$\Omega^n_P = \left(\frac{1}{z^{n+1}} + O(1) \right) dz \tag{3.1.30}$$

with respect to some local parameter z, $z(P) = 0$, $n = 1, 2, \dots$.

2. An Abelian differential of the third kind Ω_{PQ} has a pair of simple poles at the points P and Q with residues $+1$ and -1 respectively.

Example 3.1.28. We construct elementary Abelian differentials on a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z)$. Suppose that a point P which is not a branch point takes the form $P = (a, w_a = \sqrt{P_{2g+1}(a)})$. An Abelian differential of the second kind $\Omega_P^{(1)}$ has the form

$$\Omega_P^{(1)} = \left(\frac{w + w_a}{(z-a)^2} + \frac{P'_{2g+1}(a)}{2w_a(z-a)} \right) \frac{dz}{2w} \tag{3.1.31}$$

(with respect to the local parameter z-a). The differentials $\Omega_P^{(n)}$ can be obtained as follows:

$$\Omega_P^n = \frac{1}{n!} \frac{d^{n-1}}{da^{n-1}} \Omega_P^1. \tag{3.1.32}$$

If $P = (z_i, 0)$ is one of the branch points, then

$$\Omega_P^n = \frac{dz}{2(z-z_i)^{k+1}} \text{ for } n = 2k, \quad \Omega_P^n = \frac{dz}{2(z-z_i)^{k+1}w} \text{ for } n = 2k+1. \tag{3.1.33}$$

Finally, if $P = \infty$, then

$$\Omega_P^{(n)} = -\frac{1}{2} z^{k-1} dz \text{ for } n = 2k, \quad \Omega_P^n = -\frac{1}{2} z^{g+k-1} \frac{dz}{w} \text{ for } n = 2k+1. \tag{3.1.34}$$

We now construct differentials of the third kind. Suppose that the point P and Q have the form $P = (a, w_a = \sqrt{P_{2g+1}(a)})$ and $Q = (b, w_b = \sqrt{P_{2g+1}(b)})$. Then

$$\Omega_{PQ} = \left(\frac{w + w_a}{z-a} - \frac{w + w_b}{z-b} \right) \frac{dz}{2w} \tag{3.1.35}$$

If $Q = +\infty$ then

$$\Omega_{PQ} = \frac{w + w_a}{z-a} \frac{dz}{2w}. \tag{3.1.36}$$

Accordingly, we see that for a hyperelliptic Riemann surface it is possible to represent all the Abelian differentials without appealing to Theorem 3.1.27.

Exercise 3.1.29: Deduce from Theorem 3.1.27 that a Riemann surface $\mathcal{S}$ of genus 0 is rational. *Hint.* Show that for any points $P, Q \in \mathcal{S}$ the function $f = \exp \int \Omega_{PQ}$ is single valued and meromorphic on $\mathcal{S}$ and gives a biholomorphic isomorphism $f : \mathcal{S} \to \mathbb{P}^1$.

The period of a meromorphic differential ω along the cycle γ is defined if the cycle does not pass through poles of this differential. The period $\int_\gamma \omega$ depends only on the homology class of γ on the surface $\mathcal{S}$, with the poles of ω with nonzero residue deleted. For example, the periods of the differential Ω_{PQ} of the third kind along a cycle not passing through the points P and Q are determined to within integer multiples of $2\pi i$. In speaking of the periods of meromorphic differentials we shall assume that the cycles do not pass through the poles of the differential.

Lemma 3.1.30. *Suppose that the differentials Ω_1 and Ω_2 on a Riemann surface $\mathcal{S}$ have the same poles and principal parts, and the same periods with respect to the cycles $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$. Then these differentials coincide.*

Proof. The difference $\omega_1 - \omega_2$ is a holomorphic differential that has zero α-periods. Therefore, it is identically zero (see Lecture 3.1.2). The lemma is proved. □

Definition 3.1.31. *A meromorphic differential ω is said to be normalized with respect to a canonical basis of cycles $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ if it has zero α-periods.*

Any meromorphic differential ω can be turned into a normalized differential by adding a holomorphic differential $\sum_{k=1}^{g} c_k \omega_k$. Indeed the condition that $\Omega = \omega + \sum c_k \omega_k$ is normalised, namely

$$\int_{\alpha_j} \omega + \sum_{k=1}^{g} c_k \int_{\alpha_j} \omega_k = 0, \quad j = 1, \dots, g,$$

defines the constants $c_1, \dots, c_g$ uniquely.

By Lemma 3.1.30, a normalized meromorphic differential is uniquely determined by its poles and by the principal parts at the poles. In what follows we assume that meromorphic differentials are normalized. We obtain formulas that will be useful for the β-periods of such differentials by arguments like those in the proof of Lemma 3.1.16.

Lemma 3.1.32. *The following formulas hold for the β-periods of normalized differentials $\Omega_P^{(n)}$ and Ω_{PQ}*

$$\oint_{\beta_k} \Omega_P^{(n)} = 2\pi i \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \psi_k(z)|_{z=0}, \quad k = 1, \dots, g, \ n = 1, 2, \dots, \tag{3.1.37}$$

where z is a particular local parameter in a neighbourhood of P, $z(P) = 0$, and the functions $\psi_k(z)$ are determined by the equality $\omega_k = \psi_k(z)dz$ and $\omega_1, \dots, \omega_g$ is a normalized basis of holomorphic differentials with respect to the canonical homology basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$,

$$\oint_{\beta_k} \Omega_{PQ} = 2\pi i \int_Q^P \omega_k, \quad i = 1, \dots, g, \tag{3.1.38}$$

where the integration from Q to P in the last integral does not intersect the cycles $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$.

Proof. We encircle the point P with a small circle C oriented anticlockwise; deleting the interior of this circle from the surface $\mathcal{S}$, we get a domain $\mathcal{S}'$ with $\partial \mathcal{S}' = -C$. Let us apply the arguments of Lemma 3.1.16 to the pair of differentials $\omega = \omega_k$, $\omega' = \Omega_P^{(n)}$. Denote by u_i the primitive

$$u_k(Q) = \int_{P_0}^{Q} \omega_k \tag{3.1.39}$$

which is single-valued on the Poincare' polygon $\tilde{\mathcal{S}}$ of the surface $\mathcal{S}$. We have that

$$0 = \int\int_{\mathcal{S}'} \omega \wedge \omega' = \int_{\partial \tilde{\mathcal{S}}'} u_k \Omega_P^{(n)} = \sum_{j=1}^{g} (A_j B_j' - A_j' B_j) - \oint_C u_k \Omega_P^{(n)} \tag{3.1.40}$$

(the boundary $\partial\tilde{\mathcal{S}}'$ differs from the boundary $\partial\tilde{\mathcal{S}}$ by $(-C)$). Here the α and β-periods of ω_k and Ω_P^N have the form

$$A_j = \delta_{kj}, \ \ B_j = B_{kj}, \quad A'_j = 0, \ \ B'_j = \oint_{\beta_j} \Omega_P^{(n)}.$$

From this,

$$\oint_{\beta_k} \Omega_P^{(n)} = \oint_C u_k \Omega_P^{(n)} = 2\pi i \operatorname*{Res}_P (u_k \Omega_P^{(n)}) = 2\pi i \operatorname*{Res}_{z=0} \left[\left(\int_{P_0}^P + \int_0^z \psi_k(\tau) d\tau \right) \frac{dz}{z^{n+1}} \right] \tag{3.1.41}$$

Computation of the residue on the right-hand side of this equality leads to (3.1.37).
We now prove (3.1.38). Let C and C' small circles around P and Q respectively. Deleting the interior of this circles from the surface $\mathcal{S}$, we get a domain $\mathcal{S}'$ with $\partial\mathcal{S}' = -C - C'$. Let us apply the arguments of Lemma 3.1.16 to the pair of differentials $\omega = \omega_k$, $\omega' = \Omega_{PQ}$. Denote by u_i the primitive of ω_i. By analogy with (3.1.40) and (3.1.41) we have that

$$\oint_{\beta_k} \Omega_{PQ} = 2\pi i \oint_C u_k \Omega_{PQ} + 2\pi i \oint_{C'} u_k \Omega_{PQ}$$

Since the differential Ω_{PQ} has a simple pole in P and Q with residue ± 1 respectively, the above integrals are equal to

$$\oint_{\beta_k} \Omega_{PQ} = u_k(P) - u_k(Q) = \int_{P_0}^P \omega_k - \int_{P_0}^Q \omega_k = \int_Q^P \omega_k$$

where we assume that the point P_0 lies in the interior of $\mathcal{S}'$. The lemma is proved. □

Exercise 3.1.33: Prove the following equality, which is valid for any quadruple of distinct points $P_1, \ldots, P_4$ on a Riemann surface:

$$\int_{P_2}^{P_1} \Omega_{P_3 P_4} = \int_{P_4}^{P_3} \Omega_{P_1 P_2}. \tag{3.1.42}$$

Exercise 3.1.34: Consider the series expansion of the differentials $\Omega_P^{(n)}$ in a neighbourhood of the point P

$$\Omega_P^{(n)} = \left(\frac{1}{z^{n+1}} + \sum_{j=0}^{\infty} c_j^{(n)} z^j \right) dz. \tag{3.1.43}$$

Prove the following symmetry relations for the coefficients $c_j^{(k)}$:

$$k c_{j-1}^{(k)} = j c_{k-1}^{(j)}, \quad k, j = 1, 2 \ldots. \tag{3.1.44}$$

Exercise 3.1.35: Prove that a meromorphic differential of the second kind ω is uniquely determined by its poles, principal parts, and the real normalization condition

$$\Im \oint_{\mathcal{S}} \omega = 0 \tag{3.1.45}$$

for any cycle $\mathcal{S}$. Formulate and prove an analogous assertion for differentials of the third kind (with purely imaginary residues).

3.1.4 The Jacobi variety, Abel's theorem

Let $e_1, \ldots, e_g$ be the standard basis in the space $\mathbb{C}^g$, $e_j = (0, \ldots, 1, \ldots, 0)$, with one on the j-entry. Given $2g$ row vectors $\lambda_k \in \mathbb{C}^g$, $k = 1, \ldots, 2g$, with $\lambda_k = \sum_{j=1}^g \lambda_{kj} e_j$, we construct the $2g \times g$ matrix Λ having in the k-row the vector λ_k

$$\Lambda_{kj} = (\lambda_k)_j. \tag{3.1.46}$$

The matrix Λ generates a lattice in $\mathbb{C}^g$ of maximal rank, if its row vectors are linearly independent over the real numbers.

Consider in $\mathbb{C}^g$ the integer period lattice L generated by the vectors (3.1.46). The vectors in this lattice can be written in the form

$$L = \{v \in \mathbb{C}^g \mid v = \sum_{k=1}^{2g} m_k \lambda_k, \quad (m_1, \ldots, m_{2g}) \in \mathbb{Z}^{2g}\} \tag{3.1.47}$$

We assume that L generates a lattice of maximal rank in $\mathbb{C}^g$. Then the quotient of $\mathbb{C}^g$ by this lattice is the $2g$-dimensional torus

$$T^{2g} = \mathbb{C}^g / L \tag{3.1.48}$$

namely a g-dimensional complex manifold. Changing the basis in $\mathbb{C}^g$, namely $e_k \to e_k M$, with $M \in GL(g, \mathbb{C})$, the matrix $\Lambda \to \Lambda M$. Furthermore, the same lattice is given by vectors $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{2g})$ with

$$\tilde{\lambda}_k = \sum_{k=1}^{2g} n_{kj} \lambda_j$$

with $N = \{n_{kj}\}_{k,j=1}^{2g} \in SL(2g, \mathbb{Z})$. Therefore $\Lambda \to N\Lambda$. Summarizing, two matrices Λ and $\tilde{\Lambda}$ represent the same torus if

$$\tilde{\Lambda} = N \Lambda M, \quad M \in GL(g, \mathbb{C}), \ \ N \in SL(2g, \mathbb{Z}). \tag{3.1.49}$$

If we assume that the lattice generated by Λ has maximal rank, we can always choose Λ in such a way that

$$\Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}$$

with $\Lambda_1 \in GL(g, \mathbb{C})$. Therefore, by (3.1.49) the two matrices Λ and $\Lambda \Lambda_1^{-1} = \begin{pmatrix} I_g \\ \Lambda_2 \Lambda_1^{-1} \end{pmatrix}$ with I_g the g-dimensional identity, represent the same torus.

Let $B = (B_{jk})$ be an arbitrary complex symmetric $g \times g$ matrix with positive-definite imaginary part (as shown in Lecture 3.1.2, the period matrices of Riemann surfaces have this property). We consider the vectors

$$e_1, \ldots, e_g, \quad e_1 B, \ldots, e_g B. \tag{3.1.50}$$

Lemma 3.1.36. *The vectors (3.1.50) are linearly independent over* $\mathbb{R}$.

Proof. Assume that these vectors are dependent over $\mathbb{R}$:

$$(\rho_1 e_1 + \cdots + \rho_g e_g) + (\mu_1 e_1 + \cdots + \mu_g e_g) B = 0, \ \ \rho_i, \ \mu_j \in \mathbb{R}.$$

Separating out the real part of this equality we get that $\Im((\mu_1 e_1 + \cdots + \mu_g e_g)B) = 0$. But the matrix $\Im(B)$ is non-singular, which implies $\mu_1 = \cdots = \mu_g = 0$. Hence also $\rho_1 = \cdots = \rho_g = 0$. The lemma is proved. $\square$

Consider in $\mathbb{C}^g$ the integer period lattice generated by the vectors (3.1.50). The vectors in this lattice can be written in the form

$$m + nB, \quad m, n \in \mathbb{Z}^g. \tag{3.1.51}$$

By Lemma 3.1.36 the quotient of $\mathbb{C}^g$ by this lattice is a torus of maximal rank:

$$T^{2g} = T^{2g}(B) = \mathbb{C}^g/\{m + nB\}. \tag{3.1.52}$$

Definition 3.1.37. *Suppose that $B = (B_{jk})$ is a period matrix of a Riemann surface $\mathcal{S}$ of genus g. The torus $T^{2g}(B)$ in (3.1.52), constructed from this period matrix is called the Jacobi variety (or Jacobian) of the surface $\mathcal{S}$ and denoted by $J(\mathcal{S})$.*

Remark 3.1.38. What happens with the torus $J(\mathcal{S})$ when the canonical basis of cycles on $\mathcal{S}$ changes? Let $\alpha = (\alpha_1, \ldots, \alpha_g)^t$ and $\beta = (\beta_1, \ldots, \beta_g)^t$ be the column vectors of the canonical homology basis. Let α' and β' be a new canonical homology basis related to α and β by the symplectic transformation

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

Let $\omega = (\omega_1, \ldots, \omega_g)$ be the canonical homology basis of holomorphic differentials with respect to the basis α and β, namely

$$\int_\alpha \omega = I_g, \quad \int_\beta \omega = B$$

where I_g is the g dimensional identity matrix. Then

$$\int_{\alpha'} \omega = \int_{a\alpha + b\beta} \omega = aI_g + bB,$$

$$\int_{\beta'} \omega = \int_{c\alpha + d\beta} \omega = cI_g + dB.$$

Observe that $aI_g + bB$ is non singular, since it is the matrix of α-periods of the holomorphic differentials. So the canonical basis of holomorphic differentials $\omega' = (\omega'_1, \ldots, \omega'_g)$ with respect to the basis α' and β' is given by

$$\omega' = \omega(aI_g + bB)^{-1}$$

This implies that the corresponding period matrix

$$B' = \int_{\beta'} \omega' = (cI_g + dB)(aI_g + bB)^{-1}. \tag{3.1.53}$$

From (3.1.49) it follows that the tori $T^{2g}(B)$ and $T^{2g}(B')$ are isomorphic. Accordingly, the Jacobian $J(\mathcal{S})$ changes up to isomorphism when the canonical basis changes.

We consider the primitives ("Abelian integrals") of the basis of holomorphic differentials:

$$u_k(P) = \int_{P_0}^{P} \omega_k, \quad k = 1, \ldots, g, \tag{3.1.54}$$

where P_0 is a fixed point of the Riemann surface. The vector-valued function

$$\mathcal{A}(P) = (u_1(P), \ldots, u_g(P)) \tag{3.1.55}$$

is called the Abel mapping (the path of integration is chosen to be the same in all the integrals $u_1(P), \ldots, u_g(P)$).

Lemma 3.1.39. *The Abel mapping is a well-defined holomorphic mapping*

$$\mathcal{S} \to J(\mathcal{S}). \tag{3.1.56}$$

Proof. (cf. Example 3.1.28). A change of the path of integration in the integrals (3.1.54) leads to a change in the values of these integrals according to the law

$$u_k(P) \to u_k(P) + \oint_\gamma \omega_k, \quad k = 1, \ldots, g,$$

where γ is some cycle on $\mathcal{S}$. Decomposing it with respect to the basis of cycles, $\gamma \simeq \sum m_j \alpha_j + \sum n_j \beta_j$ we get that

$$u_k(P) \to u_k(P) + m_k + \sum_j B_{kj} n_j, \quad k = 1, \ldots, g.$$

The increment on the right-hand side is the kth coordinate of the period lattice vector $m + nB$ where $m = (m_1, \ldots, m_g)$, $n = (n_1, \ldots, n_g)$. The lemma is proved. □

The Jacobi variety together with the Abel mapping (3.1.56) is used for solving the following problem: what points of a Riemann surface can be the zeros and poles of meromorphic functions? We have the Abel's theorem.

Theorem 3.1.40 (Abel's Theorem)**.** *The points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ (some of the points can repeat) on a Riemann surface $\mathcal{S}$ are the respective zeros and poles of some function meromorphic on $\mathcal{S}$ if and only if the following relation holds on the Jacobian:*

$$\mathcal{A}(P_1) + \cdots + \mathcal{A}(P_n) \equiv \mathcal{A}(Q_1) + \cdots + \mathcal{A}(Q_n). \tag{3.1.57}$$

Here and below, the sign $\equiv$ will mean equality on the Jacobi variety (congruence modulo the period lattice (3.1.51)). We remark that the relation (3.1.57) does not depend on the choice of the initial point P_0 of the Abel map (3.1.54).

Proof. 1) Necessity. Suppose that a meromorphic function f has the respective points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ as zeros and poles, where each zero and pole is written the number of times corresponding to its multiplicity. Consider the logarithmic differential $\Omega = d(\log f)$. Since $f = \mathrm{const} \exp \int_{P_0}^{P} \Omega$, is a meromorphic function, the integral in the exponent does not depend on the path of integratio. It follows that all the periods of this differential Ω are integer multiples of $2\pi i$. On the other hand, we represent it in the form

$$\Omega = \sum_{j=1}^{n} \Omega_{P_j Q_j} + \sum_{s=1}^{g} c_s \omega_s, \tag{3.1.58}$$

where $\Omega_{P_j Q_j}$ are normalized differentials of the third kind (see Lecture 3.1.3) and $c_1, \dots, c_g$ are constant coefficients. Let us use the information about the periods of the differential. We have that

$$2\pi i n_k = \oint_{a_k} \Omega = c_k, \quad n_k \in \mathbb{Z},$$

which gives us $c_k = 2\pi i n_k$. Further,

$$2\pi i m_k = \oint_{b_k} \Omega = 2\pi i \sum_{j=1}^{n} \int_{Q_j}^{P_j} \omega_k + 2\pi i \sum_{s=1}^{g} n_s B_{sk}$$

(we used the formula (3.1.38)). From this,

$$u_k(P_1) + \dots + u_k(P_n) - u_k(Q_1) - \dots - u_k(Q_n) = \sum_{j=1}^{n} \int_{Q_j}^{P_j} \omega_k = m_k - \sum_{s=1}^{g} n_s B_{sk}. \tag{3.1.59}$$

The right-hand side is the kth coordinate of the vector $m + nB$ of the period lattice (3.1.51), where $m = (m_1, \dots, m_g)$, $n = (n_1, \dots, n_g)$. The necessity of the condition (3.1.57) is proved.

2) Sufficiency. Suppose that

$$u_k(P_1) + \dots + u_k(P_n) - u_k(Q_1) - \dots - u_k(Q_n) = m_k - \sum_{s=1}^{g} n_s B_{sk}. \tag{3.1.60}$$

Consider the function

$$f(P) = \exp\left[\sum_{j=1}^{g} \int_{P_0}^{P} \Omega_{P_j Q_j} + \sum_{j=1}^{g} c_j \int_{P_0}^{P} \omega_j\right]$$

where $\Omega_{P_j Q_j}$ are the normalised third kind differentials with poles in P_j and Q_j and c_j are constants. The function is a single valued meromorphic function if the integrals in the exponent do not depend on the path of integration. Let us study the behaviour of f when $P \to P + \alpha_k$:

$$f(P) \to f(P) \exp\left[\sum_{j=1}^{g} c_j \int_{\alpha_k} \omega_j\right].$$

In order to have a single valued function the constant $c_k = 2\pi n_k$, $n_k \in \mathbb{N}$. Next let us consider the behaviour of f when $P \to P + \beta_k$:

$$f(P) \to f(P) \exp\left[\sum_{j=1}^{g} \int_{\beta_k} \Omega_{P_j Q_j} + \sum_{j=1}^{g} n_j \int_{\beta_k} \omega_j\right] = f(P) \exp\left[2\pi i \sum_{j=1}^{g} \int_{Q_j}^{P_j} \omega_k + 2\pi i \sum_{j=1}^{g} n_j \int_{\beta_k} \omega_j\right]$$

Using the relation (3.1.60) one obtains that $f(P) \to f(P)\exp[2\pi i m_k] = f(P)$ which shows that $f(P)$ is a meromorphic function on $\mathcal{S}$. □

Example 3.1.41. We consider the elliptic curve

$$w^2 = 4z^3 - g_2 z - g_3. \tag{3.1.61}$$

For this curve the Jacobi variety $J(\mathcal{S})$ is a two-dimensional torus, and the Abel mapping (which coincides with (**??**)) is an isomorphism (see Example 3.1.22). Abel's theorem becomes the following assertion from the theory of elliptic functions: the sum of all the zeros of an elliptic function is equal to the sum of all its poles to within a vector of the period lattice.

Example 3.1.42. (also from the theory of elliptic functions). Consider an the elliptic function of the form $f(z,w) = az + bw + c$, where a, b, and c are constants. It has a pole of third order at infinity (for $b \neq 0$). Consequently, it has three zeros P_1, P_2, and P_3. In other words, the line $az + bw + c = 0$ intersects the elliptic curve (3.1.61) in three points (see the figure). We choose ∞ as the initial point for the Abel mapping, i.e., $u(\infty) = 0$. Let $u_i = u(P_i)$, $i = 1, 2, 3$. In other words,

$$P_i = (\wp(u_i), \wp'(u_i)), \quad i = 1, 2, 3,$$

where $\wp(u)$ is the Weierstrass function corresponding to the curve (3.1.61). Applying Abel's theorem to the zeros and poles of f, we get that

$$u_1 + u_2 + u_3 = 0.$$

Conversely, according to the same theorem, if $u_1 + u_2 + u_3 = 0$, i.e. $u_3 = -u_2 - u_1$ then the points P_1, P_2 and P_3 lie on a single line. Writing the condition of collinearity of these points and taking into account the evenness of $\wp$ and oddness of $\wp'$, we get the addition theorem for Weierstrass functions:

$$\det \begin{vmatrix} 1 & \wp(u_1) & \wp'(u_1) \\ 1 & \wp(u_2) & \wp'(u_2) \\ 1 & \wp(u_1+u_2) & -\wp'(u_1+u_2) \end{vmatrix} = 0. \tag{3.1.62}$$

3.1.5 Divisors on a Riemann surface. The canonical class. The Riemann-Roch theorem

Definition 3.1.43. *A divisor D on a Riemann surface is defined to be a (formal) integral linear combination of points on it:*

$$D = \sum_{i=1}^{n} n_i P_i, \quad P_i \in \mathcal{S}, \quad n_i \in \mathbb{Z}. \tag{3.1.63}$$

For example, for any meromorphic function f the divisor (f) of its zeros $P_1, \ldots, P_k$ and poles $Q_1, \ldots, Q_l$ of multiplicities $m_1, \ldots, m_k$, and $n_1, \ldots, n_l$, respectively is defined

$$(f) = m_1 P_1 + \cdots + m_k P_k - n_1 Q_1 - \cdots - n_l Q_l. \tag{3.1.64}$$

Observe that given f and g two meromorphic functions

$$(fg) = (f) + (g), \quad (f/g) = (f) - (g).$$

Definition 3.1.44. *Divisors of meromorphic functions are also called principal divisors.*

Another useful notation for the divisor of a meromoprhic function is given by

$$(f) = \sum_P \mathrm{mult}_P(f) \cdot P$$

where we recall that the multiplicity of f in P is the minimum coefficient present in the Laurent expansion in a neighbourhood of the point P namely $\mathrm{mult}_P f = \min_{n \in \mathbb{Z}}\{n, |\alpha_n \neq 0\}$ where the Laurent expansion of f in P is $\sum_n \alpha_n z^n$. Such definition does not depend on the choice of the local coordinates. The set of all divisors on $\mathcal{S}$, $\mathrm{Div}(\mathcal{S})$, obviously form an Abelian group (the zero is the empty divisor).

Definition 3.1.45. *The degree* $\deg D$ *of a divisor of the form (3.1.63) is defined to be the number*

$$\deg D = \sum_{i=1}^{N} n_i. \tag{3.1.65}$$

The degree is a linear function on the group of divisors. For instance,

$$\deg(f) = 0. \tag{3.1.66}$$

Definition 3.1.46. *Two divisors D and D' are said to be linearly equivalent, $D \simeq D'$ if their difference is a principal divisor.*

Linearly equivalent divisors have the same degree in view of (3.1.66). For example, on $\mathbb{P}^1$ any divisor of zero degree is principal, and two divisors of the same degree are always linearly equivalent.

Example 3.1.47. The divisor (ω) of any Abelian differential ω on a Riemann surface $\mathcal{S}$ is well-defined by analogy with (3.1.64). If ω' is another Abelian differential, then $(\omega) \simeq (\omega')$. Indeed, their ratio $f = \omega/\omega'$ is a meromorphic function on $\mathcal{S}$, and $(\omega) - (\omega') = (f)$. We remark that any differential in a coordinate chart $\phi_\alpha : U_\alpha \to V_\alpha$, with $\phi_\alpha(P) = z_\alpha$ take the form

$$\omega = h_\alpha(z_\alpha)dz_\alpha, \quad \omega' = h'_\alpha(z_\alpha)dz_\alpha$$

where h_α and h'_α are meromorphic functions. The ratio $g_\alpha = h_\alpha/h'_\alpha$ is a meromorphic function of V_α. Now define $f := g_\alpha \circ \phi_\alpha$ which is a meromorphic function on U_α. It is easy to check that f is well defined and independent from the coordinate chart.

Definition 3.1.48. *The linear equivalence class of divisors of Abelian differentials is called the canonical class of the Riemann surface $\mathcal{S}$. We denote it by $K_{\mathcal{S}}$.*

For example, the divisor $-2\infty = (dz)$ can be taken as a representative of the canonical class $K_{\mathbb{P}^1}$.

We reformulate Abel's theorem in the language of divisors. Note that the Abel map extends linearly to the whole group of divisors. Abel's theorem obviously means that a divisor D is principal if and only if the following two conditions hold:

1. $\deg D = 0$;

2. $\mathcal{A}(D) \equiv 0$ on $J(\mathcal{S})$,

where

$$\mathcal{A}(D) = \sum_{j=1}^{M} (\mathcal{A}(P_j) - \mathcal{A}(Q_j)), \quad D = \sum_{j=1}^{M} (P_j - Q_j),$$

with $\mathcal{A}$ the Abel map defined in (3.1.55).

Let us return to the canonical class. We compute it for a hyperelliptic surface $w^2 = P_{2g+2}(z)$. Let $P_1, \ldots, P_{2g+2}$ be the branch points of the Riemann surface, and P_{∞^+} and P_{∞^-} its point at infinity. We have that

$$(dz) = P_1 + \cdots + P_{2g+2} - 2P_{\infty^+} - 2P_{\infty^-}.$$

Thus the degree of the canonical class on this surface is equal to $2g - 2$. We prove an analogous assertion for an arbitrary Riemann surface. For the purpose we need the following lemma.

Lemma 3.1.49. *Let $f : \mathcal{S} \to X$ be a holomorphic map between Riemann surfaces $\mathcal{S}$ and X and ω a meromorphic one form on X, then for any fixed point $P \in \mathcal{S}$*

$$mult_P f^*\omega = (1 + mult_{f(P)}(\omega)) mult_P(f) - 1 \tag{3.1.67}$$

where $f^*\omega$ denotes the pull back of ω via f. We recall that the multiplicity of f in P is the unique integer m such that there is local a coordinate near $P \in \mathcal{S}$ and $f(P) \in X$ such that f takes the form $\tau \to \tau^m$.

Proof. Suppose that the map f can be represented near the point P and $f(P)$ with centred local coordinates τ and τ' as $\tau \to \tau' = \tau^m$. Suppose that near the point $f(P)$ the one form ω takes the form $\omega = g(\tau')d\tau'$ with $g(\tau') = \sum_{k \geqslant n} \alpha_k \tau'^k$. Then, the one form $f^*\omega$, near the point P, takes the form

$$f^*\omega = g(\tau^m) m \tau^{m-1} d\tau = \sum_{k \geqslant n} \alpha_k \tau^{mk+m-1} d\tau.$$

Looking at the coefficient in the exponent, one has the claim of the lemma. □

Definition 3.1.50. *Let $f : \mathcal{S} \to X$ be a holomorphic map between Riemann surfaces. The branch point divisor W_f is the divisor on $\mathcal{S}$ defined by*

$$W_f = \sum_{P \in \mathcal{S}} [mult_P(f) - 1] P. \tag{3.1.68}$$

For example, let us consider the Riemann surface $\mathcal{S}$ of the curve $C := \{z, w) \in \mathbb{C}^2 | F(z, w) = 0\}$ and consider the projection $\pi_z : C \to \mathbb{C}$ such that $\pi_z(z, w) = z$. Such map can be extended to a holomorphic function $\hat{z} : \mathcal{S} \to \mathbb{P}^1$. Let $P_1, \ldots, P_N$ be the ramification points of $\hat{z}$ with multiplicity $b_1, \ldots, b_N$ respectively. The branch point divisor is $W_{\hat{z}} = b_1 P_1 + \ldots b_N P_N$.

Definition 3.1.51. *Let $f : \mathcal{S} \to X$ be a holomoprhic map between Riemann surfaces and let $Q \in X$. The inverse image of the divisor nQ, $n \in \mathbb{Z} \backslash \{0\}$, denoted $f^*(nQ)$ is defined as*

$$f^*(nQ) = n \sum_{P \in f^{-1}(Q)} mult_p(f) \cdot P. \tag{3.1.69}$$

Applying (3.1.67), (3.1.68) and (3.1.69), we arrive to the relation between divisors

$$(f^*\omega) = W_f + f^*(\omega), \tag{3.1.70}$$

where $f^*(\omega)$ is the inverse image of the divisor (ω) of the one form ω.

Lemma 3.1.52. *The canonical class of the surface $\mathcal{S}$ has the form*

$$K_{\mathcal{S}} = W_f + f^*(K_{\mathbb{P}^1}). \tag{3.1.71}$$

Here f^* denotes the inverse image of a divisor in the class $K_{\mathbb{P}^1}$ with respect to the holomorphic function $f : \mathcal{S} \to \mathbb{P}^1$.

Proof. . This follows immediately from (3.1.70). □

Corollary 3.1.53. *The degree of the canonical class $K_{\mathcal{S}}$ of a Riemann surface $\mathcal{S}$ of genus g is equal to $2g-2$.*

Proof. We have from (3.1.71) that $\deg K_{\mathcal{S}} = \deg W_f - 2\deg f$, where $\deg W_f$ is the total multiplicity of the ramification points of the map f. By the Riemann-Hurwitz formula (2.1.4), $\deg W_f = 2g + 2\deg f - 2$. The corollary is proved. □

The divisor (3.1.63) is *positive* if all multiplicities n are non negative numbers An *effective* divisor is a divisor linearly equivalent to a positive divisor. Divisors D and D' are connected by the inequality $D \geqslant D'$ if their difference $D - D'$ is a positive divisor.

With each divisor D we associate the linear space of meromorphic functions

$$L(D) = \{f \mid (f) + D \geqslant 0\}. \tag{3.1.72}$$

If D is a positive divisor, then this space consists of functions f having poles only at points of D, with multiplicities not greater than the multiplicities of these points in D. If $D = D_+ - D_-$, where D_+ and D_- are positive divisors, then the space $L(D)$ consists of the meromorphic functions with poles possible only at points of D_+, with multiplicities not greater than the multiplicities of these points in D , and with zeros at all points of D_- (at least), with multiplicities not less than the multiplicities of these points in D.

Lemma 3.1.54. *If the divisors D and D' are linearly equivalent, then the spaces $L(D)$ and $L(D')$ are isomorphic.*

Proof. Let $D - D' = (g)$, where g is a meromorphic function. If $f \in L(D)$, then $f' = fg \in L(D')$. Indeed,

$$(f') + D' = (f) + (g) + D' = (f) + D \geqslant 0.$$

Conversely, if $f' \in L(D')$, then $f = g^{-1}f' \in L(D)$. The lemma is proved. □

We denote the dimension of the space $L(D)$ by

$$l(D) = \dim L(D). \tag{3.1.73}$$

By Lemma 3.1.54, the function $l(D)$ (as well as the degree $\deg D$) is constant on linear equivalence classes of divisors. We make some simple remarks about the properties of this important function.

Remark 3.1.55. For the zero (empty) divisor, $l(0) = 1$. If $\deg D < 0$, then $l(D) = 0$.

Remark 3.1.56. A divisor D is effective if and only if $l(D) > 0$. Indeed, replacing D by a positive divisor D' linearly equivalent to it, we see that the space $L(D')$ contains the constants. Conversely, if $l(D) > 0$, then D is effective. Indeed, if the meromorphic function f is such that $D' = (f) + D > 0$, then the divisor D', which is linearly equivalent to D is positive.

Remark 3.1.57. The number $l(D) - 1$ is often denoted by $|D|$. According to Remark 3.1.56 $|D| \geqslant 0$ for effective divisors. The number $|D|$ admits the following intuitive interpretation. Let us assume $D \geqslant 0$. We show that $|D| \geqslant m$ if and only if for any points $P_1, \ldots, P_m$ there is a divisor $D' \simeq D$ containing the points $P_1, \ldots, P_m$ (the presence of coinciding points among $P_1, \ldots, P_m$ is taken into account by their multiple occurrence in D').

If $l(D) \geqslant m + 1$, then there are linearly independent functions $f_1, \ldots, f_m \in L(D)$ such that the function $f = \sum_{j=1}^{m} c_j f_j - c_0$, where c_j, $j = 1, \ldots, m$ are arbitrary constants, has zeros in $P_1, \ldots, P_m$, namely

$$f(P_j) = 0, \quad j = 1, \ldots, m.$$

This system can be written in the form

$$\begin{pmatrix} f_1(P_1) & f_2(P_1) & \ldots & f_m(P_1) \\ f_1(P_2) & f_2(P_2) & \ldots & f_m(P_2) \\ \ldots & \ldots & \ldots & \\ f_1(P_m) & f_2(P_m) & \ldots & f_m(P_m) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} c_0 \\ c_0 \\ \vdots \\ c_0 \end{pmatrix}$$

It is a system of inhomogeneous linear equations for the constants $c_1, \ldots, c_m$ which has a solution for any choice of the points $P_1, \ldots, P_m$ since the functions $f_1, \ldots, f_m \in L(D)$ are linearly independent. Note that a similar inhomogeneous linear equations can be obtained when the points $P_1, \ldots, P_m$ are not all distinct.

We conclude that the divisor $D' = (f) + D \geqslant 0$ contains the arbitrary points $P_1, \ldots, P_m$, and $D' \simeq D$.

Viceversa suppose that there is a positive divisor D' containing the arbitrary points $P_1, \ldots, P_m$ and such that $D' \simeq D$. Then there is a meromorphic function f such that $(f) = D' - D$, or $(f) + D = D' > 0$. It follows that $f \in L(D)$ and f has zeros in arbitrary points $P_1, \ldots, P_k$. We write f is the form $f = \sum_{j=1}^{m} c_j f_j - c_0$ where $f_j \in L(D)$. If the function f has zeros at arbitrary points $P_1, \ldots, P_k$ it follows that the system of equations

$$f(P_j) = 0, \quad j = 1, \ldots, m,$$

must be solvable for any set of points $P_1, \ldots, P_m$, but this is possible only if the functions $f_1, \ldots, f_m$ are linearly independent and different from the constant, which means that $l(D) \geqslant m + 1$. One therefore says that $|D|$ is the number of mobile points in the divisor D.

Remark 3.1.58. Let $K = K_{\mathcal{S}}$, be the canonical class of a Riemann surface. We mention an interpretation that will be important later for the space $L(K - D)$ for an arbitrary divisor D. First, if $D = 0$, the empty divisor, then the space $L(K)$ is isomorphic to the space of holomorphic differentials on $\mathcal{S}$. Indeed, choose a representative $K_0 > 0$ in the canonical class, taking K_0 to be the zero divisor of some holomorphic differential ω_0, $K_0 = (\omega_0)$. If $f \in L(K_0)$, i.e. $(f) + (\omega_0) \geqslant 0$, then the divisor $(f\omega_0)$ is positive, i.e., the differential $f\omega_0$ is holomorphic. Conversely, if ω is any holomorphic differential, then the meromorphic function $f = \omega/\omega_0$ lies in $L(K_0)$.

It follows from the above considerations and Theorem 3.1.12 that

$$l(K) = g.$$

Lemma 3.1.59. *For a positive divisor D the space $L(K - D)$ is isomorphic to the space*

$$\Omega(D) = \{\omega \in H^1(\mathcal{S}) \mid (\omega) - D \geqslant 0\}.$$

Proof. We choose a representative $K_0 > 0$ in the canonical class, taking K_0 to be the zero divisor of some holomorphic differential ω_0, $K_0 = (\omega_0)$. If $f \in L(K_0 - D)$, then $(f) + (\omega_0) - D \geqslant 0$, namely the differential $f\omega_0$ is holomorphic and has zeros at the points of D, i.e., $f\omega_0 \in \Omega(D)$. Conversely, if $\omega \in \Omega(D)$, then $f = \omega/\omega_0 \in L(K_0 - D)$. The assertion is proved. □

The main way of getting information about the numbers $l(D)$ is the Riemann-Roch Theorem.

Theorem 3.1.60 (Riemann Roch Theorem)**.** *For any divisor D*

$$l(D) = 1 + \deg D - g + l(K - D). \tag{3.1.74}$$

Proof. For surfaces $\mathcal{S}$ of genus 0 (which are isomorphic to $\mathbb{P}^1$ in view of Problem 6.1) the Riemann-Roch theorem is a simple assertion about rational functions (verify!). By Remarks 3.1.55 and 3.1.58 (above) the Riemann-Roch theorem is valid for $D = 0$.

For Riemann surfaces $\mathcal{S}$ of positive genus we first prove (3.1.74) for positive divisors $D > 0$. Let $D = \sum_{k=1}^{m} n_k P_k$ where all the $n_k > 0$ and $P_k \neq P_j$ for $k \neq j$. We first verify the arguments when all the $n_k = 1$ for $k = 1, \dots, m$ and $m = \deg D$. Let $f \in L(D)$ be a nonconstant function.

We consider the Abelian differential $\omega = df$. It has double poles and zero residues at the points $P_1, \dots, P_m$ and does not have other singularities. Therefore, it is representable in the form

$$\Omega = df = \sum_{k=1}^{m} c_k \Omega_{P_k}^{(1)} + \phi$$

where $\Omega_{P_k}^{(1)}$ are normalized differentials of the second kind (see Lecture 3.1.3), $c_1, \dots, c_m$ are constants, and the differential ϕ is holomorphic. Since the function $f(P) = \int_{P_0}^{P} \Omega$ is single-valued on $\mathcal{S}$, the integral $\int_{P_0}^{P} \Omega$ is independent from the path of integration. This implies that

$$\oint_{\alpha_i} \Omega = 0, \quad \oint_{\beta_i} \Omega = 0, \quad i = 1, \dots, g. \tag{3.1.75}$$

From the vanishing of the α-periods of the meromorphic differentials $\Omega_{P_k}^{(1)}$ we get that $\phi = 0$ (see Corollary 3.1.18). From the vanishing of the β-period we get, by (3.1.37) with $n = 1$, that

$$0 = \oint_{\beta_i} \Omega = 2\pi i \sum_{k=1}^{m} c_k \psi_{ik}(z_k)|_{z_k=0}, \quad i = 1, \dots, g, \tag{3.1.76}$$

where z_k is a local parameter in a neighbourhood of P_k, $z_k(P_k) = 0$, $k = 1, \ldots, m$, and the basis of holomorphic differentials are written in a neighbourhood of P_k in the form $\omega_i = \psi_{ik}(z_k)dz_k$. Defining $\omega_i(P_k) := \psi_{ik}(0)$, we write the system (3.1.76) in the form

$$\begin{pmatrix} \omega_1(P_1) & \omega_1(P_2) & \ldots & \omega_1(P_m) \\ \omega_2(P_1) & \omega_2(P_2) & \ldots & \omega_2(P_m) \\ \ldots & \ldots & \ldots & \ldots \\ \omega_g(P_1) & \omega_g(P_2) & \ldots & \omega_g(P_m) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \ldots \\ c_m \end{pmatrix} = 0, \tag{3.1.77}$$

We have obtained a homogeneous linear system of $m = \deg D$ equations in the coefficients $c_1, \ldots, c_m$. The nonzero solutions of this systems are in a one-to-one correspondence with the non constant functions f in $L(D)$, where f can be reproduced from a solution $c_1, \ldots, c_m$ of the system (3.1.76) in the form

$$f(P) = \sum_{k=1}^{m} c_k \int_{P_0}^{P} \Omega^{(1)}_{P_k}.$$

Thus $l(D) = 1 + \deg D - \text{rank}\rho$ where ρ is the matrix of holomorphic differentials in (3.1.77) (the 1 is added because the constant function belong to the space $L(D)$). On the other hand the rank of the matrix ρ has another interpretation. Consider the holomorphic differential $\omega = \sum_{j=1}^{g} r_j \omega_j$. Such differential ω belongs to the space $\Omega(D)$ if

$$\omega(P_k) = 0, \quad k = 1, \ldots, m.$$

The above system of equations can be written in the equivalent form

$$\begin{pmatrix} r_1 & r_2 & \ldots & r_g \end{pmatrix} \begin{pmatrix} \omega_1(P_1) & \ldots & \omega_1(P_m) \\ \ldots & \ldots & \ldots \\ \omega_g(P_1) & \ldots & \omega_g(P_m) \end{pmatrix} = 0. \tag{3.1.78}$$

The number of solutions of this system is equal to $g - \text{rank}\rho$ and it is in one to one correspondence with the linearly independent holomorphic differentials in $\Omega(D)$. Therefore $\dim\Omega(D) = g - \text{rank}\rho$. On the other hand we have obtained that

$$l(D) = 1 + \deg D - \text{rank}\rho$$

so that combining the two equations we obtain

$$l(D) = 1 + \deg D - g + \dim\Omega(D) = 1 + \deg D - g + l(K - D)$$

where the second identity is due to the fact that the space $\Omega(D)$ and $L(K - D)$ are isomorphic for positive divisors. Accordingly the Riemann-Roch theorem has been proved in this case.

We explain what happens when the positive divisor D has multiple points. For example suppose that $D = n_1 P_1 + P_2 + \cdots + P_m$. Then $\Omega = df = \sum_{j=1}^{n_1} c_1^j \Omega^{(j)}_{P_1} + \sum_{k=2}^{m} c_k \Omega^{(1)}_{P_k}$ and the system (3.1.76) can be written in the form

$$\sum_{j=1}^{n_1} c_1^j \, \frac{1}{j!} \frac{d^{j-1}\psi_{i1}}{dz_1^{j-1}}\bigg|_{z_1=0} + \sum_{k=2}^{m} c_k \psi_{ik}(z_k)|_{z_k=0}, = 0\,.$$

This is a system of homogeneous equations is the variables $c_1^1, \dots, c_1^{n_1}, c_2, \dots, c_m$. If the rank of the coefficient matrix of this system is denoted (as above) by rankρ, the dimension of the space $L(D)$ is equal to $l(D) = 1 + \deg D - \mathrm{rank}\rho$ while the dimension of the space $\Omega(D)$ is equal to $g - \mathrm{rank}\rho$ so that $l(D) = 1 + \deg D - g + \dim\Omega(D)$. We have proved the Riemann-Roch theorem for all positive divisors and hence for all effective divisors, which (accordingly to Remark 3.1.56) are distinguished by the condition $l(D) > 0$. Next we note that the relation in this theorem can be written in the form

$$l(D) - \frac{1}{2}\deg D = l(K - D) - \frac{1}{2}\deg(K - D), \tag{3.1.79}$$

which is symmetric with respect to the substitution $D \to K - D$. Therefore the theorem is proved for all divisors D such that D or $K - D$ is equivalent to a positive divisor. If neither D nor $K - D$ are equivalent to a positive divisor, then $l(D) = 0$ and $l(K - D) = 0$ and the Riemann-Roch theorem reduces in this case to the equality

$$\deg D = g - 1. \tag{3.1.80}$$

Let us prove this equality. We represent D in the form $D = D_+ - D_-$, where D_+ and D_- are positive divisors and $\deg D_- > 0$. It follows from the validity of the Riemann-Roch theorem for D_+ that $l(D_+) \geqslant \deg D_+ - g + 1 = \deg D + \deg D_- - g + 1$. Therefore if $\deg D \geqslant g$, then $l(D_+) \geqslant 1 + \deg D_-$. Then the space $L(D_+)$ contains a nonzero function f vanishing on D_-, i.e. $f \in L(D_+ - D_-) = L(D)$. This contradicts the condition $l(D) = 0$. Similarly, suppose $\deg(K - D) \geqslant g$ and $K - D = \tilde{D}_+ - \tilde{D}_-$ with $\tilde{D}_+$ and $\tilde{D}_-$ positive divisors. Then $l(\tilde{D}_+) \geqslant \deg \tilde{D}_+ - g + 1 = \deg(K - D) + \deg \tilde{D}_- - g + 1$ or

$$l(\tilde{D}_+) \geqslant \deg(\tilde{D}_-) + 1,$$

which implies that there exists a nonzero function $f \in L(\tilde{D}_+)$ and vanishing in $\tilde{D}_-$, namely $f \in L(\tilde{D}_+ - \tilde{D}_-) = L(K - D)$. This contradicts the condition $l(K - D) = 0$. We conclude that

$$\deg D < g, \quad \deg(K - D) < g$$

which is equivalent to $\deg D = g - 1$. The theorem is proved. □

3.1.6 Some consequences of the Riemann-Roch theorem. The structure of surfaces of genus 1. Weierstrass points. The canonical embedding

Corollary 3.1.61. *If $\deg D \geqslant g$, then the divisor D is effective.*

Corollary 3.1.62. *The Riemann inequality*

$$l(D) \geqslant 1 + \deg D - g, \tag{3.1.81}$$

holds for $\deg D \geqslant g$.

Definition 3.1.63. *A positive divisor D is called special if*

$$dim\Omega(D) > 0.$$

We remark that any effective divisor of degree less then g is special since $l(D) > 0$ and by Riemann-Roch theorem this implies $\dim\Omega(D) > 0$.

Corollary 3.1.64. *If $\deg D > 2g - 2$, then D is nonspecial.*

Proof. For $\deg D > 2g-2$ we have that $\deg(K-D) < 0$, hence $l(K-D) = 0$ (see Remark 3.1.55). The corollary is proved. □

Exercise 3.1.65: Suppose that $k \geqslant g$; let the Abel mapping $A : \mathcal{S} \to J(\mathcal{S})$ (see Lecture 3.1.4) be extended to the kth-power mapping

$$A^k : \underbrace{\mathcal{S} \times \cdots \times \mathcal{S}}_{k \text{ times}} \to J(\mathcal{S})$$

by setting $A^k(P_1, \ldots, P_k) = A(P_1) + \cdots + A(P_k)$ (it can actually be assumed that A^k maps into $J(\mathcal{S})$ the kth symmetric power $S^k\mathcal{S}$, whose points are the unordered collections $(P_1, \ldots, P_k)$ of points of $\mathcal{S}$). Prove that the special divisors of degree k are precisely the critical points of the Abel mapping A^k. Deduce from this that a divisor D with $\deg D \geqslant g$ in general position is nonspecial.

Remark 3.1.66. Let $\deg D = 0$, then if D is equivalent to a divisor of a meromorphic function, then $l(D) = 1$ otherwise $l(D) = 0$. Let $\deg D = 2g-2$, then if D is equivalent to the canonical divisor, then $l(D) = g$ otherwise $l(D) = g-1$. Furthermore if $\deg D > 2g-2$, then by Riemann Roch theorem one has $l(D) = 1 + \deg D - g$. If $0 \leqslant \deg D \leqslant g-1$ the minimum value of $l(D)$ is zero while for $g \leqslant \deg D \leqslant 2g-2$, $\min(l(D)) = 1 - g + \deg D$.

The values of $l(D)$ for $0 \leqslant \deg D \leqslant 2g-2$ are estimated by the Clifford theorem.

Theorem 3.1.67. *If* $0 \leqslant \deg D \leqslant 2g-2$, *then*

$$l(D) \leqslant 1 + \frac{1}{2}\deg D. \tag{3.1.82}$$

Proof. If $l(D) = 0$ or $l(K-D) = 0$, the proof of the theorem is straightforward. Let us assume that $l(D) > 0$ and $l(K-D) > 0$ and consider the map $L((D) \times L(K-D) \to L(K)$ given by $(f,h) \to fh$ where $(f,h) \in L((D) \times L(K-D)$. Let V be the subspace in $L(K)$ which is the image of this map. Then one has

$$g = l(K) \geqslant \dim V = l(D)l(K-D) \geqslant l(D) + l(K-D) - 1$$

where in the last equality we use the identity which holds for real numbers a and b bigger then one: $(a-1)(b-1) \geqslant 0$ and so $ab \geqslant a + b - 1$.

Therefore

$$g \geqslant l(D) + l(K-D) - 1 = 2l(D) + g - 2 - \deg D,$$

which implies (3.1.82). □

Let us make a plot of the possible values of $l(D)$ using Clifford theorem and the above observations.

We now present examples of the use of the Riemann-Roch theorem in the study of Riemann surfaces.

Example 3.1.68. Let us show that any Riemann surface $\mathcal{S}$ of genus $g = 1$ is isomorphic to an elliptic surface $w^2 = P_3(z)$. Let P_0 be an arbitrary point of $\mathcal{S}$. Here $2g-2 = 0$, therefore, any positive divisor is nonspecial. We have that $l(2P_0) = 2$, hence there is a nonconstant function z in $l(2P_0)$, i.e., a function having a double pole at P_0. Further $l(3P_0) = 3$, hence there is a function

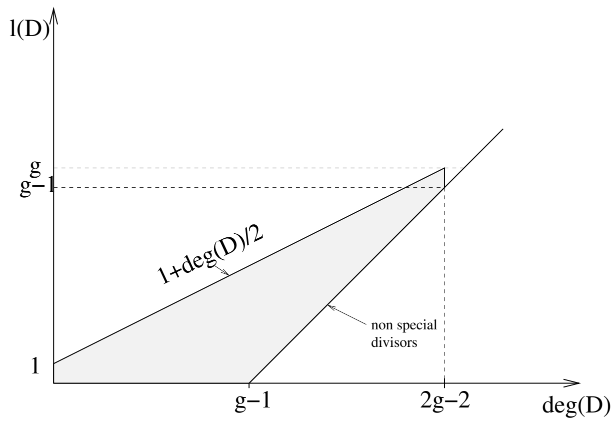


Figure 3.5: The values of $l(D)$ as a function of $\deg D$. One can see that the value of $l(D)$ of a special divisors is located between the two lines.

$w \in l(3P_0)$ that cannot be represented in the form $w = az+b$. This function has a pole of order three at P_0. Finally, since $l(6P_0) = 6$, the functions $1, z, z^2, z^3, w, w^2, wz$ which lie in $l(6P_0)$ are linearly dependent. We have that

$$a_1 w^2 + a_2 wz + a_3 w + a_4 z^3 + a_5 z^2 + a_6 z + a_7 = 0. \tag{3.1.83}$$

The coefficient a_1 is nonzero (verify). Making the substitution

$$w \to w - \left(\frac{a_2}{2a_1} z + \frac{a_3}{2a_1} \right)$$

we get the equation of an elliptic curve from (3.1.83).

Example 3.1.69 (Riemann count of the moduli space of Riemann surface)**.** Consider a Riemann surface $\mathcal{S}$ of genus g and a meromorphic function of degree $n > 2g-2$. Such function represents $\mathcal{S}$ as a n-sheeted covering of the complex plane, branched over a number of points with total branching number b_f equal to

$$b_f = 2n + 2g - 2$$

where the Riemann-Hurwitz formula has been used. Generically the ramification points have branching number equal to one so that b_f is also equal to the ramification points of the Riemann surface with respect to the map f. From the Riemann existence theorem, given the branch points $z_1, \ldots, z_{b_f}$ and a permutation associated to each branch point such that the corresponding monodromy group is a transitive sub-group of the permutation group S_n, one can construct a Riemann surface $\mathcal{S}$ up to isomorphism. Let us count how many distinct surfaces one can obtain.

Any meromorphic function of degree n on $\mathcal{S}$ represents $\mathcal{S}$ as a n-sheeted covering of the complex plane. Let D_∞ be the divisor of poles of f. Since the degree of f is equal to n then $\deg D_\infty = n$. Furthermore from Riemann-Roch theorem

$$l(D_\infty) = n + 1 - g.$$

So the freedom of choosing the function f is given by the position of the poles, and this gives n parameters, and the number of functions having poles in D_∞, which is equal to $n+1-g$. The total number of parameters in choosing the meromorphic function of degree n is $2n+1-g$. So the total number of parameters for describing a curve of genus g is the number of branch points b_f minus the parameters for describing the meromorphic function f, namely

$$2n+2g-2-(2n+1-g)=3g-3.$$

Definition 3.1.70 (Weierstrass points)**.** *A point P_0 of a Riemann surface $\mathcal{S}$ of genus g is called a Weierstrass point if $l(kP_0)>1$ for some $k\leqslant g$.*

It is clear that in the definition of a Weierstrass point it suffices to require that $l(gP_0)>1$ when $g\geqslant 2$. There are no Weierstrass points on a surface of genus $g=1$. On hyperelliptic Riemann surfaces of genus $g>1$ all branch points are Weierstrass points, since there exist functions with second-order poles at the branch points (see Lecture **??**).

Definition 3.1.71. *A Riemann surface is called hyperelliptic if and only if it admits a non constant meromorphic function of degree 2.*

The use of Weierstrass points can be illustrated in the next exercise.

Exercise 3.1.72: Let $\mathcal{S}$ be a Riemann surface of genus $g>1$, and P_0 a Weierstrass point of it, with $l(2P_0)>1$. Prove that $\mathcal{S}$ is hyperelliptic. Prove that the surface is also hyperelliptic if $l(P+Q)>1$ for two points P and Q.

Exercise 3.1.73: Let $\mathcal{S}$ be a hyperellitpic Rieamnn surface and z a function of degree two. Prove that any other function f of degree two is a Moebius transformation of z.

We show that there exist Weierstrass points on any Riemann surface $\mathcal{S}$ of genus $g>1$.

Lemma 3.1.74. *Suppose that z is a local parameter in a neighbourhood P_0, $z(P_0)=0$; assume that locally the basis of holomorphic differentials has the form $\omega_i=\psi_i(z)dz$, $i=1,\dots,g$. Consider the determinant*

$$W(z)=\det\begin{pmatrix}\psi_1(z) & \psi_1'(z) & \dots & \psi_1^{(g-1)}(z)\\ \dots & \dots & & \dots\\ \psi_g(z) & \psi_g'(z) & \dots & \psi_g^{(g-1)}(z)\end{pmatrix}. \tag{3.1.84}$$

The point P_0 is a Weierstrass point if and only if $W(0)=0$.

Proof. If P_0 is a Weierstrass point, i.e., $l(gP_0)>1$, then $l(K-gP_0)>0$ by the Riemann-Roch theorem. Hence, there is a holomorphic differential with a g-fold zero at P_0 on $\mathcal{S}$. The condition that there be such a differential can be written in the form $W(0)=0$ (cf. the proof of the Riemann-Roch theorem). The lemma is proved. □

Lemma 3.1.75. *Under a local change of parameter $z=z(w)$ the quantity W transforms according to the rule $\tilde{W}(w)=\left(\dfrac{dz}{dw}\right)^{\frac{1}{2}g(g+1)}W(z)$.*

Proof. Suppose that $\omega_i = \psi_i(z)dz = \tilde{\psi}_i(w)dw$. Then each $\tilde{\psi}_i = \psi_i \dfrac{dz}{dw}$, $i = 1, \dots, g$. This implies that the derivatives $d^k\tilde{\psi}_i/dw^k$ can be expressed for each i in terms of the derivatives $d^l\psi_i/dz^l$ by means of a triangular transformation of the form

$$\frac{d^k\tilde{\psi}_i}{dw^k} = \left(\frac{dz}{dw}\right)^{k+1} \frac{d^k\psi_i}{dz^k} + \sum_{j=1}^{k-1} c_j \frac{d^j\psi_i}{dz^j}, \quad i = 1, \dots g$$

(the coefficients c_s in this formula are certain differential polynomials in $z(w)$). The statement of the Lemma readily follows from the transformation rule. ◻

Let us define the *weight* of a Weierstrass point P_0 as the multiplicity of zero of $W(z)$ at this point. According to the previous Lemma the definition of weight does not depend on the choice of the local parameter.

The proof of existence of Weierstrass points for $g > 1$ can be easily obtained from the following statement.

Lemma 3.1.76. *The total weight of all Weierstrass points on the Riemann surface $\mathcal{S}$ of genus g is equal to $(g-1)\,g\,(g+1)$.*

Proof. Let us consider the ratio

$$W(z)/\psi_1^N(z).$$

Here $N = \frac{1}{2}\,g(g+1)$. According to lemma (3.1.75), the above ratio does not depend on the choice of the local parameter and, hence, it is a meromorphic function on $\mathcal{S}$. This function has poles of multiplicity N at the zeroes of the differential ω_1 (the total number of all poles is equal to $2g-2$). Therefore this function must have $N\,(2g-2) = (g-1)\,g\,(g+1)$ zeroes (as usual, counted with their multiplicities). These zeroes are the Weierstrass points. ◻

Let us do few more remarks about the Weierstrass points. Given a point $P_0 \in \mathcal{S}$, let us consider the dimension $l(k\,P_0)$ as a function of the integer argument k. This function has the following properties. According to figure (3.5) we have

$$1 \leqslant l(k\,P_0) \leqslant g, \quad 1 \leqslant k \leqslant 2g-1.$$

In particular $l\,((2g-1)P_0) = g$. It follows that while k increases $2g-2$ times the function $l(k\,P_0)$ increases only $g-1$ times. The next lemma shows that the function $l(k\,P_0)$ is a piece-wise constant function where each step has size equal to one.

Lemma 3.1.77.

$$l(k\,P_0) = \begin{cases} l\,((k-1)P_0) + 1, & \textit{if there exists a function with a pole of order } k \textit{ at } P_0 \\ l\,((k-1)P_0)\,, & \textit{if such a function does not exist} \end{cases}$$

Proof. The space $L(k\,P_0)$ is larger then the space $L((k-1)P_0)$ therefore $l(k\,P_0) \geqslant l((k-1)P_0)$. On the other hand, $\dim\Omega(kP_0) \leqslant \dim\Omega((k-1)P_0)$. From the Riemann Roch theorem one has

$$l(k\,P_0) - l((k-1)P_0) = 1 + \dim\Omega(kP_0) - \dim\Omega((k-1)P_0)$$

which, when combined with the above two inequalities, gives the statement. ◻

When $l(kP_0) = l((k-1)P_0)$ we will say that the number k is a *gap* at the point P_0. From the previous remarks it follows the following *Weierstrass gap theorem*:

Theorem 3.1.78. *There are exactly g gaps $1 = a_1 < ... < a_g < 2g$ at any point P_0 of a Riemann surface of genus g.*

The gaps have the form $a_i = i$, $i = 1, \ldots, g$, for a point P_0 in general position (which is not a Weierstrass point). Namely for a non Weierstrass point the function $l(kP_0)$ is non-zero only for $k > g$ and one has $l(kP_0) = 1 + k - g$ for $k > g$. A Weierstrass point P_0 is called normal if the Weierstrass gap sequence takes the form $1, 2, \ldots, g-1, g+1$ where g is the genus of the surface. Namely a meromorphic function with only a pole in P_0 has order at least equal to g. Normal Weierstrass points are generic. A Weierstrass point P_0 is called hyperelliptical is the Weierstrass gap sequence takes the form $1, 3, 5, \ldots, 2g-1$. In this case a meromorphic function with only a pole in P_0 has order equal to two.

Exercise 3.1.79: Show that every compact Riemann surface of genus g is conformally equivalent to a $(g+1)-$sheeted covering surface of the complex plane.

Exercise 3.1.80: Prove that for branch points of a hyperelliptic Riemann surface of genus g the gaps have the form $a_i = 2i - 1$, $i = 1, \ldots, g$. Prove that a hyperelliptic surface does not have other Weierstrass points. Next suppose that the hyperelliptic Riemann surface has genus 2 and let P_0 be a Weierstrass point. Show that there exist meromorphic functions z and w with only a pole in P_0 and such that

$$w^2 + a_1 wz + a_2 wz^2 + a_3 z^5 + a_4 z^4 + a_5 z^3 + a_6 z^2 + a_7 z + a_8 = 0.$$

Exercise 3.1.81: Prove that any Riemann surface of genus 2 is hyperelliptic.

Exercise 3.1.82: Let $\mathcal{S}$ be a hyperelliptic Riemann surface of the form $w^2 = P_{2g+l}(z)$. Prove that any birational (biholomorphic) automorphism $\mathcal{S} \to \mathcal{S}$ has the form $(z, w) \to (\frac{az+b}{cz+d}, \pm w)$, where the linear fractional transformation leaves the collection of zeros of $P_{2g+2}(z)$ invariant.

Example 3.1.83 (The canonical embedding)**.** . Let $\mathcal{S}$ be an arbitrary Riemann surface of genus $g \geqslant 2$. We fix on $\mathcal{S}$ a canonical basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$; let $\omega_1, \ldots, \omega_g$ be the corresponding normal basis of holomorphic differentials. This basis gives a canonical mapping $\mathcal{S} \to \mathbb{P}^{g-1}$ according to the rule

$$P \to (\omega_1(P) : \omega_2(P) : \cdots : \omega_g(P)). \tag{3.1.85}$$

Indeed, it suffices to see that all the differentials $\omega_1, \ldots, \omega_g$ cannot simultaneously vanish at some point of the surface. If P were a point at which any holomorphic differential vanished, i.e., $l(K - P) = g$, (see Remark 3.1.58), then $l(P)$ would be $= 2$ in view of the Riemann-Roch theorem, and this means that the surface $\mathcal{S}$ is rational (verify!). Accordingly (3.1.85) really is a mapping $\mathcal{S} \to \mathbb{P}^{g-1}$; it is obviously well-defined.

Lemma 3.1.84. *If $\mathcal{S}$ is a non hyperelliptic surface of genus $g \geqslant 3$, then the canonical mapping (3.1.85) is a smooth embedding. If $\mathcal{S}$ is a hyperelliptic surface of genus $g \geqslant 2$, then the image of the canonical mapping is a rational curve, and the map itself is a two-sheeted covering.*

Proof. We prove that the mapping (3.1.85) is an embedding. Assume not: assume that the points P_1 and P_2 are merged into a single point by this mapping. This means that the rank of the matrix

$$\begin{pmatrix} \omega_1(P_1) & \omega_1(P_2) \\ \dots & \dots \\ \omega_g(P_1) & \omega_g(P_2) \end{pmatrix}$$

is equal to 1. But then $l(P_1 + P_2) > 1$ (see the proof of the Riemann-Roch theorem). Hence, there exists on $\mathcal{S}$ a nonconstant function with two simple poles at P_1 and P_2 i.e., the surface $\mathcal{S}$ is hyperelliptic. The smoothness is proved similarly: if it fails to hold at a point P, then the rank of the matrix

$$\begin{pmatrix} \omega_1(P) & \omega_1'(P) \\ \dots & \dots \\ \omega_g(P) & \omega_g'(P) \end{pmatrix}$$

is equal to 1. Then $l(2P) > 1$, and the surface is hyperelliptic. Finally, suppose that $\mathcal{S}$ is hyperelliptic. Then it can be assumed of the form $w^2 = P_{2g+1}(z)$. Its canonical mapping is determined by the differentials (4.2.37). Performing a projective transformation of the space $\mathbb{P}^{g-1}$ with the matrix (c_{jk}) (see the formula (4.2.37)), we get the following form for the canonical mapping:

$$P = (z, w) \to (1 : z : \dots : z^{g-1}) \tag{3.1.86}$$

Its properties are just as indicated in the statement of the lemma. The lemma is proved. □

Exercise 3.1.85: Suppose that the Riemann surface $\mathcal{S}$ is given in $\mathbb{P}^2$ by the equation

$$\sum_{i+j=4} a_{ij} \xi^i \eta^j \zeta^{4-i-j} = 0, \tag{3.1.87}$$

and this curve is non-singular in $\mathbb{P}^2$ (construct an example of such a non-singular curve). Prove that the genus of this surface is equal to 3 and the canonical mapping is the identity up to a projective transformation of $\mathbb{P}^2$. Prove that $\mathcal{S}$ is a non hyperelliptic surface. Prove that any non hyperelliptic surface of genus 3 can be obtained in this way.

The range $\mathcal{S}' \subset \mathbb{P}^{g-1}$ of the canonical mapping is called the canonical curve.

Exercise 3.1.86: Prove that any hyperplane in $\mathbb{P}^{g-1}$ intersects the canonical curve $\mathcal{S}'$ in $2g - 2$ points (counting multiplicity).

Chapter 4

Jacobi inversion problem and theta-functions

4.1 Statement of the Jacobi inversion problem. Definition and simplest properties of general theta functions

In Lecture 3.1.2 we saw that inversion of an elliptic integral leads to elliptic functions. For a surface of genus $g > 1$ the Inversion of integrals of Abelian differentials is not possible since any such differential has zeros (at least $2g-2$ zeros). Instead of the problem of inverting a single Abelian integral, Jacobi proposed for hyperelliptic surfaces of genus two of the form $w^2 = P_5(z)$ the problem of solving the system

$$\begin{aligned} \int_{P_0}^{P_1} \frac{dz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{dz}{\sqrt{P_5(z)}} &= \eta_1 \\ \int_{P_0}^{P_1} \frac{zdz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{zdz}{\sqrt{P_5(z)}} &= \eta_2 \end{aligned} \tag{4.1.1}$$

where η_1, η_2 are given numbers from which the location of the points $P_1 = (z_1, w_1)$, $P_2 = (z_2, w_2)$ is to be determined. It is clear, moreover, that P_1 and P_2 are determined from (4.1.1) only up to permutation. Jacobi's idea was to express the symmetric functions of P_1 and P_2 as functions of η_1 and η_2. He noted also that this will give meromorphic functions of η_1 and η_2 whose period lattice is generated by the periods of the basis of holomorphic differentials $dz/\sqrt{P_5(z)}$ and $zdz/\sqrt{P_5(z)}$. This Jacobi inversion problem was solved by Göepel and Rosenhain by means of the apparatus of theta functions of two variables. The generalization of the Jacobi inversion problem to arbitrary Riemann surfaces and its solution are due to Riemann. We give a precise statement of the Jacobi inversion problem. Let $\mathcal{S}$ be an arbitrary Riemann surface of genus g, and fix a canonical basis of cycles $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ on $\mathcal{S}$; as above let $\omega_1, \ldots, \omega_g$ be be the corresponding basis of normalized holomorphic differentials. Recall (see Lecture 3.1.4) that the Abel mapping has the form

$$\mathcal{A} : \mathcal{S} \to J(\mathcal{S}), \quad \mathcal{A}(P) = (u_1(P), \dots, u_g(P)), \tag{4.1.2}$$

where $J(\mathcal{S})$ is the Jacobi variety,

$$u_i(P) = \int_{P_0}^{P} \omega_i, \tag{4.1.3}$$

P_0 is a particular point of $\mathcal{S}$, and the path of integration from P_0 to P is the same for all $i = 1, \dots, g$. Consider the gth symmetric power

$$S^g(\mathcal{S}) = \underbrace{\mathcal{S} \times \cdots \times \mathcal{S}}_{g \text{ times}} / S_g$$

the symmetric group of g elements. The unordered collections $(P_1, \dots, P_g)$ of g points of $\mathcal{S}$ are the points of the manifold $S^g(\mathcal{S})$. The meromorphic functions on $S^g(\mathcal{S})$ are the meromorphic symmetric functions of g variables $P_1, \dots, P_g$, $P_j \in \mathcal{S}$. The Abel mapping (4.1.2) determines a mapping

$$\mathcal{A}^{(g)} : S^g(\mathcal{S}) \to J(\mathcal{S}), \quad \mathcal{A}^g(P_1, \dots, P_g) = \mathcal{A}(P_1) + \cdots + \mathcal{A}(P_g), \tag{4.1.4}$$

which we also call the Abel mapping.

Lemma 4.1.1. *If the divisor $D = P_1 + \cdots + P_g$ is nonspecial, then in a neighbourhood of a point $\mathcal{A}^{(g)}(P_1, ..., P_g) \in J(\mathcal{S})$ the mapping $A^{(g)}$ has a single-valued inverse.*

Proof. Suppose that all the points are distinct; let $z_1, \dots, z_g$ be local parameters in neighbourhoods of the respective points $P_1, \dots, P_g$ with $z_k(P_k) = 0$ and $\omega_i = \psi_{ik}(z_k)dz_k$ the normalized holomorphic differentials in a neighbourhood of P_k. The Jacobi matrix of the mapping (4.1.4) has the following form at the points $(P_1, \dots, P_g)$

$$\begin{pmatrix} \psi_{11}(z_1 = 0) & \dots & \psi_{1g}(z_g = 0) \\ \dots & \dots & \dots \\ \psi_{g1}(z_1 = 0) & \dots & \psi_{gg}(z_g = 0) \end{pmatrix}.$$

If the rank of this matrix is less than g, then $l(K-D) > 0$, i.e., the divisor D is special by the Riemann-Roch theorem. The case when not all the points $P_1, \dots, P_g$ are distinct is treated similarly. We now prove that the inverse mapping is single-valued. Assume that the collection of points $(P'_1, \dots, P'_g)$ is also carried into $A^{(g)}(P_1, \dots, P_g)$. Then the divisor $D' = P'_1 + \cdots + P'_g$ is linearly equivalent to D by Abel's theorem. If $D' \neq D$, then there would be a meromorphic function with poles at points of D and with zeros at points of D'. This would contradict the fact that D is nonspecial. Hence, $D' = D$, and the points $P'_1, \dots, P'_g$ differ from $P_1, \dots, P_g$ only in order. The lemma is proved. □

Since a divisor $P_1 + ... + P_g$ in general position is nonspecial (see Problem 3.1.65), the Abel mapping (4.1.4) is invertible almost everywhere. The problem of inversion of this mapping in the large is the Jacobi inversion problem. Thus, the Jacobi inversion problem can be written in coordinate notation in the form

$$\begin{cases} u_1(P_1) + \cdots + u_1(P_g) & = \eta_1 \\ \dots\dots\dots & \\ u_g(P_1) + \cdots + u_g(P_g) & = \eta_g \end{cases} \tag{4.1.5}$$

which generalizes (4.1.1). To solve this problem we need the apparatus of multi-dimensional theta functions.

4.2 Theta-functions

The g-dimensional theta-functions are defined by their Fourier serie. Let $B = (B_{jk})$ be a symmetric $g \times g$ matrix with positive-definite imaginary part and let $z = (z_1, \ldots, z_g) \in \mathbb{C}^g$ and $N = (N_1, \ldots, N_g) \in \mathbb{Z}^g$ be g-dimensional vectors. The Riemann theta function is defined by its multiple Fourier series,

$$\theta(z) = \theta(z; B) = \sum_{N \in \mathbb{Z}}^{g} \exp\left(\pi i \langle NB, N\rangle + 2\pi i \langle N, z\rangle\right), \tag{4.2.1}$$

where the angle brackets denote the Euclidean inner product:

$$\langle N, z\rangle = \sum_{k=1}^{g} N_k z_k, \quad \langle NB, N\rangle = \sum_{j,k=1}^{g} B_{kj} N_j N_k.$$

The summation in (4.2.1) is over the lattice of integer vectors $N = (N_1, \ldots, N_g)$. The obvious estimate $\Re(i\langle NB, N\rangle) \leqslant -b\langle N, N\rangle$, where $b > 0$ is the smallest eigenvalue of the matrix $\Im(B)$, implies that the series (4.2.1) defines an entire function of the variables $z_1, \ldots, z_g$.

Proposition 4.2.1. *The theta-function has the following properties.*

1. $\theta(-z\,;\,B) = \theta(z\,;\,B)$.

2. *For any integer vectors $M, K \in \mathbb{Z}^g$,*

$$\theta(z + K + MB; B) = \exp\left(-\pi i \langle MB, M\rangle - 2\pi i \langle M, z\rangle\right) \theta(z; B). \tag{4.2.2}$$

3. *It satisfies the heat equation*

$$\begin{aligned} \frac{\partial}{\partial B_{ij}} \theta(z\,;\,B) &= \frac{1}{2\pi i} \frac{\partial^2}{\partial z_i \partial z_j} \theta(z\,;\,B), \quad i \neq j \\ \frac{\partial}{\partial B_{ii}} \theta(z\,;\,B) &= \frac{1}{4\pi i} \frac{\partial^2}{\partial z_i^2} \theta(z\,;\,B). \end{aligned} \tag{4.2.3}$$

Proof. The proof of properties 1. and 3. is straightforward. Let us prove property 2. In the series for $\theta(z + K + MB)$ we make the change of summation index $N \to N - M$. The relation (4.2.2) is obtained after this transformation. □

The integer lattice $\{N + MB\}$ is called the period lattice.

Remark 4.2.2. It is possible to define the function $\theta(z)$ as an entire function of $z_1, \ldots, z_g$ satisfying the transformation law (4.2.2) (this condition determines $\theta(z)$ uniquely to within a factor).

The theta-function is an analytic multivalued function on the g-dimensional torus $T^g = \mathbb{C}^g/\{N+MB\}$. In order to construct single valued functions, i.e. meromorphic functions on the torus, one can take for example, for any two vectors $e_1, e_2 \in \mathbb{C}^g$ the product

$$\frac{\theta(z+e_1)\theta(z-e_1)}{\theta(z+e_2)\theta(z-e_2)}.$$

Indeed the above expression is by (4.2.2) a single valued function on the g-dimensional torus. In general for any two sets of g vectors $e_1, \dots e_g \in \mathbb{C}^g$, $v_1, \dots v_g \in \mathbb{C}^g$ satisfying the constraint

$$e_1 + \dots e_g = 0, \quad v_1 + \dots v_g = 0$$

the product

$$\prod_{j=1}^{g} \frac{\theta(z+e_j)}{\theta(z+v_j)},$$

is a meromorphic function on the torus (verify this!).

Let p and q be arbitrary real g-dimensional row vectors. We define the theta function with characteristics p and q:

$$\begin{aligned}\theta[p,q](z) &= \exp\left(\pi i\langle pB, p\rangle + 2\pi i\langle z+q, p\rangle\right)\theta(z+q+pB) \\ &= \sum_{N\in\mathbb{Z}^g} \exp\left(\pi i\langle (N+p)B, N+p\rangle + 2\pi i\langle z+q, N+p\rangle\right).\end{aligned} \tag{4.2.4}$$

For $p=0$ and $q=0$ we get the function $\theta(z)$. The analogue of the law (4.2.2) for the functions $\theta[p,q](z)$ has the form

$$\theta[p,q](z+K+MB) = \theta[p,q](z)\exp\left[-\pi i\langle MB, M\rangle - 2\pi i\langle M, z+q\rangle + 2\pi i\langle K, p\rangle\right]. \tag{4.2.5}$$

Observe that all the coordinates of the characteristics p and q are determined modulo 1.

Definition 4.2.3. *The characteristics p and q with all coordinates equal to 0 or $1/2$ are called half periods. A half period $[p,q]$ is said to be even if $4\langle p,q\rangle \equiv 0 \pmod 2$ and odd if $4\langle p,q\rangle \equiv 1 \pmod 2$.*

Exercise 4.2.4: Prove that the function $\theta[p,q](z)$ is even if $[p,q]$ is an even half period and odd if $[p,q]$ is an odd half period.

In particular the function $\theta(z)$ is even. For $e = q + Bp$ with $4\langle p,q\rangle \equiv 1 \pmod 2$ one has

$$\theta(e) = 0.$$

Example 4.2.5. For $g=1$ the theta-function reduces to the Jacobi theta-function $\vartheta_3(z\,;\tau)$ with parameter τ, $\Im\tau > 0$. The Jacobi theta function is defined by the series

$$\theta(z;\tau) = \sum_{-\infty<n<\infty} \exp\left(\pi i\tau n^2 + 2\pi i n z\right). \tag{4.2.6}$$

Since

$$\left|\exp\left(\pi i\tau n^2 + 2\pi i n z\right)\right| = \exp\left(-\pi\Im\tau n^2 - 2\pi n\Im z\right)$$

the series (4.2.6) converges absolutely and uniformly in the strips $|\Im(z)| \leqslant$ const and defines an entire function of z.

The series (4.2.6) can be rewritten in the form common in the theory of Fourier series:

$$\theta(z) = \sum_{-\infty<n<\infty} \exp(\pi i \tau n^2) e^{2\pi i z n} \tag{4.2.7}$$

(the function $\vartheta_3(z\,;\,\tau)$) in the standard notation; see [[4]). The function $\theta(z)$ has the following periodicity properties:

$$\theta(z+1) = \theta(z) \tag{4.2.8}$$
$$\theta(z+\tau) = \exp(-\pi i \tau - 2\pi i z)\theta(z) \tag{4.2.9}$$

The integer lattice with basis 1 and τ is called the period lattice of the theta function. The remaining Jacobi theta-functions are defined with respect to the lattice $1, \tau = b/2\pi i$ as

$$\vartheta_1(z\,;\,\tau) := \theta[\frac{1}{2}, \frac{1}{2}](z) = \sum_{-\infty<n<\infty} \exp\left[\pi i \tau \left(n+\frac{1}{2}\right)^2 + 2\pi i \left(z+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right]$$

$$\vartheta_2(z\,;\,\tau) := \theta[\frac{1}{2}, 0](z) = \sum_{-\infty<n<\infty} \exp\left[\pi i \tau \left(n+\frac{1}{2}\right)^2 + 2\pi i z \left(n+\frac{1}{2}\right)\right]$$

$$\vartheta_4(z\,;\,\tau) := \theta[0, \frac{1}{2}](z) = \sum_{-\infty<n<\infty} \exp\left[\pi i \tau n^2 + 2\pi i \left(z+\frac{1}{2}\right) n\right].$$

The functions $\vartheta_2(z\,;\,\tau)$, $\vartheta_3(z\,;\,\tau)$ and $\vartheta_4(z\,;\,\tau)$ are even functions of z while $\vartheta_1(z\,;\,\tau)$ is odd. So for $g = 1$, the theta-function $\theta(z\,;\,\tau) = \vartheta_3(z\,;\,\tau) = 0$ for $z = \dfrac{1+\tau}{2}$.

Exercise 4.2.6: Prove that the zeros of the function $\theta(z)$ form an integer lattice with the same basis $1, \tau$ and with origin at the point $z_0 = \dfrac{1+\tau}{2}$.

By multiplying theta function (4.2.4) we obtain higher order theta functions. The function $f(z)$ is said to be a nth order theta function with characteristics p and q if it is an entire function of $z_1, \ldots, z_g$ and transforms according to the following law under translation of the argument by a vector of the period lattice

$$f(z+N+MB) = \exp\left[-\pi i n \langle MB, M\rangle - 2\pi i n\langle M, z+q\rangle + 2\pi i \langle p, N\rangle\right] f(z). \tag{4.2.10}$$

Exercise 4.2.7: Prove that the nth order theta functions with given characteristics q, p form a linear space of dimension n^g. Prove that a basis in this space is formed by the functions

$$\theta[\frac{p+\mathcal{S}}{n}, q](nz\,;\,nB), \tag{4.2.11}$$

where the coordinates of the vector $\mathcal{S}$ run independently through all values from 0 to $n-1$.

Under a change of the homology basis $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$ under a symplectic transformation

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

The period matrix transforms as (see 3.1.53)

$$B' = \int_{\beta'} \omega' = (cI_g + dB)(aI_g + bB)^{-1}.$$

Denote by R the matrix

$$R = aI_g + bB \tag{4.2.12}$$

The transformed values of the argument of the theta-function and of the characteristics are determined by

$$\begin{gathered} z = z'R \\ \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \frac{1}{2}\mathrm{diag}\begin{pmatrix} cd^t \\ ab^t \end{pmatrix}. \end{gathered} \tag{4.2.13}$$

Here the symbol diag means the vectors of diagonal elements of the matrices ab^t and cd^t. We have the equality

$$\theta[p', q'](z'\,;\, B') = \chi \sqrt{\det R} \exp\left\{ \frac{1}{2} \sum_{i \leqslant j} z_i z_j \frac{\partial \log \det R}{\partial B_{ij}} \right\} \theta[p, q](z\,;\, B), \tag{4.2.14}$$

where χ is a constant independent from z and B. See [19] for a proof.

Exercise 4.2.8: Prove the formula (4.2.14) for $g = 1$. *Hint*. Use the Poisson summation formula (see [20],[19]: if

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

is the Fourier transform of a sufficiently nice function $f(x)$, then

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

Theta function are connected by a complicated system of algebraic relations, which are called addition theorems. These are basically relations between formal Fourier series (see [19]). We present one of these relations. Let

$$\hat{\theta}[n](z; B) = \theta[\frac{n}{2}, 0](2z\,;\, 2B),$$

according to (4.2.11) this is a basis of second order theta functions.

Lemma 4.2.9. *The following identity holds:*

$$\theta(z+w)\theta(z-w) = \sum_{n\in(\mathbb{Z}_2)^g} \hat{\theta}[n](z)\hat{\theta}[n](w). \tag{4.2.15}$$

The expression $n \in (Z_2)^g$ means that the summation is over the g-dimensional vectors n whose coordinates all take values in 0 or 1.

Proof. Let us first analyze the case $g = 1$. The formula (4.2.15) can be written as

$$\theta(z+w)\theta(z-w) = \hat{\theta}(z)\hat{\theta}(w) + \hat{\theta}[1](z)\hat{\theta}[1](w) \tag{4.2.16}$$

where

$$\theta(z) = \sum_k \exp(\pi i b k^2 + 2\pi i k z), \quad \hat{\theta}(z) = \sum_k \exp(2\pi i b k^2 + 4\pi i k z),$$

$$\hat{\theta}[1](z) = \sum_k \exp(\left[2\pi i b(\frac{1}{2}+k)^2 + 4\pi i(k+1/2)z\right], \quad \Im(b) > 0.$$

The left-hand side of (4.2.16) has then the form

$$\sum_{k,l} \exp\left[\pi i b(k^2+l^2) + 2\pi i k(z+w) + 2\pi i l(z-w)\right]. \tag{4.2.17}$$

We introduce new summation indices m and n by setting $m = (k+l)/2$ and $n = (k-l)/2$. The numbers m and n simultaneously are integers or half integers. In these variables the sum (4.2.17) takes the form

$$\sum \exp[2\pi i b m^2 + 4\pi i m z + 2\pi i b n^2 + 4\pi i n w]. \tag{4.2.18}$$

We break up this sum into two parts. The first part will contain the terms with integers m and n, while in the second part m and n are both half-integers. In the second part we change the notation from m to $m + \frac{1}{2}$ and from n to $n + \frac{1}{2}$. Then m and n are integers, and the expression (4.2.14) can be written in the form

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \exp[2\pi i b m^2 + 4\pi i m z] \exp[2\pi i b n^2 + 4\pi i n w] + \\
&\sum_{m,n\in\mathbb{Z}} \exp[2\pi i b(m+\frac{1}{2})^2 + 4\pi i(m+\frac{1}{2})z] \exp[2\pi i b(n+\frac{1}{2})^2 + 4\pi i(n+\frac{1}{2})w] = \\
&\hat{\theta}(z)\hat{\theta}(w) + \hat{\theta}[1](z)\hat{\theta}[1](w).
\end{aligned}$$

The lemma is proved for $g = 1$. In the general case $g > 1$ it is necessary to repeat the arguments given for each coordinate separately. The lemma is proved. □

Exercise 4.2.10: Suppose that the Riemann matrix B has a block-diagonal form $B = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}$, where B' and B'' are $k \times k$ and $l \times l$ Riemann matrices, respectively with $k + l = g$. Prove that the corresponding theta function factors into the product of two theta function

$$\begin{gathered}
\theta(z\,; B) = \theta(z'\,; B')\theta(z''\,; B''), \\
z = (z_1, \ldots, z_g), \;\; z' = (z_1, \ldots, z_k), \;\; z'' = (z_{k+1}, \ldots, z_g).
\end{gathered} \tag{4.2.19}$$

Notte that the period matrix of a Riemann surface never has a block diagonal structure.

4.2.1 The Riemann theorem on zeros of theta functions and its applications

To solve the Jacobi inversion problem we use the Riemann θ-function $\theta(z) = \theta(z\,;B)$ on the Riemann surface $\mathcal{S}$. As usual we assume that $\alpha_1, \dots \alpha_g$ and $\beta_1, \dots, \beta_g$ is a canonical homology basis. The basis of holomorphic differentials $\omega_1, \dots, \omega_g$ is normalized

$$\int_{\alpha_j} \omega_k = \delta_{jk}, \quad \int_{\beta_j} \omega_k = B_{jk}.$$

Even though $\theta(z\ B)$ is not single-valued on $J(\mathcal{S})$, the set of zeros is well defined because of (4.2.2). The set of zeros of $\theta(z\,|\,B)$ is an analytic set of codimension one in $J(\mathcal{S})$. Let $e = (e_1, \dots, e_g) \in \mathbb{C}^g$ be a given vector. We consider the function $F : \mathcal{S} \to \mathbb{C}$ defined as

$$F(P) = \theta(A(P) - e), \tag{4.2.20}$$

where the Abel map A

$$A(P) = \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right),$$

is a holomorphic map of maximal rank of $\mathcal{S}$ into $J(\mathcal{S})$. Because of the periodicity properties of the theta-function (4.2.2), the function $F(P)$ transforms in the following way:

- $$F(P + \alpha_j) = F(P) \tag{4.2.21}$$
- $$F(P + \beta_j) = F(P) \exp \left[-\pi i B_{jj} - 2\pi i \int_{P_0}^{P} \omega_j + 2\pi i e_j \right]. \tag{4.2.22}$$

The study of the zeros of $F(P)$ is thus the study of the intersection of $A(\mathcal{S}) \subset J(\mathcal{S})$ with the set of zeros of $\theta(z\,;B)$ which form a well defined compact analytic sub-variety of the torus $J(\mathcal{S})$. Since $\mathcal{S}$ is compact, there are only two possibilities. Either $F(P)$ is identically zero on $\mathcal{S}$ or else $F(P)$ has only a finite number of zeros. The function $F(P)$ is single-valued and analytic on the cut surface $\tilde{\mathcal{S}}$ (the Poincaré polygon). Assume that it is not identically zero. This will be the case if, for example $\theta(e) \neq 0$.

Lemma 4.2.11. *If $F(P) \not\equiv 0$, then the function $F(P)$ has g zeros on $\tilde{\mathcal{S}}$ (counting multiplicity).*

Proof. To compute the number of zeros it is necessary to compute the logarithmic residue

$$\frac{1}{2\pi i} \oint_{\partial \tilde{\mathcal{S}}} d \log F(P) \tag{4.2.23}$$

(assume that the zeros of $F(P)$ do not lie on the boundary of $\partial\tilde{\mathcal{S}}$). We sketch a fragment of $\partial\tilde{\mathcal{S}}$ (cf. the proof of lemma 3.1.16). The following notation is introduced for brevity and used below: F^+ denotes the value taken by F at a point on $\partial\tilde{\mathcal{S}}$ lying on the segment α_k or β_k and F^- the value of F at the corresponding point α_k^{-1} or β_k^{-1} (see the figure 4.1).

The notation u^+ and u^- has an analogous meaning. In this notation the integral (4.2.23) can be written in the form

$$\frac{1}{2\pi i} \oint_{\partial \tilde{\mathcal{S}}} d \log F(P) = \frac{1}{2\pi i} \sum_{k=1}^{g} \left(\int_{\alpha_k} + \int_{\beta_k} \right) [d \log F^+ - d \log F^-]. \tag{4.2.24}$$

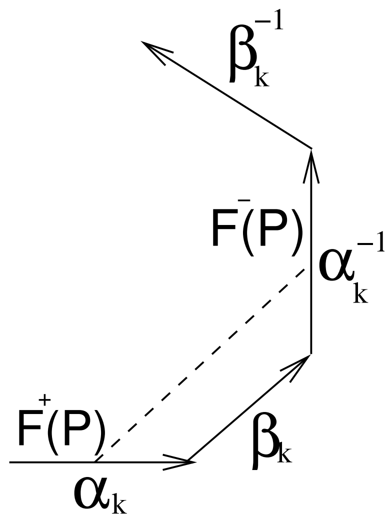


Figure 4.1: A fragment of $\tilde{S}$.

Note that if P is a point on α_k then

$$u_j^-(P) = u_j^+(P) + \int_{\beta_k} \omega_j = u_j^+(P) + B_{jk}, \quad j = 1, \ldots, g, \tag{4.2.25}$$

(cf. (3.1.17)), while if P lies on β_k, then

$$u_j^+(P) = u_j^-(P) + \int_{\alpha_k} \omega_j = u_j^-(P) + \delta_{jk}, \quad j = 1, \ldots, g, \tag{4.2.26}$$

(cfr. (3.1.18)). We get from the law of transformation (4.2.2) of the theta function or from (4.2.22), that for P on the cycle α_k one has

$$\log F^-(P) = -\pi i B_{kk} - 2\pi i u_k^+(P) + 2\pi i e_k + \log F^+(P); \tag{4.2.27}$$

while on the cycle β_k from (4.2.21) one has

$$\log F^+ = \log F^-. \tag{4.2.28}$$

From this on α_k

$$d \log F^-(P) = d \log F^+(P) - 2\pi i \omega_k(P), \tag{4.2.29}$$

and on β_k

$$d \log F^-(P) = d \log F^+(P). \tag{4.2.30}$$

Accordingly, from (4.2.29) and (4.2.29) the sum (4.2.24) can be written in the form

$$\frac{1}{2\pi i} \oint_{\partial \tilde{S}} d \log F = \sum_k \oint_{\alpha_k} \omega_k = g,$$

where we have used the normalization condition $\oint_{\alpha_k} \omega_k = 1$. The lemma is proved □

Note that although the function $F(P)$ is not a single-valued function on $\mathcal{S}$, its zeros $P_1, \dots, P_g$ do not depend on the location of the cuts along the canonical basis of cycles. Indeed, if this basis cycles is deformed then the path of integration from P_0 to P can change in the formulas for the Abel map. A vector of the form $(\oint_\gamma \omega_1, \dots, \oint_\gamma \omega_g)$ is added to the argument of the theta-function $\theta(z)$ in (4.2.20). This is a vector of period lattice $\{N + MB\}$. As a result of this the function $F(P)$ can only be multiplied by a non-zero factor in view of (4.2.2).

Now we will show now that the g zeros of $F(P)$ give a solution of the Jacobi inversion problem for a suitable choice of the vector e.

Theorem 4.2.12. *Let $e \in \mathbb{C}^g$, suppose that $F(P) = \theta(A(P) - e) \not\equiv 0$ and $P_1, \dots, P_g$ are its zeros on $\mathcal{S}$. Then on the Jacobi variety $J(\mathcal{S})$*

$$A^g(P_1, \dots, P_g) = e + \mathcal{K}, \tag{4.2.31}$$

where $\mathcal{K} = (\mathcal{K}_1, \dots, \mathcal{K}_g)$ is the vector of Riemann constants,

$$\mathcal{K}_j = -\frac{1 + B_{jj}}{2} + \sum_{l \neq j} \left(\oint\limits_{\alpha_l} \omega_l(P) \int_{P_0}^{P} \omega_j \right), \quad j = 1, \dots, g. \tag{4.2.32}$$

Proof. Consider the integral

$$\zeta_j = \frac{1}{2\pi i} \oint\limits_{\partial \tilde{\mathcal{S}}} u_j(P) d \log F(P). \tag{4.2.33}$$

This integral is equal to the sum of the residues of the integrands i.e.,

$$\zeta_j = u_j(P_1) + \cdots + u_j(P_g), \tag{4.2.34}$$

where $P_1, \dots, P_g$ are the zeros of $F(P)$ of interest to us. On the other hand, this integral can be represented by analogy with the proof of Lemma 4.2.11 in the form

$$\begin{aligned}
\zeta_j &= \frac{1}{2\pi i} \sum_{k=1}^{g} \left(\int_{\alpha_k} + \int_{\beta_k} \right) \left(u_j^+ d \log F^+ - u_j^- d \log F^-) \right) \\
&= \frac{1}{2\pi i} \sum_{k=1}^{g} \int_{\alpha_k} [u_j^+ d \log F^+ - (u_j^+ + B_{jk})(d \log F^+ - 2\pi i \omega_k)] \\
&+ \frac{1}{2\pi i} \sum_{k=1}^{g} \int_{\beta_k} u_j^+ d \log F^+ - (u_j^+ - \delta_{jk}) d \log F^+] \\
&= \frac{1}{2\pi i} \sum_{k=1}^{g} \left[\int_{\alpha_k} 2\pi i u_j^+ \omega_k - B_{jk} \int_{a_k} d \log F^+ + 2\pi i B_{jk} \right] + \frac{1}{2\pi i} \int_{b_j} d \log F^+,
\end{aligned}$$

in the course of computation we used formula (4.2.25)-(4.2.30). The function F takes the same values at the endpoints of α_k, therefore

$$\int_{\alpha_k} d \log F^+ = 2\pi i n_k,$$

where n_k is an integer. Further let Q_j and $\tilde{Q}_j$ be the initial and terminal point of β_j. Then

$$\begin{aligned}&\int_{\beta_j} d\log F^+ = \log F^+(\tilde{Q}_j) - \log F^+(Q_j) = \\ &= \log\theta(A(Q_j+\beta_j)-e) - \log\theta(A(Q_j)-e) = -\pi i B_{jj} + 2\pi i e_j - 2\pi i u_j(Q_j),\end{aligned}$$

The expression for ζ_j can now be written in the form

$$\begin{aligned}\zeta_j &= u_j(P_1) + \cdots + u_j(P_g) = \\ &= e_j - \frac{1}{2}B_{jj} - u_j(Q_j) + \sum_k \int_{a_k} u_j\omega_k + \sum_k B_{jk}(-n_k+1).\end{aligned} \tag{4.2.35}$$

The last two terms can be thrown out, they correspond to the j-coordinate of some vector of the period lattice. Thus the relation (4.2.35) coincides with the desired relation (4.2.31) if it is proved that the constant in this equality reduces to (4.2.32), i.e.

$$-\frac{1}{2}B_{jj} - u_j(Q_j) + \sum_k \int_{\alpha_k} u_j\omega_k = \mathcal{K}_j, \quad j = 1, \ldots, g.$$

To get rid of the term $u_j(Q_j)$ we transform the integral

$$\oint_{\alpha_j} u_j\omega_j = \frac{1}{2}[u_j^2(Q_j) - u_j^2(R_j)],$$

where R_j is the beginning of α_j and Q_j is its end (which is also the beginning of b_j). Further $u_j(Q_j) = u_j(R_j) + 1$. We obtain

$$\oint_{\alpha_j} u_j\omega_j = \frac{1}{2}[2u_j(Q_j) - 1],$$

hence

$$-u_j(Q_j) + \sum_{k=1}^{g} \int_{\alpha_k} u_j\omega_k = -\frac{1}{2} + \sum_{k\neq j, k=1}^{g} \int_{\alpha_k} u_j\omega_k.$$

The theorem is proved. □

Remark 4.2.13. We observe that the vector of Riemann constant depends on the choice of the base point P_0 of the Abel map. Indeed let $\mathcal{K}_{P_0}$ be the vector of Riemann constants with base point P_0. Then $\mathcal{K}_{Q_0}$ is related to $\mathcal{K}_{P_0}$ by

$$\mathcal{K}_{Q_0} = \mathcal{K}_{P_0} + (g-1)\int_{Q_0}^{P_0} \omega. \tag{4.2.36}$$

Example 4.2.14. The vector of Riemann constants can be easily calculated for hyperelliptic Riemann surfaces. In particular let us consider the curve $w^2 = \prod_{i=1}^{5}(z - z_i)$ of genus $g = 2$, and choose a basis of cycles as indicated in the figure 4.2. A normal basis of holomorphic differentials

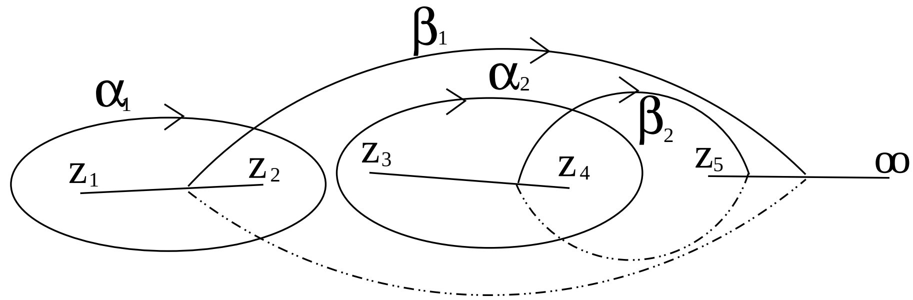


Figure 4.2: Homology basis.

has the form

$$\omega_j = \frac{\prod_{k=1}^{2} c_{jk} z^{k-l} dz}{w}, \quad j = 1, 2, \tag{4.2.37}$$

where the constants c_{jk} are uniquely determined by

$$\int_{\alpha_k} \omega_j = \delta_{jk} .$$

We chose as base point of the Abel map the point $P_0 = (\infty, \infty)$. We need to compute

$$\left(\oint_{\alpha_2} \omega_2(P) \int_{P_0}^{P} \omega_1 \right), \quad \left(\oint_{\alpha_1} \omega_1(P) \int_{P_0}^{P} \omega_2 \right).$$

Using the fact that

$$\begin{aligned}\oint_{\alpha_2} \omega_2(P) \int_{P_0}^{P} \omega_1 &= \oint_{\alpha_2} \omega_2(P) \int_{P_0}^{z_4} \omega_1 + \int_{z_3}^{z_4} \omega_2(z, w) \int_{z_4}^{(z,w)} \omega_1 - \int_{z_3}^{z_4} \omega_2(z, -w) \int_{z_4}^{(z,-w)} \omega_1 \\ &= \oint_{\alpha_2} \omega_2(P) \int_{P_0}^{z_4} \omega_1 = \int_{P_0}^{z_4} \omega_1 = (-\frac{1}{2} - \frac{B_{12}}{2})\end{aligned}$$

one obtains

$$\mathcal{K}_1 = -\frac{1 + B_{11}}{2} - \frac{1}{2} - \frac{B_{12}}{2} = -1 - \frac{B_{11} + B_{12}}{2}$$

In the same way calculating

$$\begin{aligned}\oint_{\alpha_1} \omega_1(P) \int_{P_0}^{P} \omega_2 &= \oint_{\alpha_1} \omega_1(P) \int_{P_0}^{z_2} \omega_2 + \int_{z_1}^{z_2} \omega_1(z, w) \int_{z_2}^{(z,w)} \omega_2 - \int_{z_1}^{z_2} \omega_1(z, -w) \int_{z_2}^{(z,-w)} \omega_2 \\ &= \oint_{\alpha_1} \omega_1(P) \int_{P_0}^{z_2} \omega_2 = -B_{21}/2\end{aligned}$$

one obtains that

$$\mathcal{K}_2 = -\frac{1 + B_{22} + B_{21}}{2}$$

Observe that the vector $\mathcal{K}$ can be written in the form

$$\mathcal{K} = \left(0, \frac{1}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}\right) B$$

Namely, given the odd characteristic

$$p = \left(\frac{1}{2}, \frac{1}{2}\right), \quad q = \left(0, \frac{1}{2}\right),$$

one has that $\mathcal{K} = q + pB$. From this expression it follows that

$$\theta(\mathcal{K}) = 0.$$

It is a general result not restricted to this particular example that $\theta(z)|_{z=\mathcal{K}} = 0$.

Corollary 4.2.15. *Let D a positive divisor of degree g. If the function*

$$\theta(\mathcal{A}(P) - \mathcal{A}(D) + \mathcal{K})$$

does not vanish identically on $\mathcal{S}$ then its divisor of zeros coincides with D.

Accordingly, if the function $\theta(\mathcal{A}(P) - e)$ is not identically equal to zero on $\mathcal{S}$, then its zeros give a solution of the Jacobi inversion problem (4.1.5) for the vector $\eta = e + \mathcal{K}$. We have shown that the map (4.1.4) $\mathcal{A}^g : \mathcal{S}_g \to J(\mathcal{S})$ is a local homeomorphism in a neighbourhood of a non special positive divisor D of degree g. Since $\theta(z) \not\equiv 0$ for $z \in J(\mathcal{S})$, then $\theta(\mathcal{A}(D))$ does not vanish identically on open subsets of $S_g\mathcal{S}$. In the next subsection, we characterize the zero set of the θ-function. The zeros of the theta-function form an analytic subvariety of $J(\mathcal{S})$. The collection of these zeros forms the theta divisor in $J(\mathcal{S})$.

4.3 The Theta Divisor

In this section we study the set of zeros of the theta functions and in particular the Riemann vanishing theorem which prescribes in a rather detail manner the set of zeros of the theta-function on $\mathbb{C}^g$.

Theorem 4.3.1. *Let $e \in \mathbb{C}^g$, then $\theta(e) = 0$ if and only if $e = \mathcal{A}(D) - \mathcal{K}$ where D is a positive divisor of degree $g - 1$ and $\mathcal{K}$ is the vector of Riemann constants (4.2.32).*

Remark 4.3.2. For a positive divisor D of degree $g - 1$, the expression $A(D) - \mathcal{K}$ does not depend on the base point of the Abel map. The theorem 4.3.1 says that the theta-function vanishes on a $g - 1$-dimensional variety parametrized by $g - 1$ points of $\mathcal{S}$, namely the theta function vanishes on $\mathcal{A}(\mathcal{S}_{g-1}) - \mathcal{K}$.

Proof. We first prove sufficiency. Let $P_1 + \cdots + P_g$ be a non special divisor and $v = A(P_1 + \cdots + P_g) - \mathcal{K}$. Let us consider $F(P) = \theta(A(P) - v)$. Either F is identically zero or not. In the former case for each $k = 1, \dots g$

$$F(P_k) = \theta(A(P_1 + \cdots + \hat{P}_k + \cdots + P_g) - \mathcal{K}) = 0,$$

where we use the symbol $\hat{P}_k$ to mean that P_k does not appear in the divisor. So for $e = A(P_1 + \cdots + \hat{P}_k + \cdots + P_g) - \mathcal{K}$ we have $\theta(e) = 0$.

In the latter case $F(P) \not\equiv 0$, we have that F has precisely g zeros on $\mathcal{S}$ due to lemma 4.2.11. Let $Q_1, \dots Q_g$ be the zeros of F, then according to theorem 4.2.12 one has

$$A(Q_1 + \cdots + Q_g) = v + \mathcal{K} = A(P_1 + \cdots + P_g).$$

Since $P_1 + \cdots + P_g$ is not special, it follows from the Riemann-Roch and the Abel theorems that $Q_1 + \cdots + Q_g = P_1 + \cdots + P_g$. Therefore also in this case $F(P_k) = \theta(A(P_1 + \cdots + \hat{P}_k + \cdots + P_g) - \mathcal{K}) = 0$ for $k = 1, \dots, g$. Since the set of non-special divisor of degree g is dense in $S^{(g)}\mathcal{S}$, the divisors of the form $P_1 + \cdots + \hat{P}_k + \cdots + P_g$ form a dense subset of $S^{(g-1)}\mathcal{S}$. Since the function $\theta(z)$ is continuous, it follows that $\theta(z)$ is identically zero on $W_{g-1} - \mathcal{K}$, where in general $W_n \subset J(\mathcal{S})$, is the Abel image of $S^{(n)}\mathcal{S}$ for $n \geqslant 1$.

Conversely, let $\theta(e) = 0$. Then by Jacobi inversion theorem, since θ is not identically zero on $J(\mathcal{S})$. Then there exists an integer s, $1 \leqslant s \leqslant g$, so that

$$\theta(A(\tilde{D}_1 - \tilde{D}_2) - e) = 0, \quad \forall \tilde{D}_1, \tilde{D}_2 \in S^{(s-1)}\mathcal{S}$$

but

$$\theta(A(D_1 - D_2) - e) \neq 0, \quad D_1, D_2 \in S^{(s)}\mathcal{S}.$$

Let $D_1 = P_1 + \cdots + P_s$ and $D_2 = Q_1 + \cdots + Q_s$ where we assume that the points of the divisors are mutually distinct. Now let us consider the function

$$F(P) = \theta(A(P) + A(P_2 + \cdots + P_s) - A(Q_1 + \cdots + Q_s) - e)$$

Since $F(P_1) \neq 0$, this function is not identically zero on $\mathcal{S}$. Therefore, by theorem 4.2.12 it has g zeros on $\mathcal{S}$. These zeros are by construction $Q_1, \dots, Q_s$ plus some other $g - s$ points $T_{s+1}, \dots, T_g$. By theorem 4.2.12 one has

$$A(Q_1 + \cdots + Q_s + T_{s+1}, + \cdots + T_g) - \mathcal{K} = A(Q_1 + \cdots + Q_s) - A(P_2 + \cdots + P_s) + e$$

or equivalently

$$e = A(P_2 + \cdots + P_s + T_{s+1}, + \cdots + T_g) - \mathcal{K}$$

which is a point in $W_{g-1} - \mathcal{K}$. □

Regarding the zeros of the theta-function it is possible to prove a little bit more then stated in the previous theorems. Let $D \in S^{(g-1)}\mathcal{S}$ and let $e = A(D) - \mathcal{K}$. Then

$$\mathrm{mult}_{z=e}\theta(z) = l(D).$$

where $l(D)$ is the dimension of the space $L(D)$. The proof of this identity can be found in [20].

Remark 4.3.3. The vector of Riemann constants has a characterisation in terms of divisors. Indeed there is a non positive divisor Δ of degree $g-1$ such that its Abel image coincides with $\mathcal{K}$, namely $A(\Delta) = \mathcal{K}$. Furthermore let D be a positive divisor of degree $g-1$, then the vector

$$e = A(D) - \mathcal{K}$$

is a zero of the theta-function, namely $\theta(e) = 0$. By the parity of the theta-function one has $\theta(-e) = 0$. It follows by theorem 4.3.1 that

$$-e = A(D') - \mathcal{K}$$

where D' is a positive divisor of degree $g-1$. Then summing up the two relations we obtain

$$2\mathcal{K} = A(D + D')$$

where $D + D'$ is a positive divisor of degree $2g-2$. Since $D + D'$ has arbitrary $g-1$ points in it, it follows from remark 3.1.57 that $l(D + D') \geqslant g$ which is equivalent, by Riemann-Roch theorem, to $l(K - D - D') \geqslant 1$. Since $\deg(D + D') = 2g-2$ and $\deg(K - D - D') = 0$, one has $l(K - D - D') = 1$ which implies $K = D + D'$, namely we have shown that

$$2\mathcal{K} = \mathcal{A}(K)\,. \tag{4.3.1}$$

Using the characterization of the theta-divisor one can complete the description of the function $F(P)$.

Lemma 4.3.4. *Let $F(P) = \theta(A(P) - e)$ where $e = A(D) - \mathcal{K}$, $D \in S^{(g)}\mathcal{S}$ and $\mathcal{K}$ the vector of Riemann constants defined in (4.2.32). Then*

1. *$F(P) \equiv 0$ iff the divisor D is special;*
2. *$F(P) \not\equiv 0$ iff $\dim\Omega(D) = 0$, i.e. the divisor D is not special. In this last case D is the divisor of zeros of $F(P)$.*

Proof. Let's prove part 1. of the lemma. Let $F(P) \equiv 0$, then by theorem 4.3.1 there is a positive divisor $\tilde{D}$ of degree $g-1$ so that

$$A(D) - \mathcal{K} - A(P) = A(\tilde{D}) - \mathcal{K}.$$

By Abel theorem, the identity holds if and only if D and $\tilde{D} + P$ are linearly equivalent, that is there is a meromorphic function in $L(D)$ with a zero in an arbitrary point $P \in \mathcal{S}$. This is possible only if $l(D) > 1$ or equivalently $\dim\Omega(D) > 0$, namely D is special. Conversely, if $D \in S^g\mathcal{S}$ is special then $l(D) > 1$ and therefore there is a function $f \in L(D)$ with an arbitrary zero in a point $P \in \mathcal{S}$ so that $(f) = P + \tilde{D} - D$. where $\tilde{D} \in S^{(g-1)}\mathcal{S}$. It follows by Abel theorem that $A(P) - A(D) + \mathcal{K} = -A(\tilde{D}) + \mathcal{K}$, then by theorem 4.3.1, one has $\theta(A(\tilde{D}) - \mathcal{K}) = 0$.

Now let us prove part 2. of the lemma. Suppose now that D is not special, then $F(P) \not\equiv 0$ and by theorem 4.2.12, the divisors of zeros of $F(P)$ coincides with D. □

Corollary 4.3.5. *Let $e = A(D) - \mathcal{K}$ with $D \in S^{g-1}\mathcal{S}$. Them the function $F(P) = \theta(A(P) - e)$ vanishes identically if and only if* **dim**$\Omega(D + P_0) \geqslant 1$ *(Check!!) where P_0 is the base point of the Abel map.*

Proof. Let P_0 be the base point of the Abel map, then $A(P-P_0) = A(P)$. Suppose $F(P) \equiv 0$, then by theorem 4.3.1 there exists a positive divisor $\tilde{D}$ of degree $g-1$ such that

$$A(P-P_0) - A(D) + \mathcal{K} = -A(\tilde{D}) + \mathcal{K}$$

which implies that $A(D+P_0) = A(\tilde{D}+P)$. By Abel theorem, there is a nontrivial meromorphic function h with divisor

$$(h) = \tilde{D} + P - D - P_0$$

for all $P \in \mathcal{S}$. This implies that $l(D+P_0) \geqslant 2$ or equivalently, $D+P_0$ is a special divisor. Viceversa suppose that $\dim\Omega(D+P_0) \geqslant 1$, then $l(D+P_0) > 1$ so that $L(D+P_0)$ is generated by $\{1, h\}$ where h is a meromorphic function. So there is a nontrivial meromorphic function with poles in $D+P_0$ and having zero in an arbitrary point P (take for example the function $h - h(P)$) and some other $g-1$ points given by the divisor $\tilde{D}$. It follows that

$$A(D+P_0) = A(\tilde{D}+P)$$

or equivalently

$$A(P-P_0) - A(D) + \mathcal{K} = -A(\tilde{D}) - \mathcal{K}$$

which implies by theorem 4.3.1 that $0 = \theta(-A(\tilde{D}) - K) = \theta(A(P-P_0) - A(D) - \mathcal{K}) = \theta(A(P) - A(D) - \mathcal{K})$ where we recall that P_0 is the base point of the Abel map. □

The zeros of the theta function (the points of the theta divisor) form a variety of dimension $2g-2$ (for $g \geqslant 3$). If we delete from $J(\mathcal{S})$, the theta divisor, then we get a connected $2g$-dimensional domain. We get that the Jacobi inversion problem is solvable for all points of the Jacobian $J(\mathcal{S})$ and uniquely solvable for almost all points. Thus the collection $(P_1, \ldots, P_g) = (A^{(g)})^{-1}(\eta)$ of points of the Riemann surface $\mathcal{S}$ (without consideration of order) is a single valued function of a point $\eta = (\eta_1, \ldots \eta_g) \in J(\mathcal{S})$ (which has singularities at points of the theta divisor.) To find an analytic expression for this function we take an arbitrary meromorphic function $f(P)$ on $\mathcal{S}$. Then the specification of the quantities $\eta_1, \ldots, \eta_g$ uniquely determines the collection of values

$$f(P_1), \ldots, f(P_g), \quad A^{(g)}(P_1, \ldots, P_g) = \eta. \tag{4.3.2}$$

Therefore, any symmetric function of $f(P_1), \ldots, f(P_g)$ is a single-valued meromorphic function of the g variables $\eta = (\eta_1, \ldots, \eta_g)$, that is $2g$-fold periodic with period lattice $\{2\pi i M + BN\}$. All these functions can be expressed in terms of a Riemann theta function. The following elementary symmetric functions has an especially simple expression:

$$\sigma_f(\eta) = \sum_{j=1}^{g} f(P_j). \tag{4.3.3}$$

From Theorem 4.2.31 and the residue formula we get for this function the representation

$$\begin{aligned}\sigma_f(\eta) &= \frac{1}{2\pi i} \oint_{\partial\tilde{S}} f(P) d\log\theta(A(P) - \eta + \mathcal{K}) \\ &\quad - \sum_{f(Q_k)=\infty} \operatorname*{Res}_{P=Q_k} f(P) d\log\theta(A(P) - \eta + \mathcal{K}),\end{aligned} \tag{4.3.4}$$

the second term in the right hand side is the sum of the residue of the integrand over all poles if $f(P)$. As in the proof of Lemma 4.2.11 and Lemma 4.2.12, it is possible to transform the first term in (4.3.4) by using the formulas (4.2.29) and (4.2.30). The equality (4.3.4) can be written in the form

$$\sigma_f(\eta) = \frac{1}{2\pi i} \sum_k \int_{a_k} f(P)\omega_k - \sum_{f(a_k)=\infty} \operatorname*{Res}_{P=Q_k} f(P) d\log\theta(A(P) - \eta + \mathcal{K}). \tag{4.3.5}$$

Here the first term is a constant independent of η. We analyze the computation of the second term (the sum of residue) using an example.

Example 4.3.6. $\mathcal{S}$ is an hyperelliptic Riemann surface of genus g given by the equation $w^2 = P_{2g+1}(z)$, and the function f has the form $f(z,w) = z$, the projection on the z-plane. This function on $\mathcal{S}$ has a unique two-fold pole at ∞. We get an analytic expression for the function σ_f constructed according to the formula (4.3.3). In other words if $P_1 = (z_1, w_1), \ldots, P_g = (z_g, w_g)$ is a solution of the inversion problem $A(P_1) + \cdots + A(P_g) = \eta$, then

$$\sigma_f(\eta) = z_1 + \cdots + z_g. \tag{4.3.6}$$

We take ∞ as the base point P_0 (the lower limit in the Abel mapping). According to (4.3.5) the function $\sigma_f(\eta)$ has the form

$$\sigma_f(\eta) = c - \operatorname*{Res}_{\infty} \left[z d\log\theta(A(P) - \eta + \mathcal{K}) \right].$$

Let us compute the residue. Take $\tau = z^{-\frac{1}{2}}$ as a local parameter in a neighbourhood of ∞. Suppose that the holomorphic differentials ω_i have the form $\omega_i = \psi_i(\tau)d\tau$ in a neighbourhood of ∞. Then

$$\begin{aligned} d\log\theta(A(P) - \eta + \mathcal{K}) &= \sum_{i=1}^{g} [\log\theta(A(P) - \eta + \mathcal{K}]_i \omega_i(P) = \\ &= \sum_{i=1}^{g} [\log\theta(A(P) - \eta + \mathcal{K})]_i \psi_i(\tau) d\tau \end{aligned}$$

where $[\ldots]_i$ denotes the partial derivative with respect to the ith variable. By the choice of the base point $P_0 = \infty$, the decomposition of the vector-valued function $A(P)$ in a neighbourhood of ∞ has the form

$$A(P) = \tau U + O(\tau^2),$$

where the vector $U = (U_1, \ldots, U_g)$ has the form

$$U_j = \psi_j(0), \quad j = 1, \ldots, g.$$

From these formulas we finally get

$$\sigma_f(\eta) = -(\log\theta(\eta - \mathcal{K}))_{i,j} U_i U_j + c = -\partial_x^2 \log\theta(xU + \eta - \mathcal{K})|_{x=0} + c, \tag{4.3.7}$$

where $(\log\theta(\eta - \mathcal{K}))_{i,j}$ denotes derivative with respect to the $i - th$ and $j - th$ argument of the *theta*-function and c is a constant.

We shall show in the next Section that the function

$$u(x,t) = \frac{\partial^2}{\partial x^2} \log \theta(Ux + Wt - \eta + \mathcal{K}) + c$$

where $W_k = \frac{1}{3}\psi''(0)$ solves the Korteweg de Vries equation

$$u_t = \frac{1}{4}(6uu_x + u_{xxx}).$$

Exercise 4.3.7: Suppose that a hyperelliptic Riemann surface of genus g is given by the equation $w^2 = P_{2g+2}(z)$. Denotes its points at infinity by P_- and P_+. Chose P_- as the base point P_0 of the Abel mapping. Take $f(z,w) = z$ as the function f. Prove that the function $\sigma_f(\eta)$ has the form

$$\sigma_f(\eta) = \left(\log \frac{\theta(\eta - \mathcal{K} - A(P_+))}{\theta(\eta - \mathcal{K})}\right)_j U_j + c \tag{4.3.8}$$

where the vector $U = (U_1, \dots, U_g)$ has the form

$$U_j = \psi_j(0), \quad j = 1, \dots, g, \tag{4.3.9}$$

where the basis of holomorphic differentials have the form

$$\omega_j(P) = \psi_j(\tau)d\tau, \quad \tau = z^{-1}, \quad P \to \infty.$$

Exercise 4.3.8: Let $\mathcal{S}$ be a Riemann surface $w^2 = P_5(z)$ of genus 2. Consider the two systems of differential equations:

$$\frac{dz_1}{dx} = \frac{\sqrt{P_5(z_1)}}{z_1 - z_2}, \quad \frac{dz_2}{dx} = \frac{\sqrt{P_5(z_2)}}{z_2 - z_1} \tag{4.3.10}$$

$$\frac{dz_1}{dt} = \frac{z_2\sqrt{P_5(z_1)}}{z_1 - z_2}, \quad \frac{dz_2}{dt} = \frac{z_1\sqrt{P_5(z_2)}}{z_2 - z_1}. \tag{4.3.11}$$

Each of these systems determined a law of motion of the pair of points

$$P_1 = (z_1, \sqrt{P_5(z_1)}), \quad P_2 = (z_2, \sqrt{P_5(z_2)})$$

on the Riemann surface $\mathcal{S}$. Prove that under the Abel mapping (4.1.1) these systems pass into the systems with constant coefficients

$$\frac{d\eta_1}{dx} = 0, \quad \frac{d\eta_2}{dt} = 1$$

$$\frac{d\eta_1}{dt} = -1, \quad \frac{d\eta_2}{dt} = 0.$$

In other words, the Abel mapping (4.1.1) is simply a substitution integrating the equations (4.3.10) and (4.3.11).

4.4 Holomorphic line bundles and divisors

In this section we show the equivalence between holomorphic line bundles and divisors on a compact Riemann surface $\mathcal{S}$.

4.4.1 Holomorphic line bundle

Let $\{U_\alpha\}_{\alpha\in A}$ an open covering of a compact Riemann surface $\mathcal{S}$. Let

$$\mathcal{O}^*(U) \subset \mathcal{O}(U) \subset \mathcal{M}(U)$$

be the set of nowhere vanishing holomorphic, holomorphic and meromorphic functions on $U \subset \mathcal{S}$.

Definition 4.4.1. *A complex line bundle over the Riemann surface $\mathcal{S}$ is a complex manifold L and a holomorphic map $\pi : L \to \mathcal{S}$ such that*

- $L_P := \pi^{-1}(P) \simeq P \times \mathbb{C}$. *$L_P$ is called the fiber of L*
- *for a covering $\{U_\alpha\}_{\alpha\in A}$ of $\mathcal{S}$ the triples $\{P, U_\alpha, v_\alpha\}_{\alpha\in\mathcal{A}}$ with $P \in U_\alpha$ and $v_\alpha \in \mathbb{C}$ satisfy the equivalence relation*
$$\{P, U_\alpha, v_\alpha\} \simeq \{Q, U_\beta, v_\beta\} \longleftrightarrow P = Q \in U_\alpha \cap U_\beta \neq \emptyset, \quad v_\alpha = g_{\alpha\beta}(P)v_\beta$$
where $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ is called transition function

The functions $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfy the cocycle condition

$$g_{\alpha\beta}(P)g_{\beta\gamma}(P)g_{\gamma\alpha}(P) = 1, \quad \forall P \in U_\alpha \cap U_\beta \cap U_\gamma,$$

and

$$g_{\alpha\beta}(P)g_{\beta\alpha}(P) = 1.$$

The line bundle with $g_{\alpha\beta} = 1$ for all $\alpha, \beta \in \mathcal{A}$ is called trivial.

Definition 4.4.2. *Two line bundles L and L' with transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ define isomorphic line bundles i it exists $f_\alpha \in \mathcal{O}^*(U_\alpha)$ so that*

$$g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}. \tag{4.4.12}$$

One can give to the set of line bundles over $\mathcal{S}$ the structure of a group where the multiplication is given by tensor product and inverse by dual bundle, namely if L and L' are give by $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ then

$$L \otimes L' \sim \{g_{\alpha\beta}g'_{\alpha\beta}\}, \quad L^* \sim \{g^{-1}_{\alpha\beta}\}$$

The group of line bundles over $\mathcal{S}$ is called the Picard group of $\mathcal{S}$ and denoted by $Pic(\mathcal{S})$[1].

A section of L is a map $\psi : \mathcal{S} \to L$ such that $\psi(P) \in L_P$ with $P \in \mathcal{S}$. For the trivial bundle $L = \mathbb{C} \times \mathcal{S}$ every section is of the form $\psi(P) = (f(P), P)$ for some holomorphic or meromorphic function f in $\mathcal{S}$. A set of meromorphic functions $f_\alpha \in \mathcal{M}(U_\alpha)$ such that $f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$

[1]More precisely the group of line bundles coincides with the first cohomology group $H^1(\mathcal{S}, \mathcal{O}^*)$ and this group last is called the Picard group of $\mathcal{S}$.

$\forall\alpha,\beta\in A$, is a meromorphic section of the line bundle L. Indeed by defining it transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ are

$$g_{\alpha\beta}(P) := \frac{f_\alpha(P)}{f_\beta(P)}, \quad P\in U_\alpha\cap U_\beta.$$

one can immediately see that $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ satisfy the cocycle condition.

The divisor of the meromorphic section $\{f_\alpha\}_{\alpha\in A}$ is well defined as

$$(f_\alpha) = (f_\alpha)|_{U_\alpha}.$$

We now describe a basic correspondence between divisors and line bundles. Let $D\in\mathrm{Div}(\mathcal{S})$ with $D=\sum_i n_iP_i$ and let U_α be a covering such that each open set U_α contains at most a point of D. Let $f_\alpha\in\mathcal{M}^*(U_\alpha)$ be meromorphic functions, such that the divisor of f_α is precisely the part of D lying in U_α, for example if $P_i\in U_\alpha$ and z_α is a centred coordinate near P_i, then $f_\alpha = z_\alpha^{n_i}$

$$(f_\alpha) = D|_{U_\alpha} = n_iP_i.$$

Then the functions

$$g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}\in\mathcal{O}^*(U_\alpha\cap U_\beta)$$

satisfy the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \frac{f_\alpha}{f_\beta}\frac{f_\beta}{f_\gamma}\frac{f_\gamma}{f_\alpha} = 1.$$

The line bundle constructed in this way is called the line bundle associated to the divisor D and it is denoted by $L[D]$. It is well defined. Indeed if (z'_α, U_α) is another chart and $f'_\alpha = (z'_\alpha)^{n_i}$ then $h_\alpha = \dfrac{f_\alpha}{f'_\alpha}\in\mathcal{O}^*(U_\alpha)$ and

$$g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta} = g_{\alpha\beta}\frac{h_\beta}{h_\alpha}$$

Therefore according to Definition 4.4.2 $g'_{\alpha\beta}$ and $g_{\alpha\beta}$ define isomorphic line bundles.

The degree of the divisor is called the degree of the line bundle and is denoted by $\deg L[D]$.

The map $\mathrm{Div}(\mathcal{S})\to\mathrm{Pic}(\mathcal{S})$ given by $D\to L[D]$ is a homomorphism of groups. Indeed, given two divisors D and D' with local data $\{f_\alpha\}$ and $\{f'_\alpha\}$ respectively, then the local data for $D+D'$ is given by $\{f_\alpha f'_\alpha\}$. It follows that $L[D+D'] = L[D]\otimes L[D']$.

If D is the divisor of a meromorphic function f, namely $D=(f)$, then we can take as a local data over any cover U_α the functions $f_\alpha := f|_{U_\alpha}$. The transition functions $g_{\alpha\beta} = f_\alpha/f_\beta = 1$ so $L[D]$ is trivial. Conversely, if D is given by local data $\{f_\alpha\}$ and the line bundle $L[D]$ is trivial, then there exists functions $h_\alpha\in\mathcal{O}^*(U_\alpha)$ such that

$$\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$$

so that $f_\alpha h_\alpha^{-1} = f_\beta h_\beta^{-1}$ is a global meromorphic function on $\mathcal{S}$ with divisor D.

Lemma 4.4.3. *The divisors D and D' are linearly equivalent iff the holomorphic line bundles $L[D]$ and $L[D']$ are isomorphic*

Proof. Let $h \in \mathcal{M}(\mathcal{S})$ so that $(h) = D' - D$. Choose a covering of $\mathcal{S}$ so that each point of D and D' belongs only to one U_α. If f_α is a meromorphic section of $L[D]$, then $h|_{U_\alpha} f_\alpha$ is a meromorphic section of $L[D']$ which implies (4.4.12). Conversely, let f_α and f'_α be meromorphic sections of isomorphic line bundles $L[D]$ and $L[D']$. Then it exists $h_\alpha \in \mathcal{O}^*(U_\alpha)$ so that

$$\frac{h_\alpha f_\alpha}{h_\beta f_\beta} = \frac{f'_\alpha}{f'_\beta}$$

that is $\dfrac{h_\alpha f_\alpha}{f'_\alpha}$ is a meromorphic function with divisor $D - D'$, which gives $D \sim D'$. ◻

Summarizing, for each divisor $D \in \mathrm{Div}(\mathcal{S})$ we can associate a line bundle $L[D]$. Conversely, given a line bundle L and a meromorphic section f_α we see that $g_{\alpha\beta} = f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $L = L[(f)]$. In particular, L is the line bundle associated to a divisor D on $\mathcal{S}$ if and only if it has a non vanishing meromorphic section.

Lemma 4.4.4. *Every holomorphic line bundle on a compact Riemann surface $\mathcal{S}$ admits a meromorphic section.*

We do not prove this lemma. Therefore, the map $\mathrm{Div}(\mathcal{S}) \to \mathrm{Pic}(\mathcal{S})$ given by $D \to L[D]$ is also and isomorphism of groups. We can then summarize the results of lemmas 4.4.3 and lemma 4.4.4.

Theorem 4.4.5. *The Picard group $\mathrm{Pic}(\mathcal{S})$ is isomorphic to the group of divisors $\mathrm{Div}(\mathcal{S})$ modulo linear equivalence.*

We give now a geometric interpretation of the Riemann-Roch theorem. Denote by $h^0(L)$ the dimension of the space of holomorphic sections of L and by $\deg L$ the degree of the line bundle, i.e. the degree of the divisor D associated to L. Furthermore, we denote by K the canonical line bundle associated to the canonical divisor K. Its transition functions are

$$g_{\alpha\beta} = \frac{dz_\alpha}{dz_\beta}$$

where (z_α, U_α) is a chart of $\mathcal{S}$.

Theorem 4.4.6. *Let L be an holomorphic line bundle over a Riemann surface $\mathcal{S}$ of genus g. Then*

$$h^0(L) = \deg L + 1 - g + h^0(KL^{-1}) \tag{4.4.13}$$

Proof. We just show that the space of holomorphic section of $L[D]$ is isomorphic to the space $L(D)$ defined in (3.1.72). Indeed, let be ϕ a meromorphic section of $L[D]$ with divisor D and h a holomorphic section of $L[D]$. Then h/ϕ is a meromorphic function on $\mathcal{S}$ and $\left(\dfrac{h}{\phi}\right) \geqslant 0$, therefore $h/\phi \in L(D)$. Conversely, given $f \in L(D)$ then f/ϕ is a holomorphic section of $L[D]$. In the same way one can show that the space of holomorphic section of $L[K-D]$ is isomorphic to the space $L(K-D)$. Then the relation (4.4.13) follows immediately from the Riemann-Roch theorem 3.1.60. ◻

Among the line bundles, the spin bundles deserve special attention.

Definition 4.4.7. *A holomorphic line bundle $L = L[D]$ with* $\deg D = g-1$, *and such that the*

$$2D = K,$$

where K is the canonical divisor, is called holomorphic spin bundle. Its holomorphic section are called spinors or theta-charateristics.

Remark 4.4.8. We observe that the Riemann-Roch theorem does not provide any information on such divisors. The dimension of the space of holomorphic sections of the corresponding line bundles is obtained using the theory of theta-functions.

Example 4.4.9. Let $e = q + pB$ be an odd half integer charactheristic. Then $\theta(e) = 0$ and

$$e = \mathcal{A}(D) - \mathcal{K}, \quad 0 = 2e = 2\mathcal{A}(D) - 2\mathcal{K}$$

where by Theorem 4.3.1 $D = P_1 + \cdots + P_{g-1}$ is a positive divisor of degree $g-1$. But we also know from Remark 4.3.3 that $2\mathcal{K} = \mathcal{A}(K)$ so that $2D = K$.

On the other hand differentiating $\theta(\mathcal{A}(D) - \mathcal{K}) \equiv 0$ with respect to P_k we obtain

$$\sum_{i=1}^{g} \frac{\partial\theta(\mathcal{A}(D) - \mathcal{K})}{\partial z_i}\omega_i(P_k) = 0$$

So we have found that

$$\omega = \sum_{i=1}^{g} \frac{\partial\theta(e)}{\partial z_i}\omega_i(P)$$

is a holomorphic differential with zeros in D. Since $2D = K$ we have that ω has double zeros in D.

Proposition 4.4.10. *There exists 4^g non equivalent holomorphic spin bundles on a Riemann surface of genus g.*

Proof. Let D be the divisor of the spin bundle. Observe that for any base point P_0, the Abel map gives the identity

$$2A_{P_0}(D) = A_{P_0}(K).$$

From Remark 4.3.3 one obtains

$$2A_{P_0}(D) - 2\mathcal{K}_{P_0} = 0.$$

Therefore there is a half integer characteristics $e = q + pB$, $q = (q_1, \ldots, q_g)$ and $p = (p_1, \ldots, p_g)$, with q_j and p_j in $\mathbb{Z}_2$ such that

$$e = A_{P_0}(D_e) - \mathcal{K}_{P_0}.$$

Since there are 4^g half-periods e, it follows from the Jacobi inversion theorem, that there exists 4^g non equivalent divisor D_e such that $2D_e = K$. ◻

We observe that 0 is an even half integer characteristics. Therefore, there is a divisor D_0 such that

$$0 = A_{P_0}(D_0) - \mathcal{K}_{P_0},$$

namely, the vector of Riemann constants $\mathcal{K}_{P_0} = A_{P_0}(D_0)$. This relation gives the clear dependence of the vector of Riemann constants on the choice of the base point and the canonical homology basis. Since $\theta(0; B) \neq 0$ it follows from Theorem 4.3.1, that the corresponding divisor D_0 is not a positive divisor.

Lemma 4.4.11 (Fay)**.** *The dimension of the space of holomorphic sections of the spin bundle $L[D_e]$, where e is an half integer characteristics is given by*

$$h^0(L[D_e]) = mult_{z=e}\theta(z;B).$$

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