

Integrable Systems and Riemann Surfaces

Lecture Notes (preliminary version)

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1 KdV equation and Schrödinger operator

1.1 Integrability of Korteweg – de Vries equation

Let $u := u(x, t)$ be a function on $x \in \mathbb{R}$ depending on a *time* parameter t . We will denote u_x, u_{xx} etc. the derivatives with respect to x , u_t will stand for the time derivative. With these notations the Korteweg – de Vries (KdV) equation is the following partial differential equation

$$u_t = 6uu_x - u_{xxx}. \tag{1.1.1}$$

Using linear transformations ($x \mapsto \alpha x, t \mapsto \beta t$ and $u \mapsto \gamma u$) we can change the coefficients as we want. Another standard form often used in physics is

$$u_t + uu_x + u_{xxx} = 0.$$

A deep relationship of the KdV equation to the spectral theory of the Schrödinger operator

$$L = -\partial_x^2 + u(x) \tag{1.1.2}$$

was discovered in 1967 by Gardner, Green, M. Kruskal and R.Miura (we will often write ∂_x instead of the derivative operator $\frac{d}{dx}$). Namely, let the potential $u = u(x, t)$ of L depend on t according to the KdV equation. The key observation is the following: the spectrum of the

operator remains invariant under time (the so-called *isospectrality*). To show this invariance, we use the following remarkable identity.

Theorem 1.1.1. *The KdV equation is equivalent to the following operator equation*

$$L_t = [L, A] \quad (1.1.3)$$

where L is the Schrödinger operator (1.1.2) and

$$A = 4\partial_x^3 - 6u\partial_x - 3u_x. \quad (1.1.4)$$

Here

$$[L, A] = LA - AL$$

is the commutator of the differential operators.

Proof Note that the Schrödinger operator depends on time through its potential $u = u(x, t)$. So the time derivative in the left hand side of (1.1.3) reduces to

$$L_t = u_t.$$

Therefore to complete the proof of the Theorem it suffices to establish validity of the following

Lemma 1.1.2. *The commutator of the differential operators L and A is the operator of multiplication by a function:*

$$[L, A] = 6uu_x - u_{xxx}. \quad (1.1.5)$$

Proof We have

$$\begin{aligned} [L, A] &= [-\partial_x^2 + u, 4\partial_x^3 - 6u\partial_x - 3u_x] = \\ &= 6[\partial_x^2, u\partial_x] + 3[\partial_x^2, u_x] + 4[u, \partial_x^3] - 6[u, u\partial_x] \end{aligned}$$

since ∂_x^i and ∂_x^j commutes and u for all i and j and the operators of multiplication by functions u, u_x etc. commute too. Let us compute the commutators applying them to a sample function f :

$$\begin{aligned} [\partial_x^2, u\partial_x]f &= u_{xx}f_x + 2u_x f_{xx} + \cancel{u f_{xxx}} - \cancel{u f_{xxx}}, \\ [\partial_x^2, u_x]f &= u_{xxx}f + 2u_{xx}f_x + \cancel{u_x f_{xx}} - \cancel{u_x f_{xx}}, \\ [u, \partial_x^3]f &= \cancel{u f_{xxx}} - \cancel{u f_{xxx}} - 3u_x f_{xx} - 3u_{xx}f_x - u_{xxx}f, \\ [u, u\partial_x]f &= u(u\partial_x f) - u\partial_x(uf) = \cancel{u^2 f_x} - \cancel{u^2 f_x} - uu_x f = -uu_x f; \end{aligned}$$

so we have

$$\begin{aligned} [\partial_x^2, u\partial_x] &= u_{xx}\partial_x + 2u_x\partial_x^2, \\ [\partial_x^2, u_x] &= u_{xxx} + 2u_{xx}\partial_x, \\ [u, \partial_x^3] &= -3u_x\partial_x^2 - 3u_{xx}\partial_x - u_{xxx}, \\ [u, u\partial_x] &= -uu_x. \end{aligned}$$

Applying these results we obtain:

$$\begin{aligned} [L, A] &= 6(\cancel{u_{xx}\partial_x} + \cancel{2u_x\partial_x^2}) + 3(u_{xxx} + \cancel{2u_{xx}\partial_x}) - \\ &\quad - 4(\cancel{3u_x\partial_x^2} + \cancel{3u_{xx}\partial_x} + u_{xxx}) + 6uu_x = \\ &= -u_{xxx} + 6uu_x = u_t; \end{aligned}$$

where the last equality coincides with the KdV equation. \square

The operator equation (1.1.3) is called *Lax representation* of the KdV equation.

Now we are ready to prove isospectrality.

Corollary 1.1.3. *Let λ be an eigenvalue of the Schrödinger operator L satisfying (1.1.3) and $\psi \in L^2(-\infty, +\infty)$ the corresponding eigenfunction,*

$$L\psi = \lambda\psi, \quad (\psi, \psi) := \int_{-\infty}^{+\infty} |\psi|^2 dx < +\infty.$$

Then $\dot{\lambda} = 0$.

Here the dot above stands for the time derivative, so

$$\dot{L} \equiv L_t = 6uu_x - u_{xxx}.$$

Proof. Differentiating the equation $L\psi = \lambda\psi$ in time we obtain

$$\dot{L}\psi + L\dot{\psi} = \dot{\lambda}\psi + L\dot{\psi}.$$

Replacing \dot{L} with $[L, A]$ and reorganizing terms, we have

$$\lambda(\dot{\psi} + A\psi) = L(\dot{\psi} + A\psi) + \dot{\lambda}\psi.$$

Taking the inner product by ψ we obtain

$$\lambda(\psi, \dot{\psi} + A\psi) = (\psi, L(\dot{\psi} + A\psi)) + \dot{\lambda}(\psi, \psi)$$

and using the fact that L is self-adjoint, we move it to the left of the inner product and we cancel two terms. So we obtain $\dot{\lambda}(\psi, \psi) = 0$, i.e. $\dot{\lambda} = 0$. \square

In other words, any eigenvalue of discrete spectrum of L is a *first integral* of the KdV equation. When saying this we consider KdV as a dynamical system in a suitable space of functions $u(x)$ (in this setting the space of smooth functions on the real line rapidly decreasing at infinity, see next section for the precise description of the functional space). The integral curve $u(x, t)$ passing through the given point $u_0(x)$ of the functional space is obtained by solving the Cauchy problem

$$\begin{aligned} u_t &= 6uu_x - u_{xxx} \\ u(x, 0) &= u_0(x). \end{aligned} \tag{1.1.6}$$

One can easily derive isospectrality also for periodic functions (see Section 1.6 below).

1.2 Elements of scattering theory for the Schrödinger operator

Informally speaking the *scattering* describes the result of passing of plane waves $\psi \sim e^{\pm ikx}$ through the field of the potential $u(x)$, from $x = -\infty$ to $x = +\infty$. The simplest way to define the scattering is the case of a compact support potential,

$$u(x) = 0 \quad \text{for } |x| > N.$$

In this case we have a pair of linearly independent solutions

$$L\phi_{1,2}(x, k) = k^2\phi_{1,2}(x, k), \quad \phi_{1,2}(x, k) = e^{\pm ikx} \quad \text{for } x < -N$$

and

$$L\psi_{1,2}(x, k) = k^2\psi_{1,2}(x, k), \quad \psi_{1,2}(x, k) = e^{\mp ikx} \quad \text{for } x > N$$

for any real $k \neq 0$. The *scattering matrix* is the transition matrix

$$\begin{aligned} \phi_1(x, k) &= a_{11}(k)\psi_1(x, k) + a_{21}(k)\psi_2(x, k) \\ \phi_2(x, k) &= a_{12}(k)\psi_1(x, k) + a_{22}(k)\psi_2(x, k) \end{aligned}$$

between these two bases in the space of solutions of the second order ordinary linear differential equation

$$-\psi'' + u(x)\psi = k^2\psi.$$

As the space and the bases depend on k , the transition matrix depends on k either. It is easy to see that this matrix is unimodular and satisfies

$$a_{22}(k) = \bar{a}_{11}(k), \quad a_{21}(k) = \bar{a}_{12}(k), \quad k \in \mathbb{R}$$

(bar stands for the complex conjugation), see below for the details.

Let us now explain how to extend this definition for the case of non-localized potentials decaying at $|x| \rightarrow \infty$.

Let $u(x)$ be a smooth real function on the real line $x \in (-\infty, +\infty)$ (for the moment not depending on t) such that $|u(x)| \rightarrow 0$ for $|x| \rightarrow +\infty$. Moreover we assume that

$$\int_{-\infty}^{+\infty} (1 + |x|)|u(x)|dx < +\infty.$$

Under these assumptions the discrete spectrum of the operator

$$L = -\frac{d^2}{dx^2} + u(x)$$

consists of a finite number of negative eigenvalues

$$\begin{aligned} \lambda_1 &< \dots < \lambda_n < 0 \\ L\psi_s &= \lambda_s\psi_s \quad \text{with } \psi_s \in \mathcal{L}^2(-\infty, +\infty) \\ \text{that is, } &\int_{-\infty}^{+\infty} \psi_s^2(x)dx < +\infty. \end{aligned}$$

The continuous spectrum of the operator L coincides with the positive real line. The so-called *Jost solutions*¹ are defined in the following way. Let λ be a positive real number; let us choose a basis in the two-dimensional space of solutions of $L\psi = \lambda\psi$. Introduce $k \in \mathbb{R}$ such that $k^2 = \lambda$ and fix two solutions for every k (to one λ corresponds two k , positive and negative). The solutions $\psi_1(x, k)$ and $\psi_2(x, k)$ are chosen to satisfy

$$\psi_1(x, k) \sim e^{-ikx} + o(1) \quad \text{for } x \rightarrow +\infty \quad \text{and} \quad \psi_2(x, k) \sim e^{ikx} + o(1) \quad \text{for } x \rightarrow +\infty. \quad (1.2.1)$$

Lemma 1.2.1. *For every $k \in \mathbb{R}$, there exist exactly two solutions with the chosen asymptotical behaviour (1.2.1). Moreover, the solution ψ_2 extends analytically to the upper half plane ($\Im k > 0$) and*

$$\psi_2(x, k)e^{-ikx} = 1 + O\left(\frac{1}{k}\right) \quad \text{for } |k| \rightarrow +\infty$$

Proof. We use the Picard's method reducing the differential equation

$$\psi'' + k^2\psi = u\psi \quad (1.2.2)$$

plus the asymptotic conditions of the form (1.2.1) to an integral equation.

1. Solve the homogeneous equation $\psi'' + k^2\psi = 0$: $\psi = a_1e^{ikx} + a_2e^{-ikx}$.
2. Use variation of constants to solve the inhomogeneous equation $\psi'' + k^2\psi = f$ with $f = u\psi$: let $a_j = a_j(x)$, these functions have to be determined from the linear system

$$\begin{cases} a_1'e^{ikx} - a_2'e^{-ikx} &= \frac{1}{ik}f \\ a_1'e^{ikx} + a_2'e^{-ikx} &= 0 \end{cases}$$

The solution reads $a_1' = \frac{1}{2ik}e^{-ikx}f$ and $a_2' = -\frac{1}{2ik}e^{ikx}f$. So

$$\begin{aligned} a_1(x) &= a_1^0 + \frac{1}{2ik} \int_{x_0}^x f(y)e^{-iky} dy \\ a_2(x) &= a_2^0 - \frac{1}{2ik} \int_{x_0}^x f(y)e^{iky} dy \end{aligned}$$

and

$$\psi(x) = \frac{1}{2ik} \int_{x_0}^x e^{ik(x-y)} u(y)\psi(y) dy - \frac{1}{2ik} \int_{x_0}^x e^{ik(y-x)} u(y)\psi(y) dy + a_1^0 e^{ikx} + a_2^0 e^{-ikx}.$$

3. We fix the basepoint $x_0 = +\infty$ and set the integration constants as $a_1^0 = 1$, $a_2^0 = 0$, to have the desired behaviour at infinity

$$\psi \sim e^{ikx}.$$

¹They are also the generalized eigenfunctions of the continuous spectrum of L

We arrive at the following integral equation for the function $\psi(x) := \psi_2(x, k)$:

$$\psi(x) = e^{ikx} - \int_x^{+\infty} \frac{\sin k(x-y)}{k} u(y) \psi(y) dy. \quad (1.2.3)$$

The solution to (1.2.3) is represented by the sum of a uniformly convergent series

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots, \quad \psi_0 = e^{ikx}$$

$$\psi_{n+1}(x) = - \int_x^{+\infty} \frac{\sin k(x-y)}{k} u(y) \psi_n(y) dy$$

$$|\psi_n(x)| \leq \frac{1}{n!} U^n(x) \quad \text{where} \quad U(x) := \frac{1}{k} \int_x^{+\infty} |u(y)| dy.$$

It is easy to see that the solution $\psi(x)$ satisfies the differential equation (1.2.2) and

$$|\psi(x) - e^{ikx}| \leq |e^{U(x)} - 1| \rightarrow 0 \quad \text{for} \quad x \rightarrow +\infty.$$

Observe that $\psi_1 := \bar{\psi}_2$. So the above considerations also prove existence and uniqueness of the solution $\psi_1(x)$.

We will now prove analyticity of $\psi_2(x, k)$ for $\Im k > 0$. Replace $\psi(x)$ with $e^{ikx} \chi(x)$ and prove analyticity of $\chi(x)$ satisfying:

$$\begin{aligned} \chi(x) &= 1 - \int_x^{+\infty} e^{-ikx} \frac{\sin k(x-y)}{k} u(y) e^{iky} \chi(y) dy \\ &= 1 - \int_x^{+\infty} \frac{1 - e^{2ik(y-x)}}{2ik} u(y) \chi(y) dy; \end{aligned}$$

and now $|e^{2ik(y-x)}| = e^{2\Im k(x-y)} \rightarrow 0$ for $|k| \rightarrow \infty$, $\Im k > 0$ since $x - y < 0$. Solving the above integral equation by iteration we easily prove the needed analyticity in k of the solution χ . \square

We found that for every k there is a unique solution of $L\psi = k^2\psi$, with the prescribed asymptotic behaviour as $x \rightarrow +\infty$. Similarly we can prove existence and uniqueness of two solutions $\phi_1(x, k)$, $\phi_2(x, k)$ with the following behaviour at $x \rightarrow -\infty$:

$$\phi_1 \sim e^{-ikx} \quad \text{and} \quad \phi_2 \sim e^{ikx}.$$

Using similar arguments we can also prove that ϕ_1 extends analytically to $\Im k > 0$.

Lemma 1.2.2. *The functions $\psi := \psi_2$ and $\bar{\psi}$, for $k \neq 0$, form a basis in the space of solutions. Similarly for $\phi := \phi_1$ and $\bar{\phi}$.*

Proof. We compute the Wronskian of $\bar{\psi}$ and ψ , i.e.

$$W(\psi, \bar{\psi}) := \psi' \bar{\psi} - \bar{\psi}' \psi.$$

This does not depend on x so we can compute it for $x \rightarrow +\infty$: $W(\psi, \bar{\psi}) = 2ik \neq 0$. \square

In particular we got two bases in the space of solutions, $(\bar{\psi}, \psi)$ and $(\phi, \bar{\phi})$. We can define the transition matrix between these two bases (expressing ϕ in terms of ψ):

$$\phi(x, k) = a(k)\bar{\psi}(x, k) + b(k)\psi(x, k).$$

Taking conjugates we obtain $\bar{\phi}(x, k) = \bar{b}(k)\bar{\phi}(x, k) + \bar{a}(k)\psi(x, k)$. This gives the *scattering matrix*

$$\begin{pmatrix} a(k) & b(k) \\ \bar{b}(k) & \bar{a}(k) \end{pmatrix}$$

$$(\phi, \bar{\phi}) = (\bar{\psi}, \psi) \begin{pmatrix} a(k) & b(k) \\ \bar{b}(k) & \bar{a}(k) \end{pmatrix}.$$

Lemma 1.2.3. *The determinant of the scattering matrix is 1 (i.e. the scattering matrix is unimodular):*

$$|a(k)|^2 - |b(k)|^2 = 1.$$

Proof. This follows from the fact that the Wronskians of $(\bar{\psi}, \psi)$ and $(\phi, \bar{\phi})$ are the same, since W is an invariant skew symmetric bilinear form. \square

Lemma 1.2.4. *$a(k)$ can be analytically extended to $\Im k > 0$.*

Proof. The Wronskian of ψ and ϕ is $a(k)W(\psi, \bar{\psi}) + b(k)W(\psi, \psi)$ so that $a(k) = \frac{1}{2ik}W(\psi, \phi)$ and this Wronskian can be analytically extended to $\Im k > 0$. \square

Moreover, in the upper half plane, $a(k) \sim 1 + O(\frac{1}{k})$ as $|k| \rightarrow \infty$. Therefore, $a(k)$ has at most a finite number of zeroes in the upper half plane. We will see that these zeroes are related to the discrete spectrum.

Lemma 1.2.5. *We have $a(k) = 0$ if and only if there exists a solution to $L\psi = k^2\psi$ that is exponentially decaying at infinity (and therefore is in $L^2(-\infty, +\infty)$).*

Proof. If $a(k) = 0$, then $W(\psi, \phi) = 0$, so ϕ is proportional to ψ . But $\phi \sim e^{-ikx}$ as $x \rightarrow -\infty$ and $\psi \sim e^{ikx}$ as $x \rightarrow +\infty$. In the upper half plane $\Im k > 0$ we have

$$|e^{-ikx}| = e^{\Im kx} \rightarrow 0 \quad \text{for } x \rightarrow -\infty.$$

A similar exponential decay takes place for $|e^{ikx}| = e^{-\Im kx}$ for $x \rightarrow +\infty$. \square

So zeroes of $a(k)$ correspond to eigenvalues of the discrete spectrum: $a(k) = 0$ if and only if $\lambda = k^2$ is a point of the discrete spectrum. Since λ is real negative, so k must be an imaginary number, with positive imaginary part. Denote these zeroes $i\kappa_1, \dots, i\kappa_n$, with $\kappa_1 > \dots > \kappa_n > 0$ for some $n \geq 0$ (the discrete spectrum is empty for $n = 0$). Then $\lambda_i := -\kappa_i^2$ are the eigenvalues of the discrete spectrum of L . The eigenfunctions of the discrete spectrum are $\phi_s(x) := \phi(x, i\kappa_s)$ and we have

$$\phi_s(x) = \begin{cases} e^{\kappa_s x} & x \rightarrow -\infty \\ b_s e^{-\kappa_s x} & x \rightarrow +\infty \end{cases}$$

for some $b_s \in \mathbb{R}$. One can show that the signs of the real constants b_1, \dots, b_n alternate:

$$(-1)^{s-1} b_s > 0, \quad s = 1, \dots, n.$$

So, from the original problem we derive these *scattering data*:

1. the reflection coefficient $r(k) := \frac{b(k)}{a(k)}$, for $k \in \mathbb{R}$;
2. $\kappa_1, \dots, \kappa_n$
3. b_1, \dots, b_n .

Example 1.2.6. Let us consider the Schrödinger operator with delta-potential

$$L = -\partial_x^2 + \alpha \delta(x), \quad \alpha \in \mathbb{R}.$$

Here $\delta(x)$ is the Dirac delta-function:

$$\int_{-\infty}^{\infty} f(x)\delta(x) dx = f(0)$$

for any smooth function $f(x)$ rapidly decreasing at infinity. For two continuous functions $f_1(x), f_2(x)$ on \mathbb{R} smooth outside $x = 0$ the following simple identity holds true

$$\begin{aligned} & \int_{-\infty}^{\infty} [f_1(x)Lf_2(x) - f_2(x)Lf_1(x)] dx \\ &= f_1(0) [\alpha f_2(0) - f_2'(0+) + f_2'(0-)] - f_2(0) [\alpha f_1(0) - f_1'(0+) + f_1'(0-)]. \end{aligned}$$

So the eigenfunctions ψ of the operator L must satisfy [?]

$$\begin{aligned} -\psi''(x) &= \lambda \psi(x) \quad \text{for } x \neq 0 \\ \psi'(0+) - \psi'(0-) &= \alpha \psi(0). \end{aligned} \tag{1.2.4}$$

It is easy to see that for any negative α the operator L has exactly one eigenvalue of the discrete spectrum

$$\lambda = -\frac{\alpha^2}{4}, \quad \psi = e^{\frac{\alpha}{2}|x|}.$$

For $\alpha > 0$ the discrete spectrum is empty. The generalized eigenfunctions of the continuous spectrum can also be constructed explicitly: for any $k \in \mathbb{R}$

$$\begin{aligned} \phi(x, k) &= \frac{2k + i\alpha}{2k} e^{-ikx} - \frac{i\alpha}{2k} e^{ik|x|} \\ \psi(x, k) &= \frac{2k + i\alpha}{2k} e^{ikx} - \frac{i\alpha}{2k} e^{ik|x|}. \end{aligned}$$

Indeed, these functions satisfy (1.2.4) with $\lambda = k^2$ and

$$\begin{aligned} \phi(x, k) &= e^{-ikx}, \quad x < 0 \\ \psi(x, k) &= e^{ikx}, \quad x > 0. \end{aligned}$$

This gives

$$a(k) = \frac{2k + i\alpha}{2k}, \quad b(k) = -\frac{i\alpha}{2k}.$$

Thus the reflection coefficient of the potential $u(x) = \alpha \delta(x)$ is equal to

$$r(k) = -\frac{i\alpha}{2k + i\alpha}.$$

For negative α one has to add the numbers

$$\kappa = -\frac{\alpha}{2}, \quad b = 1$$

associated with the discrete spectrum in order to complete the list of scattering data.

We have constructed the *scattering map*

$$\{\text{potential } u(x)\} \mapsto \{\text{scattering data } (r(k), \kappa_1, \dots, \kappa_n, b_1, \dots, b_n)\}$$

It will be later shown that, under certain analytic assumptions about the reflection coefficients, the scattering map is invertible (see the next section). Let us now describe the time dependence of the scattering data assuming that the potential $u = u(x, t)$ depends on time t according to the KdV equation.

We have seen that the KdV equation $u_t = 6uu_x - u_{xxx}$ is equivalent to $\dot{L} = [L, A]$ where $L = -\partial_x^2 + u$ and $A = 4\partial_x^3 - 6u\partial_x - 3u_x$.

The next theorem describes the scattering data.

Theorem 1.2.7. *If $u := u(x, t)$ satisfies the KdV equation, then*

1. $\dot{r}(k) = 8ik^3r(k)$,
2. $\dot{\kappa}_s = 0$,
3. $\dot{b}_s = 8\kappa_s^3b_s$,

for $s \in \{1, \dots, n\}$. So we have:

1. $r(k) = r_0(k)e^{8ik^3t}$,
2. $\kappa_s = \kappa_s(0)$,
3. $b_s = b_s(0)e^{8\kappa_s^3b_s t}$.

Proof. In the first lecture using the KdV equation represented in the form $\dot{L} = [L, A]$ and differentiating $L\psi = \lambda\psi$ in time, we derived the following identity

$$L(\dot{\psi} + A\psi) = \lambda(\dot{\psi} + A\psi) + \dot{\lambda}\psi.$$

We use these formulas in the following.

We take $\lambda := k^2$ so that $k \in \mathbb{R}$ is fixed; by definition, $\dot{\lambda} = 0$, so $\dot{\psi} + A\psi$ is again an eigenfunction for λ and must be a linear combination of $\bar{\psi}$ and ψ :

$$\dot{\psi} + A\psi = \alpha\bar{\psi} + \beta\psi,$$

with $\alpha := \alpha(k)$ and $\beta := \beta(k)$. The behaviour of this new solution at $x \rightarrow +\infty$ is $-4ik^3 e^{ikx}$; the behaviour of the right side is $\alpha e^{-ikx} + \beta e^{ikx}$ so we must have $\alpha = 0$ and $\beta = -4ik^3$. In the same way we get these results:

$$\begin{aligned}\dot{\psi} + A\psi &= -4ik^3\psi, & \dot{\bar{\psi}} + A\bar{\psi} &= 4ik^3\bar{\psi}, \\ \dot{\phi} + A\phi &= 4ik^3\phi, & \dot{\bar{\phi}} + A\bar{\phi} &= -4ik^3\bar{\phi}.\end{aligned}$$

We consider $\phi = a\bar{\psi} + b\psi$ and take time derivative: $\dot{\phi} = \dot{a}\bar{\psi} + \dot{b}\psi + a\dot{\bar{\psi}} + b\dot{\psi}$. So

$$\dot{\phi} + A\phi = \dot{a}\bar{\psi} + \dot{b}\psi + a(\dot{\bar{\psi}} + A\bar{\psi}) + b(\dot{\psi} + A\psi)$$

and substituting what we found before, we have $4ik^3(a\bar{\psi} + b\psi) = \dot{a}\bar{\psi} + \dot{b}\psi + 4ik^3 a\bar{\psi} - 4ik^3 b\psi$. Elaborating this we get differential equation for a and b , obtaining $\dot{a} = 0$ and $\dot{b} = 8ik^3 b$, from which we get $\dot{r}(k)$.

For the last statement, we do the same trick with $\dot{\phi}_s + A\phi_s = 4\kappa_s^3 \phi_s$. \square

1.3 Inverse scattering

The direct scattering problem is to compute the scattering data from the given $u(x)$; so the inverse scattering is the problem to reconstructing $u(x)$ from the scattering data $(r(k), \kappa_1, \dots, \kappa_n, b_1, \dots, b_n)$. Let us discuss the properties of the scattering data.

Starting from the reflection coefficient $r(k)$:

1. it is a function defined on the real line satisfying the symmetry $r(-k) = r(\bar{k})$. Indeed, the substitution $k \mapsto -k$ exchanges the roles of ψ with $\bar{\psi}$, and ϕ with $\bar{\phi}$; moreover, $r(k) \sim O(\frac{1}{k})$ for $|k| \rightarrow +\infty$;
2. $|r(k)| < 1$, for every $k \in \mathbb{R} \setminus \{0\}$, since the scattering matrix is unimodular;
3. the Fourier transform

$$\hat{r}(x) := \frac{1}{2\pi i} \int_{-\infty}^{+\infty} r(k) e^{ikx} dk;$$

satisfies $\int_{-\infty}^{+\infty} (1 + |x|) |\hat{r}(x)| dx < +\infty$.

For the discrete spectrum $\kappa_1, \dots, \kappa_n$ and the b_1, \dots, b_n we do not have many constraints: $\kappa_1 > \dots > \kappa_n > 0$ are real, b_1, \dots, b_n are real and non zero and we will see that sign $b_s = (-1)^{s-1}$.

The first question now is how to reconstruct the functions $a(k)$ and $b(k)$ from the scattering data. We have

$$|a(k)| = \frac{1}{\sqrt{1 - |r(k)|^2}},$$

so we must find the argument of $a(k)$. We define

$$\tilde{a}(k) := a(k) \frac{\prod(k + ik_s)}{\prod(k - ik_s)};$$

this is again analytic in the upper half plane (since zeroes of the denominator cancel with zeroes of $a(k)$); for $k \in \mathbb{R}$, the modulus of $\tilde{a}(k)$ is equal to the modulus of $a(k)$, since the rational function

$$\frac{\prod(k + ik_s)}{\prod(k - ik_s)}$$

is unimodular for $k \in \mathbb{R}$. Moreover, $\tilde{a}(k)$ has no zeroes in the upper half plane and still behaves like $1 + O(\frac{1}{k})$. We can reconstruct now the argument of $a(k)$ using the Cauchy integral applied to $\log \tilde{a}(k)$:

$$\arg \tilde{a}(k) = -\frac{1}{\pi} \text{v.p.} \int_{-\infty}^{+\infty} \frac{\log |\tilde{a}(k')|}{k' - k} dk'$$

and then

$$\arg a(k) = \frac{1}{i} \sum \log \frac{k - ik_s}{k + ik_s} - \frac{1}{\pi} \text{v.p.} \int_{-\infty}^{+\infty} \frac{\log |a(k')|}{k' - k} dk'.$$

The next will be:

1. define

$$F(x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} r(k) e^{ikx} dk + \sum_{s=1}^n \frac{b_s e^{-\kappa_s x}}{ia'(i\kappa_s)}$$

where $a'(k) := \frac{d}{dk} a(k)$;

2. solve the Gelfand-Levitan-Marchenko integral equation for the function $K = K(x, y)$:

$$K(x, y) + F(x + y) + \int_{-\infty}^{+\infty} K(x, z) F(z + y) dz = 0;$$

3. prove that $u(x) = -2 \frac{d}{dx} K(x, x)$.

This procedure comes from the theory of the so-called transformation operators: as an example, start from $L_0 := -\partial_x^2$ and go to $-\partial_x^2 + u(x)$; from the basis of solution of L_0 , $(e^{ikx})_{k \in \mathbb{R}}$ we can go to the basis $(\psi(x, k))_{k \in \mathbb{R}}$ of solutions for L . Remarkably the matrix of the transformation operator is (upper) triangular! The following general statement from the theory of Fourier integrals is useful for establishing the triangularity.

Lemma 1.3.1. *If $f(k)$ is analytic in the lower half plane and behaves like $O(\frac{1}{k})$ for $|k| \rightarrow +\infty$, then the Fourier transform*

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(k) e^{ikx} dk$$

is zero for $x < 0$, and viceversa.

Proof. The shift $k \mapsto k - ia$ with $a > 0$ changes the exponential from e^{ikx} to e^{ikx+ax} . Such a shift does not change the integral. Therefore the modulus $|\hat{f}(x)|$ for negative x admits an upper estimate as small as we want. \square

Let us denote $\psi_- := \bar{\psi}$. It admits an analytic continuation into the *lower* half plane $\Im k < 0$. Moreover the product

$$\chi_-(x, k) := \psi_-(x, k)e^{ikx}$$

admits an asymptotic expansion of the form

$$\chi_-(x, k) \sim 1 + O\left(\frac{1}{k}\right), \quad |k| \rightarrow \infty, \quad \Im k < 0.$$

Denote

$$A(x, y) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iky} (\chi_-(x, k) - 1) dk$$

the Fourier transform of $\chi_-(x, k) - 1$ with respect to k . Due to Lemma

$$A(x, y) = 0 \quad \text{for } y < 0.$$

Now taking the inverse transform we get

$$\chi_-(x, k) = 1 + \int_0^{+\infty} A(x, y) e^{-ikx} dy,$$

where the integral starts from 0 thanks to the lemma. Finally,

$$\begin{aligned} \psi_-(x, k) &= e^{-ikx} + \int_0^{+\infty} A(x, y) e^{-ik(x+y)} dy = \\ &= e^{-ikx} + \int_x^{+\infty} A(x, \tilde{y} - x) e^{-ik\tilde{y}} d\tilde{y}, \end{aligned}$$

changing variable ($\tilde{y} := y + x$). Denoting $K(x, y) := A(x, y - x)$, we get

$$\psi_-(x, k) = e^{-ikx} + \int_x^{+\infty} K(x, y) e^{-iky} dy.$$

Applying complex conjugation and $k \mapsto -k$ to this formula, the only thing that changes is the conjugation of the kernel $K(x, y)$; therefore it must be real for $y \geq x$.

Let us derive the GLM equation. From $\phi(x, k) = a(k)\bar{\psi}(x, k) + b(k)\psi(x, k)$, multiplying by $\frac{e^{iky}}{a(k)}$ and integrating with respect to k , we obtain

$$\int_{-\infty}^{+\infty} \frac{\phi(x, k)}{a(k)} e^{iky} dk = \int_{-\infty}^{+\infty} (\psi_-(x, k) + r(k)\psi(x, k)) e^{iky} dk.$$

Since these integrals will be not well defined, we subtract something:

$$\int_{-\infty}^{+\infty} \left(\frac{\phi(x, k)}{a(k)} - e^{ikx} \right) e^{iky} dk = \int_{-\infty}^{+\infty} (\psi_-(x, k) - e^{ikx} + r(k)\psi(x, k)) e^{iky} dk.$$

In the left hand side we have the fraction which has a finite number of simple poles (not yet proved, but will be); after the subtraction we have the desired behaviour at infinity ($O(\frac{1}{k})$) and so we can express the left hand side as a sum of residues for $k \in \{i\kappa_1, \dots, i\kappa_n\}$:

$$2\pi i \sum \frac{\phi(x, i\kappa_s)}{a'(i\kappa_s)} e^{-\kappa_s y}.$$

Now, $\phi(x, i\kappa_s) \sim b_s e^{i\kappa_s x}$ for $x \rightarrow +\infty$; but $\phi(x, i\kappa_s) = b_s \psi_-(x, -i\kappa_s)$ and

$$\psi_-(x, -i\kappa_s) = e^{\kappa_s x} + \int_x^{+\infty} K(x, y) e^{-\kappa_s z} dz.$$

Finally, the left hand side is

$$2\pi i \sum_{s=1}^n \frac{b_s e^{-\kappa_s x}}{a'(i\kappa_s)} + 2\pi i \int_x^{+\infty} K(x, z) \sum \frac{b_s e^{-\kappa_s(z+y)}}{a'(i\kappa_s)} dz.$$

From property of the Fourier transform on the right hand side, we justify the additional terms in the GLM formula.

From the integral equation for $\chi_-(x, k)$ we see that

$$\chi_-(x, k) = 1 + \frac{1}{2ik} \int_x^{+\infty} u(x) dx + O\left(\frac{1}{k^2}\right)$$

Comparing with

$$\chi_-(x, k) = 1 + \int_0^{+\infty} A(x, y) e^{-iky} dy$$

What we would like to prove is that $\frac{1}{2} \int_x^{+\infty} u(x) dx = K(x, x)$.

1.4 Dressing operator

We recall briefly some properties of the Fourier transform of an integrable function $f(x)$ with $x \in (-\infty, +\infty)$. The Fourier transform of f is

$$\hat{f}(k) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{ikx} dx,$$

where the coefficient is a normalization one that can be changed if needed. The inverse Fourier transform is

$$f(x) := \int_{-\infty}^{+\infty} \hat{f}(k) e^{-ikx} dk;$$

infact,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dy e^{-ik(x-y)} f(y) = \int_{-\infty}^{+\infty} dy f(y) \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik(x-y)} dk$$

and $\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik(x-y)} dy =: \delta(x-y)$.

We study now the decay at ∞ . If $f \in \mathcal{C}^{(m)}(\mathbb{R})$, then $\hat{f}(k) \sim O(|k|^{-m})$. Indeed, by integration by parts:

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(x)}{ik} de^{ikx} = \left[\frac{1}{2\pi ik} f(x) e^{ikx} \right]_{-\infty}^{+\infty} - \frac{1}{2\pi ik} \int_{-\infty}^{+\infty} f'(x) e^{ikx} dx.$$

We define now the transformation (or dressing) operator. We have $\psi(x, k) = e^{ikx} + \int_x^{+\infty} K(x, y) e^{iky} dy$ and $\psi_-(x, k) = e^{-ikx} + \int_x^{+\infty} K(x, y) e^{-iky} dy$. What is behaviour of ψ for $|k| \rightarrow +\infty$? We can repeat the same argument as before:

$$\psi(x, k) = e^{ikx} + \left[\frac{1}{ik} K(x, y) e^{iky} \right]_x^{+\infty} - \frac{1}{ik} \int_x^{+\infty} K_y(x, y) e^{iky} dy$$

but now the first term do not vanish since the lower limit is x . Iterating these arguments we can expand asymptotically $\psi(x, k)$ (obviously if K is infinitely differentiable in y). So we have

$$\psi(x, k) = \left(1 + \frac{\xi_1(x)}{ik} + \frac{\xi_2(x)}{(ik)^2} + \dots \right) e^{ikx}.$$

We observe that $\frac{1}{ik}$ is the integral of the exponential, so we may write

$$\frac{e^{ikx}}{ik} = \partial_x^{-1}(e^{ikx})$$

so that we have

$$\psi(x, k) = (1 + \xi_1(x)\partial_x^{-1} + \xi_2(x)\partial_x^{-2} + \dots)e^{ikx}.$$

The operator between parenthesis is called the *dressing operator* P . If ψ is a solution for the Schrödinger operator L , then $\psi = P\psi_0$, where ψ_0 is a solution for $L_0 = -\partial_x^2$. Then we could say that $L = PL_0P^{-1}$.

Now we use the dressing operator to obtain a different way to derive of the GLM equation. We have

$$\frac{\phi(x, k)}{a(k)} = \psi_-(x, k) + r(k)\psi(x, k);$$

since $a(k) \sim 1 + O(\frac{1}{k})$ and $\phi(x, k) \sim e^{-ikx}$, for $|k| \rightarrow +\infty$, then, as we did before, we must subtract e^{-ikx} to both side to obtain an integrable function. Then we can take the integral in k :

$$\int_{-\infty}^{+\infty} \frac{\phi(x, k)}{a(k)} - e^{-ikx} dk = \int_{-\infty}^{+\infty} \psi_-(x, k) - e^{-ikx} + r(k)\psi(x, k) dk.$$

We multiply both side by e^{iky} so that the left hand side is the sum of the residues relative to the k_s :

$$2\pi i \sum_{s=1}^n \frac{\phi(x, ik_s) - \frac{a(ik_s)e^{k_s x}}{a'(ik_s)}}{a'(ik_s)} e^{-k_s y};$$

but now we can rewrite it using the dressing operator. Substituting $\phi(x, ik_s)$ with $b_s\psi(x, ik_s)$ and $\psi(x, ik_s)$ with $e^{-k_s x} + \int_x^{+\infty} K(x, z) e^{-k_s z} dz$, we obtain this form for the left hand side:

$$2\pi i \sum_{s=1}^n \frac{b_s}{a'(ik_s)} e^{-k_s(x+y)} + 2\pi i \int_x^{+\infty} K(x, z) \sum_{s=1}^n \frac{b_s}{a'(ik_s)} e^{-k_s(z+y)} dz.$$

As for the right hand side,

$$(\psi_-(x, k) - e^{-ikx})e^{iky} = \int_x^{+\infty} K(x, z)e^{-ik(z-y)} dz,$$

so integrating by k we have $2\pi \int_x^{+\infty} K(x, z)\delta(z-y)dz = 2\pi K(x, y)$. We have to add the last part of the right hand side, $\int_x^{+\infty} r(k)\psi(x, k)e^{iky}dk$, that we change again with the dressing operator: it is

$$\begin{aligned} \int_{-\infty}^{+\infty} r(k)e^{ik(y+x)}dk &= \int_{-\infty}^{+\infty} r(k)dk \int_{-\infty}^{+\infty} K(x, z)e^{ik(z+y)}dz = \\ &= 2\pi \hat{r}(x+y) + 2\pi \int_x^{+\infty} K(x, z)\hat{r}(z+y)dz. \end{aligned}$$

Summing up, we can divide all terms by 2π , obtaining the equation

$$\begin{aligned} i \sum_{s=1}^n \frac{b_s}{a'(ik_s)} e^{-k_s(x+y)} + i \int_x^{+\infty} K(x, z) \sum_{s=1}^n \frac{b_s}{a'(ik_s)} e^{-k_s(z+y)} dz = \\ = K(x, y) + \hat{r}(x+y) + \int_x^{+\infty} K(x, z)\hat{r}(z+y)dz; \end{aligned}$$

after moving i to the denominator, we get

$$\begin{aligned} \hat{r}(x+y) + \sum_{s=1}^n \frac{b_s}{ia'(ik_s)} e^{-k_s(x+y)} + \\ + K(x, y) + \int_x^{+\infty} K(x, z) \left(\hat{r}(z+y) + \sum_{s=1}^n \frac{b_s}{ia'(ik_s)} e^{-k_s(z+y)} \right) dz = 0. \end{aligned}$$

The sum of the first two terms is what we called before $F(x)$; after this substitution we have the GLM equation.

To derive the formula $u(x) = -2\frac{d}{dx}K(x, x)$, we observe that $\xi_1(x) = -K(x, x)$, so substituting ψ with $(1 + \frac{\xi_1(x)}{ik} + \dots)e^{ikx}$ in $\psi'' + u\psi = k^2\psi$ gives the formula.

The last thing that was left to prove is that the zeroes of $a(k)$ are simple, i.e. $a'(ik_s) \neq 0$. Starting from $L\phi = k^2\phi$, derivating by k we have $L\phi' = k^2\phi' - 2k\phi$ so $(L + k_s^2)\phi'(x, ik_s) = -2ik_s\phi(x, ik_s)$. Multiplying by $\phi(x, ik_s)$, integrating in x and denoting $\phi := \phi(x, ik_s)$ and $\phi' := \phi'(x, ik_s)$, we obtain

$$\int_{-\infty}^{+\infty} (L + k_s^2)\phi\phi' dx = -2ik_s \int_{-\infty}^{+\infty} \phi^2 dx.$$

Integrating by parts we have

$$[\phi\phi'_x - \phi_x\phi']_{-\infty}^{+\infty} = -2ik_s \int_{-\infty}^{+\infty} \phi^2 dx.$$

We know that ϕ^2 is exponential decaying at ∞ ; as for the left hand side, ϕ has the same property and

$$\phi' \sim \begin{cases} xe^{k_s x} & x \rightarrow -\infty \\ a'(ik_s)e^{k_s x} + \dots & x \rightarrow +\infty \end{cases}$$

where the omitted terms are exponentially decaying. After working on the previous identity, we finally get

$$ia'(ik_s)b_s = \int_{-\infty}^{+\infty} \phi^2(x, ik_s) dx \in \mathbb{R}^+.$$

With this last statement we also proved that the sign of b_s are alternating.

1.5 Particular case: reflectionless potential

To solve the initial value problem for KdV in the class of rapidly decreasing initial data $u_0(x)$, we have to: solve the scattering problem (find $r(k)$ for the given potential $u_0(x)$, the k_s and the b_s); define the function

$$F(x, t) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} r(k)e^{ikx+8ik^3t} dk + \sum_{s=1}^n \frac{b_s e^{-k_s x + 8k_s^3 t}}{ia'(ik_s)};$$

then solve the GLM equation

$$K(x, y, t) + F(x + y, t) + \int_x^{+\infty} K(x, z, t)F(z + y, t) dz = 0;$$

finally compute $u(x, t) = -2\frac{d}{dx}K(x, x, t)$.

This is not really all computable. We'll try to solve a particular case, in which $r(k) = 0$ (this case was previously solved by Bargmann in 1949). In this case, the first term of $F(x, t)$ vanish and the integral equation can be solved explicetly. Forgetting for the moment about time dependance, we have

$$F(x) = \sum_{s=1}^n \frac{b_s}{ia'(ik_s)} e^{-k_s x}$$

and the fraction is a real positive coefficient that we'll denote with c_s . Then

$$\int_x^{+\infty} K(x, z) \sum_{s=1}^n e^{-k_s(z+y)} dz = \sum_{s=1}^n \tilde{c}_s(x) e^{-k_s y},$$

where $\tilde{c}_s(x) = c_s \int_x^{+\infty} K(x, z) e^{-k_s z} dz$. We look then for solutions like $K(x, y) = \sum_{i=1}^n K_i(x) e^{-k_i y}$: substituting what we know in the GLM equation we get

$$\sum_{i=1}^n K_i(x) e^{-k_i y} + \sum_{i=1}^n c_i e^{-k_i(x+y)} + \sum_{i=1}^n \int_x^{+\infty} K_i(x) e^{-k_i z} \sum_{j=1}^n c_j e^{-k_j(z+y)} dz = 0$$

and this last term is equal to

$$- \left[\sum_{i,j=1}^n K_i(x) \frac{e^{-(k_i+k_j)z}}{k_i+k_j} \right]_x^{+\infty} c_k e^{-k_j y}.$$

Now we can put the equation in the form of a linear system of n equations

$$K_i + \sum_{j=1}^n c_j \frac{e^{-(k_i+k_j)x}}{k_i+k_j} K_j = -c_i e^{-k_i x}, \quad i \in \{1, \dots, n\}.$$

So the matrix of the system is A such that

$$A_{i,j} = \delta_{i,j} + c_i \frac{e^{-(k_i+k_j)x}}{k_i+k_j}$$

and we can solve the system with Kramer's rule $K_i = \frac{\det A^{(j)}}{\det A}$.

After this we have to compute $K(x, x)$:

$$K(x, x) = \sum_{i=1}^n K_i(x) e^{-k_i x} = \sum_{i=1}^n \frac{\det A^{(j)}}{\det A} e^{-k_i x}.$$

We define another matrix \tilde{A} incorporating the exponential so that $K(x, x) = \sum_{j=1}^n \frac{\det \tilde{A}^{(j)}}{\det \tilde{A}}$: to do that, $\tilde{A}^{(j)}$ is obtained from A substituting the j -th column in this way:

$$\left(\begin{array}{c} c_1 \frac{e^{-(k_1+k_j)x}}{k_1+k_j} \\ 1 + c_j \frac{e^{-2k_j x}}{2k_j} \\ \vdots \\ c_n \frac{e^{-(k_n+k_j)x}}{k_n+k_j} \end{array} \right)_i \mapsto \left(\begin{array}{c} -c_1 e^{-(k_1+k_j)x} \\ -c_j e^{-2k_j x} \\ \vdots \\ -c_n e^{-(k_n+k_j)x} \end{array} \right)_i.$$

We observe that the substitution is really a differentiation so that we get

$$K(x, x) = \frac{\frac{d}{dx} \det A}{\det A} = \frac{d}{dx} \log \det A(x)$$

and then

$$u(x) = -2 \frac{d^2}{dx^2} \log \det A(x).$$

Studying the reflectionless case, we saw that $F(x)$ assume the form $\sum_{i=1}^n c_i e^{-k_i x}$ with c_i positive real constants. This allows us to solve the GLM equation as a system of n linear equation for the n unknowns $K_1(x), \dots, K_n(x)$. Then we arrive to the potential $u(x)$ as $-2 \frac{d^2}{dx^2} \log \det A$.

Now we'll inspect the dependency on time. We recall that c_s was defined as $-\frac{b_s}{ia(ik_s)}$: the numerator does depend on t , but the denominator does not. Including time dynamics we get $c_s \mapsto c_s e^{8k_s^3 t}$.

Exercise 1.5.1: Denote $\omega_i := k_i(x - 4k_i^2 t)$; let

$$\tilde{A}_{i,j} = \delta_{i,j} + \frac{c_i e^{-(\omega_i + \omega_j)}}{k_i + k_j};$$

prove that $u(x, t) = -2 \frac{d^2}{dx^2} \log \det \tilde{A}(x, t)$.

In the case $n = 1$, we have

$$A = 1 + \frac{ce^{-2kx+8k^3t}}{2k}$$

so that

$$\begin{aligned} \frac{d}{dx} \log A &= -\frac{ce^{-2kx+8k^3t}}{1 + \frac{ce^{-2kx+8k^3t}}{2k}} = \\ &= -2k \frac{\frac{c}{2k} e^{-2kx+8k^3t} + 1 - 1}{1 + \frac{ce^{-2kx+8k^3t}}{2k}} = -2k + \frac{2k}{1 + \frac{ce^{-2kx+8k^3t}}{2k}}; \end{aligned}$$

deriving again,

$$\begin{aligned} \frac{d^2}{dx^2} \log A &= \frac{2kce^{-2kx+8k^3t}}{\left(1 + \frac{ce^{-2kx+8k^3t}}{2k}\right)^2} = \frac{2kce^{-2\omega}}{e^{-2\omega} \left(e^\omega + \frac{c}{2k} e^{-\omega}\right)^2} = \\ &= \frac{2k\cancel{c}}{\frac{\cancel{c}}{2k} \left(\sqrt{\frac{2k}{c}} e^\omega + \sqrt{\frac{c}{2k}} e^{-\omega}\right)^2} = \frac{k^2}{\cosh^2(k(x - 4k^2t - x_0))} \end{aligned}$$

where $x_0 = \frac{1}{2k} \log \frac{c}{2k}$.

Then the solution of KdV is

$$u(x, t) = -\frac{2k^2}{\cosh^2(k(x - 4k^2t - x_0))}$$

which is called the *soliton solution*. This solution is moving to the right with constant speed $4k^2$; x_0 is just interpreted as a phase shift. For fixed t , the graph of the solution resemble the opposite of a bell; as time goes on, the bell travel to the right. Another way to see this is to find solution to the KdV in the form $u = u(x - ct)$.

Substituting in the KdV, we get $-cu' = 6uu' - u$, that is $-cu = 3u^2 - u'' + a$, i.e. $u'' = 3u^2 + cu + a$. This is the equation for the motion of a particle in a specific cubic potential $V := V(u)$ defined by $u'' = -\frac{\partial V(u)}{\partial u}$: the potential is then

$$V(u) = -u^3 - c\frac{u^2}{2} + au + \text{const.}$$

We can solve this equation using the conserving of energy:

$$\frac{(u')^2}{2} + V(u) = E$$

and we get the elliptic integral

$$\int \frac{du}{\sqrt{2(E - V(u))}} = x - x_0.$$

Now we have to use particular values of the integration constants to impose the decaying behaviour of the solution. In particular we have to choose the right energy level E in such

a way the particle can pass over the mountain of potential given by the cubic polynomial. If we choose a lower energy level, then the solution is periodic and is called a *cnoidal wave*; if the energy level is greater, then the solution does not decay at infinity.

If $n > 1$ we have a nonlinear interaction of solitons with different x_0 ; asymptotically they look like a sum of noninteracting solitons; but for finite time there may be nontrivial interactions deriving from the different velocities. In particular, at the beginning the amplest bell will be on the left (since its x_0 is lower), but at infinity it will be on the right (since its velocity is greater). If we denote with \tilde{x}_0^i the phase at a particular time, we get

$$\tilde{x}_0^i - x_0^i = \sum_{j|x_j < x_i} \Delta x^{i,j}, \quad \text{with } \Delta x^{i,j} := \frac{1}{2k_i} \log \left(\frac{k_i - k_j}{k_i + k_j} \right)^2.$$

1.6 Bloch spectrum of the Schrödinger operator with a periodic potential

We now consider smooth real periodic potentials

$$u(x+T) = u(x)$$

of the Schrödinger operator.

Definition 1.6.1. *The Bloch spectrum (or stability zone) is the set*

$$\lambda \in \mathbb{C} \quad \text{such that} \quad \exists \quad \text{solution to } L\psi = \lambda\psi, \text{ bounded } \forall x \in \mathbb{R}.$$

Theorem 1.6.2.

1. *The Bloch spectrum is a collection of (finite or infinite) real intervals $[\lambda_1, \lambda_2], [\lambda_3, \lambda_4], \dots$ with $\lambda_1 < \lambda_2 < \lambda_3 < \dots$. If there are only a finite number of intervals, then the last one is $[\lambda_{2n+1}, +\infty]$.*
2. *Consider the Riemann surface*

$$\Gamma := \{(\lambda, \nu) \in \mathbb{C}^2 \mid \nu^2 = \prod_{i \geq 1} (\lambda - \lambda_i)\}$$

(we will explain later how to manage the infinite interval case); then for any $x \in \mathbb{R}$, there exists a function $\psi(x, P)$ meromorphic in $P \in \Gamma$ such that:

- *its poles are not in some interval $(\lambda_{2i-1}, \lambda_{2i})$;*
- *the restriction of $\psi(x, P)$ to the internal part of the Bloch spectrum (i.e., $P = (\lambda, \nu)$ is such that $\lambda \in (\lambda_{2i-1}, \lambda_{2i})$) is a pair of independent solutions $\psi_t(x, \lambda)$ to $L\psi_t = \lambda\psi_t$ bounded for every $x \in \mathbb{R}$;*
- *$\psi(x, \lambda)$ has exponential behaviour at infinity.*

Example 1.6.3. If $u(x) = u_0$, then $L\psi = \lambda\psi$ corresponds to $\psi'' = (u_0 - \lambda)\psi$ and its solutions if $\lambda \neq u_0$ are $\psi_{\pm}(x, \lambda) := e^{\pm i\sqrt{\lambda - u_0}x}$. At least one solution is bounded for every $x \in \mathbb{R}$ if and only if $\sqrt{\lambda - u_0}$ is real; this means that $\lambda \in \mathbb{R}$ and $\lambda \geq u_0$. Hence the

Bloch spectrum is constituted by only one interval $[u_0, +\infty]$. If $\lambda > u_0$, both solutions are bounded; instead if $\lambda = u_0$, $\psi_1 = 1$ and $\psi_2 = x$ are the two solutions. The solutions ψ_{\pm} are not analytic on all the complex plane for λ , since we have a square root. But it is analytic on the Riemann surface given by $\nu^2 = \lambda - u_0$: infact we can rewrite the two functions as a single function $\psi_{\pm}(x, \nu) = e^{i\nu x}$. What we are doing is take two sheaves over the inner part of the interval glued at $\lambda = u_0$; the two-valued function ψ (considered as a function of λ) becomes a single-valued function (considered as a function of ν). In general, for every bounded zone in the Bloch spectrum, we obtain a circle.

We now define the *monodromy operator* as $\hat{T}\psi(x) = \psi(x + T)$. If we fix λ , we have a 2-dimensional solutions space to $L\psi = \lambda\psi$; we denote $\mu_{\pm}(\lambda)$ the two eigenvalues of \hat{T} in the solutions space (depending on λ).

Lemma 1.6.4. *The Bloch spectrum contains $\lambda \in \mathbb{C}$ if and only if there exists an eigenvalue $\mu(\lambda)$ of \hat{T} such that $|\mu(\lambda)| = 1$.*

Proof. If $|\mu(\lambda)| > 1$, then $|\psi(x + nT)| = |\mu(\lambda)|^n |\psi(x)|$ goes to infinity on the right; if $|\mu(\lambda)| < 1$, it goes to infinity to the left. \square

We have now to choose a basis for the space of solutions to $L\psi = \lambda\psi$. We fix $x_0 \in \mathbb{R}$ and let $c := c(x, x_0, \lambda)$ and $s := s(x, x_0, \lambda)$ be such that $c|_{x_0} = 1$, $c'|_{x_0} = 0$, $s|_{x_0} = 0$, $s'|_{x_0} = 1$. Any other solution y such that $y|_{x_0} = y_0$ and $y'|_{x_0} = y'_0$ is represented as $cy_0 + sy'_0$.

Example 1.6.5. If $u(x) = 0$, then $c = \cos \sqrt{\lambda}(x - x_0)$ and $s = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(x - x_0)$. In this case both functions are entire functions in λ (since cosine is an even function and sine is odd).

Lemma 1.6.6. *The functions c and s are always entire functions in λ .*

Proof. The function c as a function of x is determined by

$$c(x) = \cos \sqrt{\lambda}(x - x_0) + \int_{x_0}^x \frac{\sin \sqrt{\lambda}(x - y)}{\sqrt{\lambda}} u(y) c(y) dy;$$

for s , we have

$$\frac{\sin \sqrt{\lambda}(x - x_0)}{\sqrt{\lambda}} + \int_{x_0}^x \frac{\sin \sqrt{\lambda}(x - y)}{\sqrt{\lambda}} u(y) s(y) dy;$$

expanding this functions we get the analyticity. \square

We define the *monodromy matrix* as

$$T(x_0, \lambda) = \begin{pmatrix} c(x_0 + T, x_0, \lambda) & s(x_0 + T, x_0, \lambda) \\ c'(x_0 + T, x_0, \lambda) & s'(x_0 + T, x_0, \lambda) \end{pmatrix}$$

so that

$$(c(x + T, x_0, \lambda), s(x + T, x_0, \lambda)) = (c(x, x_0, \lambda), s(x, x_0, \lambda))T(x_0, \lambda).$$

As a corollary we get that the monodromy matrix T is composed by entire functions in $\lambda \in \mathbb{C}$.

We begin the study of the solution of the Schrödinger operator L in the case of a periodic data u of period T . For a given parameter $\lambda \in \mathbb{C}$ and a given $x_0 \in \mathbb{R}$, we defined a basis $(c = c(x, x_0, \lambda), s = s(x, x_0, \lambda))$ of the space of solutions of $L\psi = \lambda\psi$, given by two entire function in λ . We also defined the monodromy operator $\hat{T}y(x) = y(x + T)$ and the monodromy matrix $T(x_0, \lambda)$; the entries of this matrix are also entire in λ ; moreover, T is unimodular, because in the space of solutions we have an antisymmetric bilinear form (the Wronskian) which do not depend on x and in particular on the shift by a period. Let μ an eigenvalue of the monodromy operator and ψ an eigenvector (also called a Bloch function), so that $\psi(x + T) = \mu\psi(x)$.

To find μ we have to find the eigenvalues of $T(x_0, \lambda)$, i.e. the roots of $\det(\mu I - T) = 0$. To write down this equation, denote with Δ the half trace of T : it depends only on λ (and not on x_0), since changing the choosen basis (c, s) conjugate T with an invertible matrix A , but this do not change Δ . It is also an entire function and we can write the characteristic polynomial as $\mu^2 + 2\Delta(\lambda)\mu + 1 = 0$, since T is unimodular. So, the solutions are $\mu_{\pm}(\lambda) = \Delta(\lambda) \pm i\sqrt{1 - \Delta^2(\lambda)}$. It will be clear why we put the i in front of the square root.

In particular, if $\lambda \in \mathbb{R}$, then $\Delta(\lambda) \in \mathbb{R}$ (since c and s are real if the initial data is real) and this tell us something about the module of Δ :

1. if $|\Delta(\lambda)| > 1$, then from $\mu_+\mu_- \in \mathbb{R}$ and $\mu_+ + \mu_- = 2\Delta$, we get $|\mu_+| > 1$ and $|\mu_-| < 1$, so λ is outside the Bloch spectrum;
2. if $|\Delta(\lambda)| \leq 1$, then from $\mu_- = \bar{\mu}_+$ we have $|\mu_+|^2 = 1$, hence λ is in the Bloch spectrum.

Lemma 1.6.7. *If $|\mu(\lambda)| = 1$, then $\lambda \in \mathbb{R}$.*

Proof. There exists $\psi(x)$ such that $L\psi = \lambda\psi$ and $\psi(x + T) = \mu\psi(x)$. In particular $\psi(x_0 + T) = \mu\psi(x_0)$ and $\psi'(x_0 + T) = \mu\psi'(x_0)$. The complex conjugate function $\bar{\psi}$ satisfies $L\bar{\psi} = \bar{\lambda}\bar{\psi}$, $\bar{\psi}(x_0 + T) = \bar{\mu}\bar{\psi}(x_0)$ and $\bar{\psi}'(x_0 + T) = \bar{\mu}\bar{\psi}'(x_0)$. Therefore

$$\begin{aligned} (\lambda - \bar{\lambda}) \int_{x_0}^{x_0+T} |\psi|^2 dx &= \int_{x_0}^{x_0+T} (\bar{\psi} L\psi - \psi L\bar{\psi}) \\ &= [\psi \bar{\psi}' - \bar{\psi} \psi']_{x_0}^{x_0+T} = (|\mu|^2 - 1) (\psi(x_0)\bar{\psi}'(x_0) - \bar{\psi}(x_0)\psi'(x_0)). \end{aligned}$$

The right hand side vanishes, since $|\mu| = 1$. Hence $\bar{\lambda} = \lambda$. □

In other words, the Bloch spectrum is equal to the set of $\lambda \in \mathbb{R}$ such that $|\Delta(\lambda)| \leq 1$.

Example 1.6.8. If $u = u_0$, we already saw that

$$c = \cos \sqrt{\lambda - u_0}(x - x_0) \quad \text{and} \quad s = \frac{1}{\sqrt{\lambda - u_0}} \sin \sqrt{\lambda - u_0}(x - x_0).$$

Then $\Delta(\lambda) = \cos \sqrt{\lambda - u_0}T$. The square root is real and positive if and only if λ is real and greater or equal than u_0 . For λ real and less than u_0 , the cosine becomes a hyperbolic cosine, and we can draw the graph of Δ depending on real λ : before u_0 it comes from above, reaching $\Delta = 1$ for $\lambda = u_0$, then it oscillates between $\Delta = -1$ and $\Delta = 1$. The points where it reaches this bounds are the ones with $\lambda = u_0 + (\frac{\pi n}{T})^2$ and they are the spectrum of $L = -\partial_x^2 + u_0$. In particular, for n even we have periodic eigenvectors and for n odd we have antiperiodic eigenvectors (i.e., $\psi(x + T) = -\psi(x)$).

We now look to deformations of constant potential. If we start from $u = u_0$, then after a deformation sure it cannot happen that an eigenvector (periodic or antiperiodic) vanish, i.e. the graph cannot be included in $|\Delta(\lambda)| < 1$. Instead, if it rises above $|\Delta(\lambda)| = 1$, the Bloch spectrum splits in some number of intervals, potentially infinite. Before, at a point with $|\Delta(\lambda)| = 1$, we had two equal eigenvalues corresponding to the same energy level; after the deformation, the two eigenvalues split in two different ones.

We have to justify something anyway:

1. that in intervals of the Bloch spectrum the graph of Δ has to be monotonic, even after a deformation;
2. roots of $|\Delta(\lambda)| = 1$ are at most double (i.e. we have a simple maximum or a simple minimum, like before the deformation, or a transversal intersection, like after).

For generic λ (outside some isolated points) there are two linearly independent Bloch functions, i.e. two roots $\mu_{\pm}(\lambda)$ and two functions $\psi_{\pm}(x, \lambda)$ (the functions are not really unique, but they're determined up to normalization, for example they may be such that $\psi(x_0, \lambda) = 1$).

Consider the log derivative:

$$i\chi_{\pm}(x, \lambda) = \frac{\psi'_{\pm}(x, x_0, \lambda)}{\psi_{\pm}(x, x_0, \lambda)}.$$

It does not depends on x_0 (since choosing x_0 changes both the solutions and its derivative by a common factor). We write now the Riccati equation

$$i\chi' - \chi^2 = u - \lambda.$$

Lemma 1.6.9. *Let $\lambda \in \mathbb{R}$, then $\Im\chi = \frac{1}{2} \frac{(\Re\chi)'}{\Re\chi}$.*

Proof. From Riccati, we get $(\Re\chi)' - 2\Im\chi\Re\chi = 0$. □

Lemma 1.6.10.

$$\psi_{\pm}(x, x_0, \lambda) = c(x, x_0, \lambda) + \frac{\pm i\sqrt{1 - \Delta^2(\lambda)} + \frac{1}{2}(T_{2,2} - T_{1,1})}{T_{1,2}} s(x, x_0, \lambda).$$

Proof. From $\psi(x, x_0, \lambda) = a c(x, x_0, \lambda) + b s(x, x_0, \lambda)$, let $x = x_0$ then $1 = \psi = a$ (from the normalization); then $T \begin{pmatrix} 1 \\ b \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ b \end{pmatrix}$. □

Corollary 1.6.11.

$$\chi_{\pm}(x, \lambda) = \frac{\pm\sqrt{1 - \Delta^2(\lambda)} + \frac{1}{2}(T_{2,2}(x_0, \lambda) - T_{1,1}(x_0, \lambda))}{T_{1,2}(x_0, \lambda)}.$$

Proof. From $\psi = c + bs$, we may compute b by taking derivative: $\psi' = c' + b's$; restricting to $x = x_0$, c' vanish and s' becomes 1; hence we have $i\chi(x_0, \lambda) = b$. □

This means that in the Bloch spectrum, we have

$$\begin{aligned}\Re\chi(x, \lambda) &= \frac{\sqrt{1 - \Delta^2(\lambda)}}{T_{1,2}(x_0, \lambda)}, \\ \Im\chi(x, \lambda) &= \frac{T_{2,2}(x_0, \lambda) - T_{1,1}(x_0, \lambda)}{2T_{1,2}(x_0, \lambda)}.\end{aligned}$$

The functions $\mu_{\pm}(\lambda)$ are not analytic, since there is a square root. Hence they have branch points at the roots of $|\Delta(\lambda)| = 1$. If λ is not a branch point, there are locally two solutions μ_+ and μ_- ; moreover, there are locally two meromorphic eigenvectors ψ_+ and ψ_- (they may have poles). If λ_0 is a branch point, then $\Delta(\lambda_0) = 1$ or $\Delta(\lambda_0) = -1$; walking along a small loop around λ_0 , the two eigenvalues could interchange; but if the multiplicity is odd (i.e. it is 1, since it is less or equal to 2) that can't happen. Let us study the poles of ψ_{\pm} : they can be at points λ such that $T_{1,2}(x_0, \lambda) = 0$.

Lemma 1.6.12.

1. Roots of $T_{1,2}(x_0, \lambda) = 0$ are real.
2. They are not in the inner part of the Bloch spectrum.

Proof. The reality of the roots is related to the self-adjointness: recall that $T_{1,2}(x_0, \lambda) = s(x_0 + T, x_0, \lambda)$; we imposed that $s(x_0, x_0, \lambda) = 0$, so if $T_{1,2} = 0$ we have $s(x_0 + T, x_0, \lambda) = 0$. Then λ is an eigenvalue of the Dirichlet spectrum on $[x_0, x_0 + T]$.

Now if $T_{1,2} = 0$, the unimodularity says that $T_{1,1}T_{2,2} = 1$ and the reality of λ says that $\frac{1}{2}|T_{1,1} + T_{2,2}| \leq 1$. The case where the poles are in the border are precisely when $T_{1,1}$ and $T_{2,2}$ are both 1 or both -1 . This allows also roots of $T_{1,2}$ in case of a maximum or a minimum at $|\Delta(\lambda)| = 1$ (so that this point is inside the Bloch spectrum); but if λ_0 is a double root of $|\Delta(\lambda)| = 1$, then also $T_{1,1} = T_{2,2}$ so that ψ_{\pm} has no poles at λ_0 (provided that $T_{1,2}$ have only simple root at λ_0). \square

Example 1.6.13. Recall the case $u = u_0$; then $T_{1,2}$ roots are simple; when perturbing, we open some real gaps between the intervals of the Bloch spectrum, and the root still remains in these gaps and cannot merge into roots of higher order.

We saw the behaviour of the function $\Delta(\lambda)$ for $\lambda \in \mathbb{R}$. Points where the graph intersects $\Delta(\lambda) = 1$ are the eigenvalues of the periodic problem and we have $\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \dots$, where the less or equal is equal if the corresponding gap is reduced to a point. If instead $\Delta(\mu) = -1$, μ is an eigenvalue for the antiperiodic problem and we have again $\mu_1 \leq \mu_2 < \mu_3 \leq \dots$.

Remark 1.6.14. The set $\{\lambda_0, \lambda_1, \dots\}$ and the set $\{\mu_1, \mu_2, \dots\}$ are not independent. In other words, we may fix one of the two and the other will be determined. Indeed, from the theory of entire function, fixed for example the first set, we can express Δ in terms of an infinite product:

$$\Delta(\lambda) - 1 = \frac{\pi}{T}(\lambda - \lambda_0) \prod_{k \geq 1} \frac{\lambda_n - \lambda}{\left(\frac{2\pi k}{T}\right)^2}.$$

1.7 Properties of the monodromy matrix

We saw that the eigenvalues of T , $\mu = \mu_{\pm}(\lambda)$ are determined from the equation $\mu^2 - 2\Delta(\lambda) + 1 = 0$. Two branches collide (i.e., $\mu_+ = \mu_-$) when $|\Delta(\lambda)| = 1$. Branch points are subdivided in:

1. $1 - \Delta^2(\lambda)$ has a simple root at λ_0 ;
2. $1 - \Delta^2(\lambda)$ has a double root at λ_0 .

Indeed we will see that cannot happen a root of multiplicity greater than 2. What is the local behaviour of $\mu(\lambda)$ when λ is near λ_0 ? In the second case, we have the Taylor expansion $1 - \Delta^2(\lambda) = a(\lambda - \lambda_0)^2 + b(\lambda - \lambda_0)^3 + \dots$ with $a \neq 0$; then

$$\sqrt{1 - \Delta^2(\lambda)} = \sqrt{a}(\lambda - \lambda_0) \left(1 + \frac{b}{2a}(\lambda - \lambda_0) + \dots \right)$$

or the same expression with a minus. In other words, we have two functions with the same eigenvalue, so we have no branch points in this case. Instead, in the first situation, i.e. if we have only one root, the Taylor expansion is $1 - \Delta^2(\lambda) = a(\lambda - \lambda_0) + b(\lambda - \lambda_0)^2 + \dots$ and

$$\sqrt{1 - \Delta^2(\lambda)} = \pm \sqrt{a} \sqrt{\lambda - \lambda_0} \left(1 + \frac{b}{2a}(\lambda - \lambda_0) + \dots \right).$$

Doing a small loop around λ_0 , then $\sqrt{\lambda - \lambda_0} \mapsto \sqrt{\lambda - \lambda_0} e^{\frac{i\phi}{2}}$ and at $\phi = 2\pi$, the square root becomes negative.

Properties:

1. all roots of $1 - \Delta^2(\lambda)$ are at most double;
2. $\Delta(\lambda)$ is monotone and increasing on the intervals $(\mu_{2i}, \lambda_{2i-1})$ and monotone and decreasing on the intervals $(\lambda_{2i}, \mu_{2i+1})$;
3. $T_{1,2}(x_0, \lambda)$ has just one simple root in every gap (i.e. in every interval of the form $[\lambda_{2i-1}, \lambda_{2i}]$ and in every interval $[\mu_{2i-1}, \mu_{2i}]$); in particular if some gap is closed, i.e. $\lambda_{2i-1} = \lambda_{2i}$, then this is a simple root of $T_{1,2}$.

We defined the Bloch functions for a generic $\lambda \in \mathbb{C}$ as $\psi_{\pm}(x, x_0, \lambda)$ such that $\psi_{\pm}(x + T, x_0, \lambda) = \mu_{\pm}(\lambda)\psi_{\pm}(x, x_0, \lambda)$, normalized in such a way that $\psi_{\pm}(x_0, x_0, \lambda) = 1$. We also wrote $\psi_{\pm}(x, x_0, \lambda) = c(x, x_0, \lambda) + i\chi_{\pm}(x_0, \lambda)s(x, x_0, \lambda)$ where $\chi_{\pm}(x, \lambda) = \frac{1}{2} \frac{\psi'_{\pm}}{\psi_{\pm}}$ does not depend on the normalization. We also saw that we can write

$$\chi_{\pm}(x, \lambda) = \frac{\pm \sqrt{1 - \Delta^2(\lambda)} + \frac{i}{2}(T_{1,1}(x, \lambda) - T_{2,2}(x, \lambda))}{T_{1,2}(x, \lambda)}.$$

The functions s and c are entire in λ , but χ_{\pm} is not since it has a square root. But we can see χ as a two valued meromorphic function in λ with branch points at the simple zeroes of $1 - \Delta^2(\lambda) = 0$. In particular, poles of $\chi_{\pm}(x_0, \lambda)$ can be only at zeroes of $T_{1,2}(x_0, \lambda)$.

If the gap is reduced to a point, then $1 - \Delta^2(\lambda)$ has a double root, its square root has a simple root; also $T_{1,2}(x_0, \lambda)$ has a simple root; moreover, $T_{1,1}(x_0, \lambda) = T_{2,2}(x_0, \lambda)$: infact,

at λ we have two independent eigenvectors for the same eigenvalues and this means that the monodromy matrix is plus or minus the identity matrix. So the simple zero of the denominator cancels with the simple zero of the numerator. We have proved that zeroes of χ_{\pm} may happen only on nondegenerate gaps.

we are going to prove a stronger statement: if we have two branches $\psi_{\pm}(x, x_0, \lambda)$, only one branch may have pole at a given λ . This is clear from the next lemma.

Lemma 1.7.1.

$$\psi_+(x, x_0, \lambda)\psi_-(x, x_0, \lambda) = \frac{T_{1,2}(x, \lambda)}{T_{1,2}(x_0, \lambda)}$$

Proof. We have $i\chi_{\pm}(x, \lambda) = \frac{d}{dx} \log \psi_{\pm}$, so

$$\frac{d}{dx} \log(\psi_+\psi_-) = i(\chi_+(x, \lambda) + \chi_-(x, \lambda)) = -\frac{T_{1,1}(x, \lambda) - T_{2,2}(x, \lambda)}{T_{1,2}(x, \lambda)}.$$

But we have seen that if $\lambda \in \mathbb{R}$, $\Im\chi = \frac{1}{2} \frac{d}{dx} \log(\Re\chi)$. So, inside the Bloch zones, we have

$$\Re\chi = \frac{\sqrt{1 - \Delta^2(\lambda)}}{T_{1,2}(x, \lambda)} \quad \Im\chi = \frac{1}{2} \frac{T_{1,1}(x, \lambda) - T_{2,2}(x, \lambda)}{T_{1,2}(x, \lambda)}.$$

Applying all this, we have

$$\frac{d}{dx} \log(\psi_+\psi_-) = -\frac{d}{dx} \log \frac{\sqrt{1 - \Delta^2(\lambda)}}{T_{1,2}(x, \lambda)} = +\frac{d}{dx} \log T_{1,2}(x, \lambda).$$

Now we have

$$\log \psi_+\psi_- = \int_{x_0}^x \frac{d}{dx} \log T_{1,2}(x, \lambda) dx = \log T_{1,2}(x, \lambda) - \log T_{1,2}(x_0, \lambda).$$

□

1.8 Differentiating with respect to the spectral parameter

We define the *Bloch quasimomentum* $p(\lambda)$ in such a way that $\mu(\lambda) = e^{ip(\lambda)T}$, i.e. $p(\lambda) = \frac{1}{iT} \log \mu(\lambda)$, or $\Delta(\lambda) = \cos p(\lambda)T$. It is called quasimomentum since $\psi_{\pm}(x, x_0, \lambda) = e^{\pm ip(\lambda)(x-x_0)} \phi_{\pm}(x, x_0, \lambda)$, where ϕ_{\pm} are periodic on x .

Remark 1.8.1. The Bloch quasimomentum is determined up to change of sign and shifts by $\frac{2\pi n}{T}$.

Lemma 1.8.2.

$$\frac{dp(\lambda)}{d\lambda} = \frac{1}{T} \int_{x_0}^{x_0+T} \frac{T_{1,2}(x, \lambda)}{2\sqrt{1 - \Delta^2(\lambda)}} dx$$

Proof. We use a trick: from

$$0 = -y_1'' + u_1 y_1, \tag{1.8.1}$$

$$0 = -y_2'' + u_2 y_2, \tag{1.8.2}$$

we have the identity

$$\frac{d}{dx}(y_1 y_2' - y_1' y_2) = (u_2 - u_1) y_1 y_2.$$

So

$$[y_1 y_2' - y_1' y_2]_{x_0}^{x_0+T} = \int_{x_0}^{x_0+T} (u_2(x) - u_1(x)) y_1(x) y_2(x) dx.$$

Now we choose $u_1 := u - \lambda$ and $u_2 := u - \lambda + \delta u$; moreover we fix λ so that $|\Delta_1(\lambda)| \neq 1$ (i.e. we can choose just one analytic branch of the quasimomentum); then for small δu also $|\Delta_2(\lambda)| \neq 1$. Also, let $y_1 := \psi_{1,-}$ and $y_2 := \psi_{2,+}$; then $\psi_{1,-}(x+T, x_0, \lambda) = e^{-ip_1(\lambda)T} \psi_{1,-}(x, x_0, \lambda)$ and $\psi_{2,+}(x+T, x_0, \lambda) = e^{ip_2(\lambda)T} \psi_{2,+}(x, x_0, \lambda)$. Substituting all this into the first equation, we get in the left hand side $(y_2'(x_0) - y_1'(x_0))(e^{i(p_2 - p_1)T} - 1)$ (the exponential comes from x_0+T , the 1 from x_0) and in the right hand side $\int_{x_0}^{x_0+T} \delta u(x) \psi_{1,-}(x) \psi_{2,+}(x) dx$. So approximately, we have

$$(\psi_+'(x_0) - \psi_-'(x_0))(i\delta p(\lambda)T) \simeq \int_{x_0}^{x_0+T} \delta u(x) \psi_-(x) \psi_+(x) dx;$$

now we replace the first parenthesis by

$$i\chi_+ - i\chi_- = \frac{2i\sqrt{1 - \Delta^2(\lambda)}}{T_{1,2}(x_0, \lambda)}$$

so that, after simplifying $T_{1,2}(x_0, \lambda)$ with the one coming from $\psi_-(x)\psi_+(x)$, we have

$$\delta p = -\frac{1}{T} \int_{x_0}^{x_0+T} \delta u(x) \frac{T_{1,2}(x, \lambda)}{2\sqrt{1 - \Delta^2(\lambda)}} dx.$$

Choosing $\delta u(x) = -d\lambda$, we get the result. \square

The previous lemma proves property two (monotonicity of $\Delta(\lambda)$); infact in a zone, $T_{1,2}$ has always the same sign (it has zeroes only on the gaps); and the denominator is always positive. Another corollary is that

$$\frac{\delta p}{\delta u(x)} = -\frac{T_{1,2}(x, \lambda)}{2\sqrt{1 - \Delta^2(\lambda)}}.$$

With a little work, we may derive from this lemma also the first property (the simplicity of the roots).

Theorem 1.8.3 (Sturm). *Let y_1, y_2, u_1 and u_2 be real functions satisfying (1.8.1) for all $x \in [a, b]$, and let $u_1(x) \geq u_2(x)$. Assume that $y_1(x)$ has n zeroes $a < x_1 < x_2 < \dots < x_n \leq b$ and that $\begin{bmatrix} y_1' \\ y_1 \end{bmatrix}_a \geq \begin{bmatrix} y_2' \\ y_2 \end{bmatrix}_a$. Then also $y_2(x)$ has at least n zeroes on $(a, b]$. Moreover, $y_2(x)$ has at least n zeroes on (a, x_n) if $u_1(x) > u_2(x)$ for every $x \in [a, x_n]$.*

We can apply Sturm theorem to $\psi(x, \lambda')$ and $\psi(x, \lambda'')$, where (λ', λ'') is a gap in the Bloch spectrum. We obtain that $\psi(x, \lambda')$ has n zeroes on $[x_0, x_0 + T]$ and $\psi(x, \lambda'')$ has at least $n + 1$ zeroes on the same interval.

1.9 Finite gap case

We request now that the gaps are a finite number, i.e. $\lambda_1, \dots, \lambda_{2n+1}$ are simple eigenvalues of the (anti)periodic problem, n is the number of gaps. We recall that in every gap there is exactly one zero of $T_{1,2}(x_0, \lambda)$ that is called $\gamma_j(x_0)$. Then in our situation, $\gamma_j(x_0) \in [\lambda_{2j}, \lambda_{2j+1}]$. The other zeroes are trivial since they must be in the only point of the degenerate gaps.

We introduce two polynomials:

$$R(\lambda) := \prod_{s=1}^{2n+1} (\lambda - \lambda_s),$$

$$P(\lambda, x) := \prod_{j=1}^n (\lambda - \gamma_j(x)).$$

Lemma 1.9.1. *For a finite gap potential:*

$$\chi_{\pm}(x, \lambda) = \frac{\pm \sqrt{R(\lambda)} - \frac{i}{2} \frac{d}{dx} P(\lambda, x)}{P(\lambda, x)}.$$

Proof. At a double zero λ_0 of $1 - \Delta^2(\lambda)$ we have that $\sqrt{1 - \Delta^2(\lambda)}$ is an analytic function with a simple zero at λ_0 ; we have seen that zeroes of the numerator and of the denominator cancel; once cancelled, we have

$$\chi_{\pm}(x, \lambda) = c(x, \lambda) \frac{\pm \sqrt{R(\lambda)} - \frac{i}{2} (\tilde{T}_{1,1} - \tilde{T}_{2,2})}{P(\lambda, x)},$$

where c is an entire function without zeroes. At $|\lambda| \rightarrow +\infty$, we have $i\chi' - \chi^2 = u - \lambda$, so that $\chi \sim \sqrt{\lambda} + O(\frac{1}{\sqrt{\lambda}})$. So $\Re \chi = \sqrt{\lambda} + O(\frac{1}{\sqrt{\lambda}})$ and $\Im \chi = O(\frac{1}{\sqrt{\lambda}})$; then

$$\frac{\sqrt{R(\lambda)}}{P(\lambda)} = \frac{\lambda^{\frac{2n+1}{2}} \prod_{s=1}^{2n+1} (1 - \lambda_s)^{\frac{1}{2}}}{\lambda^n \prod_{j=1}^n \left(1 - \frac{\gamma_j(\lambda)}{\lambda}\right)}.$$

Grouping the λ , we have $\sqrt{\lambda}$ so that $c \sim O(1)$ and applying Liouville theorem, $c = 1$.

As an exercise, prove that $-\frac{i}{2} \frac{d}{dx} P(\lambda, x) = \frac{i}{2} (\tilde{T}_{1,1} - \tilde{T}_{2,2})$, using that $\Im \chi = \frac{i}{2} (\log \Re \chi)'$. \square

In the case of finite gaps, we obtain a Riemann surface $\nu^2 = \prod_{s=1}^{2n+1} (\lambda - \lambda_0)$. In this case, we can compactify with a point at infinity, obtaining a sphere with n handles.

Corollary 1.9.2.

1. For every x , $\chi(x, \lambda)$ is an algebraic function on the compactification of Γ ;
2. $\psi_{\pm}(x, x_0, \lambda)$ are meromorphic functions on $\Gamma \setminus \{\infty\}$ and have simple poles at $\lambda = \gamma_1(x_0), \dots, \gamma_n(x_0)$ and $\psi_{\pm}(x, x_0, \lambda) = e^{\pm i \sqrt{\lambda}(x-x_0)} (1 + O(\frac{1}{\sqrt{\lambda}}))$.

How to compute finite gap potentials?

Lemma 1.9.3. *The zeroes of $T_{1,2}$ satisfies*

$$\gamma'_j = -\frac{2i\sqrt{R(\gamma_j)}}{\prod_{k \neq j} (\gamma_j - \gamma_k)}.$$

Lemma 1.9.4.

$$u(x) = -2 \sum_{j=1}^n \gamma_j(x) + \sum_{s=1}^{2n+1} \lambda_s.$$

Example 1.9.5. If $n = 0$, then $\chi = \sqrt{\lambda - \lambda_0}$ and $u = \lambda_1$.

Example 1.9.6. If $n = 1$ then the associated Riemann surface is an elliptic curve: $\nu^2 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$. Let $\gamma := \gamma_1$ be the only nontrivial zero; then $u = -2\gamma + \lambda_1 + \lambda_2 + \lambda_3$ and integrating by quadrature $\gamma' = -2i\sqrt{(\gamma - \lambda_1)(\gamma - \lambda_2)(\gamma - \lambda_3)}$ we have

$$\int_{x_0}^x \frac{dy}{\sqrt{(\gamma - \lambda_1)(\gamma - \lambda_2)(\gamma - \lambda_3)}}.$$

Last time we explored the time dependence of a finite gap potential: we are looking for solution to the Cauchy problem where $u(x, 0) = u_0(x)$ is a finite gap potential. This implies that $u(x, t)$ still is finite gap with the same spectrum. If $Ly = \lambda y$ with λ fixed, we have seen that $\dot{y} + Ay$ satisfies again the same equation, with the same λ . Applying this argument to $y = c$ or $y = s$, we get that $\dot{c} + Ac$ and $\dot{s} + As$ are linear combinations of c and s , i.e.

$$\begin{aligned} \dot{c} + Ac &= V_{1,1}c + V_{1,2}s, \\ \dot{s} + As &= V_{2,1}c + V_{2,2}s. \end{aligned}$$

The matrix V depends obviously on x_0 and λ .

Lemma 1.9.7.

$$V = V(x_0, \lambda) = \begin{pmatrix} u' & -2(u' + 2\lambda) \\ u'' - 2(u - \lambda)(u + 2\lambda) & -u' \end{pmatrix}.$$

Proof. Consider $\dot{y} + Ay$; then $y'' = (u - \lambda)y$ and $y = (u - \lambda)y' + u'y$. Easily we have $Ay = -2(u + 2\lambda)y' + u'y$. So

$$[\dot{c} + Ac]_{x=x_0} = [\dot{c} - \cancel{2(u + 2\lambda)c'} + u'c]_{x=x_0} = [V_{1,1}c + \cancel{V_{1,2}s}]_{x=x_0}.$$

Moreover $c = 1$ and $\dot{c} = 0$ so that $V_{1,1} = u'$. Applying the same argument to s we get $V_{1,2} = -2(u + 2\lambda)s'$. The second row is obtained deriving the two expressions used before by x . \square

Lemma 1.9.8. *Consider the monodromy matrix $T(x_0, \lambda)$; then $\dot{T} = [T, V]$.*

Proof. We have $(c(x + T), s(x + T)) = (c(x), s(x))T$; moreover $(c(x), s(x)) = (c(x), s(x))V$. So $(c(x + T), s(x + T))$ can be computed in two ways: in the first it is $(c(x + T), s(x + T))V = (c(x), s(x))TV$; in the second it is $(c(x), s(x))T + (c(x), s(x))\dot{T} = (c(x), s(x))VT + (c(x), s(x))\dot{T}$. Then

$$(c(x), s(x))[T, V] = (c(x), s(x))\dot{T}.$$

\square

Corollary 1.9.9. *The characteristic polynomial of T does not depend on time (in other words, the eigenvalues of T does not depend on time).*

Proof. We know that the determinant of T is always 1 by unimodularity; we only need to prove that also the trace is constant; but trace commutes with derivative, so $\frac{d}{dt} \text{tr } T = \text{tr } \dot{T} = \text{tr } [T, V] = 0$. \square

In particular, the quasi-momentum does not depend on time.

Exercise 1.9.10: Prove that

$$\frac{d}{dx_0} T(x_0, \lambda) = [T, U], \quad \text{with } U = \left[\begin{array}{cc} 0 & -1 \\ \lambda - u(x) & 0 \end{array} \right]_{x=x_0}.$$

Observe that U is just the matrix form of $Ly = \lambda y$.

Lemma 1.9.11. *For $j \in \{1, \dots, n\}$ we have*

$$\dot{\gamma}_j = -\frac{8i(\gamma_j + \frac{u}{2})\sqrt{R(\gamma_j)}}{\prod_{k \neq j} (\gamma_j - \gamma_k)},$$

where $u = -2 \sum_{k=1}^n \gamma_k + \sum_{s=1}^{2n+1} \lambda_s$.

Proof. We can derive this equation from the expression of $\dot{T}_{1,2}$ in $[T, V] = \dot{T}$, for $\lambda = \gamma_j$. We can then express $T_{1,2}(\lambda) = \prod_{k=1}^n (\lambda - \gamma_k) \tilde{T}_{1,2}(\lambda)$ there $\tilde{T}_{1,2}$ has all zeroes on the squeezed gaps, so that its zeroes do not depend on time. Deriving this and computing at $\lambda = \gamma_j$, we have $-\dot{\gamma}_j \prod_{k \neq j} (\gamma_j - \gamma_k) \tilde{T}_{1,2}(\lambda)$. Substituting this in the first equation we have

$$\dot{\gamma}_j = \text{Res}_0^{\lambda=\gamma_j} \frac{T_{2,2} - T_{1,1}}{T_{1,2}} V_{1,2}(\gamma_j).$$

Recalling the expression of χ , we have

$$\text{Res}_0^{\lambda=\gamma_j} \frac{T_{1,1} - T_{2,2}}{T_{1,2}} = -2i \text{Res}_0^{\lambda=\gamma_j} \frac{\sqrt{1 - \Delta^2}}{T_{1,2}} = -2i \text{Res}_0^{\lambda=\gamma_j} \frac{\sqrt{R(\lambda)}}{P(\lambda)}.$$

\square

We claim that in the solutions of periodic KdV equation, the set of finite gap potentials is dense in the set of solutions. Indeed, one can prove the following statement: expand u in the Fourier series, so that $u(x) = \sum u_n e^{\frac{ikx}{T}}$; then the length of the n -th gap decays as $|u_n|$; therefore, the smoother is the potential, the faster is the decay of the gaps. This means that the deformation of the potential needed to close the gaps from some point to infinity is small.

1.10 The theory of KdV hierarchy-1. Recursion relations and generating functions.

We will now include KdV equation in an infinite family of pairwise commuting flows represented by evolutionary PDEs of the form

$$\begin{aligned}\frac{\partial u}{\partial t_n} &= K_n \left(u, u_x, u_{xx}, \dots, u^{(2n+1)} \right), \quad n = 0, 1, 2, \dots \\ \frac{\partial}{\partial t_m} \frac{\partial u}{\partial t_n} &= \frac{\partial}{\partial t_n} \frac{\partial u}{\partial t_m}.\end{aligned}$$

All equations of this so-called *KdV hierarchy* will admit Lax representation

$$\frac{\partial L}{\partial t_n} = [L, A_n]$$

with the same Schrödinger operator L ; the differential operators A_n of the order $2n + 1$ will be constructed below.

For $n = 1$ one obtains the KdV equation itself

$$\dot{u} = 6 u u' - u''', \quad (1.10.1)$$

$$\dot{u} := \frac{\partial u}{\partial t}, \quad u' := \frac{\partial u}{\partial x}.$$

The commutation representation (3.10.9) reads

$$\dot{L} = [L, A], \Leftrightarrow [\partial_t + A, L] = 0 \quad (1.10.2)$$

$$L = -\partial_x^2 + u \quad (1.10.3)$$

$$A = A_1 = 4\partial_x^3 - 3(u\partial_x + \partial_x u). \quad (1.10.4)$$

We will first rewrite the commutation representation (1.10.2) in the matrix form. The basic idea is to restrict (1.10.2) onto the space of solutions to the Sturm - Liouville equation

$$L y = \lambda y. \quad (1.10.5)$$

Here λ is an arbitrary complex parameter. We will call it *spectral parameter* although all references to the theory will be rather formal at the moment.

The following simple statement is very important for the restriction we are looking for.

Lemma 1.10.1. *Let y be a solution of equation (1.10.5). Assume that the operator L depends on the time t according to eq. (1.10.2) for some A . Then the function $y_t + A y$ solves the same equation (1.10.5), i.e.,*

$$L(y_t + A y) = \lambda(y_t + A y). \quad (1.10.6)$$

Proof. Taking time derivative of (1.10.5) we obtain

$$\dot{L} y + L \dot{y} = \lambda \dot{y}.$$

Substituting $\dot{L} = LA - AL$ yields

$$(LA - AL)y + L\dot{y} = LAy - \lambda Ay + L\dot{y} = \lambda \dot{y}.$$

This coincides with (1.10.6). The lemma is proved.

We choose a basis y_1, y_2 in the two-dimensional space of solutions of (1.10.5). The most convenient basis is specified by the following Cauchy data at a normalization point $x = x_0$:

$$\begin{aligned} y_1(x_0) &= 1, & y_1'(x_0) &= 0 \\ y_2(x_0) &= 0, & y_2'(x_0) &= 1. \end{aligned} \tag{1.10.7}$$

These solutions to (1.10.5) will be denoted $y_1(x, x_0, \lambda)$ and $y_2(x, x_0, \lambda)$.

Lemma 1.10.2. *If $u(x)$ is a smooth function near $x = x_0$ then the solutions $y_1(x, x_0, \lambda)$ and $y_2(x, x_0, \lambda)$ are entire analytic functions in λ .*

This is a standard fact of the theory of ordinary linear differential equations - see, e.g., [18].

Assuming, as above, that the time dependence of the operator L is determined by the equation (1.10.2) with some operator A we conclude that, due to Lemma 1.10.1, the solutions $(\partial_t + A)y_1$ and $(\partial_t + A)y_2$ can be represented as a linear combination of the same functions y_1 and y_2 :

$$\begin{aligned} (\partial_t + A)y_1 &= v_{11}y_1 + v_{21}y_2 \\ (\partial_t + A)y_2 &= v_{12}y_1 + v_{22}y_2 \end{aligned} \tag{1.10.8}$$

The coefficients $v_{ij} = v_{ij}(x_0, \lambda)$ depend on x_0 and on λ (and, of course, on the time, but we will suppress the explicit time-dependence for the sake of simplicity of notations). In the matrix form (1.10.8) reads

$$(\partial_t + A)(y_1, y_2) \equiv ((\partial_t + A)y_1, (\partial_t + A)y_2) = (y_1, y_2)V, \tag{1.10.9}$$

where

$$V = V(x_0, \lambda) = \begin{pmatrix} v_{11}(x_0, \lambda) & v_{12}(x_0, \lambda) \\ v_{21}(x_0, \lambda) & v_{22}(x_0, \lambda) \end{pmatrix}. \tag{1.10.10}$$

Let us explain the algorithm of computing the coefficients $v_{ij}(x_0, \lambda)$.

Lemma 1.10.3. *Let $A = A(x, \partial_x)$ be any linear differential operator. Then there exist polynomials $p(x, \lambda)$, $q(x, \lambda)$ in λ with x -dependent coefficients such that*

$$Ay = q(x, \lambda)y + p(x, \lambda)y' \tag{1.10.11}$$

for any solution $y = y(x, \lambda)$ of (1.10.5).

Proof. Using (1.10.5) the derivatives y'' , y''' etc. can be expressed as linear combinations of y and y' :

$$y'' = (u - \lambda)y, \quad y''' = u'y + (u - \lambda)y', \quad y^{IV} = [u'' + (u - \lambda)^2]y + 2u'y', \dots \quad (1.10.12)$$

This proves the lemma.

Example 1.10.4. For the operator A given in (1.10.4) one obtains

$$q(x, \lambda) = u'(x), \quad p(x, \lambda) = -2(u(x) + 2\lambda).$$

We are now able to compute the matrix $V = (v_{ij})$ for a given operator A .

Lemma 1.10.5. *The matrix V in (1.10.9) for an operator A has the form*

$$V = V(x_0, \lambda) = \begin{pmatrix} q & p \\ q' + (u - \lambda)p & q + p' \end{pmatrix}_{x=x_0} \quad (1.10.13)$$

where the polynomials $q(x, \lambda)$, $p(x, \lambda)$ are determined by the operator A from the equation (1.10.11).

Proof. Due to (1.10.8), (1.10.11) we have

$$\partial_t y_1(x, x_0, \lambda) + q(x, \lambda)y_1(x, x_0, \lambda) + p(x, \lambda)y_1'(x, x_0, \lambda) = y_1(x, x_0, \lambda)v_{11}(x_0, \lambda) + y_2(x, x_0, \lambda)v_{21}(x_0, \lambda) \quad (1.10.14)$$

$$\partial_t y_2(x, x_0, \lambda) + q(x, \lambda)y_2(x, x_0, \lambda) + p(x, \lambda)y_2'(x, x_0, \lambda) = y_1(x, x_0, \lambda)v_{12}(x_0, \lambda) + y_2(x, x_0, \lambda)v_{22}(x_0, \lambda).$$

Substituting $x = x_0$ we obtain

$$v_{11}(x_0, \lambda) = q(x_0, \lambda), \quad v_{12}(x_0, \lambda) = p(x_0, \lambda)$$

since the time derivatives $\partial_t y_1$ and $\partial_t y_2$ vanish at $x = x_0$ due to the time-independent initial conditions (1.10.7). We obtain the first row of the matrix V . Let us now take x -derivatives of (1.10.14). We obtain

$$\partial_t y_1' + [q' + p(u - \lambda)]y_1 + (q + p')y_1' = y_1'v_{11} + y_2'v_{21}$$

$$\partial_t y_2' + [q' + p(u - \lambda)]y_2 + (q + p')y_2' = y_1'v_{12} + y_2'v_{22}$$

where, as above, the coefficients p , q , u and their x -derivatives in the left hand side depend on x and λ and the coefficients v_{ij} in the right hand side depend on x_0 and λ . At the point $x = x_0$ the derivatives $\partial_t y_1'$ and $\partial_t y_2'$ vanish again due to (1.10.7). This gives the second row of the matrix V . The lemma is proved.

Example 1.10.6. For the operator A of the form (1.10.4) (i.e., for the KdV equation itself) the matrix V has the form

$$V(x_0, \lambda) = \begin{pmatrix} u' & -2(u + 2\lambda) \\ u'' - 2(u - \lambda)(u + 2\lambda) & -u' \end{pmatrix}_{x=x_0}. \quad (1.10.15)$$

We will now derive the matrix reformulation of the Lax representation of (1.10.2). Let us introduce the 2×2 matrix

$$U = U(x, \lambda) = \begin{pmatrix} 0 & -1 \\ \lambda - u & 0 \end{pmatrix}. \quad (1.10.16)$$

This matrix appears in the vector form of the Sturm - Liouville equation (1.10.5). I.e., if y is a solution to $Ly = \lambda y$ then

$$\mathbf{y} := \begin{pmatrix} y \\ y' \end{pmatrix}$$

is a solution to

$$\mathcal{L}(\lambda)\mathbf{y} = 0, \quad \mathcal{L}(\lambda) = \partial_x + U(\lambda). \quad (1.10.17)$$

Conversely, if $\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}$ solves (1.10.17) then $z = y'$ and $Ly = \lambda y$.

Theorem 1.10.7. *If the time dependence of L is determined by (1.10.2) with some linear differential operator A then the matrices $U = U(x, \lambda)$ and $V = V(x, \lambda)$ given in (1.10.16) and (1.10.15) resp. satisfy the equation*

$$V_x(\lambda) - U_t(\lambda) = [V(\lambda), U(\lambda)] \quad (1.10.18)$$

identically in λ .

Remark 1.10.8. Introducing λ -dependent matrix operators $\mathcal{A}(\lambda)$ as in (1.10.17) and

$$\mathcal{A}(\lambda) = \partial_t + V(\lambda) \quad (1.10.19)$$

we can rewrite (1.10.18) as the commutativity

$$[\mathcal{L}(\lambda), \mathcal{A}(\lambda)] = 0. \quad (1.10.20)$$

To prove the theorem we first rewrite Lemma 1.10.1 in the matrix form.

Lemma 1.10.9. *If \mathbf{y} solves*

$$\mathcal{L}(\lambda)\mathbf{y} = 0$$

then $\mathcal{A}(\lambda)\mathbf{y}$ is a solution to the same equation.

Proof. The vector \mathbf{y} has the form

$$\mathbf{y} = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad Ly = \lambda y.$$

Thus

$$\begin{aligned} \mathcal{A}(\lambda)\mathbf{y} &\equiv \partial_t \mathbf{y} + V(\lambda)\mathbf{y} = \begin{pmatrix} \partial_t y \\ \partial_t y' \end{pmatrix} + \begin{pmatrix} qy + py' \\ [q' + (u - \lambda)p]y + (q + p')y' \end{pmatrix} \\ &= \begin{pmatrix} \partial_t + Ay \\ [\partial_t y + Ay]' \end{pmatrix} \equiv \begin{pmatrix} z \\ z' \end{pmatrix}, \quad \text{where } z = \partial_t y + Ay. \end{aligned}$$

We already know that $Lz = \lambda z$. Hence

$$\mathcal{L}(\lambda) \begin{pmatrix} z \\ z' \end{pmatrix} = 0.$$

The lemma is proved.

Proof of the theorem. Due to Lemma 1.10.9 an arbitrary solution \mathbf{y} of $\mathcal{L}(\lambda)\mathbf{y} = 0$ also satisfies $\mathcal{L}(\lambda)\mathcal{A}(\lambda)\mathbf{y} = 0$. Hence it also satisfies

$$[\mathcal{L}(\lambda), \mathcal{A}(\lambda)]\mathbf{y} = 0. \quad (1.10.21)$$

But the commutator in the left hand side is an operator of multiplication by the matrix

$$[\partial_x + U(\lambda), \partial_t + V(\lambda)] = V_x - U_t + [U(\lambda), V(\lambda)]. \quad (1.10.22)$$

Since (1.10.21) holds true for an arbitrary solution \mathbf{y} of (1.10.17), we conclude that the matrix (1.10.22) must be an identical zero. The theorem is proved.

So, for the equations of the KdV hierarchy instead of the Lax-type representation (1.10.2) we have obtained a zero curvature representation (1.10.18) with the matrices U, V depending on the spectral parameter λ . In our case both U and V are polynomials in λ . Such commutation representation with polynomial, rational, or even more complicated dependence on the spectral parameter is one of the most efficient tools of constructing, integrating and studying the nonlinear integrable equations.

We have not proved yet that, conversely, any nonlinear equation admitting a representation (1.10.18) with the matrix $U(\lambda)$ as in (1.10.16) and with a polynomial in λ matrix $V(\lambda)$ is an equation of KdV hierarchy (i.e., that it admits a Lax representation (1.10.2) with some differential operator A). The main step in the proof of this converse statement is a constructive answer to the following question: how to describe all matrix polynomials $V(\lambda)$ satisfying (1.10.18)? The answer can be obtained in the following way. Let us look for the matrix $V(\lambda)$ in the form

$$V = V_0\lambda^{N+1} + V_1\lambda^N + \cdots + V_N\lambda + V_{N+1} \quad (1.10.23)$$

with indeterminate matrix coefficients depending on x . Substituting this polynomial into (1.10.18) we will derive recursion relations for the matrix coefficients.

In the case under consideration it is convenient to reduce the procedure to the scalar case. Observe first that the trace of the matrix $V(\lambda)$ does not depend on x :

$$(\operatorname{tr} V)' = 0.$$

Indeed, from (1.10.18) one has

$$(\operatorname{tr} V)' = \operatorname{tr} \dot{U} + \operatorname{tr} [V, U] = 0.$$

Adding an appropriate scalar matrix to V we can assume that

$$\operatorname{tr} V(\lambda) = 0.$$

Lemma 1.10.10. *Any traceless polynomial solution (1.10.23) of (1.10.18) can be represented in the form*

$$V = \begin{pmatrix} \frac{1}{2} v' & -v \\ \frac{1}{2} v'' - (u - \lambda) v & -\frac{1}{2} v' \end{pmatrix} \quad (1.10.24)$$

where

$$v = v_0 \lambda^N + v_1 \lambda^{N-1} + \cdots + v_N \quad (1.10.25)$$

is a polynomial with the coefficients v_1, v_2, \dots, v_N depending on x . The function u and the polynomial v satisfy the equation

$$\dot{u} + \frac{1}{2} v''' - 2(u - \lambda) v' - u' v = 0. \quad (1.10.26)$$

Proof. Explicitly the matrix equation (1.10.18) for a traceless matrix

$$V(\lambda) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix}$$

reads

$$\begin{pmatrix} v'_{11} & v'_{12} \\ v'_{21} & -v'_{11} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix} + \begin{pmatrix} v_{12}(u - \lambda) - v_{21} & 2v_{11} \\ -2v_{11}(u - \lambda) & v_{21} - (u - \lambda)v_{12} \end{pmatrix} = 0.$$

Denoting

$$v := -v_{12}$$

we obtain

$$\begin{aligned} v_{11} &= -\frac{1}{2} v' \\ v_{21} &= \frac{1}{2} v'' - v(u - \lambda). \end{aligned}$$

From the left bottom corner of the matrix equation one readily obtains (1.10.26). The lemma is proved.

Remark 1.10.11. The result of the lemma means that the matrix V satisfying (1.10.18) must have the form (1.10.13) with

$$p = -v, \quad q = \frac{1}{2} v'.$$

Exercise 1.10.12: Prove that any polynomial in λ traceless matrix $V(\lambda)$ satisfying (1.10.18) has the form (1.10.13), (1.10.11) for a differential operator

$$A = -v_0 \partial_x^{2N+1} + \sum_{k=0}^{2N-1} a_k \partial_x^k$$

where the leading coefficient v_0 is defined in (1.10.25).

Substituting the polynomial (1.10.25) into (1.10.26) we obtain

Corollary 1.10.13. *The coefficients of the polynomial $v = v(x, \lambda)$ satisfy the recursion relations*

$$\begin{aligned} v'_0 &= 0 \\ 2v'_{k+1} &= -\frac{1}{2}v'''_k + 2uv'_k + u'v_k, \quad k = 0, 1, \dots, N-1. \end{aligned} \quad (1.10.27)$$

The equation (1.10.26) can be rewritten in the form

$$u_t + \frac{1}{2}v'''_N - 2uv'_N - u'v_N = 0. \quad (1.10.28)$$

Choosing $v_0 = 1$ one obtains the following values of the first few coefficients

$$\begin{aligned} v_1 &= \frac{1}{2}u + c_1 \\ v_2 &= \frac{1}{8}(3u^2 - u'') + \frac{1}{2}c_1u + c_2 \end{aligned}$$

etc. Here c_1, c_2 are integration constants.

Remark 1.10.14. The recursion relations (1.10.27) can be recast in the form

$$v_{k+1} = \frac{1}{2}\partial_x^{-1}Mv_k + c_{k+1} \quad (1.10.29)$$

where ∂_x^{-1} is the integration operator, c_{k+1} is an integration constant and

$$M = -\frac{1}{2}\partial_x^3 + (u\partial_x + \partial_x u). \quad (1.10.30)$$

The equation (1.10.28) then reads

$$u_t = Mv_N. \quad (1.10.31)$$

For $N = 0$ the equation (1.10.31) reads

$$u_t = u'. \quad (1.10.32)$$

For $N = 1$ it follows

$$u_t = \frac{1}{4}(6uu' - u''') + c_1u'. \quad (1.10.33)$$

This is equivalent to the KdV equation. For $N = 2$ one obtains the first higher analogue of KdV:

$$u_t = \frac{1}{16}[u^V - 10uu''' + 30u^2u' - 20u'u''] + \frac{1}{2}c_1(6uu' - u''') + c_2u'. \quad (1.10.34)$$

This is a linear combination of the first three equations of the KdV hierarchy.

We will prove now that for any N the coefficients v_i of the polynomial $v(\lambda)$ can be found from (1.10.29) in the form of polynomial in u, u', u'' etc. (the so-called *differential polynomials*) with constant coefficients. To this end we introduce a generating function for these coefficients.

Theorem 1.10.15. 1) The equation

$$-\frac{1}{2} w''' + 2(u - \lambda) w' + u' w = 0 \quad (1.10.35)$$

has a solution in the form of formal series in inverse powers of λ

$$w = 1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \dots \quad (1.10.36)$$

where the coefficients w_k are polynomials in $u, u', \dots, u^{(2k-2)}$. These polynomials are uniquely determined by the normalization condition

$$w_k|_{u=0} = 0, \quad k = 1, 2, \dots \quad (1.10.37)$$

2) Define the polynomials $v^{[k]}(\lambda)$ putting

$$v^{[k]}(\lambda) = [\lambda^k w(\lambda)] \bmod \lambda^{-1} = \lambda^k + w_1 \lambda^{k-1} + \dots + w_k \quad (1.10.38)$$

where the series $w(\lambda)$ is uniquely determined by (1.10.36), (1.10.37). Then any polynomial $v(\lambda)$ of degree N satisfying (1.10.28) has the form

$$v(\lambda) = \sum_{k=0}^N c_{N-k} v^{[k]}(\lambda) \quad (1.10.39)$$

where c_0, c_1, \dots, c_N are arbitrary constants, $c_0 \neq 0$.

Proof. Plug the series (1.10.36) into equation (1.10.35) and collect the coefficient of λ^{-k} . This gives the recursion relation coinciding with (1.10.27)

$$2 w'_{k+1} = -\frac{1}{2} w'''_k + 2 u w'_k + u' w_k, \quad k \geq 0 \quad (1.10.40)$$

(we denote $w_0 = 1$). Let us show that this recursion admits a solution in the form of differential polynomials. To do this we will find a first integral of the differential equation (1.10.35). First we prove

Lemma 1.10.16. The equation (1.10.35) is equivalent to commutativity of the matrix

$$W(\lambda) = \begin{pmatrix} \frac{1}{2} w'(\lambda) & -w(\lambda) \\ \frac{1}{2} w'' - (u - \lambda) w & -\frac{1}{2} w'(\lambda) \end{pmatrix} \quad (1.10.41)$$

with the operator $\mathcal{L}(\lambda)$:

$$[\mathcal{L}(\lambda), W(\lambda)] = 0 \quad \Leftrightarrow \quad W'(\lambda) = [W(\lambda), U(\lambda)]. \quad (1.10.42)$$

Proof is analogous to the proof of Lemma 1.10.10.

Corollary 1.10.17. For a solution w of (1.10.35) the expression

$$c(\lambda) := \det W(\lambda) = \frac{1}{2} w'' w - \frac{1}{4} w'^2 - (u - \lambda) w^2 \quad (1.10.43)$$

does not depend on x .

Proof follows from (1.10.42).

The first integral $c(\lambda)$ must be a series of the form

$$c(\lambda) = \lambda + c_0 + \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \dots \quad (1.10.44)$$

with some constants c_0, c_1, \dots . From (1.10.43) a new recursion for the coefficients of the series $w(\lambda)$ readily follows

$$2 w_{n+1} = \sum_{k+l=n} \left[-\frac{1}{2} w_k'' w_l + \frac{1}{4} w_k' w_l' + u w_k w_l \right] - \sum_{k=1}^n w_k w_{n-k+1} + c_n, \quad n = 0, 1, \dots \quad (1.10.45)$$

The solution w_1, w_2, \dots to this recursion is clearly unique for arbitrary given constants c_0, c_1, \dots . Moreover, it is a differential polynomial in $u, u', \dots, u^{(2n)}$. For the solution satisfying the normalization (1.10.37) one sets to zero all the integration constants $c_0 = c_1 = \dots = 0$. The Theorem is proved.

Exercise 1.10.18: Let $w_c(\lambda)$ be a solution to (1.10.43) with a given integration constants (1.10.44). Prove that it is obtained from the normalized solution $w(\lambda)$ by

$$w_c(\lambda) = \sqrt{1 + \frac{c_0}{\lambda} + \frac{c_1}{\lambda^2} + \dots} w(\lambda). \quad (1.10.46)$$

We are now ready to define the k -th equation of the KdV hierarchy by one of the following two equivalent representations

$$\frac{\partial u}{\partial t_k} = M w_k \quad (1.10.47)$$

$$= 2 \partial_x w_{k+1} \quad (1.10.48)$$

This an evolution PDE with the right hand side depending on $u, u', \dots, u^{(2k+1)}$. The zero curvature representation of this equation reads

$$V_x^{[k]}(\lambda) - U_{t_k}(\lambda) = \left[V^{[k]}(\lambda), U(\lambda) \right] \quad (1.10.49)$$

$$V^{[k]}(\lambda) = \begin{pmatrix} \frac{1}{2} v'(\lambda) & -v(\lambda) \\ \frac{1}{2} v'' - (u - \lambda) v & -\frac{1}{2} v'(\lambda) \end{pmatrix}$$

where $v(\lambda) = v^{[k]}(\lambda) = \lambda^k + w_1 \lambda^{k-1} + \dots + w_k$.

Any other evolutionary equation admitting the zero curvature representation (1.10.18) with a matrix $V(\lambda)$ polynomial in λ is a linear combination with constant coefficients of the equations (1.10.47):

$$\begin{aligned} u_t &= M \sum_{k=0}^N c_{N-k} w_k \\ &= 2 \partial_x \sum_{k=0}^N c_{N-k} w_{k+1} \end{aligned} \tag{1.10.50}$$

$$V_x(\lambda) - U_t(\lambda) = [V(\lambda), U(\lambda)]$$

$$V(\lambda) = \sum_{k=0}^N c_{N-k} V^{[k]}(\lambda).$$

The first few equations of the KdV hierarchy along with the matrices $V^{[k]}(\lambda)$ realizing the zero curvature representation are listed below.

$$u_{t_0} = u', \quad V^{[0]}(\lambda) = \begin{pmatrix} 0 & -1 \\ \lambda - u & 0 \end{pmatrix} \tag{1.10.51}$$

$$u_{t_1} = \frac{1}{4} (6 u u' - u'''), \quad V^{[1]}(\lambda) = \begin{pmatrix} u' & -2(u + 2\lambda) \\ u'' - 2(u - \lambda)(u + 2\lambda) & -u' \end{pmatrix} \tag{1.10.52}$$

$$u_{t_2} = \frac{1}{16} [u^V - 10 u u''' + 30 u^2 u' - 20 u' u''], \tag{1.10.53}$$

$$V^{[2]}(\lambda) = \begin{pmatrix} \frac{1}{4} \lambda u' + \frac{1}{16} (6 u u' - u''') & -[\lambda^2 + \frac{1}{2} \lambda u + \frac{1}{8} (3 u^2 - u'')] \\ \frac{1}{4} \lambda u'' + \frac{1}{16} (6 u'^2 + 6 u u'' - u^{IV}) & -\frac{1}{4} \lambda u' - \frac{1}{16} (6 u u' - u''') \\ +(\lambda - u) [\lambda^2 + \frac{1}{2} \lambda u + \frac{1}{8} (3 u^2 - u'')] & \end{pmatrix}.$$

We are going now to prove that that the equations (1.10.47) (or (1.10.48)) commute pairwise. The main tools for this is provided by

Lemma 1.10.19. *Let the t -dependence of u be determined by the zero curvature equation (1.10.18) with a polynomial matrix $V(\lambda)$. Then the matrix $W(\lambda)$ defined in Lemma 1.10.16 satisfies the equation*

$$[W(\lambda), \partial_t + V(\lambda)] = 0 \quad \Leftrightarrow \quad \partial_t W(\lambda) = [W(\lambda), V(\lambda)]. \tag{1.10.54}$$

Proof. Recall that the zero curvature equation (1.10.18) is equivalent to the commutativity

$$[\mathcal{L}(\lambda), \mathcal{A}(\lambda)] = 0, \quad \mathcal{A}(\lambda) = \partial_t + V(\lambda).$$

Using Jacobi identity

$$[\mathcal{L}(\lambda), [W(\lambda), \mathcal{A}(\lambda)]] + [\mathcal{A}(\lambda), [\mathcal{L}(\lambda), W(\lambda)]] + [W(\lambda), [\mathcal{A}(\lambda), \mathcal{L}(\lambda)]] = 0$$

and (1.10.42) we conclude that the matrix

$$\tilde{W}(\lambda) := [W(\lambda), \mathcal{A}(\lambda)] = -W_t(\lambda) + [W(\lambda), V(\lambda)]$$

is another solution to the commutativity equation (1.10.42). Clearly the coefficients of the matrix series $\tilde{W}(\lambda)$ are differential matrix valued polynomials in u vanishing identically for $u \equiv 0$. Moreover, it is easy to see that

$$\tilde{W}(\lambda)_{12} = O\left(\frac{1}{\lambda}\right).$$

Due to uniqueness we conclude that $\tilde{W}(\lambda) = 0$. The lemma is proved.

Corollary 1.10.20. *In the situation of Lemma 1.10.19 the t -derivative of the series $w(\lambda)$ defined in (1.10.35) - (1.10.38) can be found from the equation*

$$w_t(\lambda) = w(\lambda)v_x(\lambda) - w_x(\lambda)v(\lambda) \quad (1.10.55)$$

where $v(\lambda) = -(V(\lambda))_{12}$.

Proof. Using the formula (1.10.24) for the matrix $V(\lambda)$ we calculate the commutator in (1.10.54). This gives (1.10.55).

Corollary 1.10.21. *All the equations (1.10.48) of the KdV hierarchy commute pairwise.*

Proof. Commutativity of equations

$$\begin{aligned} u_{t_k} &= 2w'_{k+1} \\ u_{t_l} &= 2w'_{l+1} \end{aligned}$$

will follow from a stronger equation

$$\partial_{t_l} w_{k+1} = \partial_{t_k} w_{l+1}.$$

Indeed, from (1.10.55) it follows that

$$\partial_{t_m} w_n = \sum_{p+q=n+m} (w_p w'_q - w'_p w_q).$$

The two choices $(m, n) = (l, k+1)$ and $(m, n) = (k+1, l)$ yield the same result. The corollary is proved.

Corollary 1.10.22. *The linear operators*

$$\mathcal{A}_k(\lambda) = \partial_{t_k} + V^{[k]}(\lambda), \quad \mathcal{A}_l(\lambda) = \partial_{t_l} + V^{[l]}(\lambda) \quad (1.10.56)$$

commute pairwise for any k, l :

$$[\mathcal{A}_k(\lambda), \mathcal{A}_l(\lambda)] = 0 \quad \Leftrightarrow \quad \partial_{t_k} V^{[l]}(\lambda) - \partial_{t_l} V^{[k]}(\lambda) = [V^{[l]}(\lambda), V^{[k]}(\lambda)]. \quad (1.10.57)$$

Proof. Using commutativity

$$[\mathcal{L}(\lambda), \mathcal{A}_k(\lambda)] = 0, \quad [\mathcal{L}(\lambda), \mathcal{A}_l(\lambda)] = 0$$

and Jacobi identity one derives that

$$\left[\mathcal{L}(\lambda), \partial_{t_k} V^{[l]}(\lambda) - \partial_{t_l} V^{[k]}(\lambda) + [V^{[k]}(\lambda), V^{[l]}(\lambda)] \right] = 0.$$

The matrix

$$\partial_{t_k} V^{[l]}(\lambda) - \partial_{t_l} V^{[k]}(\lambda) + [V^{[k]}(\lambda), V^{[l]}(\lambda)] \quad (1.10.58)$$

being polynomial in λ must be a linear combination of the matrices $V^{[m]}(\lambda)$. As the matrix (1.10.58) vanishes for $u \equiv 0$, all the coefficients of the linear combination must be equal to zero. The corollary is proved.

1.11 Stationary equations of KdV hierarchy, spectral curves and eigenfunctions of the Schrödinger operator

Let us fix a flow of the KdV hierarchy represented in the zero curvature form (1.10.50) with some matrix $V(\lambda)$. The stationary points of this flow gives an invariant submanifold for all flows of the KdV hierarchy (1.10.52) due to commutativity. The equations for the stationary points $u_t = 0$ can be written in the form

$$\sum_{k=-1}^N c_{N-k} w_{k+1} = 0 \quad (1.11.1)$$

where we have added one more constant c_{N+1} obtained from integration of the equation

$$2 \partial_x \sum_{k=0}^N c_{N-k} w_{k+1} = 0 \quad \Rightarrow \quad \sum_{k=0}^N c_{N-k} w_{k+1} = -c_{N+1}.$$

Without loss of generality we may normalize $c_0 = 1$. So, the stationary equation (1.11.1) is an ODE of the order $2N$ depending on $N+1$ constants c_1, \dots, c_{N+1} . This equation admits *Lax representation* on λ -matrices

$$V_x(\lambda) = [V(\lambda), U(\lambda)]. \quad (1.11.2)$$

With the stationary equation (1.11.2) we associate the *spectral curve* defined by the characteristic equation

$$\Gamma : \det(i\nu \cdot 1 - V(\lambda)) = -\nu^2 + R(\lambda) = 0, \quad R(\lambda) = \det V(\lambda). \quad (1.11.3)$$

Theorem 1.11.1. *The spectral curve associated with the stationary equation (1.11.2) does not depend on x .*

Proof. Denote

$$Y = Y(x, x_0, \lambda)$$

the fundamental matrix solution to the equation

$$\mathcal{L}(\lambda)Y = 0, \quad Y|_{x=x_0} = 1.$$

Then the solution $V(x, \lambda)$ to the linear differential equation (1.11.2) with a given initial datum $V(x_0, \lambda)$ can be written in the form

$$V(x, \lambda) = Y(x, x_0, \lambda)V(x_0, \lambda)Y^{-1}(x, x_0, \lambda).$$

Hence

$$\det(i\nu - V(x, \lambda)) = \det(i\nu - V(x_0, \lambda)).$$

The Theorem is proved.

This simple but very important statement gives a possibility to construct first integrals of the stationary equations (1.11.2).

Example 1.11.2. For $N = 1$ the stationary equation reads

$$\frac{1}{4} (3u^2 - u'') + c_1 u + c_2 = 0. \quad (1.11.4)$$

The Lax representation is given by the matrix

$$V(\lambda) = \begin{pmatrix} \frac{u'}{4} & -\lambda - \frac{u}{2} - c_1 \\ \frac{u''}{4} - (\lambda + \frac{u}{2} + c_1)(u - \lambda) & -\frac{u'}{4} \end{pmatrix}. \quad (1.11.5)$$

The determinant of this matrix

$$\det V(\lambda) = \lambda^3 + 2c_1\lambda^2 + (c_2 + c_1^2)\lambda + J \quad (1.11.6)$$

gives the first integral of the ODE (1.11.4)

$$J = -\frac{1}{16}u'^2 + \frac{1}{8}u^3 + \frac{1}{4}c_1u^2 + \frac{1}{2}u + c_1c_2. \quad (1.11.7)$$

Of course, this is nothing but the energy integral for the equation (1.11.4) written in the form of Euler - Lagrange equation for the functional

$$S = \int L(u, u'; c_1, c_2) dx, \quad L(u, u'; c_1, c_2) = \frac{1}{4}u^3 + \frac{1}{8}u'^2 + \frac{1}{2}c_1u^2 + c_2u. \quad (1.11.8)$$

Example 1.11.3. For $N = 2$ the stationary equation is a 4th order ODE

$$\frac{1}{16} [u^{IV} - 10u u'' - 5u'^2 + 10u^3] + \frac{1}{4}c_1(3u^2 - u'') + c_2u + c_3 = 0. \quad (1.11.9)$$

The $V(\lambda)$ matrix has the usual form (1.10.24) with

$$v(\lambda) = \lambda^2 + \left(\frac{u}{2} + c_1\right)\lambda + \left(-\frac{1}{8}u'' + \frac{3}{8}u^2 + \frac{1}{2}c_1u + c_2\right). \quad (1.11.10)$$

Here one obtains two first integrals from

$$\det V(\lambda) = \lambda^5 + 2c_1\lambda^4 + (2c_2 + c_1^2)\lambda^3 + (c_3 + 2c_1c_2)\lambda^2 + J_1\lambda + J_2, \quad (1.11.11)$$

$$J_1 = \frac{1}{64} \left(5u^4 - 10uu'^2 - u''^2 + 2u'u''' \right) \quad (1.11.12)$$

$$\begin{aligned} & + \frac{1}{16}c_1(2u^3 - u'^2) + \frac{1}{4}c_2u^2 + c_1c_3 + c_2^2 \\ J_2 = & \frac{3}{32}u^5 - \frac{15}{128}u^2u'^2 - \frac{5}{64}u^3u'' - \frac{1}{128}u'^2u'' + \frac{1}{64}uu''^2 + \frac{3}{64}uu'u''' - \frac{1}{256}u''^3 \\ & + \frac{1}{32}c_1(7u^4 - 5uu'^2 - 3u^2u'' + u'u''') + \frac{1}{16}c_2(4u^3 + u'^2 - 2uu'') + \frac{1}{16}c_1^2(2u^3 - u'^2) \\ & + \frac{1}{4}c_1c_2u^2 + \frac{1}{2}c_1c_3u + \frac{1}{8}c_3(3u^2 - u'') + c_2c_3. \end{aligned} \quad (1.11.13)$$

One can check that (1.11.9) is the Euler - Lagrange equation for the functional

$$S = \int L(u, u', u''; c_1, c_2, c_3) dx \quad (1.11.14)$$

$$L(u, u', u''; c_1, c_2, c_3) = \frac{1}{32} \left(u''^2 + 10uu'^2 + 5u^4 \right) + \frac{1}{8}c_1(u'^2 + 2u^3) + \frac{1}{2}c_2u^2 + c_3u.$$

We have constructed a map

$$\text{spec} : \mathbb{C}^{3N+1} \rightarrow \mathbb{C}^{2N+1} \quad (1.11.15)$$

from the space \mathbb{C}^{3N+1} with the coordinates

$$(u, u', \dots, u^{2N-1}, c_1, \dots, c_{N+1}) \in \mathbb{C}^{3N+1} \quad (1.11.16)$$

to the $(2N + 1)$ -dimensional space \mathbb{C}^{2N+1} of hyperelliptic curves of the form

$$\begin{aligned} \Gamma : \nu^2 = R(\lambda), \quad R(\lambda) = \lambda^{2N+1} + a_1\lambda^{2N-1} + \dots + a_{2N+1}, \\ (a_1, \dots, a_{2N+1}) \in \mathbb{C}^{2N+1}. \end{aligned} \quad (1.11.17)$$

We have proved that the x -dependence of the vector $(u, u', \dots, u^{(2N-1)})$ lives along the fibers of the map (1.11.15). Our goal is to prove surjectivity of the map (1.11.15) and, moreover, for a generic point $\Gamma \in \mathbb{C}^{2N+1}$ the fiber is isomorphic to the Jacobian of the spectral curve:

$$\text{spec}^{-1}(\Gamma) = J(\Gamma). \quad (1.11.18)$$

To this end we will first show that the Sturm - Liouville equation $Ly = \lambda y$ with the potential satisfying (1.11.1) possesses a solution that is a BA function on the spectral curve (1.11.3).

It will be more convenient to work with the matrix realization $\mathcal{L}(\lambda)$. Recall that the Lax equation (1.11.2) is equivalent to the commutativity

$$[\mathcal{L}(\lambda), V(\lambda)] = 0. \quad (1.11.19)$$

Given a point $(\lambda, \nu) \in \Gamma$ let us choose the solution of the Sturm - Liouville equation in the form

$$\vec{\psi} = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \quad \mathcal{L}(\lambda) \vec{\psi} = 0, \quad V(\lambda) \vec{\psi} = i\nu \vec{\psi}. \quad (1.11.20)$$

If λ is not a ramification point of the hyperelliptic curve (1.11.3) then the eigenvector of the matrix $V(\lambda)$ with the eigenvalue $i\nu$ is determined uniquely up to a normalization factor; we will fix the latter by requiring that

$$\psi|_{x=x_0} = 1. \quad (1.11.21)$$

The resulting eigenvector will be denoted

$$\vec{\psi}(x, x_0, P), \quad P = (\lambda, \nu) \in \Gamma. \quad (1.11.22)$$

Theorem 1.11.4. *The eigenvector $\vec{\psi}(x, x_0, P)$ is a meromorphic vector function on $\Gamma \setminus \infty$.*

Proof. Let us introduce the fundamental matrix of the operator $\mathcal{L}(\lambda)$

$$Y(x, x_0, \lambda) = \begin{pmatrix} y_1(x, x_0, \lambda) & y_2(x, x_0, \lambda) \\ y_1'(x, x_0, \lambda) & y_2'(x, x_0, \lambda) \end{pmatrix}. \quad (1.11.23)$$

This matrix, as a function in x , satisfies

$$\mathcal{L}(\lambda) Y(x, x_0, \lambda) = 0.$$

It is unimodular, $\det Y(x, x_0, \lambda) = 1$. Moreover, it is an entire function in $\lambda \in \mathbb{C}$. Any solution to the Sturm - Liouville equation

$$\mathcal{L}(\lambda) \mathbf{y} = 0$$

with the Cauchy data

$$y|_{x=x_0} = y_0, \quad y'|_{x=x_0} = y_0'$$

can be represented in the form

$$\mathbf{y} = Y(x, x_0, \lambda) \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

Denote

$$\begin{pmatrix} 1 \\ \chi(x_0, P) \end{pmatrix} := \vec{\psi}(x_0, x_0, P).$$

Then

$$\vec{\psi}(x, x_0, P) = Y(x, x_0, \lambda) \begin{pmatrix} 1 \\ \chi(x_0, P) \end{pmatrix}, \quad P = (\lambda, \nu) \in \Gamma. \quad (1.11.24)$$

The vector

$$\begin{pmatrix} 1 \\ \chi(x_0, P) \end{pmatrix}$$

is an eigenvector of the matrix

$$V(x_0, \lambda) = \begin{pmatrix} -\frac{1}{2}v' & v \\ -\frac{1}{2}v'' + (u - \lambda)v & \frac{1}{2}v' \end{pmatrix}_{x=x_0}$$

with the eigenvalue $i\nu$. Applying the rules of linear algebra one obtains

Lemma 1.11.5. *The function $\chi(x_0, P)$ has the form*

$$\chi(x_0, P) = \frac{i\nu + \frac{1}{2}v'(x_0, \lambda)}{v(x_0, \lambda)}, \quad P = (\lambda, \nu) \in \Gamma. \quad (1.11.25)$$

This completes the proof of the theorem as the function (1.11.25) is rational on Γ and $Y(x, x_0, \lambda)$ is analytic for $\lambda \in \mathbb{C}$.

We will now study the poles of the first component $\psi(x, x_0, P)$ of the vector function $\vec{\psi}(x, x_0, P)$.

Lemma 1.11.6. *The following formula holds true for the logarithmic derivative of $\psi(x, x_0, P)$*

$$(\log \psi(x, x_0, P))' = \chi(x, P) = \frac{i\nu + \frac{1}{2}v'(x, \lambda)}{v(x, \lambda)}. \quad (1.11.26)$$

Proof. The formula (1.11.26) is obvious for $x = x_0$. Changing the normalization point x_0 multiplies the eigenvector of the matrix $V(x, \lambda)$ by a x -independent factor. Such a rescaling of $\vec{\psi}$ does not change the logarithmic derivative. This proves (1.11.26) for any x . The lemma is proved.

Corollary 1.11.7.

$$\psi(x, x_0, P) = \exp \int_{x_0}^x \frac{i\nu + \frac{1}{2}v'(y, \lambda)}{v(y, \lambda)} dy. \quad (1.11.27)$$

Corollary 1.11.8. *Denote $P_+ = (\lambda, \nu)$ and $P_- = (\lambda, -\nu)$ the two points of the spectral curve with the same value of the spectral parameter λ . Then*

$$\psi(x, x_0, P_+) \psi(x, x_0, P_-) = \frac{v(x, \lambda)}{v(x_0, \lambda)}. \quad (1.11.28)$$

Let us assume that the polynomial $v(x, \lambda)$ of the degree N is monic.

Theorem 1.11.9. *At the infinite point of the spectral curve the function $\psi(x, x_0, P)$ has the following exponential asymptotics*

$$\psi(x, x_0, P) = \left(1 + O\left(\frac{1}{k}\right)\right) e^{ik(x-x_0)} \quad (1.11.29)$$

$$P = (\lambda, \nu), \quad \lambda = k^2 \rightarrow \infty, \quad \nu = k^{2N+1} \left(1 + O\left(\frac{1}{k^2}\right)\right).$$

The function $\psi(x, x_0, P)$ has at most N poles on the spectral curve.

If all the $(2N + 1)$ roots of the polynomial $R(\lambda) = \det V(x, \lambda)$ are pairwise distinct then the number of poles of the function $\psi(x, x_0, P)$ for generic x is exactly equal to $N = \text{genus}$ of the spectral curve.

There is a natural fiber bundle over the base $\mathbb{C}^{2N+1} \ni \Gamma$: the fiber over the point Γ is the Jacobian $J(\Gamma)$. Denote \mathcal{J}_N the total space of this fiber bundle (the so-called *universal Jacobian*). We have constructed a map

$$\text{SPEC} : \mathbb{C}^{3N+1} \ni (u, u', \dots, u^{(2N-1)}, c_1, \dots, c_{N+1}) \rightarrow \mathcal{J}_N. \quad (1.11.30)$$

The map associates with a given point in the phase space of the stationary KdV equation of the order $2N$ depending on the parameters c_1, \dots, c_{N+1} the pair $(\Gamma, (D - N\infty))$ where D is the divisor of poles of the eigenfunction $\psi(x, x_0, P)$ and $\infty \in \Gamma$ is the infinite point of the hyperelliptic curve. By construction

$$\text{SPEC}(u(x), u'(x), \dots, u^{(2N-1)}(x), c_1, \dots, c_{N+1})$$

does not depend on x assuming $u(x)$ satisfies the stationary equation (1.11.1).

We have now to prove that *any* generic hyperelliptic curve of genus N with an arbitrary nonspecial divisor D of degree N belongs to the image of the map (1.11.30).

Now let us describe the time dependence of the solutions to the stationary equations (1.11.1) on the times of the KdV hierarchy (1.10.48). Let us assume that u depends on t_k according to the equation

$$u_{t_k} = 2w'_{k+1}.$$

Due to commutativity of the flows of the KdV hierarchy, the stationary manifold (1.11.1) is invariant with respect to this flow. From the zero curvature equation (1.10.57) it follows that the dependence on t_k of the matrix $V(\lambda)$ in (1.11.2) is determined by the equation

$$\partial_{t_k} V(\lambda) = \left[V(\lambda), V^{[k]}(\lambda) \right]. \quad (1.11.31)$$

Therefore the three operators $\mathcal{L}(\lambda)$, $\partial_{t_k} + V^{[k]}(\lambda)$ and $V(\lambda)$ commute pairwise. We choose the time dependence of the function $\vec{\psi}$ from the following conditions:

$$\begin{aligned} \mathcal{L}(\lambda) \vec{\psi} &= 0 \\ \partial_{t_k} \vec{\psi} + V^{[k]}(\lambda) \vec{\psi} &= 0 \\ V(\lambda) \vec{\psi} &= i\nu \vec{\psi}. \end{aligned} \quad (1.11.32)$$

The time dependence of the function ψ is obtained in the form

$$\partial_{t_k} \psi = -\frac{1}{2} v_x^{[k]} \psi + v^{[k]} \psi_x. \quad (1.11.33)$$

From this equation and from the formula (1.11.26) for the logarithmic derivative of ψ it follows that

$$\partial_{t_k} \log \psi = i\nu \frac{v^{[k]}}{v} + \frac{1}{2} \frac{v_x v^{[k]} - v_x^{[k]} v}{v}. \quad (1.11.34)$$

Exercise 1.11.10: Prove that

$$i\nu \frac{v^{[k]}}{v} + \frac{1}{2} \frac{v_x v^{[k]} - v_x^{[k]} v}{v} = i\lambda^{\frac{2k+1}{2}} + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty. \quad (1.11.35)$$

We will normalize the common eigenvector (1.11.32) in such a way that

$$\psi|_{x=x_0, t_k=t_k^0} = 1 \quad (1.11.36)$$

for some point (x_0, t_k^0) .

Theorem 1.11.11. *The normalized function ψ has at most N poles on the spectral curve Γ . At the infinite point it has the exponential asymptotics of the form*

$$\psi = \left(1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right) e^{i\sqrt{\lambda}(x-x_0) + i\lambda^{\frac{2k+1}{2}}(t-t_k^0)}, \quad \lambda \rightarrow \infty. \quad (1.11.37)$$

2 Riemann surfaces and theta-functions

2.1 Definition of a Riemann surface. Local coordinates. Branch points. Hyperelliptic Riemann surfaces. Multiplicity of a branch point.

For the geometric representation of multi-valued functions of a complex variable $w = w(z)$ it is not convenient to regard z as a point of the complex plane. For example, take $w = \sqrt{z}$. On the positive real semiaxis $z \in \mathbb{R}$, $z > 0$ the two branches $w_1 = +\sqrt{z}$ and $w_2 = -\sqrt{z}$ of this function are well defined by the condition $w_1 > 0$. This is no longer possible on the complex plane. Indeed, the two values $w_{1,2}$ of the square root of $z = r e^{i\psi}$

$$w_1 = \sqrt{r} e^{i\frac{\psi}{2}}, \quad w_2 = -\sqrt{r} e^{i\frac{\psi}{2}} = \sqrt{r} e^{i\frac{\psi+2\pi}{2}}, \quad (2.1.1)$$

interchange when passing along a cycle

$$z(t) = r e^{i(\psi+t)}, \quad t \in [0, 2\pi]$$

encircling the point $z = 0$. It is possible to select a branch of the square root as a function of z by restricting the domain of this function – for example, by making a cut from zero to infinity. We now explain another way (which will be basic for this course), using the same function \sqrt{z} as an elementary example. Consider the graph of this two-valued function in \mathbb{C}^2 with complex coordinates z, w , i.e., the points of the form $(z, +\sqrt{z})$, $(z, -\sqrt{z})$. The two branches of this graph intersect at the point $(0, 0)$, the branch point of this algebraic function. Note that this graph can be given in \mathbb{C}^2 by the single (complex) equation

$$F(z, w) = w^2 - z = 0. \quad (2.1.2)$$

The function $w = \sqrt{z}$ is a single-valued function of a point of the graph of (2.1.2): it has the form of the projection $(z, w) \rightarrow w$.

Starting from this example, we give the following preliminary definition.

Definition 2.1.1. *Let*

$$F(z, w) = \sum_{i=0}^n a_i(z) w^{n-i}$$

be a polynomial in the variables z and w . It determines an (n -valued) algebraic function $w = w(z)$. The Riemann surface Γ of this function is given in \mathbb{C}^2 by the equation $F(z, w) = 0$.

As in the example analyzed above, the multivalued function $w = w(z)$ becomes a single-valued function $w = w(P)$ of a point P of the Riemann surface Γ : if $P = (z, w) \in \Gamma$, then $w(P) = w$ (the projection of the graph on the the w -axis).

Remark 2.1.2. This definition of a Riemann surface is a simplified one. It coincides with the traditional definition only if the algebraic curve $F(z, w) = 0$ is nonsingular. We shall return to this question later (see Lecture 3).

Remark 2.1.3. Below we shall see that the function $w = w(P)$ is not only a single-valued but also an analytic (a holomorphic) function on the Riemann surface Γ considered as a complex manifold of complex dimension one.

From the algebraic point of view a Riemann surface is a (complex) algebraic curve. From the real point of view it is a two-dimensional surface in $\mathbb{C}^2 = \mathbb{R}^4$ given by the two equations

$$\left. \begin{aligned} \Re F(z, w) &= 0 \\ \Im F(z, w) &= 0 \end{aligned} \right\}.$$

In the theory of functions of a complex variable one encounters also more complicated (nonalgebraic) Riemann surfaces, where $F(z, w)$ is not a polynomial. For example, the equation $e^w - z = 0$ determines the Riemann surface of the logarithm. Such Riemann surfaces will not be considered here.

We now discuss the important property of nonsingularity for points of a Riemann surface.

Definition 2.1.4. A point $P = (z_0, w_0) \in \Gamma$ of a Riemann surface $\Gamma = \{(z, w) | F(z, w) = 0\}$ is said to be nonsingular if the complex gradient vector

$$\text{grad}_{\mathbb{C}} F|_{P_0} = \left(\frac{\partial F(z_0, w_0)}{\partial z}, \frac{\partial F(z_0, w_0)}{\partial w} \right) \quad (2.1.3)$$

does not vanish. A Riemann surface Γ is nonsingular if all its points are nonsingular.

Lemma 2.1.5 (Complex implicit function theorem). Let $F(z, w)$ be an analytic function of the variables z, w in a neighborhood of the point $P_0 = (z_0, w_0)$ such that $F(z_0, w_0) = 0$ and $\partial_w F(z_0, w_0) \neq 0$. Then there exists a unique function $w = w(z)$ such that $F(z, w(z)) = 0$ and $w(z_0) = w_0$. This function is analytic in z in some neighborhood of z_0 .

Proof. Let $z = x + iy$ and $w = u + iv$, $F = f + ig$. Then the equation $F(z, w) = 0$ can be written as the system

$$\begin{cases} f(x, y, u, v) = 0 \\ g(x, y, u, v) = 0 \end{cases} \quad (2.1.4)$$

The condition of the real implicit function theorem are satisfied for this system: the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}_{(z_0, w_0)}$$

is nonsingular because

$$\det \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \left| \frac{\partial F}{\partial w} \right|^2$$

(we use only the analyticity in w of the function $F(z, w)$). Thus, in some neighborhood of (z_0, w_0) the solution of the equation $F(z, w) = 0$ can be written as a smooth function $w = w(z)$, and this function is uniquely determined by the condition $w(z_0) = w_0$. Let us

verify its analyticity: $\frac{\partial w}{\partial \bar{z}} = 0$. Differentiating the identity $F(z, w(z)) = 0$ with respect to \bar{z} , we get that

$$\frac{\partial F}{\partial \bar{z}} + \frac{\partial w}{\partial \bar{z}} \frac{\partial F}{\partial w} + \frac{\partial \bar{w}}{\partial \bar{z}} \frac{\partial F}{\partial \bar{w}} = 0.$$

But $\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial \bar{w}} = 0$ in view of the analyticity of $F(z, w)$ and $\frac{\partial F}{\partial w} \neq 0$. From this, $\frac{\partial w}{\partial \bar{z}} = 0$. \square

Let $P_0 = (z_0, w_0)$ be a nonsingular point of the surface Γ . Suppose, for example, that the derivative $\frac{\partial F}{\partial w}$ is nonzero at this point. Then by the lemma, in a neighborhood of the point P_0 , the surface Γ admits a parametric representation of the form

$$(z, w(z)) \in \Gamma, \quad w(z_0) = w_0, \quad (2.1.5)$$

where the function $w(z)$ is holomorphic. Therefore, in this case z is a complex local coordinate also called *local parameter* on Γ in a neighborhood of $P_0 = (z_0, w_0) \in \Gamma$. Similarly, if the derivative $\frac{\partial F}{\partial z}$ is nonzero at the point $P_0 = (z_0, w_0)$, then we can take w as a local parameter (an obvious variant of the lemma), and the surface Γ can be represented in a neighborhood of the point P_0 under study in the following parametric form

$$(z(w), w) \in \Gamma, \quad z(w_0) = z_0, \quad (2.1.6)$$

where the function $z(w)$ is, of course, holomorphic. For a nonsingular Riemann surface it is possible to use both ways for representing the Riemann surface on the intersection of domains of the first and second types, i.e., at points of Γ where $\frac{\partial F}{\partial w} \neq 0$ and $\frac{\partial F}{\partial z} \neq 0$ simultaneously. The resulting *transition functions* $w = w(z)$ and, inversely, $z = z(w)$ are holomorphic. Below we consider mainly nonsingular Riemann surfaces (as already mentioned above, in this case our definition of a Riemann surface coincides with the more general one explained below). The preceding arguments show that such Riemann surfaces are complex manifolds (with complex dimension 1). The choice of the variables z or w as a local parameter is not always most convenient. We shall also encounter other ways of choosing a local parameter τ so that the point (z, w) of Γ can be represented locally in the form

$$z = z(\tau), \quad w = w(\tau) \quad (2.1.7)$$

where $z(\tau)$ and $w(\tau)$ are holomorphic functions of τ , and

$$\left(\frac{dz}{d\tau}, \frac{dw}{d\tau} \right) \neq 0. \quad (2.1.8)$$

Remark 2.1.6. It is easy to show that the nonsingularity condition implies irreducibility of the algebraic curve $F(z, w) = 0$ i.e., the impossibility of decomposing its equation into nontrivial factors $F = F_1 F_2$ where F_1 and F_2 are polynomials of positive degree (verify it!).

Let us consider a Riemann surface Γ defined in \mathbb{C}^2 by a monic polynomial

$$F(z, w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0. \quad (2.1.9)$$

Here the $a_1(z), \dots, a_n(z)$ are polynomials in z . This Riemann surface is realized as an n -sheeted covering of the z -plane. The precise meaning of this is as follows: let $\pi : \Gamma \rightarrow \mathbb{C}$ be the projection of the Riemann surface onto the z -plane given by the formula

$$\pi(z, w) = z. \quad (2.1.10)$$

Then for almost all z the preimage $\pi^{-1}(z)$ consists of n distinct points

$$(z, w_1(z)), (z, w_2(z)), \dots, (z, w_n(z)), \quad (2.1.11)$$

of the surface Γ where $w_1(z), \dots, w_n(z)$ are the n roots of (2.1.9) for given value of z . For certain values of z , some of the points of the preimage can merge. This happens at the *branch points* (z_0, w_0) of the Riemann surface where the partial derivative $F_w(z, w)$ vanishes (recall that we consider only nonsingular curves so far).

Lemma 2.1.7. *Let (z_0, w_0) be a branch point of a Riemann surface. Then there exists a positive integer $k > 1$ and k functions $w_1(z), \dots, w_k(z)$ analytic on a sector $S_{\rho, \phi}$ of the punctured disc*

$$0 < |z - z_0| < \rho, \quad \arg(z - z_0) < \phi$$

for sufficiently small ρ and any positive $\phi < 2\pi$ such that

$$F(z, w_j(z)) \equiv 0 \quad \text{for } z \in S_{\rho, \phi}, \quad j = 1, \dots, k.$$

The functions $w_1(z), \dots, w_k(z)$ are continuous in the closure $\bar{S}_{\rho, \phi}$ and

$$w_1(z_0) = \dots = w_k(z_0) = w_0.$$

Proof. By the nonsingularity assumption $F_z(z_0, w_0) \neq 0$. So the complex curve $F(z, w) = 0$ can be locally parametrized in the form $z = z(w)$ where the analytic function $z(w)$ is uniquely determined by the condition $z(w_0) = z_0$. Consider the first nontrivial term of the Taylor expansion of this function

$$z(w) = z_0 + \alpha_k(w - w_0)^k + \alpha_{k+1}(w - w_0)^{k+1} + \dots, \quad \alpha_k \neq 0.$$

Introduce an auxiliary function

$$\begin{aligned} f(w) &= \beta(w - w_0) \left[1 + \frac{\alpha_{k+1}}{\alpha_k}(w - w_0) + O((w - w_0)^2) \right]^{\frac{1}{k}} \\ &= \beta(w - w_0) \left[1 + \frac{\alpha_{k+1}}{k \alpha_k}(w - w_0) + O((w - w_0)^2) \right] \end{aligned}$$

where the complex number β is chosen in such a way that $\beta^k = \alpha_k$. The function $f(w)$ is analytic for sufficiently small $|w - w_0|$. Observe that $f'(w_0) = \beta \neq 0$. Therefore the analytic inverse function f^{-1} locally exists. The needed k functions $w_1(z), \dots, w_k(z)$ can be constructed as follows

$$w_j(z) = f^{-1} \left(e^{\frac{2\pi i(j-1)}{k}} (z - z_0)^{1/k} \right), \quad j = 1, \dots, k \quad (2.1.12)$$

where we choose an arbitrary branch of the k -th root of $(z - z_0)$ for $z \in S_{\rho, \phi}$. \square

these values are called *ramification points* of the Riemann surface. If z_0 is a branch point then the polynomial $F(z_0, w)$ has multiple roots. The multiple roots can be determined from the system

$$\left. \begin{array}{l} F(z_0, w) = 0 \\ F_w(z_0, w) = 0 \end{array} \right\}. \quad (2.1.13)$$

Recall that the projection π is a local isomorphism in a neighborhood of the points where the partial derivative $F_w \neq 0$. The ramification points on the z -plane can be determined, therefore, as the zeros of the *discriminant* $R(z)$ of $F(z, w)$ considered as a z -dependent polynomial in w :

$$R(z) = \prod_{i < j} (w_i(z) - w_j(z))^2. \quad (2.1.14)$$

The right hand side of (2.1.14) is a symmetric function of the roots hence it can be expressed as a polynomial in the coefficients $a_1(z), a_2(z), \dots, a_n(z)$. It also is the greatest common divisor of the z -dependent polynomials in w , $F(z, w)$ and $F_w(z, w)$. The discriminant can be computed as the determinant of a $(2n-1) \times (2n-1)$ matrix constructed from the coefficients of the polynomials

$$F = w^n + a_1 w^{n-1} + \dots + a_{n-1} w + a_n$$

and

$$F_w = n w^{n-1} + (n-1)a_1 w^{n-2} + \dots + a_{n-1}$$

$$R(z) = (-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_n & 0 & 0 & \dots & 0 \\ 0 & 1 & a_1 & \dots & a_{n-1} & a_n & 0 & \dots & 0 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & a_n \\ n & (n-1)a_1 & (n-2)a_2 & \dots & a_{n-1} & 0 & 0 & \dots & 0 \\ 0 & n & (n-1)a_1 & \dots & 2a_{n-2} & a_{n-1} & 0 & \dots & 0 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & a_{n-1} \end{pmatrix}. \quad (2.1.15)$$

For example, the discriminant of a cubic monic polynomial is given by the formula

$$R(z) = -\det \begin{pmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 3 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3 & 2a_1 & a_2 \end{pmatrix} = a_1^2 a_2^2 - 4a_2^3 - 4a_1^3 a_3 + 18a_1 a_2 a_3 - 27a_3^2. \quad (2.1.16)$$

One can show that the discriminant $R(z)$ is not identically equal to zero if the polynomial $F(z, w)$ is not divisible by the square of a polynomial of a positive degree. So, in the nonsingular case there are only finitely many ramification points on the complex z -plane.

Example 2.1.8. Hyperelliptic Riemann surfaces have the form

$$w^2 = P_n(z), \quad (2.1.17)$$

where $P_n(z)$ is a polynomial of degree n . These surfaces are two-sheeted coverings of the z -plane. Here $F(z, w) = w^2 - P_n(z)$. The gradient vector $\text{grad}_{\mathbb{C}} F = (-P'_n(z), 2w)$. A point $(z_0, w_0) \in \Gamma$ is singular if

$$w_0 = 0, \quad P'_n(z_0) = 0. \quad (2.1.18)$$

Together with the condition (2.1.17) for a point (z_0, w_0) to belong to Γ we get that

$$P_n(z_0) = 0, \quad P'_n(z_0) = 0, \quad (2.1.19)$$

i.e. z_0 is a multiple root of the polynomial $P_n(z)$. Accordingly, the surface (2.1.17) is nonsingular if and only if the polynomial $P_n(z)$ does not have multiple roots:

$$P_n(z) = \prod_{i=1}^n (z - z_i), \quad z_i \neq z_j, \text{ for } i \neq j. \quad (2.1.20)$$

We find the branch points of the surface (2.1.17). To determine them we have the system

$$w^2 = P_n(z), \quad w = 0,$$

which gives us n branch points $P_i = (z = z_i, w = 0)$, $i = 1, \dots, n$. In a neighborhood of any point of Γ that is not a branch point it is natural to take z as a local parameter, and $w = \sqrt{P_n(z)}$ is a holomorphic function. In a neighborhood of a branch point P_i it is convenient to take

$$\tau = \sqrt{z - z_i}, \quad (2.1.21)$$

as a local parameter. Then for points of the Riemann surface (2.1.17) we get the local parametric representation

$$z = z_i + \tau^2, \quad w = \tau \sqrt{\prod_{j \neq i} (\tau^2 + z_i - z_j)} \quad (2.1.22)$$

where the radical is a single-valued holomorphic function for sufficiently small τ ; (the expression under the root sign does not vanish), and $dw/d\tau \neq 0$ for $\tau = 0$.

We study the structure of the mapping π in (2.1.11) in a neighborhood of a branch point $P_0 = (z_0, w_0)$ of Γ . Let τ be a local parameter on Γ in a neighborhood of P_0 . It will be assumed that $z(\tau = 0) = z_0$, $w(\tau = 0) = w_0$. Then

$$\begin{aligned} z &= z_0 + a\tau^p + O(\tau^{p+1}), \\ w &= w_0 + b\tau^q + O(\tau^{q+1}), \end{aligned} \quad (2.1.23)$$

where a and b are nonzero coefficients. Since w can be taken as the local parameter in a neighborhood of P_0 it follows that $q = 1$. We get the form of the surface Γ in a neighborhood of a branch point:

$$\begin{aligned} z &= z_0 + a\tau^p + O(\tau^{p+1}), \\ w &= w_0 + b\tau + O(\tau^2), \end{aligned} \quad (2.1.24)$$

where $p > 1$. Thus, the points of the form

$$P_1(z) = (z, w_0 + \epsilon_1 c \sqrt[p]{z} + \dots), \dots, P_p(z) = (z, w_0 + \epsilon_p c \sqrt[p]{z} + \dots), \quad (2.1.25)$$

where $\epsilon_1, \dots, \epsilon_p$ are the primitive p th roots of unity and $c = ba^{-\frac{1}{p}}$ lie in the complete inverse image $\pi^{-1}(z)$ in any sufficiently small neighborhood of P_0 merging into a single point at this point itself (the dots stand for the terms of the form $o(\sqrt[p]{z})$).

Definition 2.1.9. *The number $f = p - 1$ is called the multiplicity of the branch point, or the branching index of this point.*

For example, for a hyperelliptic surface $w^2 = P_n(z)$ all the zeros $z = z_1 \dots z = z_n$ of the polynomial $P_n(z)$ give branch points of multiplicity one on the surface.

Exercise 2.1.10: Prove that the total multiplicity of all the branch points on Γ over $z = z_0$ is equal to the multiplicity of $z = z_0$ as a root of the discriminant $R(z)$.

Exercise 2.1.11: Consider the collection of n -sheeted Riemann surfaces of the form

$$F(z, w) = \sum_{i+j \leq n} a_{ij} z^i w^j \quad (2.1.26)$$

for all possible values of the coefficients a_{ij} (so-called planar curves of degree n). Prove that for a general surface of the form (2.1.26) there are $n(n-1)$ branch points and they all have multiplicity 1. In other words, conditions for the appearance of branch points of multiplicity greater than one are written as a collection of algebraic relations on the coefficients a_{ij} .

2.2 Newton polygon technique.

Let Γ be an algebraic curve given by $F(z, w) = 0$, and assume $(0, 0) \in \Gamma$, i.e. in F there is no constant term. How can we compute the Puiseux series? Consider $a_{1,0}z + a_{0,1}w$, the linear part of F ; if for example $a_{0,1} \neq 0$, then we can set $w = -\frac{a_{1,0}}{a_{0,1}}z + O(z^2)$, where the higher terms can be computed easily. It is more difficult when also the linear part is 0. In particular consider $F = F_0 + F_1$ where F_0 is the least nonzero homogeneous part (let say it has degree m). Then $F_0 = a_{m,0}z^m + \dots + a_{0,m}w^m$ and we can solve it substituting $k = \frac{w}{z}$ and solving the algebraic equation of degree m for k . Assume that all the roots k_1, \dots, k_m are pairwise distinct, then we can set $w = k_i z + O(z^{m+1})$. If the roots are not all distinct there may be a real branch point or just a singularity as a simple node. But we can take also as F_0 the least nonzero quasihomogeneous part with respect to coprime p and q , i.e. $F_0 = \sum_{pi+qj=m} a_{i,j} z^i w^j$ and all terms of F_1 has $pi + qj > m$. Then we can set $w = k_i^{\frac{1}{q}} + z^{\frac{p}{q}} + \dots$

The Newton polygon technique is the following procedure: in the (i, j) plane, mark all points where $a_{i,j} \neq 0$ and take the convex hull of this points. This is called the Newton polygon of the polynomial. We can assume without loss of generality that the Newton polygon touches both axis (otherwise we can divide by some power of w or z). Then one edge corresponds to one choose of p and q (the slope of the edge). Let us consider an example.

Example 2.2.1. Consider $F(z, w) = 2z^7 - z^8 - z^3w + (4z^2 + z^3)w^2 + (z^8 - z^4)w^3 - 4zw^4 + 7z^5w^5 + (1 - z^2)w^6 + 5z^6w^7 + z^3w^8$. There are four relevant edges in the convex hull; the first contains $2z^7 - z^3w$ where we have simply $w = 2z^4$ so that we get $w = 2z^4 + w_1$ and we can iterate to obtain another polygon in z and w_1 . The second edge corresponds to $-z^3w + 4z^2w^2$ from which we have $w = \frac{1}{4}z + \dots$. The third is $4z^2w^2 - 4zw^4 + w^6 = w^2(w^2 - 2z)^2$ so that the branches coincide two by two and we have $w_1 = w_2 = \sqrt{2}z^{\frac{1}{2}}$ and $w_3 = w_4 = -\sqrt{2}z^{\frac{1}{2}}$. To address what happens in this case, we have to substitute $w = \sqrt{2}z^{\frac{1}{2}} + w_1$ and obtain a new Newton polygon in $z^{\frac{1}{2}}$ and w_1 . It happens that the points w_1 and w_2 that coincide in the first approximation are distinct in the second; similarly for w_3 and w_4 .

In the previous lecture we worked with the assumption that $a_0(z) = 1$. More generally, if $a_0(z)$ is not constant, we have to consider as \mathbb{C}_0 the complex plane minus ramification points and zeroes of a_0 .

2.3 Riemann surfaces as two-dimensional real manifolds. Compactification of a Riemann surface. Examples. The genus of a Riemann surface. Computation of the genus for hyperelliptic surfaces. Monodromy. The Riemann-Hurwitz formula

It has already been mentioned that an arbitrary Riemann surface is a two-dimensional surface (a two-dimensional manifold) from the real point of view. What can be said about the topology of this surface? It is easy to see that this surface is connected (verify!). We show that it is oriented.

If $z = x + iy$ is a local parameter in some domain U on Γ , then x and y are real coordinates in U . Another local parameter $w = u + iv$ is connected with the first by a holomorphic change of variables $w = w(z)$, $dw/dz \neq 0$ which thus determines a smooth change of real coordinates $u = u(x, y)$, $v = v(x, y)$. The Jacobian of this change has the form

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \left| \frac{dw}{dz} \right|^2 > 0,$$

which means that the surface is oriented. The observations that Riemann surfaces are connected and oriented do not yet permit us to classify them according to topological type, because they are not compact. We now indicate a procedure for compactifying a Riemann surface Γ , i. e. , for adjoining to it some points turning it into a compact complex manifold, and hence into a closed oriented surface. We recall first how to compactify the complex z -plane \mathbb{C} . For this it is necessary to add to \mathbb{C} a single "point at infinity" ∞ . The local parameter in a neighborhood of ∞ should be taken to be $\zeta := 1/z$. The holomorphic transition functions

$$\zeta(z) = \frac{1}{z}, \quad z(\zeta) = \frac{1}{\zeta}$$

appear in the common part of the action of the local parameters z and ζ , where $z \neq 0$ and $\zeta \neq 0$. We get a surface $\bar{\mathbb{C}}$ with the topology of a sphere (the "Riemann sphere"). Topological equivalence to the standard sphere is given by stereographic projection, with one

of the poles of the sphere passing into the point ∞ . Another description of $\bar{\mathbb{C}}$ is the complex projective line $\mathbb{CP}^1 := \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 \neq 0, (z_1 : z_2) \simeq (\lambda z_1 : \lambda z_2), \lambda \in \mathbb{C}, \lambda \neq 0\}$. The equivalence $\mathbb{CP}^1 \rightarrow \bar{\mathbb{C}}$ is established as follows: $(z_1 : z_2) \rightarrow z = \frac{z_1}{z_2}$. The affine part $\{z_2 \neq 0\}$ of \mathbb{CP}^1 passes into \mathbb{C} and the point $(1 : 0)$ at infinity passes into ∞ . To compactify any (algebraic!) Riemann surface Γ , $F(z, w) = 0$, we embed it in \mathbb{CP}^2 . Here \mathbb{CP}^2 is the (complex) projective plane: the collection of nonzero complex vectors $(\xi : \eta : \zeta)$ determined up to multiplication by a nonzero complex factor,

$$(\xi : \eta : \zeta) \simeq (\lambda \xi : \lambda \eta : \lambda \zeta), \quad \lambda \neq 0.$$

This is a compact complex manifold. (Projective spaces of higher dimension are defined similarly.) The domain in \mathbb{CP}^2 given the condition $\zeta \neq 0$ is called an affine part of \mathbb{CP}^2 . The mappings

$$(\xi : \eta : \zeta) \rightarrow \left(z = \frac{\xi}{\zeta}, w = \frac{\eta}{\zeta} \right)$$

and the inverse mapping

$$(z, w) \rightarrow (z : w : 1)$$

establish an isomorphism between an affine part of \mathbb{CP}^2 and \mathbb{C}^2 . The whole projective plane is obtained from the affine part \mathbb{C}^2 by adding the part at infinity of the form $(\xi : \eta : 0) \simeq \mathbb{CP}^1 \simeq S^2$. An embedding of Γ in \mathbb{CP}^2 is defined as follows: let

$$F \left(\frac{\xi}{\zeta}, \frac{\eta}{\zeta} \right) = \frac{Q(\xi, \eta, \zeta)}{\zeta^N} \tag{2.3.1}$$

where $Q(\xi, \eta, \zeta)$ is a homogeneous polynomial in ξ, η and ζ of degree N (we assume that the fraction on the right-hand side is irreducible). A complex curve $\hat{\Gamma}$ (two dimensional surface) is given in \mathbb{CP}^2 by the homogeneous equation

$$Q(\xi, \eta, \zeta) = 0 \tag{2.3.2}$$

The finite (affine) part of the curve $\hat{\Gamma}$ (where $\zeta \neq 0$) coincides with Γ . The associated points at infinity have the form

$$Q(\xi, \eta, \zeta) = 0, \quad \zeta = 0.$$

The surface $\hat{\Gamma}$ is compact and is thus the desired compactification of the surface Γ .

Exercise 2.3.1: Prove that the curve (2.3.2) is nonsingular in \mathbb{CP}^2 if and only if

$$\text{rank} \begin{pmatrix} \xi & \eta & \zeta \\ \partial Q / \partial \xi & \partial Q / \partial \eta & \partial Q / \partial \zeta \end{pmatrix} = 2 \tag{2.3.3}$$

at all points of this curve.

Example 2.3.2. $\Gamma = \{w^2 = z\}$. A local parameter at the branch point $(z = 0, w = 0)$ is given by $\tau = \sqrt{z}$, i.e. $z = \tau^2, w = \tau$. The compactification $\hat{\Gamma}$ has the form $\hat{\Gamma} = \{\eta^2 = \xi\zeta\}$. We introduce the coordinates u, v in a neighborhood of the ideal line \mathbb{CP}^1 (with $\xi \neq 0$), setting

$$u = \frac{\eta}{\xi} = \frac{w}{z}, \quad v = \frac{\zeta}{\xi} = \frac{1}{z}, \tag{2.3.4}$$

and the ideal line has the form $v = 0$. In these coordinates the curve $\hat{\Gamma}$ can be written (locally) in the form $u^2 = v$. Its unique point at infinity is $(u = 0, v = 0)$, and $u = w/z = \sqrt{v} = 1/\sqrt{z}$ serves as a local parameter in a neighborhood of this point. In other words, in a neighborhood of the point at infinity in $\hat{\Gamma}$ we have that

$$z = \frac{1}{u^2}, \quad w = \frac{1}{u}, \quad u \rightarrow 0. \quad (2.3.5)$$

Example 2.3.3. $\Gamma = \{w^2 = z^2 - a^2\}$. The branch points are $(z = \pm a, w = 0)$ and the corresponding local parameters are $\tau_{\pm} = \sqrt{z \pm a}$. The compactification has the form $\hat{\Gamma} = \{\eta^2 = \xi^2 - a^2\zeta^2\}$. Making the substitution (2.3.4) we get the form of the curve $\hat{\Gamma}$ in a neighborhood of the ideal line: $u^2 = 1 - a^2v^2$. For $v = 0$ we get that $u = \pm 1$. Thus, there are two points at infinity, $P_{\pm} = (1 : \pm 1 : 0)$ on surface $\hat{\Gamma}$. We can take $v = l/z$ as a local parameter in a neighborhood of each of these points. The form of the surface $\hat{\Gamma}$ in a neighborhood of these points P_{\pm} is as follows:

$$z = \frac{1}{v}, \quad w = \pm \frac{1}{v} \sqrt{1 - a^2v^2}, \quad v \rightarrow 0 \quad (2.3.6)$$

where $\sqrt{1 - a^2v^2}$ is, for small v , a single-valued holomorphic function, and the branch of the square root is chosen to have value 1 at $v = 0$.

Example 2.3.4. $\Gamma = \{w^2 = P_{2n+1}(z)\}$. This example is analogous to Example 2.3.2. Here there is a single point at infinity: we can take u as a local parameter in a neighborhood of it. In a neighborhood of the point at infinity the surface $\hat{\Gamma}$ has the form

$$z = \frac{1}{u^2}, \quad w = \frac{1}{u^{2n+1}} \sqrt{\prod_{i=1}^{2n+1} (1 - z_i u)} \quad (2.3.7)$$

(here the polynomial $P_{2n+1}(z)$ has the form $P_{2n+1}(z) = \prod_{i=1}^{2n+1} (z - z_i)$; the square root is a single-valued holomorphic function of u for small u chosen to be 1 at $u = 0$).

Example 2.3.5. $\Gamma = \{w^2 = P_{2n+2}(z)\}$. This example is analogous to Example 2.3.3. Here $\hat{\Gamma}$ has two points P_{\pm} at infinity, and $v = 1/z$ can be taken as a local parameter in a neighborhood of them. The form of the surface $\hat{\Gamma}$ in a neighborhood of these points is as follows:

$$z = \frac{1}{v}, \quad w = \pm \frac{1}{v^{n+1}} \sqrt{\prod_{i=1}^{2n+2} (1 - z_i u)}, \quad (2.3.8)$$

(here $P_{2n+2}(z) = \prod_{i=1}^{2n+2} (z - z_i)$), the specification of the square root is analogous to that above).

In what follows we shall not put a hat on Γ , assuming always that the Riemann surface Γ has been suitably compactified. It is well known that connected compact (i.e., closed) oriented two-dimensional surfaces have a simple topological classification. They are all spheres with g handles, $g > 0$ (see [14]). The operation of gluing on a handle is represented in the figure. The number g of handles is called the genus of the surface. Here there are the simplest examples of a sphere with handles:

The genus of a Riemann surface is the most important characteristic of it. Let us compute the genus of the surfaces in the examples 2.3.2-2.3.5. We begin with Example 2.3.3. Delete the segment $[-a, a]$ with endpoints at the branch points from the z -plane \mathbb{C} . Off this segment it is possible to distinguish the two branches $w_{\pm} = \pm\sqrt{z^2 - a^2}$ of the two-valued function $w(z) = \sqrt{z^2 - a^2}$, that do not get mixed up with each other. In other words, the complete image $\pi^{-1}(\mathbb{C} \setminus [-a, a])$ on Γ splits into two pieces, with the mapping π an isomorphism on each of them. The branches $w_+(z)$ and $w_-(z)$ are interchanged in passing from one edge of the cut $[-a, a]$ to the other. Therefore, the surface is glued together from two identical copies of spheres with cuts according to the rule indicated in the figure. After the gluing we again obtain a sphere, i.e., the genus g is equal to zero. Example 2.3.2 is analogous to Example 2.3.3, but the cut must be made between the points 0 and ∞ , i.e. the point at infinity must be regarded as a branch point. Again the genus is equal to zero.

In Example 2.3.5 it is necessary to split up the branch points arbitrarily into pairs and make cuts (arcs) in \mathbb{C} joining the paired branch points ($n+1$ cuts in all). The surface Γ is glued together from two identical copies of a sphere with such cuts, with the edges of the corresponding cuts glued together in "cross-wise" fashion (see the figure for $n = 1$). It is not hard to see that a sphere with n handles is obtained after the gluing, i.e., the genus g is n .

In Example 2.3.4 the situation is analogous, except that in making the cuts it is necessary to take ∞ as one of the branch points. The genus g is again equal to n .

Exercise 2.3.6: Suppose that all the zeros $z_1 < \dots < z_{2n+1}$ of the polynomial $P_{2n+1}(z)$ are real. We choose the segments $[z_1, z_2], [z_3, z_4], \dots, [z_{2n+1}, \infty]$ of the real axis as the cuts for the surface $\Gamma = \{w^2 = P_{2n+1}(z)\}$. The function $w(z) = \sqrt{P_{2n+1}(z)}$ which is single-valued on each sheet of Γ formed after removal of the cycles $\pi^{-1}([z_1, z_2]), \dots, \pi^{-1}([z_{2n+1}, \infty])$ is real on the edges of these cuts on each of the sheets. Show that on each sheet the sign of the square root $\sqrt{P_{2n+1}(z)}$ on the upper edge of the cut alternates (see the figure for a possible distribution of signs).

For more complicated Riemann surfaces it is not easy to determine their topological structure. Here it is useful to exploit the monodromy group of the Riemann surface, which we now define. Let delete from \mathbb{C} the branch points z_1, \dots, z_N and delete from Γ the complete inverse images $\pi^{-1}(z_1), \dots, \pi^{-1}(z_N)$ of these points. We get a surface Γ_0 that is a n -sheeted covering of the punctured sphere $\mathbb{C} \setminus (z_1 \cup \dots \cup z_N)$. The monodromy group of the Riemann surface is the monodromy group of this covering. We recall the general definition of the monodromy group of a covering in connection with this case (see [13] for more details). Fix a point $\star \in \mathbb{C} \setminus (z_1 \cup \dots \cup z_N)$ and number the points P_1, \dots, P_n in the fiber $\pi^{-1}(\star)$ arbitrarily (these points are all distinct). Any closed contour in $\mathbb{C} \setminus (z_1 \cup \dots \cup z_N)$ beginning and ending at \star gives rise to a permutation of the points P_1, \dots, P_n of the fiber after being lifted to Γ_0 . We get a representation of the fundamental group $\pi_1(\mathbb{C} \setminus (z_1 \cup \dots \cup z_N), \star)$ (the free group with $N-1$ generators) in the group S of permutations of n elements; this is called the monodromy representation. The image of this representation in S_n is called the monodromy group. For hyperelliptic Riemann surfaces the monodromy group coincides $S_2 = Z_2$. In the general case the action of the generators of the monodromy group that correspond to circuits about branch points is determined by the branching indices.

Exercise 2.3.7: Let z_0 be the image of a branch point, and let the complete inverse image $\pi^{-1}(z_0)$ on Γ consist of the branch points P_1, \dots, P_k of multiplicity f_1, \dots, f_k , respectively

(if some point P_i is not a branch point, then we set $f_i = 0$). Prove that to a cycle in \mathbb{C} encircling z_0 once there corresponds an element in the monodromy group splitting into cycles of length $f_1 + 1, \dots, f_k + 1$. This assertion gives a purely topological definition of the multiplicities (indices) of branch points.

Remark 2.3.8. The monodromy corresponding to circuits about the point $z = \infty$ is uniquely determined by the monodromy corresponding to circuits about the images of the finite branch points. Indeed, a contour encircling only the point $z = \infty$ splits into a product of contours encircling all the finite branch points, and we get the monodromy at infinity by multiplying the corresponding elements of the monodromy groups at the finite points. For example, for the surface $w^2 = P_{2n+2}(z)$ the monodromy at infinity is trivial (the corresponding contour in the z -plane encircles an even number of branch points), i.e., this surface has no branch points at infinity. But for the surface $w^2 = P_{2n+1}(z)$ the monodromy at infinity is nontrivial, because here a contour encircling $z = \infty$ encircles an odd number of branch points. We thus see once more that the point at infinity of the surface $w^2 = P_{2n+1}(z)$ is a branch point.

Exercise 2.3.9: Prove that for a general surface of the form (2.1.26)) the monodromy group coincides with the complete symmetric group S_n . *Hint.* Show that the branch points of such a surface can be labeled by pairs of distinct numbers $i \neq j$, ($i, j = 1, \dots, n$) in such a way that a circuit about the images of the points P_{ij} and P_{ji} gives rise to a transposition of the i th and j th points of the fiber (when these points are suitably numbered).

In the conclusion of this lecture we indicate a formula (the Riemann-Hurwitz formula) expressing the genus of a Riemann surface in terms of the total multiplicity f of its branch points and the number n of sheets. This formula is

$$g = \frac{f}{2} - n + 1. \quad (2.3.9)$$

Exercise 2.3.10: Prove the Riemann-Hurwitz formula. *Hint.* triangulate the sphere $\bar{\mathbb{C}}$ in such a way that the images of the branch points are vertices of the triangulation. Let c_1, c_2 and c_3 be the numbers of vertices, edges, and triangles, respectively, of the triangulation. Lift this triangulation to the surface Γ by means of the mapping π . Let \hat{c}_0, \hat{c}_1 and \hat{c}_2 be the number of vertices edges and triangles of the lifted triangulation. Find the connection between the numbers c_i and \hat{c}_i , $i = 0, 1, 2$. Use the theorem of Euler characteristics ([14], chapt 3): $2 = c_0 - c_1 + c_2$ for a sphere and $2 - 2g = \hat{c}_0 - \hat{c}_1 + \hat{c}_2$ for the surface Γ .

2.4 Meromorphic functions on a Riemann surface. Holomorphic mappings of Riemann surfaces. Biholomorphic isomorphisms of Riemann surfaces. Examples. Remarks on singular algebraic curves

Definition 2.4.1. A function $f = f(z, w)$ is meromorphic on a Riemann surface $\Gamma = \{F(z, w) = 0\}$ if it is a rational function of z and w , i.e., has the form

$$f(z, w) = \frac{P(z, w)}{Q(z, w)} \quad (2.4.1)$$

where $P(z, w)$ and $Q(z, w)$ are polynomials, and $Q(z, w)$ is not identically zero on Γ .

The meromorphic functions on the surface Γ form a field whose algebraic structure actually bears in itself all the information about the geometry of the Riemann surface.

Definition 2.4.2. -A function f is a meromorphic function on a Riemann surface Γ if it is holomorphic in a neighborhood of any point of Γ except for finitely many points Q_1, \dots, Q_m i.e., can be represented locally in the form $f = f(\tau)$, where τ is a local parameter, and $\partial f / \partial \bar{\tau} = 0$. At the points Q_1, \dots, Q_m the function f has poles of respective multiplicities q_1, \dots, q_m i.e., in a neighborhood of any point Q_i it can be represented in the form

$$f = \tau_i^{-q_i} \tilde{f}_i(\tau_i), \quad (2.4.2)$$

where τ_i is a local parameter in a neighborhood of the point q_i , $\tau_i(Q_i) = 0$ $\tilde{f}_i(\tau_i)$ is a holomorphic function for small τ_i and $\tilde{f}_i(\tau_i) \neq 0$.

It is easy to verify that Definition 2.4.1 is unambiguous. i.e., is independent of the choice of the local parameter, and also that the definition of the multiplicity of a pole is unambiguous. It is not hard to verify also that the conditions of Definition 2.4.2 follow from the conditions of Definition 2.4.1. The following result turns out to be true.

Theorem 2.4.3. *Definitions 2.4.1 and 2.4.2 are equivalent.*

We do not give a proof of this theorem; see, for example, [27] or [6].

Example 2.4.4. A hyperelliptic Riemann surface $w^2 = P_{2n+1}(z)$. Here the coordinates z and w are single-valued functions on Γ and holomorphic in the finite part of Γ . These functions have poles at the point of Γ at infinity: z has a double pole, and w has a pole of multiplicity $2n + 1$. This follows immediately from the formula (2.3.7). If $P_{2n+1}(z) = \prod_{i=1}^{2n+1} (z - z_i)$, then the function $1/(z - z_i)$ has for each i a unique second order pole on Γ at the branch points. This follows from (2.1.22). We mention also that the function z has on Γ two simple zeros at the points $z = 0$, $w = \pm \sqrt{P_{2n+1}(0)}$ which merges into a single double zero if $P_{2n+1}(0) = 0$. The function w has $2n + 1$ simple zeros on Γ at the branch points. (The multiplicity of a zero of a meromorphic function is defined by analogy with the multiplicity of a pole.)

Example 2.4.5. . A hyperelliptic Riemann surface $w^2 = P_{2n+2}(z)$. Here again the functions z and w are holomorphic in the finite part of Γ . But these functions have two poles at infinity (in the infinite part of the surface Γ): z has two simple poles, and w has two poles of multiplicity $n + 1$. This follows from the formulas (2.3.8).

Exercise 2.4.6: Prove Theorem 2.4.3 for hyperelliptic Riemann surfaces. *Hint.* Let $f = f(z, w)$ be a meromorphic (in the sense of Definition 2.4.2) function on the hyperelliptic Riemann surface $w^2 = P(z)$. Show that the functions $f_+ = f(z, w) + f(z, -w)$ and $f_- = w^{-1}(f(z, w) - f(z, -w))$ are rational function of z .

Remark 2.4.7. 1. It is not hard to prove that there are no nonconstant holomorphic functions on (compact) Riemann surfaces. Indeed, such a function attains its maximum on Γ , and hence must be constant by the maximum principle.

Holomorphic mappings of Riemann surfaces are defined by analogy with meromorphic functions on Riemann surfaces. If $\Gamma = \{F(z, w) = 0\}$, $\tilde{\Gamma} = \{\tilde{F}(\tilde{z}, \tilde{w}) = 0\}$, then a holomorphic mapping $f : \Gamma \rightarrow \tilde{\Gamma}$ is defined by a pair of meromorphic functions $\tilde{z} = f_1(z, w)$, $\tilde{w} = f_2(z, w)$. In other words, if τ is a local parameter on Γ in a neighborhood of the point $f(P)$, then f must be written locally in the form $\tilde{\tau} = \psi(\tau)$, where ψ is a holomorphic function of τ . It follows from Theorem 2.4.3 that these two definitions are equivalent (verify!).

Example 2.4.8. Let f be a meromorphic function on Γ . It determines a mapping of Γ to \mathbb{CP}^1 where the poles pass into the point at infinity. Let us verify that this mapping is holomorphic. This is obvious in a neighborhood of regular points. Let z be a local coordinate in the finite part of \mathbb{CP}^1 , and $\zeta = 1/z$ the local coordinate at infinity $\infty \in \mathbb{CP}^1$. Assume that the function has a pole of order k at the point $P_0 \in \Gamma$, $f(P_0) = \infty \in \mathbb{CP}^1$, i.e., it can be written in terms of a local coordinate τ in the form

$$z = f(P) = \frac{c}{\tau^k} + O(\tau^{-k+1}), \quad c \neq 0,$$

where $\tau(P_0) = 0$. Then $\zeta = \frac{1}{f(P)} = c^{-1}\tau^k + O(\tau^{k+1})$, i.e., the mapping has a zero of multiplicity k at P_0 .

To prove the simplest properties of meromorphic functions on Riemann surfaces it is useful to employ arguments connected with the concept of the degree of a mapping. The key point here is the following circumstance (valid for holomorphic mappings of arbitrary complex manifolds of the same dimension). Let $f : \Gamma \rightarrow \tilde{\Gamma}$ be a holomorphic mapping of the surface Γ into $\tilde{\Gamma}$ and let $\tilde{P} \in \tilde{\Gamma}$ be a regular value of this mapping. Then the degree of f is equal to the number of inverse images of \tilde{P} . Indeed, if $f(P) = \tilde{P}$, and τ and $\tilde{\tau}$ are local parameters in neighborhoods of P and \tilde{P} , respectively, with $\tau(P) = \tilde{\tau}(\tilde{P}) = 0$, then f can be written locally as a holomorphic function $\tilde{\tau} = \psi(\tau)$ and $d\psi/d\tau \neq 0$. The Jacobian of this mapping at P is equal to $|(d\psi/d\tau)(0)|^2 > 0$, and this proves the stated assertion.

Exercise 2.4.9: Prove that for any meromorphic function on a Riemann surface Γ the number of zeros is equal to the number of poles (zeros and poles are taken with multiplicity counted).

Branch points and their multiplicities are defined for holomorphic mappings of Riemann surfaces, as is the number of sheets. The branch points are the critical points of the mapping $F : \Gamma \rightarrow \tilde{\Gamma}$.

In a neighborhood of such points F can be written in terms of local parameters in the form $\tilde{\tau} = \psi(\tau)$, where $(d\psi/d\tau)(0) = 0$. The multiplicity of a branch point is the multiplicity of the zero of the derivative $d\psi/d\tau$ at $\tau = 0$. It is clear that for $\tilde{\Gamma} = \mathbb{CP}^1$, this definition coincides with the definition in Lecture 2.1. Next, the number of sheets is the degree of the mapping F .

Exercise 2.4.10: Let g be the genus of the surface Γ , \tilde{g} the genus of $\tilde{\Gamma}$ and n the number of sheets of the mapping, and f the total multiplicity of the branch points of F . Prove the following generalization of the Riemann-Hurwitz formula (see Lecture 2.3)

$$g = \frac{f}{2} = n\tilde{g} - n + 1. \tag{2.4.3}$$

Definition 2.4.11. A mapping $F : \Gamma \rightarrow \tilde{\Gamma}$ is called a biholomorphic isomorphism if it is biholomorphic and its inverse is biholomorphic. It is not hard to derive from Theorem 2.4.3 that the class of biholomorphic isomorphisms of Riemann surfaces coincides with the class of birational isomorphisms (the mapping itself and its inverse are given by rational functions: $\tilde{z} = \tilde{z}(z, w)$, $\tilde{w} = \tilde{w}(z, w)$ and $z = z(\tilde{z}, \tilde{w})$, $w = w(\tilde{z}, \tilde{w})$). In what follows we use these two terms interchangeably.

The following is obvious but important.

Lemma 2.4.12. If the surfaces Γ and $\tilde{\Gamma}$ are biholomorphically (birationally) isomorphic, then they have the same genus.

Proof. A biholomorphic isomorphism is clearly a homeomorphism. But the genus is invariant under homeomorphisms [14]. The assertion is proved. \square

Definition 2.4.13. A Riemann surface Γ is said to be rational if it is biholomorphically isomorphic to $\mathbb{C}\mathbb{P}^1$.

The genus of a rational surface is equal to zero. It turns out (see Lecture 2.7) that this condition is also sufficient for rationality.

Exercise 2.4.14: Let Γ be a Riemann surface of genus $g > 1$. Prove that there is no meromorphic function on Γ with a single simple pole.

Example 2.4.15. The surface $w^2 = z$. This surface is rational. A birational isomorphism onto $\mathbb{C}\mathbb{P}^1$ is given by the projection $(z, w) \rightarrow w$.

Exercise 2.4.16: Show that the surface $w^2 = P_2(z)$, where $P_2(z)$ is a quadratic polynomial, is rational. An explicit form of a rational parametrization of this surface is given by the Euler substitutions known from integral calculus.

Example 2.4.17. A surface with $w^2 = P_{2g+2}(z)$ with $g > 1$ is nonrational. We show that any such surface is birationally isomorphic to some surface of the form $\tilde{w}^2 = \tilde{P}_{2g+1}(\tilde{z})$. Let z_0 be one of the zeros of the polynomial $P_{2g+2}(z)$, and let

$$\tilde{z} = \frac{1}{z - z_0}, \quad \tilde{w} = \frac{w}{(z - z_0)^{g+1}}.$$

The inverse mapping has the form

$$z = z_0 + \frac{1}{\tilde{z}}, \quad w = \frac{\tilde{w}}{\tilde{z}^{g+1}}.$$

If $P_{2g+2}(z) = (z - z_0) \prod_{i=1}^{2g+1} (z - z_i)$, then $\tilde{P}_{2g+1}(\tilde{z}) = \prod_{i=1}^{2g+1} (1 + (z_0 + z_i)\tilde{z})$. Thus, both "types" of hyperelliptic Riemann surfaces considered in Lecture 2.1 give the same class of surfaces.

In the conclusion of this lecture we return to the question of singular complex algebraic curves $\Gamma = \{F(z, w) = 0\}$. It turns out that there is always a nonsingular Riemann surface $\hat{\Gamma}$ (a complex one-dimensional manifold) such that the curve Γ is given in the $z = z(P)$, $w = w(P)$, where $z(P)$ and $w(P)$ are meromorphic functions on $\hat{\Gamma}$. The surface $\hat{\Gamma}$ can be chosen in a minimal (or universal) way in the following sense of the word. If $\hat{\Gamma}_1$ is another such surface, then its mapping to the curve Γ factors through a holomorphic mapping $\hat{\Gamma}_1 \rightarrow \hat{\Gamma}$.

Example 2.4.18. We consider what happens when a multiple zero of the polynomial $P_{2g+1}(z)$ appears in the equation $w^2 = P_{2g+1}(z)$ of a hyperelliptic curve. Let $P_{2g+1}(z) = (z-z_0)^2 \prod_{i=1}^{2g-1} (z-z_i)$, where the numbers $z_0, z_1, \dots, z_{2g+1}$ are pairwise distinct. We consider the curve

$$\Gamma : w^2 = (z - z_0)^2 \prod_{i=1}^{2g-1} (z - z_i),$$

(this curve can be thought of as coming from the nonsingular Riemann $w^2 = \prod_{i=1}^{2g-1} (z - z_i)$, by the confluence $z_{2g} \rightarrow z_0, z_{2g+1} \rightarrow z_0$) and the Riemann surface

$$\hat{\Gamma} : \hat{w}^2 = \prod_{i=1}^{2g-1} (\hat{z} - z_i),$$

of genus $g - 1$. The mapping $\hat{\Gamma} \rightarrow \Gamma$ is given by the formulas $z = \hat{z}, w = \hat{w}(\hat{z} - z_0)$. It is not hard to verify the universality property.

We do not give a general construction of the desingularization (see [22]). We point out that for singular curves $F(z, w) = 0$, the surface $\hat{\Gamma}$ (which is always nonsingular!) is called the Riemann surface of the algebraic function $w = w(z)$. Note that the collection of rational functions of z, w on the singular curve Γ can be identified in a natural way with a certain subfield of the field of meromorphic functions on the desingularization $\hat{\Gamma}$.

Example 2.4.19. The “Enriques curves” with g singularities of double point are obtained from the Riemann sphere $\mathbb{CP}^1 = \bar{\mathbb{C}}$ by identifying g pairs of points $a_1, b_1, \dots, a_g, b_g$. Thus, the rational functions on a Enriques curve are the rational functions $f(z)$ on the complex plane, $z \in \mathbb{C}$ that satisfy the conditions

$$f(a_i) = f(b_i), \quad i = 1, \dots, g. \quad (2.4.4)$$

More complicated singularities (of “beak” type) are obtained by fixing a collection of points c_1, \dots, c_k and imposing on the rational functions $f(z)$ the conditions

$$f(c_i) = 0, \quad i = 1, \dots, k. \quad (2.4.5)$$

More complicated singularities are also possible.

2.5 Differentials on a Riemann surface. Holomorphic differentials. Periods of closed differentials. Cycles on a Riemann surface, the intersection number, canonical bases of cycles. A relation between periods of closed differentials

Let $z = x + iy$ be a local parameter in some domain of a Riemann surface Γ . The differential 1-forms (also called differentials) on a Riemann surface can be written locally in the form $\omega = P(x, y)dx + Q(x, y)dy$. Introducing a basis $dz = dx + idy, d\bar{z} = dx - idy$, we can rewrite ω in the form $\omega = f dz + g d\bar{z}$. The two parts $\omega = f dz$ and $\omega = g d\bar{z}$ of this expression will be called (1, 0)- and (0, 1)-forms respectively. The decomposition of a 1-form into the sum of (1, 0) and (0, 1) forms is invariant under holomorphic changes of the local parameter (verify!).

Lemma 2.5.1. *The following relation holds*

$$d\omega = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right) dz \wedge d\bar{z} \quad (2.5.1)$$

The proof is obvious.

Corollary 2.5.2. *A (1,0)-form $\omega = f dz$ is closed if and only if the function f is holomorphic.*

Definition 2.5.3. *A differential ω is called holomorphic (or a differential of the first kind) if it can be written locally in the form $\omega = f(z)dz$, where $f(z)$ is a holomorphic function of the local parameter z .*

Example 2.5.4. Let us consider holomorphic differentials on a hyperelliptic Riemann surface

$$\Gamma = \{w^2 = P_{2g+1}(z)\}, \quad P_{2g+1}(z) = \prod_{k=1}^{2g+1} (z - z_k)$$

of genus $g \geq 1$. Let us check that the differentials

$$\eta_k = \frac{z^{k-1}dz}{w} = \frac{z^{k-1}dz}{\sqrt{P_{2g+1}(z)}}, \quad k = 1, \dots, g \quad (2.5.2)$$

are holomorphic. Indeed, holomorphicity at any finite point but branch point is obvious as the denominator does not vanish. We verify holomorphicity in a neighborhood of the i -th branch point $P_i = \{z = z_i, w = 0\}$. Choosing the local parameter τ in a neighborhood of P_i in the form $\tau = \sqrt{z - z_i}$, we get from (2.1.22) that $\eta_k = \psi_k(\tau)d\tau$, where the function

$$\psi_k(\tau) = \frac{2(z_i + \tau^2)^{k-1}}{\sqrt{\prod_{j \neq i} (\tau^2 + z_i - z_j)}}$$

is holomorphic for small τ .

At the point at infinity the differentials η_k can be written in terms of the local parameter $u = z^{-\frac{1}{2}}$ in the form $\eta_k = \phi_k(u)du$, where the functions

$$\phi_k(u) = -2u^{2(g-k)} \left[\prod_{i=1}^{2g+1} (1 - z_i u) \right]^{-\frac{1}{2}}, \quad k = 1, \dots, g$$

are holomorphic for small u (see the formulas (2.3.7)).

In the same way it can be verified that the differentials $\eta_k = z^{k-1}dz/w$, $k = 1, \dots, g$ are holomorphic on the Riemann surface $w^2 = P_{2g+2}(z)$.

Exercise 2.5.5: Suppose that the Riemann surface has the form (2.1.26), the curve (2.1.26) is nonsingular, and the equation $\sum_{i+j=n} a_{ij}\zeta^j = 0$ has n distinct roots ζ_1, \dots, ζ_n . Show that the differentials

$$\eta_{ij} = \frac{z^i w^j dz}{\partial F(z, w)/\partial w} \quad (2.5.3)$$

are holomorphic on Γ for $i + j \leq n - 3$. (The condition in the problem means that the n -th degree curve of the form

$$\sum_{i+j \leq n} a_{ij} \xi^i \eta^j \zeta^{n-i-j} = 0$$

is nonsingular in $\mathbb{C}\mathbb{P}^2$ (verify!).

We return to arbitrary closed forms ω . For any closed oriented contour (cycle) γ on Γ , the period of a closed differential ω along the contour γ , is defined as $\oint_{\gamma} \omega$. This period does not depend on the deformations of the contour γ (verify!). More generally, if the contour γ (which is not necessarily connected) is the oriented boundary of some domain Ω on the surface Γ (i.e., is homologous to zero), then the period $\oint_{\gamma} \omega$ is equal to zero. Indeed, by Stokes formula

$$\oint_{\gamma=\partial\Omega} \omega = \int \int_{\Omega} d\omega = 0.$$

Thus, the period $\oint_{\gamma} \omega$ depends only on the homology class of oriented closed contours (cycles). Recall that two cycles γ_1 and γ_2 are said to be homologous if their difference $\gamma_1 - \gamma_2 = \gamma_1 \cup (-\gamma_2)$ (where $(-\gamma_2)$ is the cycle with the opposite orientation) is the oriented boundary of some domain Ω on Γ . For example, any two cycles on a surface of genus zero are homologous.

Suppose that the genus g is ≥ 1 . We present facts from the homology theory of the surface that are needed in what follows. On such a surface it is possible to choose basis of cycle $a_1, \dots, a_g, b_1, \dots, b_g$ such that any cycle γ is homologous to a linear combination of them with integer coefficients. We write this as follows:

$$\gamma \simeq \sum_{i=1}^g m_i a_i + \sum_{i=1}^g n_i b_i, \quad m_i, n_i \in \mathbb{Z}.$$

For example, for $g = 1$ or 2 the cycles a_i and b_i can be chosen as shown in the figure 1. The intersection number $\gamma_1 \circ \gamma_2$ is defined for any two cycles γ_1 and γ_2 on Γ . Namely, suppose that all points of intersection of the cycles γ_1 and γ_2 are in pairs and the cycles at these points are not tangent to each other. At each intersection point there is an ordered reference frame consisting of the tangent vectors to the respective cycles γ_1 and γ_2 with the direction of the tangent vectors chosen to correspond to the orientation of the cycles. The intersection points are assigned the number $+1$ if the orientation of this frame coincides with that of the surface, and -1 otherwise (see the figure). The sum of these numbers ± 1 , taken over all points

of intersection of γ_1 and γ_2 is the intersection number $\gamma_1 \circ \gamma_2$. Properties of the intersection number: 1) $\gamma_1 \circ \gamma_2$ depends only on the homology classes of γ_1 and γ_2 ; 2) the scalar product $\gamma_1 \circ \gamma_2$ is bilinear, skew-symmetric, and nondegenerate. Nondegenerate means that if $\gamma_1 \circ \gamma_2 = 0$ for every cycle γ_2 , then the cycle γ_1 is homologous to zero. See [13] or [14] for proofs of these properties. A basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$ on a surface Γ of genus g can be chosen so that the pairwise intersection number have the form

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, \quad i, j = 1, \dots, g. \quad (2.5.4)$$

Figure 1: Intersection of γ_1 and γ_2 .

Such a basis will be called canonical. For example, for surfaces of genus $g = 1$ or 2 , the basis of cycles pictured in the figure above is canonical. Note that if for a cycle γ and a canonical basis $a_1, \dots, a_g, b_1, \dots, b_g$ the intersection numbers are $\gamma \circ a_i = n_i, \gamma \circ b_j = m_j, i, j = 1 \dots, g$, then the decomposition of γ in the basis has the form

$$\gamma = \sum_{i=1}^g m_i a_i - \sum_{i=1}^g n_i b_i.$$

This simple consideration is useful in practical computations with cycles on Riemann surfaces. A canonical basis of cycles on a Riemann surface Γ of genus g has another remarkable property. Let us construct the cycles a_i and b_j so that they all begin and end at a particular point $*$ of Γ and otherwise do not have common points, and let us make cuts along these cycles. As a result the surface Γ becomes a $(4g)$ -gon $\tilde{\Gamma}$ – a so-called Poincaré polygon of Γ . Indeed, the domain $\tilde{\Gamma}$ obtained as a result of the cutting is bounded by a closed contour $\partial\tilde{\Gamma}$ made up of $4g$ segments, and any cycle in $\tilde{\Gamma}$ is homologous to zero by property 2 of intersection number. Therefore, $\tilde{\Gamma}$ is a simply connected planar domain. Conversely, it is possible to glue the surface Γ together from the $(4g)$ -gon $\tilde{\Gamma}$ by identifying its sides of the same name in the way indicated in the figure. In the figure, we write a_i^{-1} and b_i^{-1} the edges of the cut along the cycles a_i and b_i , respectively, if these edges occur in the oriented boundary $\partial\tilde{\Gamma}$ with a minus sign. The segment a_i is glued together with the segment a_i^{-1} and b_i with the segment b_i^{-1} in the direction indicated by the arrows.

Example 2.5.6. Let us construct a canonical basis of cycles on the hyperelliptic surface $w^2 = \prod_{i=1}^{2g+1} (z - z_i), g \geq 1$. We represent this surface in the form of two copies of \mathbb{C} (sheets) with cuts along the segments $[z_1, z_2], [z_3, z_4], \dots, [z_{2g+1}, \infty]$. A canonical basis of cycles can be chosen as indicated on the figure for $g = 2$ (the dashed lines represent the parts of a_1 and a_2 lying on the lower sheet).

In concluding this lecture we prove a technical assertion important for what follows, a bilinear relation between the periods of closed differentials.

Figure 2: Poincaré polygon for surfaces of genus one and two.

Figure 3: Homology basis.

Lemma 2.5.7. *Let ω_1 and ω_2 be two closed differentials on a surface Γ of genus $g \geq 1$. Denote their periods with respect to a canonical basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$, by A_i, B_i and A'_i, B'_i :*

$$A_i = \int_{a_i} \omega, \quad B_i = \int_{b_i} \omega, \quad A'_i = \int_{a_i} \omega', \quad B'_i = \int_{b_i} \omega'. \quad (2.5.5)$$

Denote by $f = \int \omega$ the primitive of ω , which is single-valued on the surface $\tilde{\Gamma}$ cut along a_i, b_j , then

$$\int_{\Gamma} \omega \wedge \omega' = \oint_{\partial \tilde{\Gamma}} f \omega' = \sum_{i=1}^g (A_i B'_i - A'_i B_i). \quad (2.5.6)$$

Proof. The first of the equalities in (2.5.6) follows from Stokes' formula, since $d(f\omega') = \omega \wedge \omega'$. Let us prove the second. We have that

$$\oint_{\partial \tilde{\Gamma}} f \omega' = \sum_{i=1}^g \left(\int_{a_i} + \int_{a_i^{-1}} \right) f \omega' + \sum_{i=1}^g \left(\int_{b_i} + \int_{b_i^{-1}} \right) f \omega'.$$

To compute the i -th term in the first sum we use the fact that

$$f(P_i) - f(P'_i) = \int_{P'_i}^{P_i} \omega = -B_i \quad (2.5.7)$$

since the cycle $P'_i P_i$, which is closed on Γ , is homologous to the cycle $(-b_i)$ (see the figure; a fragment of the boundary $\partial\tilde{\Gamma}$ is pictured). Similarly, the jump of the function f in crossing the cut b_i has the form

$$)f(Q_i) - f(Q'_i) = \int_{Q'_i}^{Q_i} \omega = A_i \quad (2.5.8)$$

since the cycle $Q'_i Q_i$ on Γ is homologous to the cycle a_i . Moreover, $\omega'(P'_i) = \omega'(P_i)$ and $\omega'(Q'_i) = \omega'(Q_i)$ because the differential ω' is single-valued on Γ . We have that

$$\begin{aligned} \int_{a_i} f(P_i)\omega'(P_i) + \int_{a_i^{-1}} f(P'_i)\omega'(P'_i) &= \int_{a_i} f(P_i)\omega'(P_i) + \int_{a_i} (f(P_i) + B_i)\omega'(P_i) \\ &= -B_i \int_{a_i} \omega'(P_i) = -B_i A'_i \end{aligned}$$

where the minus sign appears because the edge a_i^{-1} occurs in $\partial\tilde{\Gamma}$ with a minus sign. Similarly,

$$\left(\int_{b_i} + \int_{b_i^{-1}} \right) f\omega' = A_i B'_i.$$

Summing these equalities, we get (2.5.6). The lemma is proved. \square

2.6 Riemann bilinear relations for periods of holomorphic differentials and their most important consequences. Elliptic functions

We derive some important consequences for periods of holomorphic differentials from the lemma proved at the end of the last lecture the so-called Riemann bilinear relations. Ev-

erywhere in this lecture we denote by $a_1, \dots, a_g, b_1, \dots, b_g$ the canonical basis of cycles on Γ .

Corollary 2.6.1. . *Let ω be a nonzero holomorphic differential on Γ , and $A_1, \dots, A_g, B_1, \dots, B_g$ its basisperiods, then*

$$\Im \left(\sum_{k=1}^g A_k \bar{B}_k \right) < 0. \quad (2.6.1)$$

Proof. Take $\omega' = \bar{\omega}$ in the lemma. Then $A'_i = \bar{A}_i$ and $B'_i = \bar{B}_i$ for $i = 1, \dots, g$. We have that

$$\frac{i}{2} \int_{\Gamma} \omega \wedge \omega' = \frac{i}{2} \int \int |f|^2 dz \wedge d\bar{z} = \int \int_{\Gamma} |f|^2 dx \wedge dy > 0.$$

Here $z = x + iy$ is a local parameter, and $\omega = f(z)dz$. In view of (2.5.6) this integral is equal to

$$\frac{i}{2} \sum_{k=1}^g A_k \bar{B}_k - \bar{A}_k B_k = -\Im \left(\sum_{k=1}^g A_k \bar{B}_k \right).$$

The corollary is proved. □

Corollary 2.6.2. *If all the a-periods of a holomorphic differential are zero, then $\omega = 0$.*

This follows immediately from Corollary 2.6.1.

Corollary 2.6.3. *The space of holomorphic differentials on a Riemann surface of genus g is no more than g -dimensional.*

The proof is obvious: any holomorphic differential is uniquely determined by its a-periods. We saw in example 2.5.4 of the last lecture that on a hyperelliptic Riemann surface Γ of genus g there are g holomorphic differentials (2.5.2), which are clearly linearly independent and, according to Corollary 2.6.3, form a basis in the space of holomorphic differentials on Γ . The following result turns out to hold.

Theorem 2.6.4. *The space of holomorphic differentials on a Riemann surface Γ of genus g has dimension g .*

See [27] for a proof.

Corollary 2.6.5. *On a surface Γ of genus g there exists a basis $\omega_1, \dots, \omega_g$ of holomorphic differentials such that*

$$\oint_{a_j} \omega_k = 2\pi i \delta_{jk}, \quad j, k = 1, \dots, g. \quad (2.6.2)$$

Proof. Let η_1, \dots, η_g be an arbitrary basis of holomorphic differentials on Γ . The matrix

$$A_{jk} = \oint_{a_j} \eta_k \quad (2.6.3)$$

is nonsingular. Indeed, otherwise there are constants c_1, \dots, c_g such that $\sum_k A_{jk}c_k = 0$. But then $\sum_k c_k\eta_k = 0$, since this differential has zero a -periods. This contradicts the independence of the differentials η_1, \dots, η_g .

$$\omega_j = 2\pi i \sum_{k=1}^g \tilde{A}_{kj} \eta_k, \quad j = 1, \dots, g, \quad (2.6.4)$$

where the matrix (\tilde{A}_{kj}) is the inverse of the matrix (A_{jk}) , $\sum_k \tilde{A}_{ik}A_{kj} = \delta_{ij}$, we get the desired basis. The corollary is proved. \square

A basis $\omega_1, \dots, \omega_g$ satisfying the conditions (2.6.2) will be called a normal basis of holomorphic differentials (with respect to a canonical basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$).

Corollary 2.6.6. *Let $\omega_1, \dots, \omega_g$ be a normal basis of holomorphic differentials, and let*

$$B_{jk} = \oint_{b_j} \omega_k, \quad j, k = 1, \dots, g. \quad (2.6.5)$$

Then the matrix (B_{jk}) is symmetric and has negative-definite real part.

Proof. Let us apply the lemma 2.5.7 to the pair $\omega = \omega_j$ and $\omega' = \omega_k$. Then $\omega \wedge \omega'$, $A_i = 2\pi i \delta_{ij}$, $B_i = B_{ij}$, $A'_i = 2\pi i \delta_{ik}$, $B'_i = B_{ik}$. By (2.5.6) we have that

$$0 = \sum_i (2\pi i \delta_{ij} B_{ik} - 2\pi i \delta_{ik} B_{ij}) = 2\pi i (B_{jk} - B_{kj}).$$

The symmetry is proved. Next, we apply Corollary 2.6.1 to the differential $\sum_{j=1}^g x_j \omega_j$ where all the coefficients x_1, \dots, x_g are real. We have that $A_k = 2\pi i x_k$, $B_k = \sum_j x_j B_{kj}$ which implies

$$\Im \left(\sum_k 2\pi i x_k \sum_j x_j B_{kj} \right) = 2\pi \sum_{k,j} (\Re(B_{kj}) x_k x_j) < 0.$$

The lemma is proved. \square

Definition 2.6.7. *The matrix (B_{jk}) is called a period matrix of the Riemann surface Γ .*

Example 2.6.8. We consider a surface Γ of the form $w^2 = P_3(z)$ of genus $g = 1$ (an elliptic Riemann surface). Let $P_3(z) = (z - z_1)(z - z_2)(z - z_3)$ and choose a basis of cycles as shown in the figure. We have that

$$\omega_1 = \omega = \frac{adz}{\sqrt{P_3(z)}}, \quad a = 2\pi i \left(\oint_{a_1} \frac{dz}{\sqrt{P_3(z)}} \right)^{-1}.$$

Note that

$$\oint_{a_1} \frac{dz}{\sqrt{P_3(z)}} = 2 \int_{z_1}^{z_2} \frac{dz}{\sqrt{P_3(z)}}.$$

The period matrix is the single number

Figure 4: Homology basis.

$$B = \oint_{b_1} \frac{adz}{\sqrt{P_3(z)}} = 2\pi i \frac{\int_{z_2}^{z_3} \frac{dz}{\sqrt{P_3(z)}}}{\int_{z_1}^{z_2} \frac{dz}{\sqrt{P_3(z)}}}, \quad \Re(B) < 0. \quad (2.6.6)$$

Consider the function ("elliptic integral")

$$u(P) = \int_{P_0}^P \omega_1, \quad (2.6.7)$$

which is single-valued and holomorphic on the surface $\tilde{\Gamma}$ which is obtained by cutting Γ along the cycles a_1 and b_1 . This function is not single-valued on Γ . When the path of integration in the integral (2.6.7) is changed, the integral changes according to the law $u(P) \rightarrow u(P) + \int_{\gamma} \omega_i$ where γ is a closed contour (cycle). Decomposing it with respect to the basis of cycles, $\gamma = ma_1 + nb_1$, m and n integers we rewrite the last formula in the form

$$u(P) \rightarrow u(P) + 2\pi im + Bn. \quad (2.6.8)$$

We define the two-dimensional torus T^2 as the quotient of the complex plane $\mathbb{C} = \mathbb{R}^2$ by the integer lattice generated by the vectors $2\pi i$ and B ,

$$T^2 = \mathbb{C} / \{2\pi im + Bn \mid m, n \in \mathbb{Z}\} \quad (2.6.9)$$

(the vectors $2\pi i$ and B are independent over \mathbb{R} because $\Re(B) < 0$). The torus T^2 is a one-dimensional compact complex manifold. By (2.6.8) the function $u(P)$ unambiguously defines a mapping $\Gamma \rightarrow T^2$. It is holomorphic everywhere on Γ : $du = \omega$ and du vanishes nowhere (verify!). It is easy to see that this is an isomorphism. The meromorphic functions on the Riemann surface Γ are thereby identified with the so-called elliptic functions – the meromorphic functions on the torus T^2 . The latter functions can be regarded as doubly periodic (with a basis of periods $2\pi i$, B) meromorphic functions of a complex variable. The absence of nonconstant holomorphic functions on Γ (see Lecture 2.4) leads to the well-known assertion that there are no nonconstant doubly periodic holomorphic functions. For comparison with the standard notation of the theory of elliptic functions we note that the mapping (2.6.7) is usually taken in the form $u(P) \rightarrow (2\pi i)^{-1}u(P)$. Then the lattice has the form $m + n\tau$, $\tau = (2\pi i)^{-1}B$, $\Im\tau > 0$.

We give the construction of the mapping $T^2 \rightarrow \Gamma$ inverse to (2.6.7). Let the torus T^2 have the form

$$T^2 = \mathbb{C}/\{2m\omega + 2n\omega' \mid m, n \in \mathbb{Z}\}, \quad \Im(\omega/\omega') > 0. \quad (2.6.10)$$

The Weierstrass elliptic function, $\wp(z)$ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{m^2+n^2 \neq 0} \left[\frac{1}{(z - 2m\omega - 2n\omega')^2} - \frac{1}{(2m\omega + 2n\omega')^2} \right] \quad (2.6.11)$$

It is not hard to verify that the (2.6.11) converges absolutely and uniformly on compact sets not containing nodes of the period lattice. Therefore, it defines a meromorphic function of z having double poles at the lattice nodes. This function is obviously doubly periodic: $\wp(z + 2k\omega + 2l\omega') = \wp(z)$, $k, l \in \mathbb{Z}$. The Laurent expansions of the functions $\wp(z)$ and $\wp'(z)$ have the following forms as $z \rightarrow 0$

$$\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{20} + \frac{g_3 z^4}{28} + \dots, \quad (2.6.12)$$

$$\wp'(z) = -\frac{2}{z^3} + \frac{g_2 z}{10} + \frac{g_3 z^3}{7} + \dots, \quad (2.6.13)$$

where

$$g_2 = 60 \sum_{m^2+n^2 \neq 0} (2m\omega + 2n\omega')^{-4} \quad (2.6.14)$$

$$g_3 = 140 \sum_{m^2+n^2 \neq 0} (2m\omega + 2n\omega')^{-6},$$

(verify!). This gives us that the Laurent expansion of the function $(\wp')^2 - 4\wp^3 - g_2\wp - g_3$ has the form $O(z)$ as $z \rightarrow 0$. Hence, this doubly periodic function is constant, and thus equal to zero. Conclusion: the Weierstrass function $\wp(z)$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 + g_2\wp + g_3. \quad (2.6.15)$$

Let us now map the torus (2.6.10) into the elliptic curve

$$W^2 = 4Z^3 - g_2Z - g_3 \quad (2.6.16)$$

by setting

$$Z = \wp(z), \quad W = \wp'(z). \quad (2.6.17)$$

This mapping is the inverse of the one constructed above.

Exercise 2.6.9: Prove that any elliptic function with period lattice $\{2m\omega + 2n\omega'\}$ can be represented as a rational function of $\wp(z)$ and $\wp'(z)$

Exercise 2.6.10: Consider the Korteweg-de Vries (KdV) equation

$$\dot{u} = 6uu' - u''' \quad (2.6.18)$$

(here $u = u(x, t)$, the dot stands for the derivative with respect to t , and the prime stands for the derivative with respect to x). Show that any (complex) periodic solution of it with the form of a traveling wave has the form

$$u(x, t) = u(x - ct) = 2\wp(x - ct - x_0) - \frac{c}{6}, \quad (2.6.19)$$

where the Weierstrass function \wp corresponds to some elliptic curve (2.6.16), and the velocity c and the phase x_0 are arbitrary.

Exercise 2.6.11: (see [12]). Look for a solution of the KdV equation in the form

$$u(x, t) = 2\wp(x - x_1(t)) + 2\wp(x - x_2(t)) + 2\wp(x - x_3(t)). \quad (2.6.20)$$

Derive for the functions $x_j(t)$ the system of differential equations

$$\ddot{x}_j = 12 \sum_{k \neq j} \wp(x_j - x_k), \quad j = 1, 2, 3, \quad (2.6.21)$$

and its integrals

$$\sum_{k \neq j} \wp'(x_j - x_k) = 0, \quad j = 1, 2, 3. \quad (2.6.22)$$

Integrate this system in quadratures.

Exercise 2.6.12: (see [31]). For the elliptic curve (2.6.16) construct a new elliptic curve $\tilde{\Gamma}$ given by a third-degree polynomial

$$\tilde{P}_3(z) = (z^2 - 3g_2) \left(z + 9\frac{g_3}{g_2} \right). \quad (2.6.23)$$

Denote by \tilde{P} the corresponding Weierstrass function. Let $\xi_{ij} = \wp(x_i(t) - x_j(t))$, $i \neq j$, where the quantities $x_i(t)$ are defined in the previous problem. Show that the functions $\xi_{12}(t)$, $\xi_{23}(t)$, and $\xi_{13}(t)$ are the roots of the cubic equation

$$4\xi^3 - g_2\xi - \frac{1}{3}g_3 + \frac{1}{2}g_2\tilde{\wp}(6i\sqrt{3g_2}t) = 0 \quad (2.6.24)$$

Remark 2.6.13. . We define the Weierstrass ζ and σ functions (which are useful in the theory of elliptic functions) from the conditions

$$\zeta'(z) = -\wp(z), \quad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z). \quad (2.6.25)$$

The series expansion of $\zeta(z)$ has the form

$$\zeta(z) = \frac{1}{z} + \sum_{w=2m\omega+2n\omega' \neq 0} \left[\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right]. \quad (2.6.26)$$

Figure 5: Homology basis.

This function has simple poles at the nodes of the period lattice. The function $\sigma(z)$ is entire. It has simple zeros at the nodes of the period lattice and can be expanded in the infinite product

$$\sigma(z) = z \prod_{w=2m\omega+2n\omega' \neq 0} \left\{ \left(1 - \frac{z}{w}\right) \exp \left[\frac{z}{w} + \frac{z^2}{2w^2} \right] \right\} \quad (2.6.27)$$

The functions $\zeta(z)$ and $\sigma(z)$ are not elliptic; under a translation of the argument by a vector of the period lattice they transform according to the law

$$\zeta(z + 2k\omega + 2l\omega') = \zeta(z) + 2k\eta + 2l\eta', \quad \eta = \zeta(\omega), \quad \eta' = \zeta(\omega'), \quad (2.6.28)$$

$$\sigma(z + 2\omega) = \sigma(z) \exp[2\eta(z + \omega)], \quad \sigma(z + 2\omega') = -\sigma(z) \exp[2\eta'(z + \omega')] \quad (2.6.29)$$

where η and η' are constants depending on the period lattice.

Exercise 2.6.14: Show that under the dilation $\omega \rightarrow \lambda\omega$, $\omega' \rightarrow \lambda\omega'$, $z \rightarrow \lambda z$ the functions \wp , ζ and σ transform according to the law $\wp \rightarrow \lambda^2\wp$, $\zeta \rightarrow \lambda^{-1}\zeta$, $\sigma \rightarrow \lambda\sigma$. In view of this assertion the properties of \wp , ζ and σ depend in essence only on the ratio $\tau = \omega/\omega'$ of the periods.

Exercise 2.6.15: Prove that the tori $T^2 = \mathbb{C}/\{m + n\tau\}$ and $T'^2 = \mathbb{C}/\{m + n\tau'\}$ are isomorphic if and only if $\tau' = \frac{a\tau + b}{d\tau + d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a unimodular integral matrix.

Exercise 2.6.16: Prove the following identity:

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} = \wp(u) - \wp(v). \quad (2.6.30)$$

Other properties of the functions \wp , ζ and σ and of other elliptic functions as well, can be found, for example, in the texts [2] and [7], or in the handbook [3].

Example 2.6.17. . Consider a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z) = \prod_{i=1}^{2g+1} (z - z_i)$ for genus $g \geq 2$, and choose a basis of cycles as indicated in the figure (there $g = 2$). A normal basis of holomorphic differentials has the form

$$\omega_j = \frac{\prod_{k=1}^g c_{jk} z^{k-l} dz}{\sqrt{P_{2g+1}(z)}}, \quad j = 1, \dots, g. \quad (2.6.31)$$

Here (c_{jk}) is the matrix inverse to the matrix (A_{jk}) where

$$A_{jk} = 2 \int_{z_{2j-1}}^{z_{2j}} \frac{z^{k-1} dz}{\sqrt{P_{2g+1}(z)}}, \quad j, k = 1, \dots, g. \quad (2.6.32)$$

2.7 Meromorphic differentials, their residues and periods

Meromorphic (Abelian) differentials on a Riemann surface differ from holomorphic differentials by the possible presence of singularities of pole type. If a surface is given in the form $F(z, w) = 0$, then the Abelian differentials have the form $\omega = R(z, w)dz$ (or, equivalently, $\omega = R_1(z, w)dw$) where $R(z, w)$ and $R_1(z, w)$ are rational functions. For example, on a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z)$ the differential $w^{-1}z^{k-1}dz$ has for $k > g$ a unique pole at infinity of multiplicity $2(k - g)$ (see Example 2.5.4). Suppose that the differential ω has a pole of multiplicity k at the point P_0 i.e., can be written in terms of a local parameter z , $z(P_0) = 0$, in the form

$$\omega = \left(\frac{c_{-k}}{z^k} + \dots + \frac{c_{-1}}{z} + O(1) \right) dz \quad (2.7.1)$$

(the multiplicity of the pole does not depend on the choice of the local parameter z).

Definition 2.7.1. *The residue $\text{Res}_{P=P_0} \omega(P)$ of the differential ω at a point P_0 is defined to be the coefficient c_{-1} .*

Lemma 2.7.2. *The residue $\text{Res}_{P=P_0} \omega(P)$ does not depend on the choice of the local parameter z .*

Proof. This residue is equal to

$$c_{-1} = \frac{1}{2\pi i} \oint_C \omega$$

where C is an arbitrary small contour encircling P_0 . The independence of this integral on the choice of the local parameter is obvious. The lemma is proved. \square

Theorem 2.7.3 (The Residue Theorem). *The sum of the residues of a meromorphic differential ω on a Riemann surface, taken over all poles of this differential, is equal to zero.*

Proof. Let P_1, \dots, P_N be the poles of ω . We encircle them by small contours C_1, \dots, C_N such that

$$\text{Res} \omega = \frac{1}{2\pi i} \oint_{C_i} \omega, \quad i = 1, \dots, N,$$

(the contours C_i run in the positive direction), and cut out the domains bounded by C_1, \dots, C_N from the surface Γ . This gives a domain Γ' with oriented boundary of the form $\partial\Gamma' = -C_1 - \dots - C_N$ (the sign means reversal of orientation). The differential ω is holomorphic on Γ' . By Stokes' formula,

$$\sum_{j=1}^N \text{Res}_{P_j} \omega = \frac{1}{2\pi i} \sum_{j=1}^N \oint_{C_j} \omega = -\frac{1}{2\pi i} \oint_{\partial\Gamma'} \omega = -\frac{1}{2\pi i} \int \int_{\Gamma'} d\omega = 0,$$

since $d\omega = 0$. The theorem is proved. \square

We present the simplest example of the use of the residue theorem: we prove that the number of zeros of a meromorphic function is equal to its number of poles (counting multiplicity). Let P_1, \dots, P_k , be the zeros of the meromorphic function f , with multiplicities m_1, \dots, m_k and let Q_1, \dots, Q_l be the poles of this function, with multiplicities n_1, \dots, n_l . Consider the logarithmic differential $d(\ln f)$. This is a meromorphic differential on Γ with simple poles at P_1, \dots, P_k with residues m_1, \dots, m_k and at the points Q_1, \dots, Q_l with residues $-n_1, \dots, -n_l$. By the residue theorem: $m_1 + \dots + m_k - n_1 - \dots - n_l = 0$, which means that the assertion to be proved is valid. One more example. For any elliptic function $f(z)$ on the torus $T^2 = \mathbb{C}/\{2m\omega + 2n\omega'\}$ the residues at the poles are defined with respect to the complex coordinate z (in \mathbb{C}). These are the residues of the meromorphic differential $f(z)dz$, since dz is holomorphic everywhere. Conclusion: the sum of the residues of any elliptic function (over all poles in a lattice parallelogram) is equal to zero. We formulate an existence theorem for meromorphic differentials on a Riemann surface Γ (see [27] for a proof).

Theorem 2.7.4 (Theorem C). . Suppose that P_1, \dots, P_N are points of a Riemann surface Γ and z_1, \dots, z_N are local parameters centered at these points, $z_i(P_i) = 0$, and the collection of principal parts is

$$\left(\frac{c_{-k_i}^{(i)}}{z_i^{k_i}} + \dots + \frac{c_{-1}^{(i)}}{z_i} \right) dz_i, \quad i = 1, \dots, N. \quad (2.7.2)$$

Assume the condition

$$\sum_{i=1}^N c_{-1}^{(i)} = 0. \quad (2.7.3)$$

Then there exists on Γ a meromorphic differential with poles at the points P_1, \dots, P_N , and principal parts (2.7.2).

Any meromorphic differential can be represented as the sum of a holomorphic differential and the following elementary meromorphic differentials.

1. Abelian differential of the second kind Ω_P^n has a unique pole of multiplicity $n + 1$ at P and a principal part of the form

$$\Omega_P^n = \left(\frac{1}{z^{n+1}} + O(1) \right) dz \quad (2.7.4)$$

with respect to some local parameter z , $z(P) = 0$, $n = 1, 2, \dots$

2. An Abelian differential of the third kind Ω_{PQ} has a pair of simple poles at the points P and Q with residues $+1$ and -1 respectively.

Example 2.7.5. We construct elementary Abelian differentials on a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z)$. Suppose that a point P which is not a branch point takes the form $P = (a, w_a = \sqrt{P_{2g+1}(a)})$. An Abelian differential of the second kind $\Omega_P^{(1)}$ has the form

$$\Omega_P^{(1)} = \left(\frac{w + w_a}{(z - a)^2} - \frac{P'_{2g+1}(a)}{2w_a(z - a)} \right) \frac{dz}{2w} \quad (2.7.5)$$

(with respect to the local parameter $z-a$). The differentials $\Omega_P^{(n)}$ can be obtained as follows:

$$\Omega_P^n = \frac{1}{n!} \frac{d^{n-1}}{da^{n-1}} \Omega_P^1. \quad (2.7.6)$$

If $P = (z_i, 0)$ is one of the branch points, then

$$\Omega_P^n = \frac{dz}{2(z-z_i)^{k+1}} \text{ for } n = 2k, \quad \Omega_P^n = \frac{dz}{2(z-z_i)^{k+1}w} \text{ for } n = 2k+1. \quad (2.7.7)$$

Finally, if $P = \infty$, then

$$\Omega_P^{(n)} = -\frac{1}{2} z^{k-1} dz \text{ for } n = 2k, \quad \Omega_P^n = -\frac{1}{2} z^{g+k-1} \frac{dz}{w} \text{ for } n = 2k+1. \quad (2.7.8)$$

We now construct differentials of the third kind. Suppose that the point P and Q have the form $P = (a, w_a = \sqrt{P_{2g+1}(a)})$ and $Q = (b, w_b = \sqrt{P_{2g+1}(b)})$. Then

$$\Omega_{PQ} = \left(\frac{w+w_a}{z-a} - \frac{w+w_b}{z-b} \right) \frac{dz}{2w} \quad (2.7.9)$$

If $Q = +\infty$ then

$$\Omega_{PQ} = \frac{w+w_a}{z-a} \frac{dz}{2w}. \quad (2.7.10)$$

Accordingly, we see that for a hyperelliptic Riemann surface it is possible to represent all the Abelian differentials without appealing to Theorem 2.7.4.

Exercise 2.7.6: Deduce from Theorem 2.7.4 that a Riemann surface Γ of genus 0 is rational. *Hint.* Show that for any points $P, Q \in \Gamma$ the function $f = \exp \int \Omega_{PQ}$ is single valued and meromorphic on Γ and gives a biholomorphic isomorphism $f : \Gamma \rightarrow \mathbb{CP}^1$.

The period of a meromorphic differential ω along the cycle γ is defined if the cycle does not pass through poles of this differential. The period $\int_\gamma \omega$ depends only on the homology class of γ on the surface Γ , with the poles of ω with nonzero residue deleted. For example, the periods of the differential Ω_{PQ} of the third kind along a cycle not passing through the points P and Q are determined to within integer multiples of $2\pi i$. In speaking of the periods of meromorphic differentials we shall assume that the cycles do not pass through the poles of the differential, and we also recall that the dependence of the period on the homology class of Γ is not single-valued (for differentials of the third kind).

Lemma 2.7.7. *Suppose that the differentials ω_1 and ω_2 on a Riemann surface Γ have the same poles and principal parts, and the same periods with respect to the cycles $a_1, \dots, a_g, b_1, \dots, b_g$. Then these differentials coincide.*

Proof. The difference $\omega_1 - \omega_2$ is a holomorphic differential that has zero a -periods. Therefore, it is identically zero (see Lecture 2.6). The lemma is proved. \square

Definition 2.7.8. *A meromorphic differential ω is said to be normalized with respect to a basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$ if it has zero a -periods.*

Any meromorphic differential ω can be turned into a normalized differential by adding a suitable holomorphic differential. By Lemma 2.7.7, a normalized meromorphic differential is uniquely determined by its poles and by the principal parts at the poles. In what follows we assume that meromorphic differentials are normalized. We obtain formulas that will be useful for the b -periods of such differentials by arguments like those in the proof of Lemma 2.5.7.

Lemma 2.7.9. *The following formulas hold for the b -periods of normalized differentials $\Omega_P^{(n)}$ and Ω_{PQ}*

$$\oint_{b_i} \Omega_P^{(n)} = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \psi_i(z)|_{z=0}, \quad i = 1, \dots, g, \quad n = 1, 2, \dots, \quad (2.7.11)$$

where z is a particular local parameter in a neighborhood of P , $z(P) = 0$, and the functions $\psi_i(z)$ are determined by the equality $\omega_i = \psi_i(z)dz$;

$$\oint_{b_i} \Omega_{PQ} = \int_Q^P \omega_i, \quad i = 1, \dots, g, \quad (2.7.12)$$

where the integration from Q to P in the last integral does not intersect the cycles $a_1, \dots, a_g, b_1, \dots, b_g$.

Proof. We encircle the point P with a small circle C ; deleting the interior of this circle from the surface Γ , we get a domain Γ' with $\partial\Gamma' = -C$. Let us apply the arguments of Lemma 2.5.7 to the pair of differentials $\omega = \omega_i, \omega' = \Omega_P^{(n)}$. Denote by u_i the primitive

$$u_i(P) = \int_{P_0}^P \omega_i \quad (2.7.13)$$

which is single-valued on the cut surface $\tilde{\Gamma}$. We have that

$$0 = \int \int_{\Gamma} \omega \wedge \omega' = \int_{\partial\tilde{\Gamma}'} u_i \Omega_P^{(n)} = \sum_{j=1}^g (A_j B_j' - A_j' B_j) - \oint_C u_i \Omega_P^{(n)} \quad (2.7.14)$$

(the boundary $\partial\tilde{\Gamma}'$ differs from the boundary $\partial\tilde{\Gamma}$ by $(-C)$). Here the a and b -periods have the form

$$A_j = 2\pi i \delta_{ij}, \quad B_j = B_{ij}, \quad A_j' = 0, \quad B_j' = \oint_{b_j} \Omega_P^{(n)}.$$

From this,

$$\oint_{b_j} \Omega_P^{(n)} = \text{Res}_P(u_i \Omega_P^{(n)}). \quad (2.7.15)$$

Computation of the residue on the right-hand side of this equality leads to (2.7.11). We now prove (2.7.12). Let γ be a path on Γ from Q to P . By Γ' denote the surface with a small neighborhood of γ removed. The boundary of this neighborhood is denoted by C .

Applying the arguments of Lemma 2.5.7 to the differentials $\omega' = \omega_i$ and $\omega = \Omega_{QP}$ we get by analogy with (2.7.14) and (2.7.15) that

$$\oint_{b_i} \Omega_{PQ} = \frac{1}{2\pi i} \oint_C u\omega_i$$

where u is the primitive of the differential Ω_{PQ} which is single-valued and holomorphic on $\tilde{\Gamma}'$. The integral on the right-hand side can be represented in the form

$$\oint_C u\omega_i = \left(\int_{C_Q} + \int_{C_P} + \int_{\gamma_+} + \int_{\gamma_-} \right) u\omega_i$$

(see the figure), where C_Q and C_P are arcs of circles of small radius ϵ . Since the function u has logarithmic singularities at Q and P , the integrals over the arcs C_Q and C_P tend to zero as $\epsilon \rightarrow 0$. Next, denote by u_+ and u_- the values of u on the corresponding edges γ_+ and γ_- . We have that the jump $u_+ - u_-$ is equal to $2\pi i$. Finally,

$$\oint_C u\omega_i = \left(\int_{\gamma_+} + \int_{\gamma_-} \right) u\omega_i = \int_{\gamma} (u_+ - u_-)\omega_i = 2\pi i \int_{\gamma} \omega_i,$$

which implies (2.7.12). The lemma is proved. \square

Exercise 2.7.10: Prove the following equality, which is valid for any quadruple of distinct points P_1, \dots, P_4 on a Riemann surface:

$$\int_{P_2}^{P_1} \Omega_{P_3P_4} = \int_{P_4}^{P_3} \Omega_{P_1P_2}. \quad (2.7.16)$$

Exercise 2.7.11: Consider the series expansion of the differentials $\Omega_P^{(n)}$ in a neighborhood of the point P

$$\Omega_P^{(n)} = \left(\frac{1}{z^{n+1}} + \sum_{j=0}^{\infty} c_j^{(n)} z^j \right) dz. \quad (2.7.17)$$

Prove the following symmetry relations for the coefficients $c_j^{(k)}$:

$$kc_{j-1}^{(k)} = jc_{k-1}^{(j)}, \quad k, j = 1, 2, \dots \quad (2.7.18)$$

Exercise 2.7.12: Prove the following relation of Legendre from the theory of elliptic functions (see Example 2.6.8 for the notation):

$$\eta\omega' - \eta'\omega = \frac{\pi i}{2}. \quad (2.7.19)$$

Exercise 2.7.13: Suppose that the surface Γ has the form $w^2 = \prod_{i=1}^{2g+1} (z - z_i)$, where all the z_i are real and $z_1 < \dots < z_{2g+1}$. Choose a basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$ as shown in the figure (for $g = 2$). Show that the normalized differential Ω_{∞}^1 has the form

$$\Omega_{\infty}^{(1)} = -\frac{z^g + \alpha_1 z^{g-1} + \dots + \alpha_g}{2w} dz \quad (2.7.20)$$

where all the coefficients are real. Denote by B_k its b -periods. Prove that all the numbers B_k are real, and

$$B_g < B_{g-1} < \cdots < B_2 < B_1 < 0. \quad (2.7.21)$$

Exercise 2.7.14: Prove that a meromorphic differential of the second kind ω is uniquely determined by its poles, principal parts, and the real normalization condition

$$\Im \oint_{\gamma} \omega = 0 \quad (2.7.22)$$

for any cycle γ . Formulate and prove an analogous assertion for differentials of the third kind (with purely imaginary residues).

2.8 The Jacobi variety, Abel's theorem

Let e_1, \dots, e_g be the standard basis in the space \mathbb{C}^g ($(e_j)_k = \delta_{jk}$). Let $B = (B_{jk})$ be an arbitrary symmetric $g \times g$ matrix with negative-definite real part (as shown in Lecture 2.6, the period matrices of Riemann surfaces have this property). We consider the vectors

$$2\pi i e_1, \dots, 2\pi i e_g, \quad B e_1, \dots, B e_g, \quad (2.8.1)$$

(here the vector $B e_j$ has coordinates $(B e_j)_k$).

Lemma 2.8.1. *The vectors (2.8.1) are linearly independent over \mathbb{R} .*

Proof. Assume that these vectors are dependent over \mathbb{R} :

$$2\pi i(\lambda_1 e_1 + \cdots + \lambda_g e_g) + B(\mu_1 e_1 + \cdots + \mu_g e_g) = 0, \quad \lambda_i, \mu_j \in \mathbb{R}.$$

Separating out the real part of this equality we get that $\Re(B(\mu_1 e_1 + \cdots + \mu_g e_g)) = 0$. But the matrix $\Re(B)$ is nonsingular, which implies $\mu_1 = \cdots = \mu_g = 0$. Hence also $\lambda_1 = \cdots = \lambda_g = 0$. The lemma is proved. \square

Consider in \mathbb{C}^g the integer period lattice generated by the vectors (2.8.1). The vectors in this lattice can be written in the form

$$2\pi i M + B N, \quad M, N \in \mathbb{Z}^g. \quad (2.8.2)$$

By Lemma 2.8.1 the quotient of \mathbb{C}^g by this lattice is the $2g$ -dimensional torus

$$T^{2g} = T^{2g}(B) = \mathbb{C}^g / \{2\pi i M + B N\}, \quad (2.8.3)$$

(a g -dimensional complex manifold – a so-called Abelian manifold).

Definition 2.8.2. *Suppose that $B = (B_{jk})$ is a period matrix of a Riemann surface Γ of genus g . The torus $T^{2g}(B)$ in (2.8.3), constructed from this period matrix is called the Jacobi variety (or Jacobian) of the surface Γ and denoted by $J(\Gamma)$.*

Remark 2.8.3. What happens with the torus $J(\Gamma)$ when the canonical basis of cycles on Γ changes? Let a'_1, \dots, a'_g and b'_1, \dots, b'_g be another canonical basis of cycles. It is connected with the first by an integer linear transformation

$$\begin{aligned} a'_i &\simeq \sum_j k_{ij} a_j + \sum_j l_{ij} b_j, \\ b'_i &\simeq \sum_j m_{ij} a_j + \sum_j n_{ij} b_j \end{aligned} \tag{2.8.4}$$

This matrix of this transformation has the form $\begin{pmatrix} k & l \\ m & n \end{pmatrix}$, where $k = (k_{ij})$, $l = (l_{ij})$, $m = (m_{ij})$ and $n = (n_{ij})$. This matrix is unimodular and also symplectic

$$\begin{pmatrix} k^t & m^t \\ l^t & n^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k & l \\ m & n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.8.5}$$

since the matrix of the intersection numbers for the basis a_i, b_i and a'_i, b'_i are the same. The new normal basis of holomorphic differentials has the form

$$\omega'_i = \sum_{j=1}^g C_{ji} \omega_j, \quad C = (C_{ij}) = 2\pi i (2\pi i k + lB)^{-1} \tag{2.8.6}$$

The new period matrix $B' = (B'_{ij})$ thus has the form

$$B' = 2\pi i (2\pi i m + nB)(2\pi i k + lB)^{-1} \tag{2.8.7}$$

It follows from these formulas that the complex linear transformation of \mathbb{C}^g with matrix C^{-1} carries the lattice (2.8.2) into an analogous lattice corresponding to the matrix B' . This gives an isomorphism $T^{2g}(B) \rightarrow T^{2g}(B')$ of the complex tori. Accordingly, the Jacobian $J(\Gamma)$ does not change when the canonical basis changes.

We consider the primitives ("Abelian integrals") of the basis of holomorphic differentials:

$$u_k(P) = \int_{P_0}^P \omega_k, \quad k = 1, \dots, g, \tag{2.8.8}$$

where P_0 is a fixed point of the Riemann surface. The vector-valued function

$$A(P) = (u_1(P), \dots, u_g(P)) \tag{2.8.9}$$

is called the Abel mapping (the path of integration is chosen to be the same in all the integrals $u_1(P), \dots, u_g(P)$).

Lemma 2.8.4. *The Abel mapping is a well-defined holomorphic mapping*

$$\Gamma \rightarrow J(\Gamma). \tag{2.8.10}$$

Proof. (cf. Example 2.7.5). A change of the path of integration in the integrals (2.8.8) leads to a change in the values of these integrals according to the law

$$u_k(P) \rightarrow u_k(P) + \oint_{\gamma} \omega_k, \quad k = 1, \dots, g,$$

where γ is some cycle on Γ . Decomposing it with respect to the basis of cycles, $\gamma \simeq \sum m_j a_j + \sum n_j b_j$ we get that

$$u_k(P) \rightarrow u_k(P) + 2\pi i m_k + \sum_j B_{kj} n_j, \quad k = 1, \dots, g.$$

The increment on the right-hand side is the k th coordinate of the period lattice vector $2\pi i M + BN$ where $M = (m_1, \dots, m_g)$, $N = (n_1, \dots, n_g)$. The lemma is proved. \square

The Jacobi variety together with the Abel mapping (2.8.10) is used for solving the following problem: what points of a Riemann surface can be the zeros and poles of meromorphic functions? We have the Abel's theorem.

Theorem 2.8.5 (Abel's Theorem). . *The points P_1, \dots, P_n and Q_1, \dots, Q_n (some of the points can repeat) on a Riemann surface Γ are the respective zeros and poles of some function meromorphic on Γ if and only if the following relation holds on the Jacobian:*

$$A(P_1) + \dots + A(P_n) \equiv A(Q_1) + \dots + A(Q_n). \quad (2.8.11)$$

Here and below, the sign \equiv will mean equality on the Jacobi variety (congruence modulo the period lattice (2.8.2)). We remark that the relation (2.8.11) does not depend on the choice of the initial point P_0 of the Abel map (2.8.8).

Proof. 1) Necessity. Suppose that a meromorphic function f has the respective points P_1, \dots, P_n and Q_1, \dots, Q_n as zeros and poles, where each zero and pole is written the number of times corresponding to its multiplicity. Consider the logarithmic differential $\Omega = d(\log f)$. Since $f = \text{const} \exp \int_{P_0}^P \Omega$, all the periods of this differential Ω are integer multiples of $2\pi i$. On the other hand, we represent it in the form

$$\Omega = \sum_{j=1}^n \Omega_{P_j Q_j} + \sum_{s=1}^g c_s \omega_s, \quad (2.8.12)$$

where $\Omega_{P_j Q_j}$ are normalized differentials of the third kind (see Lecture 2.7) and c_1, \dots, c_g are constant coefficients. Let us use the information about the periods of the differential. We have that

$$2\pi i n_k = \oint_{a_k} \Omega = 2\pi i c_k, \quad n_k \in \mathbb{Z},$$

which gives us $c_k = n_k$. Further,

$$2\pi i m_k = \oint_{b_k} \Omega = \sum_{j=1}^n \int_{Q_j}^{P_j} \omega_k + \sum_{s=1}^g B_{ks} n_s$$

(we used the formula (2.7.12)). From this,

$$u_k(P_1) + \cdots + u_k(P_n) - u_k(Q_1) - \cdots - u_k(Q_n) = \sum_{j=1}^n \int_{Q_j}^{P_j} \omega_k = 2\pi i m_k - \sum_{j=1}^g B_{kj} n_j. \quad (2.8.13)$$

The right-hand side is the k th coordinate of the vector $2\pi i M + BN$ of the period lattice (2.8.2), where $M = (m_1, \dots, m_g)$, $N = (n_1, \dots, n_g)$. The necessity of the condition (2.8.11) is proved.

2) Sufficiency. The congruence (2.8.11) can be rewritten in the form of the equalities (2.8.13) for some integer numbers $m_1, \dots, m_g, n_1, \dots, n_g$. Repeating the arguments above, we get that all the periods of the differential Ω of the form (2.8.12) with $c_s = n_s$, $s = 1, \dots, g$, are integers multiples of $2\pi i$. The function $f = \exp \int_{P_0}^P \Omega$ is thus a single-valued meromorphic function on Γ with the given zeros and poles. The theorem is proved. \square

Example 2.8.6. We consider the elliptic curve

$$w^2 = 4z^3 - g_2 z - g_3. \quad (2.8.14)$$

For this curve the Jacobi variety $J(\Gamma)$ is a two-dimensional torus, and the Abel mapping (which coincides with (2.6.7)) is an isomorphism (see Example 2.6.8). Abel's theorem becomes the following assertion from the theory of elliptic functions: the sum of all the zeros of an elliptic function is equal to the sum of all its poles to within a vector of the period lattice.

Example 2.8.7. (also from the theory of elliptic functions). Consider an the elliptic function of the form $f(z, w) = az + bw + c$, where a, b , and c are constants. It has a pole of third order at infinity (for $b \neq 0$). Consequently, it has three zeros P_1, P_2 , and P_3 . In other words, the line $az + bw + c = 0$ intersects the elliptic curve (2.8.14) in three points (see the figure). We choose ∞ as the initial point for the Abel mapping, i.e., $u(\infty) = 0$. Let $u_i = u(P_i)$, $i = 1, 2, 3$. In other words,

$$P_i = (\wp(u_i), \wp'(u_i)), \quad i = 1, 2, 3,$$

where $\wp(u)$ is the Weierstrass function corresponding to the curve (2.8.14). Applying Abel's theorem to the zeros and poles of f , we get that

$$u_1 + u_2 + u_3 = 0.$$

Conversely, according to the same theorem, if $u_1 + u_2 + u_3 = 0$, i.e. $u_3 = -u_2 - u_1$ then the points P_1, P_2 and P_3 lie on a single line. Writing the condition of collinearity of these points and taking into account the evenness of \wp and oddness of \wp' , we get the addition theorem for Weierstrass functions:

$$\det \begin{vmatrix} 1 & \wp(u_1) & \wp'(u_1) \\ 1 & \wp(u_2) & \wp'(u_2) \\ 1 & \wp(u_1 + u_2) & -\wp'(u_1 + u_2) \end{vmatrix} = 0. \quad (2.8.15)$$

2.9 Divisors on a Riemann surface. The canonical class. The Riemann-Roch theorem

Definition 2.9.1. A divisor D on a Riemann surface is defined to be a (formal) integral linear combination of points on it:

$$D = \sum_{i=1}^n n_i P_i, \quad P_i \in \Gamma, \quad n_i \in \mathbb{Z}. \quad (2.9.1)$$

For example, for any meromorphic function f the divisor (f) of its zeros P_1, \dots, P_k and poles Q_1, \dots, Q_l of multiplicities m_1, \dots, m_k , and n_1, \dots, n_l , respectively is defined

$$(f) = m_1 P_1 + \dots + m_k P_k - n_1 Q_1 - \dots - n_l Q_l. \quad (2.9.2)$$

Divisors of meromorphic functions are also called principal divisors. The divisors obviously form an Abelian group (the zero is the empty divisor). For example, for principal divisors we have $(fg) = (f) + (g)$. The degree $\deg D$ of a divisor of the form (2.9.1) is defined to be the number

$$\deg D = \sum_{i=1}^N n_i. \quad (2.9.3)$$

The degree is a linear function on the group of divisors. For instance,

$$\deg(f) = 0. \quad (2.9.4)$$

Two divisors D and D' are said to be linearly equivalent, $D \simeq D'$ if their difference is a principal divisor. Linearly equivalent divisors have the same degree in view of (2.9.4). For example, on $\mathbb{C}\mathbb{P}^1$ any divisor of zero degree is principal, and two divisors of the same degree are always linearly equivalent.

Example 2.9.2. The divisor (ω) of any Abelian differential ω on a Riemann surface Γ is well-defined by analogy with (2.9.2). If ω' is another Abelian differential, then $(\omega) \simeq (\omega')$. Indeed, their ratio $f = \omega/\omega'$ is a meromorphic function on Γ , and $(\omega) - (\omega') = (f)$.

The linear equivalence class of divisors of Abelian differentials is called the *canonical class* of the Riemann surface. We denote it by K_Γ . For example, the divisor $-2\infty = (dz)$ can be taken as a representative of the canonical class $K_{\mathbb{C}\mathbb{P}^1}$.

We reformulate Abel's theorem in the language of divisors. Note that the Abel mapping extends linearly to the whole group of divisors. Abel's theorem obviously means that a divisor D is principal if and only if the following two conditions hold:

1. $\deg D = 0$;
2. $A(D) \equiv 0$ on $J(\Gamma)$.

Let us return to the canonical class. We compute it for a hyperelliptic surface $w^2 = P_{2g+2}(z)$. Let P_1, \dots, P_{2g+2} be the branch points of the Riemann surface, and $P_{\infty+}$ and $P_{\infty-}$ its point at infinity. We have that

$$(dz) = P_1 + \dots + P_{2g+2} - 2P_{\infty+} - 2P_{\infty-}.$$

Thus the degree of the canonical class on this surface is equal to $2g - 2$. We prove an analogous assertion for an arbitrary Riemann surface.

Lemma 2.9.3. *Let $f : \Gamma \rightarrow X$ a holomorphic map between Riemann surfaces Γ and X and ω a meromorphic one form on X , then for any fixed point $P \in \Gamma$*

$$\text{ord}_P f^* \omega = (1 + \text{ord}_{f(P)}(\omega)) \text{mult}_P(f) - 1 \quad (2.9.5)$$

where $f^* \omega$ denotes the pull back of ω via f .

The proof is obvious.

Definition 2.9.4. *Let $f : \Gamma \rightarrow X$ a holomorphic map between Riemann surfaces. The branch point divisor W_f is the divisor on Γ defined by*

$$W_f = \sum_{P \in \Gamma} [\text{mult}_P(f) - 1] P. \quad (2.9.6)$$

Applying (2.9.5) and (2.9.6) we arrive at the relation

$$(f^* \omega) = W_f + f^*(\omega). \quad (2.9.7)$$

Suppose that the Riemann surface Γ is given by the equation $F(z, w) = 0$. Further, let P_1, \dots, P_N be the branch points of this surface with respective multiplicities f_1, \dots, f_N with respect to the meromorphic function $z : \Gamma \rightarrow \mathbb{CP}^1$. (see Lecture 2.1). The branch point divisor $W_z = f_1 P_1 + \dots + f_N P_N$.

Lemma 2.9.5. *The canonical class of the surface Γ has the form*

$$K_\Gamma = W_z + z^*(K_{\mathbb{CP}^1}). \quad (2.9.8)$$

Here z^* denotes the inverse image of a divisor in the class $K_{\mathbb{CP}^1}$ with respect to the meromorphic function $z : \Gamma \rightarrow \mathbb{CP}^1$.

Proof. This follows immediately from (2.9.7). □

Corollary 2.9.6. *The degree of the canonical class K_Γ of a Riemann surface Γ of genus g is equal to $2g - 2$.*

Proof. We have from (2.9.8) that $\deg K_\Gamma = f - 2n$, where f is the total multiplicity of the branch points ($f = \deg W_z$) and $n = \deg z$ is the number of sheets of the Riemann surface. But by the Riemann-Hurwitz formula (2.3.9), $f = 2g + 2n - 2$. The corollary is proved. □

The divisor (2.9.1) is *positive* if all multiplicities n are positive. An *effective* divisor is a divisor linearly equivalent to a positive divisor. Divisors D and D' are connected by the inequality $D > D'$ if their difference $D - D'$ is a positive divisor.

With each divisor D we associate the linear space of meromorphic functions

$$L(D) = \{f \mid (f) \geq -D\}. \quad (2.9.9)$$

If D is a positive divisor, then this space consists of functions f having poles only at points of D , with multiplicities not greater than the multiplicities of these points in D . If $D = D_+ - D_-$, where D_+ and D_- are positive divisors, then the space $L(D)$ consists of the meromorphic functions with poles possible only at points of D_+ , with multiplicities not greater than the multiplicities of these points in D , and with zeros at all points of D_- (at least), with multiplicities not less than the multiplicities of these points in D .

Lemma 2.9.7. *If the divisors D and D' are linearly equivalent, then the spaces $L(D)$ and $L(D')$ are isomorphic.*

Proof. Let $D - D' = (g)$, where g is a meromorphic function. If $f \in L(D)$, then $f' = fg \in L(D')$. Indeed,

$$(f') + D' = (f) + (g) + D' = (f) + D > 0.$$

Conversely, if $f' \in L(D')$, then $f = g^{-1}f' \in L(D)$. The lemma is proved. \square

We denote the dimension of the space $L(D)$ by

$$l(D) = \dim L(D). \tag{2.9.10}$$

By Lemma 2.9.7, the function $l(D)$ (as well as the degree $\deg D$) is constant on linear equivalence classes of divisors. We make some simple remarks about the properties of this important function.

Remark 2.9.8. If a divisor D is effective, then $l(D) > 0$. Indeed, replacing D by a positive divisor D' linearly equivalent to it, we see that the space $l(D')$ contains the constants. Conversely, if $l(D) > 0$, then D is effective. Indeed, if the meromorphic function f is such that $D' = (f) + D > 0$, then the divisor D' , which is linearly equivalent to D is positive.

Remark 2.9.9. For the zero (empty) divisor, $l(0) = 1$. If $\deg D < 0$, then $l(D) = 0$.

Remark 2.9.10. The number $l(D) - 1$ is often denoted by $|D|$. According Remark 2.9.8 $|D| \geq 0$ for effective divisors. The number $|D|$ admits the following intuitive interpretation. We show that $|D| > k$ if and only if for any points P_1, \dots, P_k there is a divisor $D' \simeq D$ containing the points P_1, \dots, P_k (the presence of coinciding points among P_1, \dots, P_k is taken into account by their multiple occurrence in D'). We look for a function $f \in L(D)$ such that $f(P_1) = \dots = f(P_k) = 0$. This is a system of k homogeneous linear equations in the space $L(D)$. It distinguishes in $L(D)$ a subspace of codimension $\leq k$. If $l(D) > k + 1$, then there is a nonzero function in this space. Denote by D' its set of zeros. Then $D' \simeq D$ is the desired divisor. Conversely, take a collection of points $P_1, \dots, P_k \in \Gamma$. According to the assumption about the properties of the divisor, any of these points are included in a divisor linearly equivalent to D . In other words, for all $i = 1, \dots, k + 1$ there is a nonzero function $f_i \in L(D)$ such that $f_i(P_j) = 0$ for $j \neq i$. It can be assumed that $f_i(P_i) \neq 0$ (this can be attained by a small perturbation of the points P_1, \dots, P_{k+1}). It is obvious that the functions f_1, \dots, f_{k+1} are linearly independent, from which $l(D) > k + 1$. The assertion is completely proved. One therefore says that $|D|$ is the number of mobile points in the divisor D .

Remark 2.9.11. Let $K = K_\Gamma$, be the canonical class of a Riemann surface. We mention an interpretation that will be important later for the space $L(K - D)$ for an arbitrary divisor D . First, if $D = 0$, then the space $L(K)$ is isomorphic to the space of holomorphic differentials on Γ . Indeed, choose a representative $K_0 > 0$ in the canonical class, taking K_0 to be the zero divisor of some holomorphic differential ω_0 , $K_0 = (\omega_0)$. If $f \in L(K_0)$, i.e. $(f) + (\omega_0) \geq 0$, then the divisor $(f\omega_0)$ is positive, i.e., the differential $f\omega_0$ is holomorphic. Conversely, if ω is any holomorphic differential, then the meromorphic function $f = \omega/\omega_0$ lies in $L(K_0)$.

It follows from the foregoing and Theorem 2.6.4 that

$$l(K) = g.$$

We show further that for a positive divisor D the space $L(K - D)$ is isomorphic to the space $\Omega(D)$ of holomorphic differentials having zeros at points of D with multiplicities not less than the multiplicities of these points in D . Indeed, if $f \in L(K_0 - D)$, then the differential $f\omega_0$ is holomorphic and has zeros at the points of D , i.e., $f\omega_0 \in \Omega(D)$. Conversely, if $\omega \in \Omega(D)$, then $f = \omega/\omega_0 \in L(K_0 - D)$. The assertion is proved. The main way of getting information about the numbers $l(D)$ is the Riemann-Roch Theorem.

Theorem 2.9.12 (Riemann Roch Theorem.). *For any divisor D*

$$l(D) = 1 + \deg D - g + l(K - D). \quad (2.9.11)$$

Proof. For surfaces Γ of genus 0 (which are isomorphic to $\mathbb{C}\mathbb{P}^1$ in view of Problem 6.1) the Riemann-Roch theorem is a simple assertion about rational functions (verify!). By Remarks 2.9.9 and 2.9.11 (above) the Riemann-Roch theorem is valid for $D = 0$. We prove (2.9.11) for positive divisors $D > 0$. Let $D = \sum_{k=1}^m n_k P_k$ where all the $n_k > 0$. We first verify the arguments when all the n_k are = 1, i.e., $m = \deg D$. Let $f \in L(D)$ be a nonconstant function. Denote by z_1, \dots, z_m local parameters in neighborhoods of the points P_1, \dots, P_m . We consider the Abelian differential $\omega = df$. It has double poles and zero residues at the points P_1, \dots, P_m and does not have other singularities. Therefore, it is representable in the form

$$\omega = df = \sum_{k=1}^m c_k \Omega_{P_k}^{(1)} + \psi$$

where $\Omega_{P_k}^{(1)}$ are normalized differentials of the second kind (see Lecture 2.7), c_1, \dots, c_m are constants, and the differential ψ is holomorphic. Since the function $f = \int \omega$ is single-valued on Γ , we have that

$$\oint_{a_i} \omega = 0, \quad \oint_{b_i} \omega = 0, \quad i = 1, \dots, g. \quad (2.9.12)$$

From the vanishing of the a -periods we get that $\psi = 0$ (see Corollary 2.6.2). From the vanishing of the b -period we get by (2.7.11) (with $n = 1$) that

$$0 = \oint_{b_i} \omega = \sum_{k=1}^m c_k \psi_{i,k}(z_k)|_{z_k=0}, \quad i = 1, \dots, g, \quad (2.9.13)$$

where z_k is a local parameter in a neighborhood of P_k , $z_k(P_k) = 0$, $k = 1, \dots, m$, and the basis of holomorphic differentials are written in a neighborhood of P_k in the form $\omega_i = \psi_{ik}(z)dz_k$. We have obtained a homogeneous linear system of $m = \deg D$ equations in the coefficients c_1, \dots, c_m . The nonzero solutions of this systems are in a one-to-one correspondence with the nonconstant functions f in $L(D)$, where f can be reproduced from a solution c_1, \dots, c_m of the system (2.9.13) in the form

$$f = \sum_{k=1}^m c_k \Omega_{P_k}^{(1)}.$$

Thus $l(D) = 1 + \deg D$ is the rank of the matrix of the system (2.9.13) (the 1 is added because the constant function belong to the space $L(D)$). Denoting by $\psi_{ik}(0)dz_k$ by $\omega_i(P_k)$,

we rewrite the coefficient matrix of the system (2.9.13) in the form

$$\begin{pmatrix} \omega_1(P_1) & \cdots & \omega_1(P_m) \\ \cdots & \cdots & \cdots \\ \omega_g(P_1) & \cdots & \omega_g(P_m) \end{pmatrix}. \quad (2.9.14)$$

Denote the rank of the matrix by $g - i(D)$. The number $i(D)$ admits the following obvious interpretation: it is the dimension of the solution space of the transpose system

$$\sum_{j=1}^g a_j \omega_j(P_k) = 0, \quad k = 1, \dots, m. \quad (2.9.15)$$

The solutions a_1, \dots, a_g of the system (2.9.15) are in one-to-one correspondence with the holomorphic differentials

$$\eta = a_1 \omega_1 + \cdots + a_g \omega_g, \quad (2.9.16)$$

vanishing at the points of D . In other word $i(D) = \dim \Omega(D) = \dim L(K - D)$ (see Remark 2.9.11 above). Accordingly the Riemann-Roch theorem has been proved in this case.

We explain what happens when the positive divisor D has multiple points. For example suppose that $D = n_1 P_1 + \dots$. Then $\omega = df = \sum_{j=1}^{n_1} c_1^j \Omega_{P_1}^{(j)}$, and the system (2.9.13) can be written in the form

$$\sum_{j=1}^{n_1} c_1^j \frac{1}{j!} \frac{d^{j-1} \psi_{i1}}{dz_1^{j-1}} \Big|_{z_1=0} + \cdots = 0$$

If the rank of the coefficient matrix of this system is denoted (as above) by $g - i(D)$, then the differential η in (2.9.16) constructed as above from the solution of the transpose system vanishes at the point P_1 together with the derivatives up to order $n_1 - 1$, i.e. $\eta \in \Omega(D)$. Therefore as in the case $n_k = 1$ we have that $i(D) = \dim \Omega(D)$. We have proved the Riemann-Roch theorem for all positive divisors and hence for all effective divisors, which (accordingly to Remark 2.9.8) are distinguished by the condition $l(D) > 0$. Next we note that the relation in this theorem can be written in the form

$$l(D) - \frac{1}{2} \deg D = l(K - D) - \frac{1}{2} \deg(K - D), \quad (2.9.17)$$

which is symmetric with respect to the substitution $D \rightarrow K - D$. Therefore the theorem is proved for all divisors D such that D or $K - D$ is equivalent to an integral divisor. If neither D nor $K - D$ are equivalent to an integral divisor, then $l(D) = 0$ and the Riemann-Roch theorem reduces in this case to the equality

$$\deg D = g - 1. \quad (2.9.18)$$

Let us prove this equality. We represent D in the form $D = D_+ - D_-$, where D_+ and D_- are positive divisors and $\deg D_- > 0$. It follows from the validity of the Riemann-Roch theorem for D_+ that $l(D_+) \geq \deg D_+ - g + 1 = \deg D + \deg D_- - g + 1$. Therefore if $\deg D \geq g$, then $l(D_+) \geq 1 + \deg D_-$. Then the space $L(D_+)$ contains a nonzero function vanishing on D_- , i.e. belonging to the space $L(D_+ - D_-) = L(D)$. This contradicts the condition $l(D) = 0$. Similarly, the assumption $\deg(K - D) \geq g$ leads to a contradiction. This implies (2.9.18). The theorem is proved. \square

2.10 Some consequences of the Riemann-Roch theorem. The structure of surfaces of genus 1. Weierstrass points. The canonical embedding

Corollary 2.10.1. *If $\deg D \geq g$, then the divisor D is effective.*

Corollary 2.10.2. *The Riemann inequality*

$$l(D) \geq 1 + \deg D - g, \quad (2.10.1)$$

holds for $\deg D \geq g$.

Definition 2.10.3. *The divisors D for which the Riemann inequality becomes an equality are said to be nonspecial. The remaining divisors are said to be special. Any effective divisor of degree less than g is also said to be special.*

Corollary 2.10.4. *If $\deg D > 2g - 2$, then D is nonspecial.*

Proof. For $\deg D > 2g - 2$ we have that $\deg(K - D) < 0$, hence $l(K - D) = 0$ (see Remark 2.9.9). The corollary is proved. \square

Exercise 2.10.5: Suppose that $k \geq g$; let the Abel mapping $A : \Gamma \rightarrow J(\Gamma)$ (see Lecture 2.8) be extended to the k th-power mapping

$$A^k : \underbrace{\Gamma \times \cdots \times \Gamma}_{k \text{ times}} \rightarrow J(\Gamma)$$

by setting $A^k(P_1, \dots, P_k) = A(P_1) + \cdots + A(P_k)$ (it can actually be assumed that A^k maps into $J(\Gamma)$ the k th symmetric power $S^k\Gamma$, whose points are the unordered collections (P_1, \dots, P_k) of points of Γ). Prove that the special divisors of degree k are precisely the critical points of the Abel mapping A^k . Deduce from this that a divisor D with $\deg D \geq g$ in general position is nonspecial.

Exercise 2.10.6: Let Γ be a hyperelliptic surface $w^2 = P_{2g+1}(z)$, and let the divisor D have the form $D = \sum_{j=1}^k P_j$, where $P_j = (z_j, w_j)$, $j = 1, \dots, k$, $k \geq g$. Prove that the divisor D is special if and only if $k \leq 2g - 2$ and the points P_1, \dots, P_k are not all distinct. Formulate and prove an analogous assertion for the case when D contains multiple points.

We now present examples of the use of the Riemann-Roch theorem in the study of Riemann surfaces.

Example 2.10.7. Let us show that any Riemann surface Γ of genus $g = 1$ is isomorphic to an elliptic surface $w^2 = P_3(z)$. Let P_0 be an arbitrary point of Γ . Here $2g - 2 = 0$, therefore, any positive divisor is nonspecial. We have that $A(2P_0) = 2$, hence there is a nonconstant function z in $l(2P_0)$, i.e., a function having a double pole at P_0 . Further $l(3P_0) = 3$, hence there is a function $w \in l(3P_0)$ that cannot be represented in the form $w = az + b$. This function has a pole of order three at P_0 . Finally, since $l(6P_0) = 6$, the functions $1, z, z^2, z^3, w, w^2, wz$ which lie in $l(6P_0)$ are linearly independent. We have that

$$a_1w^2 + a_2wz + a_3w + a_4z^3 + a_5z^2 + a_6z + a_7 = 0. \quad (2.10.2)$$

The coefficient a_1 is nonzero (verify). Making the substitution

$$w \rightarrow w - \left(\frac{a_2}{2a_1}z + \frac{a_3}{2a_1} \right)$$

we get the equation of an elliptic curve from (2.10.2).

Definition 2.10.8 (Weierstrass points). *A point P_0 of a Riemann surface Γ of genus g is called a Weierstrass point if $l(kP_0) > 1$ for some $k \leq g$.*

It is clear that in the definition of a Weierstrass point it suffices to require that $l(gP_0) > 1$. There are no Weierstrass points on a surface of genus $g = 1$. On hyperelliptic Riemann surfaces of genus $g > 1$ all branch points are Weierstrass points, since there exist functions with second-order poles at the branch points (see Lecture 2.4). The use of Weierstrass points can be illustrated using the example of the following assertion.

Exercise 2.10.9: Let Γ be a Riemann surface of genus $g > 1$, and P_0 a Weierstrass point of it, with $l(2P_0) > 1$. Prove that Γ is hyperelliptic. Prove that the surface is also hyperelliptic if $l(P + Q) > 1$ for two points P and Q .

We show that there exist Weierstrass points on any Riemann surface Γ of genus $g > 1$.

Lemma 2.10.10. *Suppose that z is a local parameter in a neighborhood P_0 , $z(P_0) = 0$; assume that locally the basis of holomorphic differentials has the form $\omega_i = \psi_i(z)dz$, $i = 1, \dots, g$. Consider the determinant*

$$W(z) = \det \begin{pmatrix} \psi_1(z) & \psi_1'(z) & \dots & \psi_1^{(g-1)}(z) \\ \dots & \dots & \dots & \dots \\ \psi_g(z) & \psi_g'(z) & \dots & \psi_g^{(g-1)}(z) \end{pmatrix}. \quad (2.10.3)$$

The point P_0 is a Weierstrass point if and only if $W(0) = 0$.

Proof. If P_0 is a Weierstrass point, i.e., $l(gP_0) > 1$, then $l(K - gP_0) > 0$ by the Riemann-Roch theorem. Hence, there is a holomorphic differential with a g -fold zero at P_0 on Γ . The condition that there be such a differential can be written in the form $W(0) = 0$ (cf. the proof of the Riemann-Roch theorem). The lemma is proved. \square

Lemma 2.10.11. *Under a local change of parameter $z = z(w)$ the quantity W transforms according to the rule $\tilde{W}(w) = \left(\frac{dz}{dw} \right)^N W(z)$, $N = \frac{1}{2}g(g+1)$.*

Proof. Suppose that $\omega_i = \psi_i(z)dz = \tilde{\psi}_i(w)dw$. Then each $\tilde{\psi}_i = \psi_i \frac{dz}{dw}$, $i = 1, \dots, g$. This implies that the derivatives $d^k \tilde{\psi}_i / dw^k$ can be expressed for each i in terms of the derivatives $d^l \psi_i / dz^l$ by means of a triangular transformation of the form

$$\frac{d^k \tilde{\psi}_i}{dw^k} = \left(\frac{dz}{dw} \right)^{k+1} \frac{d^k \psi_i}{dz^k} + \sum_{j=1}^{k-1} c_j \frac{d^j \psi_i}{dz^j}, \quad i = 1, \dots, g$$

(the coefficients c_s in this formula are certain differential polynomials in $z(w)$). The statement of the Lemma readily follows from the transformation rule. \square

Let us define the *weight* of a Weierstrass point P_0 as the multiplicity of zero of $W(z)$ at this point. According to the previous Lemma the definition of weight does not depend on the choice of the local parameter.

The proof of existence of Weierstrass points for $g > 1$ can be easily obtained from the following statement.

Lemma 2.10.12. *The total weight of all Weierstrass points on the Riemann surface Γ of genus g is equal to $(g - 1)g(g + 1)$.*

Proof. Let us consider the ratio

$$W(z)/\psi_1^N(z).$$

Here $N = \frac{1}{2}g(g + 1)$, like in the previous Lemma. According to the latter the ratio does not depend on the choice of the local parameter and, hence, it is a meromorphic function on Γ . This function has poles of multiplicity N at zeroes of the differential ω_1 (the total number of all poles is equal to $2g - 2$). Therefore this function must have $N(2g - 2) = (g - 1)g(g + 1)$ zeroes (as usual, counted with their multiplicities). These zeroes are the Weierstrass points. \square

Let us do few more remarks about the Weierstrass points. Given a point $P_0 \in \Gamma$, let us consider the dimension $l(kP_0)$ as a function of the integer argument k . This function has the following properties. First, $l(kP_0) = 1 + k - g$ for $k \geq 2g - 1$; in particular $l((2g - 1)P_0) = g$. Next, it grows monotonically with k , moreover,

$$l(kP_0) = \begin{cases} l((k - 1)P_0) + 1, & \text{if there exists a function with a pole of order } k \text{ at } P_0 \\ l((k - 1)P_0), & \text{if such a function does not exist} \end{cases}$$

In the second case we will say that the number k is a *gap* at the point P_0 . From the previous remarks it follows the following *Weierstrass gap theorem*:

Theorem 2.10.13. *There are exactly g gaps $a_1 < \dots < a_g \leq 2g - 1$ at any point P_0 of a Riemann surface of genus g .*

The gaps have the form $a_i = i$, $i = 1, \dots, g$, for a point P_0 in general position (which is not a Weierstrass point).

Exercise 2.10.14: Prove that the weight of a Weierstrass point is equal

$$\sum_{i=1}^g (a_i - i). \tag{2.10.4}$$

Exercise 2.10.15: Prove that for branch points of a hyperelliptic Riemann surface of genus g the gaps have the form $a_i = 2i - 1$, $i = 1, \dots, g$. Prove that a hyperelliptic surface does not have other Weierstrass points.

Exercise 2.10.16: Prove that any Riemann surface of genus 2 is hyperelliptic.

Exercise 2.10.17: Let Γ be a hyperelliptic Riemann surface of the form $w^2 = P_{2g+1}(z)$. Prove that any birational (biholomorphic) automorphism $\Gamma \rightarrow \Gamma$ has the form $(z, w) \rightarrow (\frac{az + b}{cz + d}, \pm w)$, where the linear fractional transformation leaves the collection of zeros of $P_{2g+2}(z)$ invariant.

Example 2.10.18 (The canonical embedding). . Let Γ be an arbitrary Riemann surface of genus $g \geq 2$. We fix on Γ a canonical basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$; let $\omega_1, \dots, \omega_g$ be the corresponding normal basis of holomorphic differentials. This basis gives a canonical mapping $\Gamma \rightarrow \mathbb{C}\mathbb{P}^{g-1}$ according to the rule

$$P \rightarrow (\omega_1(P) : \omega_2(P) : \dots : \omega_g(P)). \quad (2.10.5)$$

Indeed, it suffices to see that all the differentials $\omega_1, \dots, \omega_g$ cannot simultaneously vanish at some point of the surface. If P were a point at which any holomorphic differential vanished, i.e., $l(K-P) = g$, (see Remark 2.9.11), then $l(P)$ would be $= 2$ in view of the Riemann-Roch theorem, and this means that the surface Γ is rational (verify!). Accordingly (2.10.5) really is a mapping $\Gamma \rightarrow \mathbb{C}\mathbb{P}^{g-1}$; it is obviously well-defined.

Lemma 2.10.19. *If Γ is a nonhyperelliptic surface of genus $g \geq 3$, then the canonical mapping (2.10.5) is a smooth embedding. If Γ is a hyperelliptic surface of genus $g \geq 2$, then the image of the canonical embedding is a rational curve, and the mapping itself is a two-sheeted covering.*

Proof. We prove that the mapping (2.10.5) is an embedding. Assume not: assume that the points P_1 and P_2 are merged into a single point by this mapping. This means that the rank of the matrix

$$\begin{pmatrix} \omega_1(P_1) & \omega_1(P_2) \\ \dots & \dots \\ \omega_g(P_1) & \omega_g(P_2) \end{pmatrix}$$

is equal to 1. But then $l(P_1 + P_2) > 1$ (see the proof of the Riemann-Roch theorem). Hence, there exists on Γ a nonconstant function with two simple poles at P_1 and P_2 i.e., the surface Γ is hyperelliptic. The smoothness is proved similarly: if it fails to hold at a point P , then the rank of the matrix

$$\begin{pmatrix} \omega_1(P) & \omega'_1(P) \\ \dots & \dots \\ \omega_g(P) & \omega'_g(P) \end{pmatrix}$$

is equal to 1. Then $l(2P) > 1$, and the surface is hyperelliptic. Finally, suppose that Γ is hyperelliptic. Then it can be assumed form $w^2 = P_{2g+1}(z)$. Its canonical mapping is determined by the differentials (2.6.31). Performing a projective transformation of the space $\mathbb{C}\mathbb{P}^{g-1}$ with the matrix (c_{jk}) (see the formula (2.6.31)), we get the following form for the canonical mapping:

$$P = (z, w) \rightarrow (1 : z : \dots : z^{g-1}) \quad (2.10.6)$$

Its properties are just as indicated in the statement of the lemma. The lemma is proved. \square

Exercise 2.10.20: Suppose that the Riemann surface Γ is given in $\mathbb{C}\mathbb{P}^2$ by the equation

$$\sum_{i+j=4} a_{ij} \xi^i \eta^j \zeta^{4-i-j} = 0, \quad (2.10.7)$$

and this curve is nonsingular in $\mathbb{C}\mathbb{P}^2$ (construct an example of such a nonsingular curve). Prove that the genus of this surface is equal to 3 and the canonical mapping is the identity up to a projective transformation of $\mathbb{C}\mathbb{P}^2$. Prove that Γ is a non hyperelliptic surface. Prove that any non hyperelliptic surface of genus 3 can be obtained in this way.

The range $\Gamma' \subset \mathbb{C}\mathbb{P}^{g-1}$ of the canonical mapping is called the canonical curve.

Exercise 2.10.21: Prove that any hyperplane in $\mathbb{C}\mathbb{P}^{g-1}$ intersects the canonical curve Γ' in $2g - 2$ points (counting multiplicity).

Exercise 2.10.22: Suppose that $D = \sum_j P_j$ is an effective divisor. Consider the images of the points P_j , on the canonical curve Γ' . Prove that these points generate in $\mathbb{C}\mathbb{P}^{g-1}$ a hyperplane of dimension $\deg D - l(D)$.

Exercise 2.10.23 (Clifford's theorem): . Show that for any two effective divisors D and D' on a Riemann surface Γ of genus g ,

$$|D| + |D'| \leq |D + D'| \quad (2.10.8)$$

(see Remark 2.9.10), and for a special divisor D

$$|D| \leq \frac{1}{2} \deg D \quad (2.10.9)$$

with equality only in one of the following cases: $D = 0$, $D = K$, or the surface Γ is hyperelliptic.

2.11 Statement of the Jacobi inversion problem. Definition and simplest properties of general theta functions

In Lecture 2.6 we saw that inversion of an elliptic integral leads to elliptic functions. Inversion of integrals of Abelian differentials is not possible on surfaces of genus $g > 1$, since any such differential has zeros (at least $2g - 2$ zeros). Instead of the problem of inverting a single Abelian integral, Jacobi proposed for hyperelliptic surfaces $w^2 = P_5(z)$ the problem of solving the system

$$\begin{aligned} \int_{P_0}^{P_1} \frac{dz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{dz}{\sqrt{P_5(z)}} &= \eta_1 \\ \int_{P_0}^{P_1} \frac{zdz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{zdz}{\sqrt{P_5(z)}} &= \eta_2 \end{aligned} \quad (2.11.1)$$

where η_1, η_2 are given numbers from which the location of the points $P_1 = (z_1, w_1)$, $P_2 = (z_2, w_2)$ is to be determined. It is clear, moreover, that P_1 and P_2 are determined from (2.11.1) only up to permutation. Jacobi's idea was to express the symmetric functions of P_1 and P_2 as functions of η_1 and η_2 . He noted also that this will give meromorphic functions of η_1 and η_2 whose period lattice is generated by the periods of the basis of holomorphic differentials $dz/\sqrt{P_5(z)}$ and $zdz/\sqrt{P_5(z)}$. This Jacobi inversion problem was solved by Göepel and Rosenhain by means of the apparatus of theta functions of two variables. The generalization of the Jacobi inversion problem to arbitrary Riemann surfaces and its solution are due to Riemann, in whose work the theory of theta functions took, on the whole, its modern form. We give a precise statement of the Jacobi inversion problem. Let Γ be an

arbitrary Riemann surface of genus g , and fix a canonical basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$ on Γ ; as above let $\omega_1, \dots, \omega_g$ be the corresponding basis of normalized holomorphic differentials. Recall (see Lecture 2.8) that the Abel mapping has the form

$$A : \Gamma \rightarrow J(\Gamma), \quad A(P) = (u_1(P), \dots, u_g(P)), \quad (2.11.2)$$

where $J(\Gamma)$ is the Jacobi variety,

$$u_i(P) = \int_{P_0}^P \omega_i, \quad (2.11.3)$$

P_0 is a particular point of Γ , and the path of integration from P_0 to P is the same for all $i = 1, \dots, g$. Consider the g th symmetric power $S^g\Gamma$ of Γ . The unordered collections (P_1, \dots, P_g) of g points of Γ are the points of the manifold $S^g\Gamma$. The meromorphic functions on $S^g\Gamma$ are the meromorphic symmetric functions of g variables $P_1, \dots, P_g, P_j \in \Gamma$. The Abel mapping (2.11.2) determines a mapping

$$A^{(g)} : S^g\Gamma \rightarrow J(\Gamma), \quad A^{(g)}(P_1, \dots, P_g) = A(P_1) + \dots + A(P_g), \quad (2.11.4)$$

which we also call the Abel mapping.

Lemma 2.11.1. *If the divisor $D = P_1 + \dots + P_g$ is nonspecial, then in a neighborhood of a point $A^{(g)}(P_1, \dots, P_g) \in J(\Gamma)$ the mapping $A^{(g)}$ has a single-valued inverse.*

Proof. Suppose that all the points are distinct; let z_1, \dots, z_g be local parameters in neighborhoods of the respective points P_1, \dots, P_g with $z_k(P_k) = 0$ and $\omega_i = \psi_{ik}(z_k)dz_k$ the normalized holomorphic differentials in a neighborhood of P_k . The Jacobi matrix of the mapping (2.11.4) has the following form at the points (P_1, \dots, P_g)

$$\begin{pmatrix} \psi_{11}(z_1 = 0) & \dots & \psi_{1g}(z_g = 0) \\ \dots & \dots & \dots \\ \psi_{g1}(z_1 = 0) & \dots & \psi_{gg}(z_g = 0) \end{pmatrix}.$$

If the rank of this matrix is less than g , then $l(K - D) > 0$, i.e., the divisor D is special by the Riemann-Roch theorem. The case when not all the points P_1, \dots, P_g are distinct is treated similarly. We now prove that the inverse mapping is single-valued. Assume that the collection of points (P'_1, \dots, P'_g) is also carried into $A^{(g)}(P_1, \dots, P_g)$. Then the divisor $D' = P'_1 + \dots + P'_g$ is linearly equivalent to D by Abel's theorem. If $D' \neq D$, then there would be a meromorphic function with poles at points of D and with zeros at points of D' . This would contradict the fact that D is nonspecial. Hence, $D' = D$, and the points P'_1, \dots, P'_g differ from P_1, \dots, P_g only in order. The lemma is proved. \square

Since a divisor $P_1 + \dots + P_g$ in general position is nonspecial (see Problem 2.10.5), the Abel mapping (2.11.4) is invertible almost everywhere. The problem of inversion of this mapping in the large is the Jacobi inversion problem. Thus, the Jacobi inversion problem can be written in coordinate notation in the form

$$\begin{cases} u_1(P_1) + \dots + u_1(P_g) = \eta_1 \\ \dots \\ u_g(P_1) + \dots + u_g(P_g) = \eta_g \end{cases} \quad (2.11.5)$$

which generalizes (2.11.1). As already noted, to solve this problem we need the apparatus of multi-dimensional theta functions. We first define ordinary (one-dimensional) theta functions. Let b be an arbitrary number with $\Re b < 0$. A theta function is defined by the series

$$\theta(z) = \sum_{-\infty < n < \infty} \exp\left(\frac{bn^2}{2} + nz\right). \quad (2.11.6)$$

Since

$$\left| \exp\left(\frac{bn^2}{2} + nz\right) \right| = \exp\left(\frac{\Re(b)n^2}{2} + n\Re(z)\right)$$

the series (2.11.6) converges absolutely and uniformly in the strips $|\Re(z)| \leq \text{const}$ and defines an entire function of z . This is a classical Jacobi theta function. To compare this with the standard notation in the theory of elliptic functions it is useful to make a substitution, setting $b = 2\pi i\tau$, $z = 2\pi ix$. The series (2.11.6) can be rewritten in the form common in the theory of Fourier series:

$$\theta(2\pi ix) = \sum_{-\infty < n < \infty} \exp(\pi i\tau n^2) e^{2\pi i x n} \quad (2.11.7)$$

(the function $\vartheta_3(x | \tau)$ in the standard notation; see [[3]). The function $\theta(z)$ has the following periodicity properties:

$$\theta(z + 2\pi i) = \theta(z) \quad (2.11.8)$$

$$\theta(z + b) = \exp\left(-\frac{b}{2} - z\right)\theta(z) \quad (2.11.9)$$

The equality (10.8) is obvious. The equality (10.9) is also easy to prove:

$$\theta(z + b) = \sum_n \exp\left(\frac{bn^2}{2} + bn + zn\right) = \sum_n \exp\left(\frac{b(n+1)^2}{2} - \frac{b}{2} - z + z(n+1)\right) = \exp\left(-\frac{b}{2} - z\right)\theta(z).$$

The integer lattice with basis $2\pi i$, b is called the period lattice of the theta function.

Exercise 2.11.2: Prove that the zeros of the function $\theta(z)$ form an integer lattice with the same basis $2\pi i$, b and with origin at the point $z_0 = \pi i + b/2$.

Exercise 2.11.3: Prove that the Weierstrass σ -function (see Lecture 2.6) constructed from the lattice $\{2\pi im + bn\}$ is connected with the function $\theta(z)$ by the equality

$$\sigma(z - z_0) = \text{const} \exp\left(\frac{\eta(z - z_0)^2}{2\pi i} + \frac{z}{2}\right)\theta(z), \quad z_0 = \pi i \frac{b}{2}. \quad (2.11.10)$$

Deduce from this that

$$\zeta(z - z_0) = \frac{\partial}{\partial z} \log \theta(z) + \frac{\eta z}{\pi i} - \eta - \eta', \quad (2.11.11)$$

$$\wp(z - z_0) = -\frac{\partial^2}{\partial z^2} \log \theta(z) - \frac{\eta}{\pi i} \quad (2.11.12)$$

We proceed to multi-dimensional theta functions. Let $B = (B_{jk})$ be a symmetric $g \times g$ matrix with negative-definite real part. We shall call such matrices Riemann matrices. A Riemann theta function is defined by its multiple Fourier series,

$$\theta(z) = \theta(z|B) = \sum_{N \in \mathbb{Z}^g} \exp\left(\frac{1}{2}\langle BN, N \rangle + \langle N, z \rangle\right). \quad (2.11.13)$$

Here $z = (z_1, \dots, z_g)$ is a complex vector, and B is a Riemann matrix. The angle brackets denote the Euclidean inner product:

$$\langle N, z \rangle = \sum_{k=1}^g g N_k z_k, \quad \langle BN, N \rangle = \sum_{j,k=1}^g B_{kj} N_j N_k$$

The summation in (2.11.13) is over the lattice of integer vectors $N = (N_1, \dots, N_g)$. The obvious estimate $\Re(\langle BN, N \rangle) \leq -b\langle N, N \rangle$, where $-b < 0$ is the largest eigenvalue of the matrix $\Re(B)$, implies that the series (2.11.13) defines an entire function of the variables z_1, \dots, z_g .

Lemma 2.11.4. *For any integer vectors $M, K \in \mathbb{Z}^g$,*

$$\theta(z + 2\pi i K + BM) = \exp\left(-\frac{1}{2}\langle BM, M \rangle - \langle M, z \rangle\right) \theta(z). \quad (2.11.14)$$

Proof. In the series for $\theta(z + 2\pi i K + BM)$ we make the change of summation index $N \rightarrow N - M$. The relation (2.11.14) is obtained after transformations. The lemma is proved. \square

The integer lattice $\{2\pi i N + BM\}$ is called the period lattice. Let α and β be arbitrary real g -dimensional vectors. We define the theta function with characteristics α and β :

$$\begin{aligned} \theta[\alpha, \beta](z) &= \exp\left(\frac{1}{2}\langle B\alpha, \alpha \rangle + \langle z + 2\pi i\beta, \alpha \rangle\right) \theta(z + 2\pi i\beta + B\alpha) \\ &= \sum_{N \in \mathbb{Z}^g} \exp\left(\frac{1}{2}\langle B(N + \alpha), N + \alpha \rangle + \langle z + 2\pi i\beta, N + \alpha \rangle\right). \end{aligned} \quad (2.11.15)$$

For $\alpha = 0$ and $\beta = 0$ we get the function $\theta(z)$. The analogue of the law (2.11.14) for the functions $\theta[\alpha, \beta](z)$ has the form

$$\theta[\alpha, \beta](z + 2\pi i N + BM) = \exp\left[-\frac{1}{2}\langle BM, M \rangle - \langle M, z \rangle + 2\pi i(\langle \alpha, N \rangle - \langle \beta, M \rangle)\right] \theta[\alpha, \beta](z). \quad (2.11.16)$$

All the coordinates of the characteristics α and β are determined modulo 1 (verify!). The characteristics α and β with all coordinates equal to 0 or $1/2$ are called half periods. A half period $[\alpha, \beta]$ is said to be even if $4\langle \alpha, \beta \rangle \equiv 0 \pmod{2}$ and odd if $4\langle \alpha, \beta \rangle \equiv 1 \pmod{2}$.

Exercise 2.11.5: Prove that the function $\theta[\alpha, \beta](z)$ is even if $[\alpha, \beta]$ is an even half period and odd if $[\alpha, \beta]$ is an odd half period.

In particular the function $\theta(z)$ is even.

Remark 2.11.6. It is possible to define the function $\theta(z)$ as an entire function of z_1, \dots, z_g satisfying the transformation law (2.11.14) (this condition determines $\theta(z)$ uniquely to within a factor).

By multiplying theta function (2.11.15) we obtain higher order theta functions. The function $f(z)$ is said to be a n th order theta function with characteristics α and β if it is an entire function of z_1, \dots, z_g and transforms according to the following law under translation of the argument by a vector of the period lattice

$$f(z+2\pi iN+BM) = \exp \left[-\frac{n}{2} \langle BM, M \rangle - n \langle M, z \rangle + 2\pi i (\langle \alpha, N \rangle - \langle \beta, M \rangle) \right] f(z). \quad (2.11.17)$$

Exercise 2.11.7: Prove that the n th order theta functions with given characteristics α, β form a linear space of dimension n^g . Prove that a basis in this space is formed by the functions

$$\theta \left[\frac{\alpha + \gamma}{n}, \beta \right] (nz | nB), \quad (2.11.18)$$

where the coordinates of the vector γ run independently through all values from 0 to $n-1$.

Remark 2.11.8. For reference we determine the transformation law of a Riemann theta function under transformation of the Riemann matrix of the form

$$B' = 2\pi i(2\pi im + nB)(2\pi ik + lB)^{-1} \quad (2.11.19)$$

where $\begin{pmatrix} k & l \\ m & n \end{pmatrix}$ is an integer symplectic matrix (see Remark 2.8.3); it is according to this law that the period matrix of a Riemann surface transforms under changes of a canonical basis of cycles). Denote by R the matrix

$$R = 2\pi ik + lB \quad (2.11.20)$$

The transformed values of the argument and of the characteristics are determined by

$$\begin{aligned} 2\pi iz &= z'R \\ \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} &= \begin{pmatrix} n & -m \\ -l & k \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \frac{1}{2} \text{diag} \begin{pmatrix} mn^t \\ kl^t \end{pmatrix}. \end{aligned} \quad (2.11.21)$$

Here the symbol diag means the vectors of diagonal elements of the matrices mn^t and kl^t . We have the equality

$$\theta[\alpha', \beta'](z' | B') = \chi \sqrt{\det R} \exp \left\{ \frac{1}{2} \sum_{i \leq j} z_i z_j \frac{\partial \log \det R}{\partial B_{ij}} \right\} \theta[\alpha, \beta](z | B), \quad (2.11.22)$$

where χ is a constant independent of z and B . See [35] for a proof.

Exercise 2.11.9: Prove the formula (2.11.22) for $g = 1$. *Hint.* Use the Poisson summation formula (see [23]): if

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

is the Fourier transform of a sufficiently nice function $f(x)$, then

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

Theta function are connected by a complicated system of algebraic relations, the so called addition theorem. They are all relations between formal Fourier series (see [35]). We present one of these relations which will be used below. Let $\hat{\theta}[n](z) = \theta[\frac{n}{2}, 0](2z | 2B)$. (according to (2.11.18) this is a basis of second order theta function with zero characteristics).

Lemma 2.11.10. *The following identity holds:*

$$\theta(z+w)\theta(z-w) = \sum_{n \in (\mathbb{Z}_2)^g} \hat{\theta}[n](z)\hat{\theta}[n](w). \quad (2.11.23)$$

The expression $n \in (\mathbb{Z}_2)^g$ means that the summation is over the g -dimensional vectors n whose coordinates all take values in 0 or 1.

Proof. Let us first analyze the case $g = 1$. The formula (2.11.23) can be written as

$$\theta(z+w)\theta(z-w) = \hat{\theta}(z)\hat{\theta}(w) + \hat{\theta}[1](z)\hat{\theta}[1](w) \quad (2.11.24)$$

where

$$\begin{aligned} \theta(z) &= \sum_k \exp\left(\frac{1}{2}bk^2 + kz\right), & \hat{\theta}(z) &= \sum_k \exp(bk^2 + 2kz), \\ \hat{\theta}[1](z) &= \sum_k \exp\left[b\left(\frac{1}{2} + k\right)^2 + (2k+1)z\right], & \Re(b) &< 0. \end{aligned}$$

The left-hand side of (2.11.24) has then the form

$$\sum_{k,l} \exp\left[\frac{1}{2}b(k^2 + l^2) + k(z+w) + l(z-w)\right]. \quad (2.11.25)$$

We introduce new summation indices m and n by setting $m = (k+l)/2$ and $n = (k-l)/2$. The numbers m and n simultaneously are integers or half-integers. In these variables the sum (2.11.25) takes the form

$$\sum \exp[bm^2 + 2mz + bn^2 + 2nw]. \quad (2.11.26)$$

We break up this sum into two parts. The first part will contain the terms with integers m and n , while in the second part m and n are both half-integers. In the second part we change the notation from m to $m + \frac{1}{2}$ and from n to $n + \frac{1}{2}$. Then m and n are integers, and the expression (2.11.22) can be written in the form

$$\begin{aligned} &\sum_{m,n \in \mathbb{Z}} \exp[bm^2 + 2mz] \exp[bn^2 + 2nw] + \\ &\sum_{m,n \in \mathbb{Z}} \exp[b\left(m + \frac{1}{2}\right)^2 + 2\left(m + \frac{1}{2}\right)z] \exp[b\left(n + \frac{1}{2}\right)^2 + 2\left(n + \frac{1}{2}\right)w] = \\ &\hat{\theta}(z)\hat{\theta}(w) + \hat{\theta}[1](z)\hat{\theta}[1](w). \end{aligned}$$

The lemma is proved for $g = 1$. In the general case $g > 1$ it is necessary to repeat the arguments given for each coordinate separately. The lemma is proved. \square

Exercise 2.11.11: Prove the following four term product identity of Riemann. Suppose that two quadruples of g -dimensional vectors z_1, \dots, z_4 and w_1, \dots, w_4 are connected by the relation

$$(z_1, \dots, z_4) = (w_1, \dots, w_4)T, \quad (2.11.27)$$

where

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \quad (2.11.28)$$

Then the following identity holds

$$\theta(z_1)\theta(z_2)\theta(z_3)\theta(z_4) = \frac{1}{2^g} \sum_{2[\alpha, \beta] \in (\mathbb{Z}_2)^{2g}} \theta[\alpha, \beta](w_1)\theta[\alpha, \beta](w_2)\theta[\alpha, \beta](w_3)\theta[\alpha, \beta](w_4). \quad (2.11.29)$$

Exercise 2.11.12: Suppose that the Riemann matrix B has a block-diagonal form $B = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}$, where B' and B'' are $k \times k$ and $l \times l$ Riemann matrices, respectively with $k + l = g$. Prove that the corresponding theta function factors into the product of two theta function

$$\theta(z | B) = \theta(z' | B')\theta(z'' | B''), \quad (2.11.30)$$

$$z = (z_1, \dots, z_g), \quad z' = (z_1, \dots, z_k), \quad z'' = (z_{k+1}, \dots, z_g).$$

2.12 The Riemann theorem on zeros of theta functions and its applications

To solve the Jacobi inversion problem we use the Riemann θ -function $\theta(z) = \theta(z | B)$ on the Riemann surface Γ . Here $B = (B_{jk})$ is the period matrix of this surface with respect to a chosen basis of cycles. Let $e = (e_1, \dots, e_g) \in \mathbb{C}^g$ be a particular vector. We consider the function

$$F(P) = \theta(A(P) - e). \quad (2.12.1)$$

The function $F(P)$ is single-valued and analytic on the cut surface $\tilde{\Gamma}$. Assume that it is not identically zero. This will be the case if, for example $\theta(e) \neq 0$. Note that in view (2.11.14) the zeros of the theta function form a well defined compact analytic sub-variety of the torus $J(\Gamma)$. In other words for almost every vector e , the function (2.12.1) is not identically zero.

Lemma 2.12.1. *If $F(P) \neq 0$, then the function $F(P)$ has g zeros on $\tilde{\Gamma}$ (counting multiplicity).*

Proof. To compute the number of zeros it is necessary to compute the logarithmic residue

$$\frac{1}{2\pi i} \oint_{J\tilde{\Gamma}} d \log F(P) \quad (2.12.2)$$

(assume that the zeros of $F(P)$ do not lie on the boundary of $\partial\tilde{\Gamma}$). We sketch a fragment of $\partial\tilde{\Gamma}$ (cf. the proof of lemma 2.5.7). The following notation is introduced for brevity and used below: F^+ denotes the value taken by F at a point on $\partial\tilde{\Gamma}$ lying on the segment a_k or b_k and F^- the value of F at the corresponding point a_k^{-1} or b_k^{-1} (see the figure). The notation u^+ and u^- has an analogous meaning. In this notation the integral (2.12.2) can be written in the form

$$\frac{1}{2\pi i} \oint_{\partial\tilde{\Gamma}} d\log F(P) = \frac{1}{2\pi i} \sum_{k=1}^g \left(\int_{a_k} + \int_{b_k} \right) [d\log F^+ - d\log F^-]. \quad (2.12.3)$$

Note that if P is a point on a_k then

$$u_j^-(P) = u_j^+(P) + B_{jk}, \quad j = 1, \dots, g, \quad (2.12.4)$$

(cf. (2.5.7)), while if P lies on b_k , then

$$u_j^+(P) = u_j^-(P) + 2\pi i \delta_{jk}, \quad j = 1, \dots, g, \quad (2.12.5)$$

(cfr. (2.5.8)). We get from the law of transformation (2.11.14) of a theta function that on the cycle a_k

$$\log F^-(P) = -\frac{1}{2} B_{kk} - u_k^+(P) + e_k + \log F^+(P); \quad (2.12.6)$$

on the cycle b_k

$$\log F^+ = \log F^-. \quad (2.12.7)$$

From this on a_k

$$d\log F^-(P) = d\log F^+(P) - \omega_k(P), \quad (2.12.8)$$

and on b_k

$$d\log F^-(P) = d\log F^+(P). \quad (2.12.9)$$

Accordingly the sum (2.12.3) can be written in the form

$$\frac{1}{2\pi i} \oint_{\partial\tilde{\Gamma}} d\log F = \frac{1}{2\pi i} \sum_k \oint_{a_k} \omega_k = g,$$

where we have used the normalization condition $\oint_{a_k} \omega_k = 2\pi i$. The lemma is proved \square

Note that although the function $F(P)$ is not a single-valued function on Γ , its zeros P_1, \dots, P_g do not depend on the location of the cuts along the canonical basis of cycles. Indeed, if these basis cycles are deformed then the path of integration from P_0 to P can change in the formulas for the Abel map transformation. A vector of the form $(\oint_{\gamma} \omega_1, \dots, \oint_{\gamma} \omega_g)$ is added to the argument of the theta-function $\theta(z)$ in (2.12.1). This is a vector of the period lattice $\{2\pi i M + BN\}$. As a result of all this the function $F(P)$ can only be multiplied by a non zero factor in view of (2.11.14). We show now that the g zeros of $F(P)$ give a solution of the Jacobi inversion problem for a suitable choice of the vector e .

Lemma 2.12.2. *Suppose that $F(P) \neq 0$ and P_1, \dots, P_g are its zeros on Γ . Then on the Jacobi variety $J(\Gamma)$*

$$A^g(P_1, \dots, P_g) = e - \mathcal{K}, \quad (2.12.10)$$

where $\mathcal{K} = (\mathcal{K}_1, \dots, \mathcal{K}_g)$ is the vector of Riemann constants,

$$\mathcal{K}_j = \frac{2\pi i + B_{jj}}{2} - \frac{1}{2\pi i} \sum_{l \neq j} \left(\oint_{a_l} \omega_l(P) \int_{P_0}^P \omega_j \right), \quad j = 1, \dots, g. \quad (2.12.11)$$

Proof. Consider the integral

$$\zeta_j = \frac{1}{2\pi i} \oint_{\partial\bar{\Gamma}} u_j(P) d \log F(P). \quad (2.12.12)$$

On the other hand, it is equal to the sum of the residues of the integrands i.e.,

$$\zeta_j = u_j(P_1) + \dots + u_j(P_g), \quad (2.12.13)$$

where P_1, \dots, P_g are the zeros of $F(P)$ of interest to us. On the other hand, this integral can be represented by analogy with the proof of Lemma 2.12.1 in the form

$$\begin{aligned} \zeta_j &= \frac{1}{2\pi i} \sum_{k=1}^g \left(\int_{a_k} + \int_{b_k} \right) (u_j^+ d \log F^+ - u_j^- d \log F^-) \\ &= \frac{1}{2\pi i} \sum_{k=1}^g \int_{a_k} [u_j^+ d \log F^+ - (u_j^+ + B_{jk})(d \log F^+ - \omega_k)] \\ &\quad + \frac{1}{2\pi i} \sum_{k=1}^g \int_{b_k} [u_j^+ d \log F^+ - (u_j^+ - 2\pi i \delta_{jk}) d \log F^+] \\ &= \frac{1}{2\pi i} \sum_{k=1}^g \left[\int_{a_k} u_j^+ \omega_k - B_{jk} \int_{a_k} d \log F^+ + 2\pi B_{jk} \right] + \int_{b_j} d \log F^+, \end{aligned}$$

in the course of computation we used formula (2.12.4)-(2.12.9). The function F takes the same values at the endpoints of a_k , therefore

$$\int_{a_k} d \log F^+ = 2\pi i n_k,$$

where n_k is an integer. Further let Q_j and \tilde{Q}_j be the initial and terminal point of b_j . Then

$$\begin{aligned} \int_{b_j} d \log F^+ &= \log F^+(\tilde{Q}_j) - \log F^+(Q_j) + 2\pi i m_j = \\ &= \log \theta(A(Q_j) + f_j - e) - \log \theta(A(Q_j) - e) + 2\pi i m_j = \\ &\quad - \frac{1}{2} B_{jj} + e_j - u_j(Q_j) + 2\pi i m_j, \end{aligned}$$

where m_j is an integer and $f_j = (B_{1j}, \dots, B_{gj})$ is a vector of the period lattice. The expression for ζ_j can now be written in the form

$$\begin{aligned}\zeta_j &= u_j(P_1) + \dots + u_j(P_j) = \\ &= e_j - \frac{1}{2}B_{jj} - u_j(Q_j) + \frac{1}{2\pi i} \sum_k \int_{a_k} u_j \omega_k + 2\pi i m_j + \sum_k B_{jk}(-n_k + 1).\end{aligned}\quad (2.12.14)$$

The last two terms can be thrown out, they are the j -coordinate of some vector of the period lattice. Thus the relation (2.12.14) coincides with the desired relation (2.12.10) if it is proved that the constant in this equality reduces to (2.12.11), i.e.

$$-\frac{1}{2}B_{jj} - u_j(Q_j) + \frac{1}{2\pi i} \sum_k \int_{a_k} u_j \omega_k = \mathcal{K}_j, \quad j = 1, \dots, g.$$

To get rid of the term $u_j(Q_j)$ we transform the integral

$$\oint_{a_j} u_j \omega_j = \frac{1}{2}[u_j^2(Q_j) - u_j^2(R_j)],$$

where R_j is the beginning of a_j and Q_j is its end (which is also the beginning of b_j). Further $u_j(Q_j) = u_j(R_j) + 2\pi i$. We obtain

$$\oint_{a_j} u_j \omega_j = \frac{2\pi i}{2}[2u_j(Q_j) - 2\pi i],$$

hence

$$-u_j(Q_j) + \frac{1}{2\pi i} \sum_{k=1}^g \int_{a_k} u_j \omega_k = -\pi i + \frac{1}{2\pi i} \sum_{k \neq j, k=1}^g \int_{a_k} u_j \omega_k.$$

The lemma is proved. \square

Remark 2.12.3. We observe that the vector of Riemann constant depends on the choice of the base point P_0 of the Abel map. Indeed let \mathcal{K}_{P_0} the vector of Riemann constants with base point P_0 . Then \mathcal{K}_{Q_0} is related to \mathcal{K}_{P_0} by

$$\mathcal{K}_{Q_0} = \mathcal{K}_{P_0} + (g-1) \int_{P_0}^{Q_0} \omega. \quad (2.12.15)$$

Accordingly, if the function $\theta(A(P) - e)$ is not identically equal to zero on Γ , then its zeros give a solution of the Jacobi inversion problem (2.11.5) for the vector $\eta = e - \mathcal{K}$. We have shown that the map (2.11.4) $A^g : S^g \Gamma \rightarrow J(\Gamma)$ is a local homeomorphism in a neighborhood of a non special positive divisor D of degree g . Since $\theta(z) \not\equiv 0$ for $z \in J(\Gamma)$, then $\theta(A^g(D))$ does not vanish identically on open subsets of S^g . We formulate without proof the following criterion for the function $\theta(A(P) - e)$ to be identically zero (see [29]).

Theorem 2.12.4. *[Theorem D] The function $\theta(A(P) - e)$ is identically zero on Γ if and only if e admits a representation in the form*

$$e = A(Q_1) + \dots + A(Q_g) + \mathcal{K}, \quad (2.12.16)$$

where the divisor $D = Q_1 + \dots + Q_g$ is special.

In other words, the method in Lemma 2.12.2 does not give a solution to the Jacobi inversion problem if and only if this solution is not unique (see Lemma 2.11.1).

We summarize the assertions of this lecture on the zeros of a theta function.

Theorem 2.12.5. *Let $\eta = (\eta_1, \dots, \eta_g)$ be a vector such that the function $F(P) = \theta(A(P) - \eta - \mathcal{K})$ does not vanish identically on Γ . Then*

1. *on Γ the function $F(P)$ has g zeros P_1, \dots, P_g , which give a solution of the Jacobi inversion problem*

$$u_j(P_1) + \dots + u_j(P_g) = \sum_{k=1}^g \int_{P_0}^{P_k} \omega_j = \eta_j, \quad j = 1, \dots, g. \quad (2.12.17)$$

2. *the divisor $D = P_1 + \dots + P_g$ is nonspecial;*
3. *the points P_1, \dots, P_g are uniquely determined up to permutation by the system (2.12.17).*

We mention a result useful for what follows.

Corollary 2.12.6. *For a nonspecial divisor $D = P_1 + \dots + P_g$ of degree g the function $F(P) = \theta(A(P) - A^g(D) - \mathcal{K})$ has on Γ exactly g zeros $P = P_1, \dots, P = P_g$.*

(This corollary follows from Lemma 2.12.2 even without invoking the unproved Theorem 2.12.4 if in its formulation the words “nonspecial divisor” are replaced by “divisor in general position”.)

Exercise 2.12.7: Let $D = P_1 + \dots + P_n - Q_1 - \dots - Q_n$ be a divisor of degree zero on Γ . The extension $D \rightarrow A(D) = \sum_{i=1}^n (A(P_i) - A(Q_i)) \in J(\Gamma)$ of the Abel mapping to such divisors does not depend on the choice of the initial point in the Abel mapping. Prove that the correspondence establishes an isomorphism from the group of classes of divisors with zero degree modulo linear equivalence onto the Jacobian $J(\Gamma)$.

Exercise 2.12.8: Denote \mathcal{K}^Q the vector of Riemann constants evaluated with respect to the base point Q :

$$\mathcal{K}_j = \frac{2\pi i + B_{jj}}{2} - \frac{1}{2\pi i} \sum_{l \neq j} \left(\oint_{a_l} \omega_l(P) \int_Q^P \omega_j \right), \quad j = 1, \dots, g.$$

Prove that

$$\mathcal{K}^Q - \mathcal{K}^{Q'} = (g-1) A(Q - Q'). \quad (2.12.18)$$

Therefore there exists a degree $(g-1)$ Riemann divisor Δ independent on the choice of Q (but depending on the choice of the basis of cycles on Γ) such that

$$\mathcal{K}^Q = -\Delta + (g-1) Q \in J(\Gamma). \quad (2.12.19)$$

Remark 2.12.9. As already mentioned, the zeros of the theta-function form an analytic subvariety of $J(\Gamma)$. The collection of these zeros forms a theta divisor in $J(\Gamma)$.

Lemma 2.12.10. *The zeros of the theta function $\theta(e) = 0$, admits a parametric representation in the form*

$$e = A(P_1) + \cdots + A(P_{g-1}) + \mathcal{K}, \quad (2.12.20)$$

where P_1, \dots, P_{g-1} are arbitrary points of the Riemann surface.

Proof. Let $\theta(e) = 0$ and define $F(P) = \theta(A(P) - e)$. Two cases are possible.

1. $F(P) \neq 0$ on Γ . Then by Theorem 2.12.5

$$e = A(P_1) + \cdots + A(P_g) + \mathcal{K}, \quad (2.12.21)$$

where the collection of points P_1, \dots, P_g is uniquely determined. By the condition $\theta(e) = 0$, P_0 (the lower limit in the integrals) is among these points; say $P_g = P_0$. Then $A(P_0) = 0$, and it follows from (2.12.21) that

$$e = A(P_1) + \cdots + A(P_{g-1}) + \mathcal{K}.$$

2. Suppose that $F(p) \equiv 0$ on Γ . Then by Theorem, 2.12.4 it is possible to represent e in the form

$$e \equiv A(Q_1) + \cdots + A(Q_g) + \mathcal{K}, \quad (2.12.22)$$

where the divisor $D + Q_1 + \cdots + Q_g$ is special. Since D is special, there exists a meromorphic function f having poles at the points Q_1, \dots, Q_g and such that $F(P_0) = 0$. Let $D' = P_1 + \cdots + P_{g-1} + P_0$ be the zero divisor of f . By Abel's theorem, $A(D') = A(D)$. Substituting $A(D')$ in place of $A(D)$ in (2.12.22) and again using equality $A(P_0) = 0$, we conclude the proof of the lemma. □

It has already been noted that the function $F(P) = \theta(A(P) - e)$ (let $e = \eta + \mathcal{K}$) is not identically zero if $\theta(e) \neq 0$. The zeros of the theta function (the points of the theta divisor) form a variety of dimension $2g - 2$ (for $g \geq 3$) with singularities in the $2g$ -dimensional torus $J(\Gamma)$. If we delete from $J(\Gamma)$, the theta divisor, then we get a connected $2g$ -dimensional domain. We get that the Jacobi inversion problem is solvable for all points of the Jacobian $J(\Gamma)$ and uniquely solvable for almost all points. Thus the collection $(P_1, \dots, P_g) = (A^{(g)})^{-1}(\eta)$ of points if the Riemann surface Γ (without consideration of order) is a single valued function of a point $\eta = (\eta_1, \dots, \eta_g) \in J(\Gamma)$ (which has singularities at points of the theta divisor.) To find an analytic expression for these functions we take an arbitrary meromorphic function $f(P)$ on Γ . Then the specification of the quantities η_1, \dots, η_g uniquely determines the collection of values

$$f(P_1), \dots, f(P_g), \quad A^{(g)}(P_1, \dots, P_g) = \eta. \quad (2.12.23)$$

Therefore, any symmetric function of these values is a single-valued meromorphic function of the g variables $\eta = (\eta_1, \dots, \eta_g)$, that is $2g$ -fold periodic with period lattice $\{2\pi iM + BN\}$. All these functions can be expressed in terms of a Riemann theta function. The following elementary symmetric functions has an especially simple expression:

$$\sigma_f(\eta) = \sum_{j=1}^g f(P_j). \quad (2.12.24)$$

From Theorem 2.12.5 and the residue formula we get for this function the representation

$$\begin{aligned}\sigma_f(\eta) &= \frac{1}{2\pi i} \oint_{\partial\bar{\Gamma}} f(P) d\log \theta(A(P) - \eta - \mathcal{K}) \\ &\quad - \sum_{\substack{P=Q_k \\ f(Q_k)=\infty}} \operatorname{Res} f(P) d\log \theta(A(P) - \eta - \mathcal{K}),\end{aligned}\tag{2.12.25}$$

the second term in the right hand side is the sum of the residue of the integrand over all poles of $f(P)$. As in the proof of Lemma 2.12.1 and Lemma 2.12.2, it is possible to transform the first term in (2.12.25) by using the formulas (2.12.8) and (2.12.9). The equality (2.12.25) can be written in the form

$$\sigma_f(\eta) = \frac{1}{2\pi i} \sum_k \int_{a_k} f(P) \omega_k - \sum_{f(a_k)=\infty} \operatorname{Res}_{P=Q_k} f(P) d\log \theta(A(P) - \eta - \mathcal{K}).\tag{2.12.26}$$

Here the first term is a constant independent of η . We analyze the computation of the second term (the sum of residue) using an example.

Example 2.12.11. Γ is an hyperelliptic Riemann surface of genus g given by the equation $w^2 = P_{2g+1}(z)$, and the function f has the form $f(z, w) = z$, the projection on the z -plane. This function on Γ has a unique two-fold pole at ∞ . We get an analytic expression for the function σ_f constructed according to the formula (2.12.24). In other words if $P_1 = (z_1, w_1), \dots, P_g = (z_g, w_g)$ is a solution of the inversion problem $A(P_1) + \dots + A(P_g) = \eta$, then

$$\sigma_f(\eta) = z_1 + \dots + z_g.\tag{2.12.27}$$

We take ∞ as the base point P_0 (the lower limit in the Abel mapping). According to (2.12.26) the function $\sigma_f(\eta)$ has the form

$$\sigma_f(\eta) = c - \operatorname{Res}_{\infty} z d\log \theta(A(P) - \eta - \mathcal{K}).$$

Let us compute the residue. Take $\tau = z^{-\frac{1}{2}}$ as a local parameter in a neighborhood of ∞ . Suppose that the holomorphic differentials ω_i have the form $\omega_i = \psi_i(\tau) d\tau$ in a neighborhood of ∞ . Then

$$\begin{aligned}d\log \theta(A(P) - \eta - \mathcal{K}) &= \sum_{i=1}^g [\log \theta(A(P) - \eta - \mathcal{K})]_i \omega_i(P) = \\ &= \sum_{i=1}^g [\log \theta(A(P) - \eta - \mathcal{K})]_i \psi_i(\tau) d\tau\end{aligned}$$

where $[\dots]_i$ denotes the partial derivative with respect to the i th variable. By the choice of the base point $P_0 = \infty$, the decomposition of the vector-valued function $A(P)$ in a neighborhood of ∞ has the form

$$A(P) = \tau U + O(\tau^2),$$

where the vector $U = (U_1, \dots, U_g)$ has the form

$$U_j = \psi_j(0), \quad j = 1, \dots, g.$$

From these formulas we finally get

$$\sigma_f(\eta) = -\partial_x^2 \log \theta(\eta + \mathcal{K}) + c, \quad (2.12.28)$$

where $\partial_U = \sum_{j=1}^g U_j \frac{\partial}{\partial \eta_j}$ is the operator of differentiation in the direction U and c is a constant.

We shall show in Section 3.3 that the function

$$u(x, t) = \frac{\partial^2}{\partial x^2} \log \theta(Ux + Wt\mathcal{K}) + c$$

where $W_k = \frac{1}{3}\psi''(0)$ solves the Korteweg de Vries equation

$$u_t = \frac{1}{4}(6uu_x + u_{xxx}).$$

Exercise 2.12.12: Suppose that a hyperelliptic Riemann surface of genus g is given by the equation $w^2 = P_{2g+2}(z)$. Denotes its points at infinity by P_- and P_+ . Chose P_- as the base point P_0 of the Abel mapping. Take $F(z, w) = z$ as the function f . Prove that the function $\sigma_f(\eta)$ has the form

$$\sigma_f(\eta) = \partial_U \log \frac{\theta(\eta - \mathcal{K} - \Delta)}{\theta(\eta + \mathcal{K})} + c \quad (2.12.29)$$

where $\Delta = A(P_+)$ and the vector $U = (U_1, \dots, U_g)$ has the form

$$U_j = \psi_j(D), \quad j = 1, \dots, g, \quad (2.12.30)$$

where the basisholomorphic differentials have the form

$$\omega_j(P) = \psi_j(\tau)d\tau, \quad \tau = z^{-1}, \quad P \rightarrow \infty.$$

Exercise 2.12.13: Let Γ be a Riemann surface $w^2 = P_5(z)$ of genus 2. Consider the two systems of differential equations:

$$\frac{dz_1}{dx} = \frac{\sqrt{P_5(z_1)}}{z_1 - z_2}, \quad \frac{dz_2}{dx} = \frac{\sqrt{P_5(z_2)}}{z_2 - z_1} \quad (2.12.31)$$

$$\frac{dz_1}{dt} = \frac{z_2 \sqrt{P_5(z_1)}}{z_1 - z_2}, \quad \frac{dz_2}{dt} = \frac{z_1 \sqrt{P_5(z_2)}}{z_2 - z_1}. \quad (2.12.32)$$

Each of these systems determined a law of motion of the pair of points

$$P_1 = (z_1, \sqrt{P_5(z_1)}), \quad P_2 = (z_2, \sqrt{P_5(z_2)})$$

on the Riemann surface Γ . Prove that under the Abel mapping (2.11.1) these systems pass into the systems with constant coefficients

$$\begin{aligned} \frac{d\eta_1}{dx} &= 0, & \frac{d\eta_2}{dt} &= 1 \\ \frac{d\eta_1}{dt} &= -1, & \frac{d\eta_2}{dx} &= 0. \end{aligned}$$

In other words, the Abel mapping (2.11.1) is simply a substitution integrating the equations (2.12.31) and (2.12.32)

3 Baker - Akhiezer functions and differential equations

3.1 Definition of Baker - Akhiezer functions

Among the elementary functions of a complex variable the exponential functions come next in order of complexity after the rational functions. The exponential e^z is analytic in \mathbb{C} and has an essential singularity at the point $z = \infty$. If $q(z)$ is a rational function, then $f(z) = e^{q(z)}$ is analytic in $\bar{\mathbb{C}} = \mathbb{CP}^1$ everywhere except at the poles of $q(z)$, where $f(z)$ has essential singular points. In the last century Clebsch and Gordan considered a generalizing functions of exponential type to Riemann surfaces of higher genus. It turn out that for $g > 0$ such functions will have poles as rules in contrast to the usual exponential. Baker noted that such functions of exponential type can be expressed in terms of theta functions of Riemann surfaces. Akhiezer first directed attention [1] to the fact that under certain conditions functions of exponential type on hyperelliptic Riemann surfaces are eigenfunctions of second-order linear differential operators. Following the established tradition, we call functions of exponential type on Riemann surfaces Baker-Akhiezer functions. The modern way of looking at the theory of Baker Akhiezer functions crystallized as a result of studying and generalizing analytic properties of eigenfunctions of ordinary linear differential operators with periodic coefficients (see [9]-[12],[16]). The general theory of Baker-Akhiezer functions and its applications to linear differential and difference operators and to nonlinear equations was constructed by Krichever ([19]-[21]), whose approach we shall follow on the whole in Lecture3-3.3.

We give a definition of the Baker-Akhiezer functions of the simplest type, which have a unique essential singularity. Let Γ be a Riemann surface of genus g . Fix on Γ some point Q and a local parameter $z = z(P)$ in a neighborhood of this point (let the point Q itself correspond to the value $z = 0$, $z(Q) = 0$). It is convenient to introduce the reciprocal quantity $k = z^{-1}$, $k(Q) = \infty$. Further, let $q(k)$ be an arbitrary polynomial.

Definition 3.1.1. *Let $D = P_1 + \dots + P_g$ be a positive divisor of degree g on $\Gamma \setminus Q$. A Baker-Akhiezer function on Γ corresponding to the point Q , at which the local parameter is $z = k^{-1}$, the polynomial $q(k)$, and the divisor D is defined to be the function $\psi(P)$ such that:*

1. $\psi(P)$ is meromorphic on Γ everywhere except at Q , and has on $\Gamma \setminus Q$ poles only at the points P_1, \dots, P_g of D (more precisely, the divisor of $\psi|_{\Gamma \setminus Q}$ is $\geq D$);
2. the product $\psi(Q) \exp[-q(k(Q))]$ is analytic in a neighborhood of Q .

Instead of the second condition we also say that the function $\psi(P)$ has at Q an essential singularity of the form

$$\Psi(P) \simeq e^{q(k)}.$$

Such Baker-Akhiezer functions form a linear space for a given divisor D (we fix the point Q , the local parameter k^{-1} , and the polynomial $q(k)$). Denote this space by $\Lambda(D)$, by analogy with the space $L(D)$. When the divisor D varies in a class of linear equivalence, $D \simeq D'$, the space $\Lambda(D)$ is replaced by the isomorphic space $\Lambda(D')$; if $(f) = D' - D$ and $\psi \in \Lambda(D)$, then $f\psi \in \Lambda(D')$.

Example 3.1.2. Let Γ be the elliptic curve $w^2 = 4z^3 - g_2z - g_3$, and parametrize the points of this curve by the points of the torus $T^2 = \mathbb{C}/\{2m\omega + 2n\omega'\}$ (see Lecture 2.6) with $z = \wp(u)$, $w = \wp'(u)$. We take $Q = \{u = 0\}$ (the point of Γ at infinity), $k = u^{-1}$, and $q(k) = xk$, where x is a parameter. The divisor D consists of the single point $P_1 = (\wp(u_1), \wp'(u_1))$. Then the Baker-Akhiezer function which depends on x has the form

$$\Psi(P) = \Psi(x; P) = \frac{\sigma(u - u_1 - x)e^{x\zeta(u)}}{\sigma(u - u_1)\sigma(u_1 + x)}, \quad P = (\wp(u), \wp'(u)). \quad (3.1.1)$$

Indeed this function has an essential singularity of the necessary form, since $\zeta(u) = k + O(k^{-1})$ as $u = k^{-1} \rightarrow 0$. Its pole is located at the point $P = P_1$, because $\sigma(0) = 0$. The only thing that has to be verified here is that the function (3.1.1) is single-valued on Γ . In other words, we must check that the value of $\psi(P)$ does not change under the substitution $u \rightarrow u + 2m\omega + 2n\omega'$. But this follows from the laws of transformation of the function $\zeta(u)$ and $\sigma(u)$ (formulas (2.6.28) and (2.6.29)).

Exercise 3.1.3: Verify that for all u_1 , the function $\psi(x; P)$ in (3.1.1), as a function of x , is an eigenfunction for the Lamé operator

$$\left[-\frac{\partial^2}{\partial x^2} + 2\wp(x) \right] \psi(x; P) = \lambda \psi(x; P), \quad \lambda = \wp(u), \quad P = (\wp(u), \wp'(u)). \quad (3.1.2)$$

Remark 3.1.4. In the definition of the Baker-Akhiezer function the requirement $D \in \Gamma \setminus Q$ can be waived. If, for example, the point Q appears in the divisor D with multiplicity n , then as $P \rightarrow Q$ the corresponding Baker-Akhiezer function $\psi(P)$ must by definition have an asymptotic expression of the form

$$\psi(P) = \exp[q(k)](ck^n + O(k^{n-1})),$$

where c is a constant.

We return to general Riemann surfaces.

Theorem 3.1.5. *Suppose that a divisor $D = P_1 + \dots + P_g$ of degree g is nonspecial. Then the space $\Lambda(D)$ is one-dimensional for a polynomial q with sufficiently small coefficients.*

In other words for a nonspecial divisor D and a general polynomial $q(k)$ the conditions of definition 3.1.1 determine a Baker-Akhiezer function uniquely to within multiplication by a constant. We precede the proof of the theorem by an important auxiliary assertion.

Lemma 3.1.6. *A Baker-Akhiezer function $\Psi(P)$ has g zeros P'_1, \dots, P'_g on Γ . The following relation holds for the divisor $D' = P'_1 + \dots + P'_g$ of the zeros and the divisor D of the poles of this function on the Jacobi variety $J(\Gamma)$*

$$A^{(g)}(D') = A^{(g)}(D) - U_q \quad (3.1.3)$$

where $U_q = (U_{q1}, \dots, U_{qg})$ is the vector of b -periods of the normalized Abelian differentials of the second kind Ω_q with zero a -periods and with principal part at Q of the form

$$\Omega_q(P) = dq(k) + O(k^{-2})dk, \quad k = k(P) \rightarrow \infty; \quad (3.1.4)$$

$$\oint_{a_j} \Omega_q = 0, \quad j = 1, \dots, g; \quad U_{qj} = \oint_{b_j} \Omega_q, \quad j = 1, \dots, g. \quad (3.1.5)$$

Conversely if divisors D and D' of degree g satisfy (3.1.3), then they are the divisors of the poles and zeros for some Baker-Akhiezer function with poles in D , zeros in D' and an essential singularity of the form $\psi(P) \simeq \exp q(k)$ as $P \rightarrow Q$.

Proof. Consider the logarithmic differential $\Omega = d \log \psi$. This is a meromorphic differential on Γ with a pole (at least double) at Q with principal part of the form $dq(k)$, and with simple poles at the zeros and poles of ψ . Applying the residue theorem to this differential, we get that the number of zeros of ψ is equal (counting multiplicity) to the number of poles, i.e. g . The first part of the lemma is proved.

We now represent the differential $\Omega = d \log \psi$ in the form

$$\Omega = \sum_{j=1}^g \Omega_{P'_j P_j} + \Omega_q + \sum_{i=1}^g c_i \omega_i, \quad (3.1.6)$$

where $\Omega_{P'_j P_j}$ are the normalized differentials of the third kind, the differential Ω_q is defined above, $\omega_1, \dots, \omega_g$ are the basis of holomorphic differentials and c_1, \dots, c_g are constants. The conclusion of the proof of the lemma is almost identical to the proof of Abel's theorem. The condition for ψ to be single-valued on Γ can be written in the form

$$\oint_{a_k} \Omega = 2\pi n_k, \quad \oint_{b_k} \Omega = 2\pi i m_k, \quad k = 1, \dots, g, \quad (3.1.7)$$

where n_k and m_k are integers. Since the differentials $\Omega_{P'_j P_j}$ and Ω_q are normalized, the first of these conditions implies that $c_k = n_k$ for $k = 1, \dots, g$. By the formula for the periods of a differential of the third kind, the second condition gives us that

$$\sum_{j=1}^g \int_{P_j}^{P'_j} \omega_k + U_{qk} + \sum_{j=1}^g n_j B_{jk} = 2\pi i m_k, \quad k = 1, \dots, g. \quad (3.1.8)$$

This equality is the k -th coordinate of the relation (3.1.3). Conversely, if (3.1.3) holds, then (3.1.8) also holds for some integers $n_1, \dots, n_g, m_1, \dots, m_g$. This implies (3.1.7) for the differential (3.1.6) with $c_i = n_i, i = 1, \dots, g$. The function $\psi = \exp \int \Omega$ will then be single valued on Γ , have zeros and poles at the points P'_1, \dots, P'_g respectively, and have an essential singularity of the necessary form at Q , since $\int \Omega_q$ has the asymptotic expression $q(k) + O(1)$ as $P \rightarrow Q$. The lemma is proved. \square

Proof of the Theorem 3.1.5

Proof. We first prove the existence of a Baker-Akhiezer function. From the divisor D we construct a divisor D' of degree g by solving the Jacobi inversion problem (3.1.3) with respect to D' . According to the lemma, a Baker-Akhiezer function with the necessary singularities corresponds to the pair of divisors D and D' . We now prove uniqueness. Let ψ and $\tilde{\psi}$ be two Baker-Akhiezer functions with the same data. The relation (3.1.3) holds for the divisors D' and \tilde{D}' of their zeros. If the coefficients of the polynomial $q(k)$ are small, then the vector U_q is also small (verify!). Since D is nonspecial, D' and \tilde{D}' are nonspecial for a sufficiently small vector U_q in view of Lemma 2.11.1 and the fact that a divisor in general position is nonspecial. It follows from (3.1.3) and Abel's theorem that the divisors

D' and \tilde{D}' are linearly equivalent. Since they are nonspecial, they coincide. Therefore, the ratio $\psi'(P)/\psi(P)$ is a holomorphic function on Γ and hence is constant. The theorem is proved. \square

Exercise 3.1.7: Suppose that D is a divisor of degree $n \geq g$ in general position. Prove that the space $\Lambda(D)$ of Baker-Akhiezer functions with poles at the points of D (the definition of them does not differ from the Definition 3.1.1) has dimension $n - g + 1$ for a polynomial $q(k)$ with sufficiently small coefficients.

We now get an explicit formula for Baker-Akhiezer functions.

Theorem 3.1.8. *Suppose that the divisor D and the polynomial q are the same as in theorem 3.1.5. Then the Baker-Akhiezer function constructed from the Riemann surface Γ , the point Q , the local parameter k^{-1} , and the divisor D has the form*

$$\psi(P) = c \exp \left(\int_{P_0}^P \Omega_q \right) \frac{\theta(A(P) - A^{(g)}(D) + U_q - K)}{\theta(A(P) - A^{(g)}(D) - K)}. \quad (3.1.9)$$

Here c is an arbitrary constant, $P_0 \neq Q$ is an arbitrary point of Γ , the differential Ω_q and its period vector U_q are defined by the equalities (3.1.4) and (3.1.5), and K is the vector of Riemann constants. The path of integration in the integral $\int_{P_0}^P \Omega_q$ and in the Abel mapping $A(P) = \left(\int_{P_0}^P \omega_1, \dots, \int_{P_0}^P \omega_g \right)$ are chosen to be same.

Proof. We first verify that the function is single-valued on Γ . Single-valuedness can fail only because the path of integration from P_0 to P can vary. If we take another path of integration from P_0 to P , then the periods of the corresponding differentials along some cycle γ are added to the integrals $\int_{P_0}^P \Omega_q$ and $\int_{P_0}^P \omega_i$. decompose this cycle with respect to the basis of cycles: $\gamma \simeq \sum_{k=1}^g n_k a_k + \sum_{j=1}^g m_j b_j$ where n_k and m_j are integers. Then under a change of the path of integration we have that

$$\int_{P_0}^P \Omega_q \rightarrow \int_{P_0}^P \Omega_q + \sum_j m_j U_{qj} = \int_{P_0}^P \Omega_q + \langle M, U_q \rangle, \quad (3.1.10)$$

$$A(P) \rightarrow A(P) + 2\pi i N + BM. \quad (3.1.11)$$

Here $M = (m_1, \dots, m_g)$, $N = (n_1, \dots, n_g)$ are integers vectors. According to (2.11.14), under such a transformation the ratio of the theta function is multiplied by

$$\frac{\exp[-\frac{1}{2}\langle BM, M \rangle - \langle M, A(P) - A^{(g)}(D) - U_q - K \rangle]}{\exp[-\frac{1}{2}\langle BM, M \rangle - \langle M, A(P) - A^{(g)}(D) - K \rangle]} = \exp(-\langle M, U_q \rangle),$$

while the exponential term acquires the reciprocal factor $\exp \langle M, U_q \rangle > .$ The single-valuedness is proved.

Further, since D is nonspecial, the poles of the function (3.1.9) (which arise because of the zeros of the denominator) lie precisely at the points of the divisor D ; see Corollary 2.12.6. For polynomial $q(k)$ with small coefficients the numerator in (3.1.9) is not identically zero. Moreover, the function (3.1.9) has an essential singularity of the necessary form in view of the choice of Ω_q . Indeed near Q we have $\int_{P_0}^P \Omega_q = q(k) + O(1)$, $k = k(P) \rightarrow \infty$. The theorem is proved. \square

Remark 3.1.9. A function $\psi(P)$ of the form (3.1.9) depends analytically on the coefficients of the polynomial $q(k)$. Therefore, it is not identically zero only for small values of these coefficients, but also for any values of them. The same applies to Theorem 3.1.5.

Remark 3.1.10. Baker-Akhiezer functions on singular algebraic curves constructed from (nonsingular) Riemann surfaces with degeneracies of the pinched cycle are very useful for applications to differential equations. Here we consider the example of the so-called Enriques curves (see Example 2.4.19), which are obtained from surfaces of genus g by pinching all a -cycles in some canonical basis. These curves can be represented in the form of the Riemann sphere $\bar{\mathbb{C}} = \mathbb{CP}^1$ by identifying g pairs of points $a_1, b_1, \dots, a_g, b_g$. Suppose that the poles of the Baker-Akhiezer function are located at the points $z = z_1, \dots, z = z_g$ of the complex z -plane. We locate the essential singularity of this function at the point $z = \infty$, let $k = z$ and fix some polynomial $q(z)$. Then the corresponding Baker-Akhiezer function has the form

$$\psi(z) = c \frac{z^g + c_1 z^{g-1} + \dots + c_g}{\prod_{i=1}^g (z - z_i)} e^{q(z)}, \quad (3.1.12)$$

$$\psi(a_i) = \psi(b_i), \quad i = 1, \dots, g \quad (3.1.13)$$

(cfr (2.4.4)). Here c is an arbitrary constant and c_1, \dots, c_g are coefficients determined uniquely from the system of linear equations (3.1.13) for the points z_1, \dots, z_g in general position. Baker-Akhiezer functions on curves with more complicated singularities are constructed similarly.

Exercise 3.1.11: Suppose that Γ is a Riemann surface of genus g , Q is a point on it, and k^{-1} is a local parameter in a neighborhood of this point. Let P_0^\pm be any pairs of points on Γ . Then for almost any divisor D of degree $g+1$ and for almost any polynomial $q(k)$ there exists a unique (up to a factor) Baker-Akhiezer function $\psi(P)$ such that $\psi(P_0^+) = \psi(P_0^-)$ with poles at the points of D and an essential singularity at Q of the form $\psi(P) = \exp q(k(P))$.

In the situation described in this exercise it is natural to call $\psi(P)$ the Baker-Akhiezer function on the singular curve obtained from the Riemann surface Γ by gluing together the points P_0^+ and P_0^- . This singular curve can be thought of as being obtained from a Riemann surface of genus $g+1$ by pinching a cycle nonhomologous to zero (this gives a singularity, a double point). A more complicated singularity of “beak” type is obtained by subjecting the surface to a further degeneracy by letting the points P_0^+ and P_0^- approach each other and coalesce into a single point P_0 . Baker-Akhiezer functions on curves with a beak are defined as follows.

Exercise 3.1.12: Let Γ , g , Q and k be the same as in the exercise 3.1.11. Let P_0 be a point on Γ , and z a local parameter with center at this point $z(P_0) = 0$. Then for almost any divisor D of degree $g+1$ and for almost any polynomial $q(k)$ there exists a unique (up to a factor) Baker-Akhiezer function $\psi(P)$ such that $\psi(P) \simeq \exp(q(k(P)))$, with poles at the points of D and essential singularity at Q of the form $\frac{d}{dz} \psi(P)|_{P=P_0} = 0$.

3.2 Kadomtsev - Petviashvili equation and its solutions

Let Γ be an arbitrary Riemann surface of genus g . Let us fix on Γ a point Q and a local parameter k^{-1} near this point such that $k(Q) = \infty$. For the triple (Γ, Q, k) one can construct

the Baker - Akhiezer function $\psi(P)$, $P \in \Gamma$, with some nonspecial divisor of poles D and with an essential singularity at Q of the form

$$\psi(P) = \left[1 + O\left(\frac{1}{k}\right) \right] e^{kx + k^2y + k^3t} \quad (3.2.1)$$

$$k = k(P) \rightarrow \infty \quad \text{for } P \rightarrow Q.$$

In other words, we take $q(k) = kx + k^2y + k^3t$ as the polynomial $q(k)$ in the Definition 3.1.1 denoting x, y, t the coefficients of the polynomial (the parameters of the BA function). To emphasize the dependence of the BA function on the parameters we denote it $\psi(x, y, t; P)$.

From Theorem 3.1.2 we get an expression of $\psi(x, y, t; P)$ via theta-function of the Riemann surface Γ

$$\begin{aligned} \psi(x, y, t; P) = c \frac{\theta(A(P) - A^{(g)}(D) + xU + yV + tW - \mathcal{K})}{\theta(A(P) - A^{(g)}(D) - \mathcal{K})} \\ \times \exp\left(x \int_{P_0}^P \Omega_1 + y \int_{P_0}^P \Omega_2 + t \int_{P_0}^P \Omega_3\right) \end{aligned} \quad (3.2.2)$$

for an arbitrary choice of the basic cycles $a_1, \dots, a_g, b_1, \dots, b_g$ on Γ . Here $c = c(x, y, t)$ is a normalizing constant, $\Omega_1, \Omega_2, \Omega_3$ are the normalized second kind differentials on Γ with the only poles at Q having the principal parts $dk, d(k^2)$ and $d(k^3)$ resp. In the notations of Lecture 6

$$\Omega_1 = -\Omega_Q^{(1)}, \quad \Omega_2 = -2\Omega_Q^{(2)}, \quad \Omega_3 = -3\Omega_Q^{(3)}. \quad (3.2.3)$$

The vectors

$$U = (U_1, \dots, U_g), \quad V = (V_1, \dots, V_g), \quad W = (W_1, \dots, W_g)$$

are built of the b -periods of these differentials,

$$U_i = \oint_{b_i} \Omega_1, \quad V_i = \oint_{b_i} \Omega_2, \quad W_i = \oint_{b_i} \Omega_3, \quad i = 1, \dots, g. \quad (3.2.4)$$

Other entries of (3.2.2) have the same meaning like in the formula (3.1.9).

For sufficiently small x, y, t the divisor D' of zeroes of $\psi(x, y, t; P)$ does not contain the point Q (assuming the divisor D has its support on $\Gamma \setminus Q$). Hence the function $\psi(x, y, t; P)$ can be normalized in such a way that

$$\psi(x, y, t; P) = \left(1 + \frac{\xi_1}{k} + \frac{\xi_2}{k^2} + \dots \right) e^{kx + k^2y + k^3t}, \quad k = k(P) \quad (3.2.5)$$

for $P \rightarrow Q$ (this normalization determines the factor $c = c(x, y, t)$ in (3.2.2)). The coefficients ξ_1, ξ_2, \dots are certain functions of x, y, t ; we will compute them below.

For the moment we will forget the Riemann surface origin of the series (3.2.5) looking at this as a formal expansion. A simple but important statement holds for the derivatives of this expansion with respect to the parameters x, y, t .

Lemma 3.2.1. For a function ((3.2.5)) with arbitrary smooth coefficients $\xi_1 = \xi_1(x, y, t)$, $\xi_2 = \xi_2(x, y, t)$, ... the following equations hold true

$$\left[-\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2} + u \right] \psi = O\left(\frac{1}{k}\right) e^{kx+k^2y+k^3t} \quad (3.2.6)$$

$$\left[-\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} + \frac{3}{4} \left(u \frac{\partial}{\partial x} + \frac{\partial}{\partial x} u \right) + w \right] \psi = O\left(\frac{1}{k}\right) e^{kx+k^2y+k^3t} \quad (3.2.7)$$

where the functions $u = u(x, y, t)$ and $w = w(x, y, t)$ are uniquely determined from the conditions of vanishing of the coefficients of $k^n e^{kx+k^2y+k^3t}$ for $n = 0, 1, 2, 3$. These functions have the following form

$$u = -2 \frac{\partial \xi_1}{\partial x} \quad (3.2.8)$$

$$w = 3 \xi_1 \frac{\partial \xi_1}{\partial x} - \frac{3}{2} \frac{\partial^2 \xi_1}{\partial x^2} - 3 \frac{\partial \xi_2}{\partial x}. \quad (3.2.9)$$

The proof can be obtained by a straightforward computation. Denote L and A the resulting ordinary differential operators

$$L = \partial_x^2 + u \quad (3.2.10)$$

$$A = \partial_x^3 + \frac{3}{4} (u \partial_x + \partial_x u) + w \quad (3.2.11)$$

(we will often use the short notation $\partial_x := \frac{\partial}{\partial x}$).

Theorem 3.2.2. Let $\psi = \psi(x, y, t; P)$ be the BA function of the above form constructed for an arbitrary Riemann surface Γ , a point $Q \in \Gamma$, a local parameter k^{-1} with the center at the point Q and a nonspecial divisor D , normalized by the condition (3.2.5). Then ψ is a solution to the system

$$\frac{\partial \psi}{\partial y} = L \psi \quad (3.2.12)$$

$$\frac{\partial \psi}{\partial t} = A \psi \quad (3.2.13)$$

where the operators L, A are given by the formulae (3.2.8) - (3.2.11).

Proof. The functions

$$\varphi_1 := \left(-\frac{\partial}{\partial y} + L \right) \psi, \quad \varphi_2 := \left(-\frac{\partial}{\partial t} + A \right) \psi$$

satisfy all conditions of the definition of BA function with the same essential singularity $e^{kx+k^2y+k^3t}$ at the point $Q \in \Gamma$ and the same poles at the divisor D identical to those for the function $\psi(x, y, t; P)$. But from Lemma 3.2.1 it follows that the products

$$\varphi_1 e^{-kx-k^2y-k^3t} \quad \text{and} \quad \varphi_2 e^{-kx-k^2y-k^3t}$$

vanish at the point Q . Due to uniqueness of the BA function (see Theorem 3.1.5) these products must vanish identically in $P \in \Gamma$. The Theorem is proved.

Corollary 3.2.3. *The functions $u = u(x, y, t)$ and $w = w(x, y, t)$ of the form (3.2.8), (3.2.9) give a solution to the Kadomtsev - Petviashvili (KP) system*

$$\frac{3}{4}u_y = w_x \tag{3.2.14}$$

$$w_y = u_t - \frac{1}{4}(6u u_x + u_{xxx}).$$

As we will see below, the KP equations play an important role in physics of nonlinear waves. They are often written in the form of a single equation

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x} \left[u_t - \frac{1}{4}(6u u_x + u_{xxx}) \right]. \tag{3.2.15}$$

One can easily derive (3.2.15) from (3.2.14) by just eliminating w .

Proof. The conditions of compatibility of the system (3.2.12), (3.2.13). i.e., the equality of the crossed derivatives

$$\frac{\partial}{\partial t} \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} \frac{\partial \psi}{\partial t}$$

read

$$\left[-\frac{\partial}{\partial y} + L, -\frac{\partial}{\partial t} + A \right] = 0 \tag{3.2.16}$$

(we denote $[,]$ the commutator of the two operators). Let us explain how to compute this commutator. First of all, the derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}$ commute pairwise. The commutators of the derivatives with the operators of multiplication by a function can be computed like in the following two sample computations:

$$\left[\frac{\partial}{\partial y}, w \right] \psi = \frac{\partial}{\partial y}(w \psi) - w \frac{\partial}{\partial y} \psi = w_y \psi,$$

so

$$\left[\frac{\partial}{\partial y}, w \right] = w_y.$$

$$\left[\frac{\partial^2}{\partial x^2}, w \right] \psi = \frac{\partial^2}{\partial x^2}(w \psi) - w \frac{\partial^2}{\partial x^2} \psi = 2w_x \frac{\partial \psi}{\partial x} + w_{xx} \psi,$$

that is,

$$\left[\frac{\partial^2}{\partial x^2}, w \right] = 2w_x \frac{\partial}{\partial x} + w_{xx}.$$

So, after simple computations the compatibility condition (3.2.16) reads

$$\left(P_1 \frac{\partial}{\partial x} + P_2 \right) \psi = 0 \tag{3.2.17}$$

for certain differential polynomials P_1, P_2 (i.e., polynomials in u, w and their derivatives). The left hand side of the equation (3.2.17) near Q has the expansion of the form

$$\left[-\frac{\partial}{\partial y} + L, -\frac{\partial}{\partial t} + A \right] = 0 \quad \Leftrightarrow \quad \frac{\partial L}{\partial t} - \frac{\partial A}{\partial y} = [A, L] \quad (3.2.18)$$

for the operators L, A of the form (3.2.10), (3.2.11). The commutation representation (3.2.18) of KP was found in [8] (see also [17]). We will call it *zero curvature representation*.

Summarizing, for any Riemann surface Γ of genus g , any point Q on Γ , and a local parameter k^{-1} near Q , we have constructed a family of solutions of KP. The solutions are parametrized by nonspecial divisors of degree g on Γ (i.e., by generic points of the Jacobian $J(\Gamma)$).

Exercise 3.2.4: Prove that changes of the local parameter of the form

$$k \mapsto \lambda k + a + \frac{b}{k} + O\left(\frac{1}{k^2}\right) \quad (3.2.19)$$

for arbitrary complex numbers $\lambda \neq 0, a, b$ transform the solutions $u(x, y, t)$ in the following way

$$\begin{aligned} x &\mapsto \lambda x + 2\lambda a y + (3\lambda a^2 + 3\lambda^2 b)t \\ y &\mapsto \lambda^2 y + 3\lambda^2 a t \\ t &\mapsto \lambda^3 t \\ u &\mapsto \lambda^{-2} u - 2b\lambda^{-2}. \end{aligned} \quad (3.2.20)$$

Let us derive theta-functional formulae for the solutions of KP. We will use the formula (3.2.2) expressing the BA function via theta-function of Γ .

Theorem 3.2.5. *The solutions of KP constructed in Theorem 3.2.2 and Corollary 3.2.3 read*

$$u(x, y, t) = 2\partial_x^2 \log \theta(xU + yV + tW + z_0) + c \quad (3.2.21)$$

$$w(x, y, t) = \frac{3}{2} \partial_x \partial_y \log \theta(xU + yV + tW + z_0) + c_1 \quad (3.2.22)$$

Here $\theta = \theta(z)$ is the theta-function of the Riemann surface Γ wrt a basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$, the vectors U, V, W are defined in (3.2.16),

$$z_0 = -A^{(g)}(D) - \mathcal{K}$$

is an arbitrary vector, c, c_1 are some constants depending on (Γ, Q, k) and on the choice of the basis of cycles on Γ .

Proof. Due to (3.2.8), (3.2.9) it suffices to compute the coefficients ξ_1, ξ_2 of the expansion (3.2.5). It is even more convenient to use expansion of the logarithm of the BA function

$$\log \psi(x, y, t; P) = kx + k^2y + k^3t + \frac{\eta_1}{k} + \frac{\eta_2}{k^2} + \dots \quad (3.2.23)$$

where

$$\eta_1 = \xi_1, \quad \eta_2 = \xi_2 - \frac{1}{2}\xi_1^2,$$

so

$$u = -2\eta_{1x}, \quad w = -3\eta_{2x} + \frac{3}{4}u_x. \quad (3.2.24)$$

Comparing (3.2.23) with the formula (3.2.2) for $\psi(x, y, t; P)$ we obtain that the coefficients of k^{-1} and k^{-2} in the expansion of the function

$$\varphi(P) := \log \frac{\theta(A(P) - A^{(g)}(D) - \mathcal{K} + xU + yV + tW)}{\theta(A(P) - A^{(g)}(D) - \mathcal{K})} \quad (3.2.25)$$

near the point Q have the form

$$\eta_1 - cx - ay - bt \quad \text{and} \quad \eta_2 - c_1x - a_1y - b_1t$$

resp. Here the constants c, a, b and c_1, a_1, b_1 are the coefficients of k^{-1} and k^{-2} in the expansion at $P \rightarrow Q$ of the second kind integrals

$$\begin{aligned} \int_{P_0}^P \Omega_1 &= k + c_0 + \frac{c}{k} + \frac{c_1}{k^2} + \dots \\ \int_{P_0}^P \Omega_2 &= k^2 + a_0 + \frac{a}{k} + \frac{a_1}{k^2} + \dots \\ \int_{P_0}^P \Omega_3 &= k^3 + b_0 + \frac{b}{k} + \frac{b_1}{k^2} + \dots \end{aligned} \quad (3.2.26)$$

All the coefficients but a_0, b_0, c_0 do not depend on the choice of the initial point P_0 . The coefficients η_1 and η_2 do not depend on P_0 either due to (3.2.23) and to the uniqueness of the BA function. Hence we can choose $P_0 = Q$ in (3.2.25). For this choice of the initial point of the Abel map one has

$$A(Q) = 0.$$

The expansion of the Abel map near Q has the form

$$A(P) = -\frac{1}{k}U - \frac{1}{2k^2}V + O\left(\frac{1}{k^3}\right). \quad (3.2.27)$$

To derive (3.2.27) one has to use the identity

$$dA(P) = (\omega_1(P), \dots, \omega_g(P))$$

and the formulae (2.7.11) for the b -periods of the differentials $\Omega_Q^{(1)}, \Omega_Q^{(2)}, \Omega_Q^{(3)}$, and also the definitions (3.2.3), (3.2.4) of the vectors U, V, W . Hence

$$\begin{aligned} \log \varphi(P) &= \log \theta(-k^{-1}U - \frac{1}{2}k^{-2}V + O(k^{-3}) + xU + yV + tW + z_0) + \dots = \\ &= \log \theta(xU + yV + tW + z_0) - k^{-1} \partial_x \log \theta(xU + yV + tW + z_0) \quad (3.2.28) \\ &\quad - \frac{1}{2} k^{-2} (\partial_y - \partial_x^2) \log \theta(xU + yV + tW + z_0) + O(k^{-3}). \end{aligned}$$

The dots stand for some x, y, t independent terms, z_0 has the form

$$z_0 = -A^{(g)}(D) - \mathcal{K}.$$

Redenoting $c \mapsto -2c$, $c_1 \mapsto -3c_1$ we derive from (3.2.28) and (3.2.24) the formulae of the Theorem. The vector z_0 is an arbitrary generic point of $J(\Gamma)$ due to Jacobi inversion theorem. The Theorem is proved.

The solutions we have constructed satisfy KP equation for those complex values x, y, t such that

$$\theta(xU + yV + tW + z_0) \neq 0.$$

It is clear that they are complex meromorphic functions having poles when $\theta(xU + yV + tW + z_0) = 0$. An additional information about these solutions follow from the formulae (3.2.21), (3.2.22). Namely, they are quasiperiodic functions of x, y, t . Indeed, the second logarithmic derivatives

$$2 \partial_x^2 \log \theta(z) \quad \text{and} \quad \frac{3}{2} \partial_x \partial_y \log \theta(z)$$

are meromorphic single valued functions on the Jacobian torus $J(\Gamma) \ni z$. Here the differential operators ∂_x and ∂_y are defined by the formulae

$$\partial_x = \sum_{i=1}^g U_i \frac{\partial}{\partial z_i}, \quad \partial_y = \sum_{i=1}^g V_i \frac{\partial}{\partial z_i}.$$

The single-valuedness follows from the transformation law (2.11.14). Indeed, a shift of the argument by a vector of the period lattice produces the transformation

$$\log \theta(z + 2\pi i M + B N) = \log \theta(z) - \frac{1}{2} \langle B N, N \rangle - \langle N, z \rangle.$$

So, the linear in z term disappears after double differentiation. To obtain the solutions $u(x, y, t)$ and $w(x, y, t)$ one is to restrict the functions onto the straight lines directed along the vectors U, V, W .

If a commensurability condition

$$T U = 2\pi i(n_1 e_1 + \dots + n_g e_g) + B(m_1 e_1 + \dots + m_g e_g) \quad (3.2.29)$$

hold true for some number T and integers $n_1, \dots, n_g, m_1, \dots, m_g$ then the functions $u(x, y, t)$, $w(x, y, t)$ will be T -periodic in x . (In the formula (3.2.29) e_1, \dots, e_g is the standard basis in \mathbb{C}^g , B is the period matrix of the Riemann surface Γ). The conditions of periodicity in y or in t have a similar form.

Exercise 3.2.6: Prove that the periodic in x solutions to KP of the form (3.2.21), (3.2.22) are dense among all these solutions. Here we consider density wrt the uniform norm on a finite segment of the x axis.

Example 3.2.7. For the elliptic curve Γ , point Q , and the local parameter k as in Example 3.1.2 the BA function with the behavior (3.2.1) has the form

$$\psi(x, y, t; P) = \frac{\sigma(u - u_1 - x)\sigma(u_1)}{\sigma(u - u_1)\sigma(u_1 + x)} e^{x\zeta(u) + y\wp(u) - \frac{1}{2}t\wp'(u)} \quad (3.2.30)$$

$$P = (\wp(u), \wp'(u)), \quad D = (\wp(u_1), \wp'(u_1))$$

(cf. (3.1.1)). The corresponding solution to KP does not depend on y, t :

$$u = -2\wp(x + u_1).$$

Exercise 3.2.8: Let us assume that, along with the commensurability conditions (3.2.29) another representation of the same form fulfills for the vector U ,

$$T'U = 2\pi i(n'_1 e_1 + \dots + n'_g e_g) + B(m'_1 e_1 + \dots + m'_g e_g) \quad (3.2.31)$$

for a complex number T' and some integers $n'_1, \dots, n'_g, m'_1, \dots, m'_g$. Prove that, if $\Im(T'/T) > 0$ then $u(x, y, t), w(x, y, t)$ are elliptic functions in x . Prove that these functions must have the form

$$u(x, y, t) = -2 \sum_{i=1}^N \wp(x - x_i(y, t)) \quad (3.2.32)$$

$$w(x, y, t) = \frac{3}{2} \sum_{i=1}^N \frac{\partial x_i(y, t)}{\partial y} \wp(x - x_i(y, t))$$

for some N , where $x_i(y, t), i = 1, \dots, t$ are some functions of y, t , \wp is the Weierstrass \wp -function with the periods T, T' .

3.3 KP hierarchy

Let us return to general Riemann surfaces Γ . The structure of the essential singularity used in this Lecture as well as in the previous one is only the simplest among possible ones. The most straightforward and natural generalization is to use multiparametric BA functions with the essential singularity at $Q \in \Gamma$ of the form

$$\psi(x, t_2, t_3, \dots; P) = \left(1 + \sum_{i=1}^{\infty} \frac{\xi_i}{k^i}\right) e^{kx + k^2 t_2 + k^3 t_3 + \dots} \quad (3.3.1)$$

In the previous notations $t_2 = y, t_3 = t$; we will also often denote

$$t_1 = x.$$

The periods in (3.3.1) mean that only dependence on finitely many variables t_2, \dots, t_N for sufficiently large N will be under consideration. An analogue of Theorem 3.2.2 and Corollary 3.2.3 reads

Theorem 3.3.1. *The function ψ of the form (3.3.1) for any $n = 1, 2, 3, \dots$ satisfies linear differential equations*

$$\frac{\partial \psi}{\partial t_n} = A_n \psi \quad (3.3.2)$$

where

$$\begin{aligned} A_1 &= \partial_x, & A_2 &= L, & A_3 &= A, \\ A_n &= \partial_x^n + \sum_{i=2}^n u_i^n \partial_x^{n-i}, \end{aligned} \quad (3.3.3)$$

the coefficients u_2^n, \dots, u_n^n of the differential operators A_n can be expressed recursively via the coefficients ξ_1, \dots, ξ_{n-1} of the expansion (3.3.1). These coefficients satisfy an infinite system of differential equations (the so-called KP hierarchy) represented in the zero curvature form

$$\left[-\frac{\partial}{\partial t_n} + A_n, -\frac{\partial}{\partial t_m} + A_m \right] = \frac{\partial A_n}{\partial t_m} - \frac{\partial A_m}{\partial t_n} + [A_n, A_m] = 0, \quad n, m = 1, 2, 3, \dots \quad (3.3.4)$$

The coefficients u_j^k can be expressed via the theta-function of the Riemann surface Γ .

3.4 Degenerate Baker - Akhiezer functions and KP

In this section we will work out the algebro-geometric solutions to the KP equation associated with singular algebraic curves, like those constructed in Remark 3.1.10 (and even for curves with more complicated singularities) following the scheme of the Theorem 3.2.2 and Corollary 3.2.3. Indeed, in the proofs we used only the asymptotic behaviour (3.2.5) of the BA function and the uniqueness of the function. All these properties hold true for the BA functions on singular curves. Let us describe these solutions explicitly.

We define a BA function on a singular curve as a function of the form

$$\psi(x, y, t; k) = [k^N + a_1(x, y, t)k^{N-1} + \dots + a_N(x, y, t)] e^{kx + k^2y + k^3t} \quad (3.4.1)$$

where the dependence of the coefficients on x, y, t is determined from the following system of linear constraints

$$\sum_{j=1}^M \sum_{s=0}^{m_j} \alpha_{ij}^s \partial_k^s \psi(x, y, t; k)|_{k=\kappa_j} = 0, \quad i = 1, \dots, N. \quad (3.4.2)$$

The complex numbers

$$\kappa_1, \dots, \kappa_M, \quad \kappa_i \neq \kappa_j \quad \text{for } i \neq j \quad (3.4.3)$$

$$\alpha_{ij}^s, \quad i = 1, \dots, N, \quad j = 1, \dots, M, \quad s = 0, 1, \dots, m_j,$$

$$m_1 + \dots + m_M + M \geq N$$

are the parameters of our singular curve and of the divisor on it.

The constraints (3.4.2) can be rewritten as a system of linear equations for the coefficients a_1, \dots, a_N . In order to give an explicit form of these equations we introduce polynomials

$$\begin{aligned} P_{r,s}(x, y, t; k) &:= e^{-kx - k^2y - k^3t} \partial_k^s \left(k^r e^{kx + k^2y + k^3t} \right) \\ &= e^{-kx - k^2y - k^3t} \partial_x^r \partial_k^s e^{kx + k^2y + k^3t} \\ &= (\partial_k + x + 2ky + 3k^2t)^s k^r. \end{aligned} \quad (3.4.4)$$

Denote $\omega_j = \omega_j(x, y, t)$ the linear functions

$$\omega_j = \kappa_j x + \kappa_j^2 y + \kappa_j^3 t, \quad j = 1, \dots, M. \quad (3.4.5)$$

Then the constraints (3.4.2) can be written as a system of linear equations for the functions $a_k = a_k(x, y, t)$

$$\sum_{k=1}^N A_{ik}(x, y, t) a_k = b_i(x, y, t), \quad i = 1, \dots, N \quad (3.4.6)$$

where

$$A_{ik}(x, y, t) = \sum_{j=1}^M \sum_{s=0}^{m_j} \alpha_{ij}^s P_{N-k,s}(x, y, t; \kappa_j) e^{\omega_j(x, y, t)} \quad (3.4.7)$$

$$b_i(x, y, t) = - \sum_{j=1}^M \sum_{s=0}^{m_j} \alpha_{ij}^s P_{N,s}(x, y, t; \kappa_j) e^{\omega_j(x, y, t)}. \quad (3.4.8)$$

Denote

$$A(x, y, t) = (A_{ik}(x, y, t))$$

the $N \times N$ matrix of coefficients of the system (3.4.6) and $\hat{A}(x, y, t; k)$ the $(N+1) \times (N+1)$ extended matrix

$$\hat{A}(x, y, t; k) = \begin{pmatrix} k^N & k^{N-1} & \dots & 1 \\ b_1 & & & \\ \cdot & & A(x, y, t) & \\ \cdot & & & \\ b_N & & & \end{pmatrix}. \quad (3.4.9)$$

Theorem 3.4.1. *For those x, y, t such that $\det A(x, y, t) \neq 0$ the function $\psi = \psi(x, y, t; k)$ is uniquely determined by the constraints (3.4.2). It has the form*

$$\psi(x, y, t; k) = \frac{\det \hat{A}(x, y, t; k)}{\det A(x, y, t)} e^{kx + k^2y + k^3t}. \quad (3.4.10)$$

It satisfies the linear system

$$\frac{\partial \psi}{\partial y} = L \psi, \quad \frac{\partial \psi}{\partial t} = A \psi$$

with the operators L and A of the form (3.2.10), (3.2.11), where the coefficients u and w are given by the formula

$$\begin{aligned} u(x, y, t) &= 2 \partial_x^2 \log \det A(x, y, t) \\ w(x, y, t) &= \frac{3}{2} \partial_x \partial_y \log \det A(x, y, t). \end{aligned} \tag{3.4.11}$$

Corollary 3.4.2. *The functions (3.4.11) satisfy KP equations (3.2.14).*

Proof of the Theorem. Let L, A be the operators of the form (3.2.10) with the coefficients u, w defined as in (3.4.11). Choosing the coefficient $u(x, y, t)$ by

$$u(x, y, t) = -2 \partial_x a_1(x, y, t)$$

one can easily see that the function

$$\tilde{\psi}(x, y, t; k) := \frac{\partial \psi}{\partial y} - L \psi$$

has the form

$$\tilde{\psi}(x, y, t; k) = [\tilde{a}_1(x, y, t)k^{N-1} + \dots + \tilde{a}_N(x, y, t)] e^{kx+k^2y+k^3t}$$

with some coefficients $\tilde{a}_1(x, y, t), \dots, \tilde{a}_N(x, y, t)$. This form is completely analogous to (3.4.1) but the term k^N is missing from the pre-exponential factor. Let us show that the coefficients $\tilde{a}_1 = \tilde{a}_1(x, y, t), \dots, \tilde{a}_N = \tilde{a}_N(x, y, t)$ satisfy the linear *homogeneous* equations

$$\sum_{k=1}^N A_{ik}(x, y, t) \tilde{a}_k = 0, \quad i = 1, \dots, N.$$

Indeed, for an arbitrary linear differential operator $\Lambda = \Lambda(\partial_x, \partial_y, \partial_t)$ the function

$$\tilde{\psi}(x, y, t; k) := \Lambda \psi(x, y, t; k)$$

satisfies the same constraints (3.4.2) as the coefficients of the linear system of constraints do not depend on x, y, t . Applying to the operator $\Lambda = \partial/\partial y + u$ we obtain the needed linear homogeneous equations. Due to nondegenerateness $\det A(x, y, t) \neq 0$ of the coefficients matrix we obtain $\tilde{\psi} = 0$. The second linear equation $\partial\psi/\partial t = A\psi$ can be derived in a similar way.

The proof of the formulae (3.4.10) - (3.4.11) follows from the Cramer rule applied to the system (3.4.6) and from the following obvious identity

$$\partial_x \left[P_{r,s}(x, y, t; k) e^{kx+k^2y+k^3t} \right] = P_{r+1,s}(x, y, t; k) e^{kx+k^2y+k^3t}.$$

The Theorem is proved.

Exercise 3.4.3: Prove that

$$a_i = -\frac{\partial}{\partial t_i} \log \det A(x, y, t), \quad i = 1, \dots, N. \tag{3.4.12}$$

Corollary 3.4.2 follows from compatibility, identically in k , of the linear system

$$\frac{\partial \psi}{\partial y} = L \psi, \quad \frac{\partial \psi}{\partial t} = A \psi$$

like in the proof of Corollary 3.2.3.

Remark 3.4.4. For nondegeneracy of the matrix $A(x, y, t)$ of coefficients of the system of linear constraints (3.4.2) has rank N . We also note that the function $\psi(x, y, t; k)$ remains unchanged if the matrices α_{ij}^s are multiplied from the left by an arbitrary constant nondegenerate $N \times N$ matrix.

Exercise 3.4.5: For the BA function of the form

$$\psi = (k + a) e^{kx + k^2 y + k^3 t}$$

satisfying the linear constraint

$$\psi(\kappa_1) + c \psi(\kappa_2) = 0$$

obtain the following explicit formula for the function u :

$$u(x, y, t) = \frac{1}{2} (\kappa_1 - \kappa_2)^2 \operatorname{sech}^2 \frac{1}{2} [(\kappa_1 - \kappa_2)(x - x_0) + (\kappa_1^2 - \kappa_2^2) y + (\kappa_1^3 - \kappa_2^3) t] \quad (3.4.13)$$

where

$$x_0 = \frac{1}{\kappa_1 - \kappa_2} \log c.$$

For real $\kappa_1 \neq \kappa_2$ and real positive c the formula (3.4.13) gives a solitary plane wave solution of KP. More generally, taking all the multiplicities $m_j = 0$ in (3.4.2) one obtains *multisoliton* solutions describing interaction of plane waves (see below). The most known particular example of a multisoliton solution is obtained as follows.

Exercise 3.4.6: Consider the degenerate BA function of the form (3.4.1) determined by the following system of linear constraints

$$\psi(q_i) + c_i \psi(p_i) = 0, \quad i = 1, \dots, N \quad (3.4.14)$$

where $p_1, \dots, p_N, q_1, \dots, q_N$ are pairwise distinct complex numbers, c_1, \dots, c_N are arbitrary nonzero numbers. Prove that the corresponding solution to KP is given by

$$\begin{aligned} u &= 2\partial_x^2 \log \tau(x, y, t), \quad w = \frac{3}{2} \partial_x \partial_y \log \tau(x, y, t), \\ \tau(x, y, t) &= \det(A_{ij}(x, y, t)), \quad A_{ij}(x, y, t) = \left(\delta_{ij} + \rho_i \frac{e^{\lambda_i(x, y, t) - \mu_j(x, y, t)}}{p_i - q_j} \right)_{1 \leq i, j \leq N} \quad (3.4.15) \\ \lambda_i(x, y, t) &= p_i x + p_i^2 y + p_i^3 t, \quad \mu_i(x, y, t) = q_i x + q_i^2 y + q_i^3 t, \quad \rho_i = c_i (p_i - q_i) \prod_{s \neq i} \frac{p_i - q_s}{q_i - q_s}. \end{aligned}$$

Also derive the following representation for the ψ -function

$$\psi = \frac{\det \hat{A}(x, y, t; k)}{\det A(x, y, t)} e^{kx + k^2y + k^3t}$$

$$A(x, y, t) = (A_{ij}(x, y, t))_{1 \leq i, j \leq N}, \quad \hat{A}(x, y, t; k) = \left(\hat{A}_{ij}(x, y, t; k) \right)_{0 \leq i, j \leq N},$$

$$\hat{A}_{ij}(x, y, t; k) = \begin{cases} 1, & i = j = 0 \\ e^{\lambda_i(x, y, t)}, & i > 0, j = 0 \\ \frac{e^{-\mu_j(x, y, t)}}{k - q_j}, & i = 0, j > 0 \\ A_{ij}(x, y, t), & i > 0, j > 0 \end{cases} \quad (3.4.16)$$

Hint: Recast the system (3.4.14) into the form

$$\operatorname{Res}_{k=q_i} \hat{\psi}(k) + \rho_i \hat{\psi}(p_i) = 0, \quad i = 1, \dots, N \quad (3.4.17)$$

for the function

$$\hat{\psi}(k) = \frac{\psi(k)}{\prod_{i=1}^N (k - q_i)} = \left(1 + \sum_{i=1}^N \frac{r_i}{k - q_i} \right) e^{kx + k^2y + k^3t}. \quad (3.4.18)$$

Apply the Cramer rule to solving the linear constraints (3.4.17) for the unknowns

$$\hat{r}_i := r_i e^{\mu_i}, \quad i = 1, \dots, N.$$

Exercise 3.4.7: Taking the constraints (3.4.2) in the form

$$D_i \psi(k)|_{k=\kappa_i} = 0, \quad i = 1, \dots, N \quad (3.4.19)$$

for the differential operators with constant coefficients of the form

$$D_i = \sum_{s=0}^{m_i} \alpha_i^s \partial_k^s, \quad i = 1, \dots, N \quad (3.4.20)$$

prove that the resulting solutions of KP will be a rational function in x, y, t .

3.5 KP and Schur polynomials

Let us now consider a particular example of rational solutions to KP defined by the constraints of the form (3.4.19) for the BA function

$$\psi(t; k) = (k^N + a_1(t)k^{N-1} + \dots + a_N(t)) e^{k t_1 + k^2 t_2 + k^3 t_3 + \dots} \quad (3.5.1)$$

with $\kappa_1 = \kappa_2 = \dots = \kappa_N = 0$. The differential operators in (3.4.19) will be chosen in the following particular form

$$D_i = \partial_k^{n_i}, \quad i = 1, \dots, N$$

for some pairwise distinct positive integers n_1, \dots, n_N . The linear constraints for the degenerate BA function take the form

$$\partial_k^{n_i} \psi(k)|_{k=0} = 0, \quad i = 1, \dots, N. \quad (3.5.2)$$

In order to write the system more explicitly let us introduce the following *elementary Schur polynomials* $p_0(t) = 1, p_1(t), p_2(t)$ etc. as the coefficients of the following formal series in k .

$$e^{k t_1 + k^2 t_2 + k^3 t_3 + \dots} = \sum_{m \geq 0} p_m(t) k^m. \quad (3.5.3)$$

In particular,

$$p_1(t) = t_1, \quad p_2(t) = t_2 + \frac{1}{2} t_1^2, \quad p_3(t) = t_3 + t_1 t_2 + \frac{1}{6} t_1^3$$

etc. Clearly the polynomial $p_m(t)$ depends only on t_1, \dots, t_m . Note the following useful identity

$$\frac{\partial p_m(t)}{\partial t_j} = p_{m-j}(t) \quad (3.5.4)$$

(it is understood here and below that $p_m = 0$ for $m < 0$). One also has the following recursion formula

Exercise 3.5.1: Prove

$$\sum_{i=1}^m i t_i p_{m-i}(t) = m p_m(t), \quad m \geq 1. \quad (3.5.5)$$

More general Schur polynomials $p_{n_1, \dots, n_N}(t)$ in the variables t_1, \dots, t_N are defined by the following determinants

$$p_{n_1, \dots, n_N}(t) = \det \begin{pmatrix} p_{n_1-N+1} & \dots & p_{n_1-1} & p_{n_1} \\ p_{n_2-N+1} & \dots & p_{n_2-1} & p_{n_2} \\ \dots & \dots & \dots & \dots \\ p_{n_N-N+1} & \dots & p_{n_N-1} & p_{n_N} \end{pmatrix} \quad (3.5.6)$$

for arbitrary pairwise distinct positive integers n_1, \dots, n_N .

Theorem 3.5.2. *The solution $u(t_1, t_2, \dots)$ to the KP hierarchy determined by the degenerate BA function (3.5.1), (3.5.2) has the form*

$$\begin{aligned} u &= 2\partial_x^2 \log \tau(t) \\ \tau(t) &= p_{n_1, n_2, \dots, n_N}(t). \end{aligned} \quad (3.5.7)$$

Proof. The system of linear constraints (3.5.2) can be spelled out as follows:

$$\sum_{i=1}^N p_{i+n_j-N}(t) a_i = -p_{n_j-N}(t), \quad j = 1, \dots, N.$$

Applying the Cramer rule one obtains

$$a_1 = - \frac{\det \begin{pmatrix} p_{n_1-N} & p_{n_1-N+2} & \cdots & p_{n_1} \\ p_{n_2-N} & p_{n_2-N+2} & \cdots & p_{n_2} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n_N-N} & p_{n_N-N+2} & \cdots & p_{n_N} \end{pmatrix}}{\det \begin{pmatrix} p_{n_1-N+1} & p_{n_1-N+2} & \cdots & p_{n_1} \\ p_{n_2-N+1} & p_{n_2-N+2} & \cdots & p_{n_2} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n_N-N+1} & p_{n_N-N+2} & \cdots & p_{n_N} \end{pmatrix}}.$$

Using

$$\partial_x p_m(t) = p_{m-1}(t), \quad x = t_1$$

(see (3.5.4) above) one observes that the numerator in the previous formula is the x -derivative of the denominator. So

$$\xi_1 = -\partial_x \log p_{n_1, n_2, \dots, n_N}(t).$$

As $u = -2\partial_x a_1$, this completes the proof.

In order to establish the relationship between our definition of Schur polynomials and the standard one used in the theory of symmetric functions let us introduce a change of independent variables

$$(x_1, \dots, x_N) \mapsto (t_1, \dots, t_N) \tag{3.5.8}$$

$$t_m = \frac{1}{m} \sum_{s=1}^N x_s^m$$

(in the theory of KP known as *Miwa variables*).

Lemma 3.5.3. *The transformation (3.5.8) is a local diffeomorphism on the space*

$$\{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Proof. The determinant of the Jacobi matrix

$$\frac{\partial t_m}{\partial x_s} = x_s^{m-1}$$

is the Vandermonde determinant

$$\det \left(\frac{\partial t_m}{\partial x_s} \right) = \prod_{s < t} (x_s - x_t) \neq 0.$$

The generation function (3.5.3) for the elementary Schur polynomials in the Miwa variables will read

$$\sum_{j \geq 0} p_j(t) k^j = e^{\sum_{m \geq 1} t_m k^m} = \prod_{s=1}^N e^{\sum_{m \geq 1} \frac{1}{m} (k x_s)^m} = \prod_{s=1}^N \frac{1}{1 - k x_s}. \tag{3.5.9}$$

This formula realizes elementary Schur polynomials $p_1(t), \dots, p_N(t)$ as symmetric functions in x_1, \dots, x_N . The general Schur polynomials (3.5.6) are also symmetric functions in x_1, \dots, x_N . The explicit formula for these symmetric functions is given by the following

Exercise 3.5.4: Prove the following formula for the general Schur polynomials

$$p_{n_1, \dots, n_N}(t) = \frac{\det \begin{pmatrix} x_1^{n_1} & x_2^{n_1} & \dots & x_N^{n_1} \\ x_1^{n_2} & x_2^{n_2} & \dots & x_N^{n_2} \\ \dots & \dots & \dots & \dots \\ x_1^{n_N} & x_2^{n_N} & \dots & x_N^{n_N} \end{pmatrix}}{\det \begin{pmatrix} x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \\ x_1^{N-2} & x_2^{N-2} & \dots & x_N^{N-2} \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix}} \quad (3.5.10)$$

The formula (3.5.10) defines an important class of symmetric functions in the variables x_1, \dots, x_N . In order to establish a correspondence with our notations let us introduce nonnegative numbers

$$d_1 \geq d_2 \geq \dots \geq d_N$$

associated with an order decreasing set of nonnegative integers

$$n_1 > n_2 > \dots > n_N \geq 0$$

by

$$d_i = n_i - (N - i), \quad i = 1, \dots, N.$$

We also associate a Young tableau with d_1 boxes in the first row, d_2 boxes in the second row etc. According to the standard notations (see, e.g. [?]) adopted in the theory of symmetric functions the Schur polynomials (3.5.10) are labeled by Young tableaux (or, equivalently, by partitions $\{d_1, \dots, d_N\}$ of the number $d = d_1 + \dots + d_N$). They are obtained by antisymmetrization of the monomial

$$x_1^{d_1+N-1} x_2^{d_2+N-2} \dots x_N^{d_N}$$

and subsequent division by the Vandermonde determinant, so

$$S_{\{d_1, \dots, d_N\}}(x) = \frac{\det \begin{pmatrix} x_1^{d_1+N-1} & x_2^{d_1+N-1} & \dots & x_N^{d_1+N-1} \\ x_1^{d_2+N-2} & x_2^{d_2+N-2} & \dots & x_N^{d_2+N-2} \\ \dots & \dots & \dots & \dots \\ x_1^{d_N} & x_2^{d_N} & \dots & x_N^{d_N} \end{pmatrix}}{\det \begin{pmatrix} x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \\ x_1^{N-2} & x_2^{N-2} & \dots & x_N^{N-2} \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix}} \quad (3.5.11)$$

Exercise 3.5.5: Prove that the degree of the symmetric polynomial (3.5.11) is equal to

$$\deg S_{\{d_1, \dots, d_N\}}(x) = d_1 + \dots + d_N.$$

For example,

$$S_{m,0,\dots,0} = p_m, \quad S_{1,1,\dots,1} = x_1 \dots x_N.$$

The function $S_{\{d_1,\dots,d_N\}}(x)$ coincides with the characters of the linear representations of the general linear group $GL(N)$ labeled by the Young tableau with $d_1 \geq d_2 \geq \dots \geq d_N$ rows. Restricting the representation from $GL(N)$ onto the unitary subgroup $U(N) \subset GL(N)$ one obtains the representation uniquely defined by the highest weight

$$\text{diag}(\lambda_1, \dots, \lambda_N) \mapsto \lambda_1^{d_1} \dots \lambda_N^{d_N}$$

(see details in [?]). The unexpected connection between the representation theory of Lie groups and the theory of integrable systems was an important starting point for the infinite dimensional Grassmannian description of the KP hierarchy and its generalizations. We will briefly consider this theory in the next section.

3.6 Sato formulation of KP hierarchy and tau-functions

We have already outlined the scheme of including higher times in the theory of BA functions and constructing the algebro-geometric solutions to the so-called KP hierarchy. Here we represent in a more compact way the recursion relations of the KP hierarchy and introduce the important notion of *tau-function* associated with any solution to this hierarchy.

It will be convenient to use the language of *pseudodifferential operators*. By definition a pseudodifferential operator of order $n \in \mathbb{Z}$ is a symbol

$$A = \sum_{i > -\infty}^n a_i(x) \partial_x^i \tag{3.6.1}$$

where the coefficients $a_i(x)$ are arbitrary smooth functions in x on some interval of real line. The product of two pseudodifferential operators of orders m and n respectively is a pseudodifferential operator of order $m+n$ uniquely defined by the natural product structure of smooth functions and by the following commutation rule of the operator ∂_x^k and the operator of multiplication by a smooth function $f = f(x)$

$$\partial_x^k f = \sum_{i \geq 0} \binom{k}{i} f^{(i)} \partial_x^{k-i}. \tag{3.6.2}$$

Here the binomial coefficients are defined for any integer $k \in \mathbb{Z}$ by

$$\binom{k}{i} := \frac{k(k-1)\dots(k-i+1)}{i!}. \tag{3.6.3}$$

For positive integer $k > 0$ the sum truncates. It coincides then with the classical Leibnitz formula.

The resulting algebra of pseudodifferential operators will be denoted ΨDO . The subset

$$G = \left\{ 1 + \sum_{i \geq 1} g_i(x) \partial_x^{-i} \right\} \subset \Psi DO \tag{3.6.4}$$

is a subgroup. We also introduce a subset of *differential operators*

$$DO = \left\{ \sum_{i \geq 0}^{i \ll +\infty} a_i(x) \partial_x^i \right\} \subset \Psi DO. \quad (3.6.5)$$

This subset is closed with respect to the product. For any pseudodifferential operator A denote

$$A_+ \in DO \quad (3.6.6)$$

its differential part and put

$$A_- = A - A_+. \quad (3.6.7)$$

The operator A_- contains only negative powers of ∂_x .

We will now introduce a module of formal BA functions (FBA-module, to be short) over the algebra ΨDO . The elements of this module are written as formal Laurent series in $1/k$, where k is an indeterminant, with coefficients in smooth functions in x , multiplied by the exponential e^{kx}

$$FBA = \left\{ \psi = \left(\sum_{i \ll +\infty} \xi_i(x) k^i \right) e^{kx} \right\}. \quad (3.6.8)$$

The symbol $i \ll +\infty$ means that the summation truncates for some positive value of the index i . The action

$$\Psi DO \times FBA \rightarrow FBA$$

is defined by the formula

$$\left(\sum_{i \ll +\infty} a_i(x) \partial_x^i \right) e^{kx} = \left(\sum_{i \ll +\infty} a_i(x) k^i \right) e^{kx}. \quad (3.6.9)$$

Lemma 3.6.1. *The map*

$$\Psi DO \ni A \mapsto A e^{kx} \in FBA$$

establishes an isomorphism of the linear spaces

$$\Psi DO \rightarrow FBA.$$

Proof is obvious.

Let us now consider an element of the FBA module depending on parameters t_2, t_3, \dots of the form

$$\psi(\mathbf{t}, k) = \left(1 + \frac{\xi_1(\mathbf{t})}{k} + \frac{\xi_2(\mathbf{t})}{k^2} + \dots \right) e^{k t_1 + k^2 t_2 + \dots} \in FBA \otimes \mathbb{C}[[t_2, t_3, \dots]] \quad (3.6.10)$$

where

$$\mathbf{t} = (t_1, t_2, \dots), \quad t_1 = x.$$

The derivations

$$\frac{\partial}{\partial t_n} : FBA \otimes \mathbb{C}[[t_2, t_3, \dots]] \rightarrow FBA \otimes \mathbb{C}[[t_2, t_3, \dots]]$$

act on ψ -function in an obvious way. We want to formulate the conditions that ψ satisfies the linear equations of the KP hierarchy.

Denote

$$W = 1 + \xi_1 \partial_x^{-1} + \xi_2 \partial_x^{-2} + \dots \in FBA \otimes \mathbb{C}[[t_2, t_3, \dots]] \quad (3.6.11)$$

the pseudodifferential operator associated with the function ψ according to Lemma 3.6.1, i.e.,

$$\psi = W e^{k t_1 + k^2 t_2 + \dots}. \quad (3.6.12)$$

The function ψ is an eigenvector of the operator

$$L = W \partial_x W^{-1} = \partial_x + \sum_{i \geq 1} u_i(\mathbf{t}) \partial_x^{-i} \quad (3.6.13)$$

i.e.,

$$L \psi = k \psi. \quad (3.6.14)$$

The functions u_1, u_2, \dots are certain polynomials in ξ_1, ξ_2 etc. and their x -derivatives. In particular,

$$u_1 = -\partial_x \xi_1, \quad u_2 = -\partial_x \left(\xi_2 - \frac{1}{2} \xi_1^2 \right) \quad (3.6.15)$$

etc.

Let us introduce differential operators A_1, A_2, \dots by taking the differential part of L, L^2, \dots :

$$A_n := (L^n)_+, \quad n = 1, 2, \dots \quad (3.6.16)$$

In particular,

$$A_1 = \partial_x, \quad A_2 = \partial_x^2 + 2 u_1, \quad A_3 = \partial_x^3 + 3 u_1 \partial_x + 3 u_2 + 3 u_1' \quad (3.6.17)$$

etc. (here and below we use short notation $f' := \partial_x f$ for the x -derivative of a function f). Using formulae (3.6.15) one can express the coefficients of these operators via differential polynomials in ξ_1, ξ_2 etc. The meaning of these expressions is clear from the following statement.

Lemma 3.6.2. *Given a formal BA function (3.6.10), assume that the operator A_n for some positive integer n is constructed according to the procedure (3.6.11) - (3.6.16). Then the expression*

$$\frac{\partial \psi}{\partial t_n} - A_n \psi \in FBA \otimes \mathbb{C}[[t_2, t_3, \dots]]$$

has the form

$$\frac{\partial \psi}{\partial t_n} - A_n \psi = O\left(\frac{1}{k}\right) e^{k t_1 + k^2 t_2 + \dots}. \quad (3.6.18)$$

Proof. Differentiating the formula (3.6.12) with respect to the parameter t_n one obtains

$$\frac{\partial \psi}{\partial t_n} = (W_{t_n} W^{-1} + k^n) \psi.$$

Using (3.6.14) one rewrites the last formula as follows

$$\frac{\partial \psi}{\partial t_n} = (W_{t_n} W^{-1} + L^n) \psi = (A_n + B) \psi$$

where the operator

$$B := W_{t_n} W^{-1} + (L^n)_-$$

contains only negative powers of ∂_x . Clearly,

$$B \psi = O\left(\frac{1}{k}\right) e^{k t_1 + k^2 t_2 + \dots}.$$

The Lemma is proved.

Definition. We say that the ψ -function (3.6.12) satisfies the n -th equation of KP hierarchy if

$$\frac{\partial \psi}{\partial t_n} = A_n \psi, \quad (3.6.19)$$

where the operator A_n is constructed by the procedure (3.6.11) - (3.6.16).

One can consider the above definition as construction of a vector field on the space of ψ -functions of the form (3.6.12) (or, equivalently, on the space of pseudodifferential operators W of the form (3.6.11)). From the proof of the Lemma one obtains an alternative representation of this vector field:

$$\frac{\partial W}{\partial t_n} = -(L^n)_- W. \quad (3.6.20)$$

Let us denote

$$B_n := (L^n)_-, \quad n = 1, 2, \dots \quad (3.6.21)$$

Lemma 3.6.3. *The derivative of the operator L of the form (3.6.13) along the n -th equation of the KP hierarchy is given by the following Lax equation*

$$\frac{\partial L}{\partial t_n} = [A_n, L]. \quad (3.6.22)$$

Proof. Differentiating $L = W \partial_x W^{-1}$ with respect to t_n and using the formula

$$\partial(W^{-1}) = -W^{-1} \partial W W^{-1}$$

for a derivative of the inverse operator we obtain

$$\frac{\partial L}{\partial t_n} = -B_n W \partial_x W^{-1} + W \partial_x W^{-1} B_n = [L, B_n] = [A_n, L].$$

The Lemma is proved.

Let us prove commutativity of these vector fields.

Lemma 3.6.4. *The commutator of vector fields (3.6.19) acts on the ψ -function by the operator*

$$\frac{\partial A_n}{\partial t_m} - \frac{\partial A_m}{\partial t_n} + [A_n, A_m]. \quad (3.6.23)$$

Proof is obvious.

Theorem 3.6.5. *The equations of KP hierarchy commute pairwise.*

Proof. According to the Lemma it suffices to prove that the differential operator in the left hand side of (3.6.23) vanishes for every pair of integers m, n . To this end it suffices to prove the following identity:

$$\frac{\partial A_n}{\partial t_m} - \frac{\partial A_m}{\partial t_n} + [A_n, A_m] = \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_n, B_m]$$

Indeed, the left hand side of this equation contains only nonnegative powers of ∂_x while the right hand side contains only negative powers. Hence they both are equal to zero.

In order to prove the above equation we use the following simple consequence of the Lax equation (3.6.22):

$$\frac{\partial L^n}{\partial t_m} = [A_m, L^n].$$

So,

$$\begin{aligned} \frac{\partial A_n}{\partial t_m} - \frac{\partial A_m}{\partial t_n} + [A_n, A_m] &= \frac{\partial}{\partial t_m} (L^n - B_n) - \frac{\partial}{\partial t_n} (L^m - B_m) + [L^n - B_n, L^m - B_m] \\ &= [A_m, L^n] - [A_n, L^m] - [L^n, B_m] - [B_n, L^m] + \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_n, B_m] \\ &= [A_m + B_m, L^n] - [A_n + B_n, L^m] + \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_n, B_m] \\ &= \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_n, B_m]. \end{aligned}$$

The Theorem is proved.

We are now ready to define the tau-function of a given solution to the KP hierarchy. Given a solution to the KP hierarchy

$$\begin{aligned} L\psi &= k\psi, \quad L = \partial_x + \sum_{i \geq 1} u_i \partial_x^{-i} \\ \psi &= \psi(t, k) = \left(1 + \frac{\xi_1(t)}{k} + \frac{\xi_2(t)}{k^2} + \dots \right) e^{k t_1 + k^2 t_2 + \dots} \end{aligned} \tag{3.6.24}$$

$$\frac{\partial \psi}{\partial t_n} = A_n \psi, \quad A_n = (L^n)_+, \quad n = 1, 2, \dots \tag{3.6.25}$$

$$\tag{3.6.26}$$

let us introduce the functions $h_i^{(m)}(t)$ defined from the expansions

$$\frac{\partial}{\partial t_m} \log \psi = \frac{A_m \psi}{\psi} = k^m + \sum_{i \geq 1} h_i^{(m)} k^{-i}. \tag{3.6.27}$$

Note that

$$h_1^{(1)} = \partial_x \xi_1 = -u_1 = -\frac{1}{2}u. \tag{3.6.28}$$

Moreover,

$$h_1^{(m)} = \partial_m \xi_1, \quad m \geq 1. \quad (3.6.29)$$

In the above examples we have seen that the coefficient ξ_1 admitted a representation

$$\xi_1 = -\partial_x \log \tau(\mathbf{t})$$

where $\tau(\mathbf{t})$ appeared as the determinant of the linear system determining the BA function. Therefore

$$h_1^{(m)} = -\partial_x \partial_m \log \tau(\mathbf{t}), \quad m \geq 1. \quad (3.6.30)$$

We want now to define an analogue of the function $\tau(\mathbf{t})$ associated with an arbitrary solution to the KP hierarchy, to generalize the formula (3.6.30) to this general case and, moreover, to express the ψ -function in terms of this “tau-function”.

According to the definition (3.6.27) one has

$$A_m = L^m + \sum_{i \geq 1} h_i^{(m)} L^{-i}. \quad (3.6.31)$$

In particular, for $m = 1$ the formula expresses the x -derivative via the operator L :

$$\partial_x = L + \sum_{i \geq 1} h_i^{(1)} L^{-i}. \quad (3.6.32)$$

Lemma 3.6.6. *The coefficients $h_i^{(m)}$ satisfy the following equations*

$$\frac{\partial h_i^{(m)}}{\partial t_n} = \frac{\partial h_i^{(n)}}{\partial t_m} \quad (3.6.33)$$

for any $i \geq 1$.

Proof. The needed equation follows from the symmetry of the mixed derivatives

$$\frac{\partial}{\partial t_n} \frac{\partial}{\partial t_m} \log \psi = \sum_{i \geq 1} \frac{\partial h_i^{(m)}}{\partial t_n} k^{-i}.$$

According to the Lemma, the 1-form

$$\sum_{m \geq 1} h_1^{(m)}(t) dt_m$$

is closed. Therefore, locally it is differential of a function. We denote this function $-\partial_x \log \tau(t)$, i.e.,

$$h_1^{(m)} = -\partial_x \partial_m \log \tau(t), \quad m \geq 1. \quad (3.6.34)$$

In particular, comparing with (3.6.28) we obtain that

$$u(t) = 2\partial_x^2 \log \tau(t)$$

in accordance with the above examples.

We are going to express the function $\psi(t, k)$ via tau-function.

Theorem 3.6.7. 1). For any solution $\psi(\mathbf{t}, k)$ of the KP hierarchy there exists a function $\tau = \tau(\mathbf{t})$ such that

$$h_m^{(n)} = \partial_n p_m(-\tilde{\partial}) \log \tau, \quad m, n \geq 1. \quad (3.6.35)$$

where p_m is the m -th elementary Schur polynomial and

$$\tilde{\partial} := (\partial_1, \frac{1}{2}\partial_2, \frac{1}{3}\partial_3, \dots). \quad (3.6.36)$$

2). The ψ -function in (3.6.24) is expressed via its tau-function according to the following formula

$$\psi(t, k) = \frac{\tau(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, t_3 - \frac{1}{3k^3}, \dots)}{\tau(t_1, t_2, t_3, \dots)} e^{k t_1 + k^2 t_2 + k^3 t_3 + \dots}. \quad (3.6.37)$$

Observe that for $m = 1$ the formula (3.6.35) coincides with (3.6.34). Since

$$p_m(-\tilde{\partial}) = -\frac{1}{m}\partial_m + \text{derivatives in } t_1, \dots, t_{m-1}$$

one can recursively express all the *second* logarithmic derivatives $\partial^2 \log \tau / \partial t_n \partial t_m$ as certain differential polynomials of ξ_1, ξ_2, \dots . Therefore the function $\tau(\mathbf{t})$ is defined uniquely by a solution of the KP hierarchy up to multiplication by exponential of a linear function

$$\tau(\mathbf{t}) \mapsto \tau(\mathbf{t}) e^{c_1 t_1 + c_2 t_2 + \dots} \quad (3.6.38)$$

with arbitrary constants c_1, c_2, \dots .

Definition. The function $\tau(t)$ is called the *tau-function* of the solution (3.6.24) of the KP hierarchy.

We will first derive a recursion relation for the operators A_m . For convenience we put

$$A_0 = 1.$$

Lemma 3.6.8. The operators $A_m = (L^m)_+$ satisfy recursion relation

$$A_{m+1} = \partial_x \cdot A_m - \sum_{i=1}^m h_i^{(1)} A_{m-i} - h_1^{(m)}. \quad (3.6.39)$$

Proof. From (3.6.31) and (3.6.32) we derive that

$$\begin{aligned} A_{m+1} &= L \cdot L^m + \sum_{i \geq 1} h_i^{(m+1)} L^{-i} \\ &= \left(\partial_x - \sum_{i \geq 1} h_i^{(1)} L^{-i} \right) \cdot \left(A_m - \sum_{j \geq 1} h_j^{(m)} L^{-j} \right) + \sum_{i \geq 1} h_i^{(m+1)} L^{-i} \\ &= \partial_x \cdot A_m - \sum_{i \geq 1} h_i^{(1)} L^{-i} (L^m + O(\partial_x^{-1})) - h_1^{(m)} + O(\partial_x^{-1}) \\ &= \partial_x \cdot A_m - \sum_{i=1}^m h_i^{(1)} A_{m-i} - h_1^{(m)} + O(\partial_x^{-1}). \end{aligned}$$

The term $O(\partial_x^{-1})$ in the last row vanishes since all the remaining terms are purely differential operators. The Lemma is proved.

Introducing the generating function of the operators A_m

$$\mathcal{A}(z) := \sum_{m \geq 0} \frac{A_m}{z^{m+1}} \quad (3.6.40)$$

we rewrite the recursion formula in a short way:

$$\partial_x \cdot \mathcal{A}(z) = \chi(z) \mathcal{A}(z) + \sigma(z) \quad (3.6.41)$$

where

$$\chi(z) = \frac{\partial_x \psi(z)}{\psi(z)} = z + \sum_{i \geq 1} h_i^{(1)} z^{-i}, \quad \sigma(z) = \sum_{m \geq 1} \frac{h_1^{(m)}}{z^{m+1}} - 1. \quad (3.6.42)$$

Lemma 3.6.9. *The coefficients $h_m^{(1)}$ satisfy the recursion relation*

$$m h_m^{(1)} = h_1^{(m)} - \sum_{j=1}^{m-1} \partial_j h_{m-j}^{(1)}. \quad (3.6.43)$$

Proof. Acting by the operators in the both sides of (3.6.41) on $\psi(k)$ with $|z| > |k|$ one obtains, after division by $\psi(k)$

$$\partial_x \left[\frac{\mathcal{A}(z) \psi(k)}{\psi(k)} \right] = [\chi(z) - \chi(k)] \frac{\mathcal{A}(z) \psi(k)}{\psi(k)} + \sigma(z). \quad (3.6.44)$$

Since

$$\frac{\mathcal{A}(z) \psi(k)}{\psi(k)} = \frac{1}{z-k} + \sum_{m \geq 0} \frac{\alpha_m(k)}{z^{m+1}}$$

where we denote

$$\alpha_m(k) := \frac{A_m \psi(k)}{\psi(k)} - k^m = \sum_{i \geq 1} \frac{h_i^{(m)}}{k^i}$$

it follows that

$$\partial_x \left[\sum_{m \geq 0} \frac{\alpha_m(k)}{z^{m+1}} \right] = \frac{\chi(z) - \chi(k)}{z-k} + \sigma(z) + [\chi(z) - \chi(k)] \sum_{m \geq 0} \frac{\alpha_m(k)}{z^{m+1}}.$$

In the limit $z \rightarrow k$ this yields

$$\partial_x \left[\sum_{m \geq 0} \frac{\alpha_m(k)}{k^{m+1}} \right] = \frac{d}{dk} \chi(k) + \sigma(k).$$

From the last equation it follows that

$$m h_m^{(1)} = h_1^{(m)} - \partial_x \sum h_{m-j}^{(j)}.$$

Using the symmetry (3.6.33) we complete the proof.

From the recursion relation of the Lemma we can compute all the coefficients $h_m^{(1)}$. In particular,

$$2h_2^{(1)} = h_1^{(2)} - \partial_x h_1^{(1)} = -\partial_x \partial_2 \log \tau + \partial_x^3 \log \tau,$$

so

$$h_2^{(1)} = \frac{1}{2} \partial_1 (\partial_1^2 - \partial_2) \log \tau;$$

similarly,

$$h_3^{(1)} = \partial_1 \left(-\frac{1}{3} \partial_3 + \frac{1}{2} \partial_1 \partial_2 - \frac{1}{6} \partial_1^3 \right) \log \tau$$

etc. Now it is easy to derive (3.6.35) for $n = 1$ comparing the recursion relations (3.6.43) and (3.5.5). Using the symmetry (3.6.33) one extends to an arbitrary $n \geq 1$.

The proof of Theorem 3.6.7 readily follows from the the formula (3.6.35) since

$$\begin{aligned} \log \left(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, t_3 - \frac{1}{3k^3}, \dots \right) &= e^{-\frac{1}{k} \partial_1 - \frac{1}{2k^2} \partial_2 - \frac{1}{3k^3} \partial_3 - \dots} \log \tau(t) \\ &= \sum_{m \geq 0} k^{-m} p_m(-\tilde{\partial}) \log \tau(t). \end{aligned}$$

Example. For the N -soliton ψ -function (3.4.18) determined by the linear constraints (3.4.17) the formula (3.6.37) of Theorem 3.6.7 yields

$$\psi(\mathbf{t}; k) = \frac{\det \left(\delta_{ij} + \rho_i \frac{k-p_i}{k-q_j} \frac{e^{\lambda_i - \mu_j}}{p_i - q_j} \right)}{\det \left(\delta_{ij} + \rho_i \frac{e^{\lambda_i - \mu_j}}{p_i - q_j} \right)} e^{k t_1 + k^2 t_2 + k^3 t_3 + \dots} \quad (3.6.45)$$

(the notations are the same as in Exercise 3.4.6).

Exercise 3.6.10: Check directly the formula (3.6.45) by verifying validity of the linear constraints (3.4.17).

Exercise 3.6.11: Check by direct computation that the ψ -function

$$\psi(\mathbf{t}, k) = k^N \frac{p_{n_1, \dots, n_N} \left(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, \dots, t_N - \frac{1}{Nk^N} \right)}{p_{n_1, \dots, n_N}(t_1, \dots, t_N)} e^{k t_1 + k^2 t_2 + \dots}$$

satisfies the linear constraints (3.5.2) used in the construction of rational solutions to KP.

Hint: First prove the following identity for elementary Schur polynomials

$$p_m \left(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, \dots, t_N - \frac{1}{Nk^N} \right) = p_m(t_1, \dots, t_N) - \frac{1}{k} p_{m-1}(t_1, \dots, t_N).$$

We will now consider our main class of KP solutions obtained from Baker - Akhiezer functions on a Riemann surface Γ of genus g with a marked point $Q \in \Gamma$ and a chosen local

parameter $z = k^{-1}$ defined on a neighborhood of Q . Such a BA function $\psi(\mathbf{t}; P)$ is uniquely determined by a non-special divisor D of degree g . It has the asymptotic

$$\psi(\mathbf{t}; P) = \left(1 + \frac{\xi_1(\mathbf{t})}{k} + \frac{\xi_2(\mathbf{t})}{k^2} + \dots \right) e^{k t_1 + k^2 t_2 + k^3 t_3 + \dots}. \quad (3.6.46)$$

In order to express the BA function via theta-functions one has to fix a canonical basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g \in H_1(\Gamma, \mathbb{Z})$. Then the BA function reads

$$\psi(\mathbf{t}; P) = c(\mathbf{t}) e^{\sum t_i \int_{Q_0}^P \Omega_i} \frac{\theta(A(P) + \sum t_i U_i - \zeta)}{\theta(A(P) - \zeta)} \quad (3.6.47)$$

Here $\Omega_i = \Omega_i(P)$ are normalized Abelian differentials of the second kind with poles of the order $i + 1$ at Q ,

$$\Omega_i(P) = -i \frac{dz}{z^{i+1}} + \text{regular terms}, \quad P \rightarrow Q \quad (3.6.48)$$

$$\oint_{a_1} \Omega_i = 0, \dots, \oint_{a_g} \Omega_i = 0.$$

(In the previous notations $\Omega_i = -i \Omega_Q^{(i)}$.) Observe that

$$\int_{Q_0}^P \Omega_i = k^i(P) + \alpha_i + O\left(\frac{1}{k}\right), \quad P \rightarrow Q, \quad i = 1, 2, \dots \quad (3.6.49)$$

for some constants α_i . The vectors U_i are made from the b -periods of these differentials,

$$(U_i)_1 = \oint_{b_1} \Omega_i, \dots, (U_i)_g = \oint_{b_g} \Omega_i. \quad (3.6.50)$$

The Abel map

$$A : \Gamma \rightarrow J(\Gamma)$$

is defined by

$$A(P) = \left(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_g \right) \quad (3.6.51)$$

with normalized holomorphic differentials $\omega_1, \dots, \omega_g$ inside,

$$\oint_{a_k} \omega_j = 2\pi i \delta_{jk}. \quad (3.6.52)$$

The base point Q_0 and the integration paths from Q_0 to P in (3.6.47) and in (3.6.52) have to be the same. The constant vector ζ contains the information about the divisor D ,

$$\zeta = A(D) + \mathcal{K}_{Q_0} \quad (3.6.53)$$

where \mathcal{K}_{Q_0} is the vector of Riemann constants. Finally, the normalizing factor $c(\mathbf{t})$ is defined by

$$c(\mathbf{t}) = e^{-\sum_i \alpha_i t_i} \frac{\theta(A(Q) - \zeta)}{\theta(A(Q) + \sum t_i U_i - \zeta)}. \quad (3.6.54)$$

The coefficients α_i are defined in (3.6.49).

Remark 3.6.12. One can choose $Q_0 = Q$ as a base point. In that case the integrals in the exponential factor must be replaced by the principal value of the divergent integral:

$$\psi(\mathbf{t}; P) = c(\mathbf{t}) e^{\sum t_i \int_Q^P \Omega_i} \frac{\theta(A(P) + \sum t_i U_i - \zeta)}{\theta(A(P) - \zeta)} \quad (3.6.55)$$

$$\int_Q^P \Omega_i := \lim_{Q_0 \rightarrow Q} \left[\int_{Q_0}^P \omega_i + k^i(Q_0) \right]$$

$$A(P) = \int_Q^P \omega, \quad \zeta = A(D) + \mathcal{K}_Q$$

$$c(\mathbf{t}) = \frac{\theta(\zeta)}{\theta(\sum t_i U_i - \zeta)}.$$

For the sake of simplicity of the formulae we will stick to this choice of the base point.

Lemma 3.6.13. *The Abel map has the following Taylor expansion at $P \rightarrow Q$:*

$$A(P) = - \sum_{i \geq 1} \frac{1}{i k^i} U_i, \quad k = k(P). \quad (3.6.56)$$

Proof immediately follows from (2.7.11).

Let us define the infinite matrix a_{ij} from the Laurent expansions of the second kind Abelian integrals

$$\int_Q^P \Omega_i = k^i - \sum_{j \geq 1} \frac{a_{ij}}{j k^j}, \quad k = k(P), \quad P \rightarrow Q, \quad i = 1, 2, \dots \quad (3.6.57)$$

Lemma 3.6.14. *The matrix a_{ij} is symmetric.*

Proof follows from the result of Exercise 2.7.11.

Theorem 3.6.15. *The tau-function of the solution (3.6.55) reads*

$$\tau(\mathbf{t}) = e^{\frac{1}{2} \sum a_{ij} t_i t_j} \theta\left(\sum t_i U_i - \zeta\right). \quad (3.6.58)$$

Exercise 3.6.16: Let

$$k(\tilde{k}) = c_{-1} \tilde{k} + c_0 + \sum_{i \geq 1} \frac{c_i}{\tilde{k}^i}$$

be an invertible change of the local parameter, i.e., $c_{-1} \neq 0$. Introduce the following two matrices

$$b_j^i = \operatorname{Res}_{k=\infty} \frac{k^i}{\tilde{k}^j} \frac{d\tilde{k}}{\tilde{k}} \quad (3.6.59)$$

$$\Delta_{ij} = \operatorname{Res}_{k=\infty} [(k^i)_+ dk^j] \quad (3.6.60)$$

where $(\cdot)_+$ denotes the polynomial part with respect to \tilde{k} . Prove that the matrix Δ_{ij} is symmetric. Prove that the change of the local parameter yields the following transformation of the tau-function

$$\tilde{t}_i = \sum_{j \geq i} b_i^j t_j \quad (3.6.61)$$

$$\tilde{\tau}(\tilde{\mathbf{t}}) = e^{\frac{1}{2} \sum \Delta_{ij} t_i t_j} \tau(\mathbf{t}).$$

Exercise 3.6.17: Since any two pseudodifferential operators with *constant* coefficients commute, a pseudodifferential operator

$$L = \partial_x + \sum_{i \geq 1} u_i \partial_x^{-i}$$

with constant coefficients always satisfies KP hierarchy. Prove that the tau-function of this solution has the form

$$\tau(\mathbf{t}) = e^{-\frac{1}{2} \sum \Delta_{ij} t_i t_j} \quad (3.6.62)$$

where

$$\Delta_{ij} = \operatorname{Res}_{k=\infty} [(k^i)_+ dk^j] \quad (3.6.63)$$

and $k = k(\tilde{k})$ is the series inverse to

$$\tilde{k} = k + \sum_{i \geq 1} \frac{u_i}{k^i}. \quad (3.6.64)$$

The polynomial part $(k^i)_+$ in (3.6.63) is taken with respect to the variable \tilde{k} (cf. the previous exercise).

Since all unknowns of the KP hierarchy are expressed via tau-function and its derivatives, the equations of the hierarchy themselves can be recast into the form of differential equations for a *single* function $\tau(t_1, t_2, \dots)$. Indeed, substituting

$$u = 2\partial_x^2 \log \tau, \quad w = \frac{3}{2} \partial_x \partial_y \log \tau$$

into KP system (3.2.14) one obtains

$$\partial_x \left(\frac{A}{\tau^2} \right) = 0, \quad A = \tau \tau_{xxxx} - 4\tau_x \tau_{xxx} + 3\tau_{xx}^2 - 4(\tau \tau_{xt} - \tau_x \tau_t) + 3(\tau \tau_{yy} - \tau_y^2).$$

Integrating one obtains therefore a quadratic equation for τ and its derivatives

$$\tau \tau_{xxxx} - 4\tau_x \tau_{xxx} + 3\tau_{xx}^2 - 4(\tau \tau_{xt} - \tau_x \tau_t) + 3(\tau \tau_{yy} - \tau_y^2) - c\tau^2 = 0 \quad (3.6.65)$$

where c is an integration constant.

There is a remarkable way to represent all equations of the KP hierarchy in a bilinear form. To do it we need to introduce the notion of *dual* formal BA function. Let us first define an antiisomorphism

$$* : \Psi DO \rightarrow \Psi DO, \quad \partial_x^* = -\partial_x, \quad (AB)^* = B^* A^*. \quad (3.6.66)$$

Introduce the dual formal BA function by

$$\psi^*(\mathbf{t}, k) := W^{*-1} e^{-kt_1 - k^2 t_2 - \dots} = \left(1 + \frac{\xi_1^*}{k} + \frac{\xi_2^*}{k^2} + \dots\right) e^{-kt_1 - k^2 t_2 - \dots} \quad (3.6.67)$$

where the coefficients ξ_1^* , ξ_2^* etc. are defined by this equation, i.e.,

$$W^{*-1} = 1 + \sum_{i \geq 1} \xi_i^* \partial_x^{-i}.$$

Lemma 3.6.18.

$$\psi^*(\mathbf{t}, k) = \frac{\tau\left(t_1 + \frac{1}{k}, t_2 + \frac{1}{2k^2}, t_3 + \frac{1}{3k^3}, \dots\right)}{\tau(t_1, t_2, t_3, \dots)} e^{-k t_1 - k^2 t_2 - k^3 t_3 + \dots}. \quad (3.6.68)$$

Theorem 3.6.19. *The equations of KP hierarchy are equivalent to the following bilinear equation*

$$\operatorname{Res}_{k=\infty} \psi(\mathbf{t}, k) \psi^*(\mathbf{t}', k) dk = 0 \quad (3.6.69)$$

for any \mathbf{t}, \mathbf{t}' .

Let us explain how to rewrite the KP equations in terms of τ using the bilinear equations (3.6.69). We have to spell out the equation

$$\operatorname{Res}_{k=\infty} \tau\left(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, \dots\right) \tau\left(t'_1 + \frac{1}{k}, t'_2 + \frac{1}{2k^2}, \dots\right) e^{k(t_1 - t'_1) + k^2(t_2 - t'_2) + \dots} dk = 0.$$

The substitution

$$t_i = x_i - y_i, \quad t'_i = x_i + y_i, \quad i = 1, 2, \dots$$

yields

$$\begin{aligned} & \operatorname{Res}_{k=\infty} \tau\left(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, \dots\right) \tau\left(t'_1 + \frac{1}{k}, t'_2 + \frac{1}{2k^2}, \dots\right) e^{k(t_1 - t'_1) + k^2(t_2 - t'_2) + \dots} dk \\ &= \operatorname{Res}_{k=\infty} \tau\left(x_1 - y_1 - \frac{1}{k}, x_2 - y_2 - \frac{1}{2k^2}, \dots\right) \tau\left(x_1 + y_1 + \frac{1}{k}, x_2 + y_2 + \frac{1}{2k^2}, \dots\right) e^{-2k y_1 - 2k^2 y_2 - \dots} dk \\ &= \operatorname{Res}_{k=\infty} \left[e^{\sum k^{-i} \frac{\partial}{\partial y_i}} \tau(x - y) \tau(x + y) \right] e^{-2k y_1 - 2k^2 y_2 - \dots} dk \\ &= \operatorname{Res}_{k=\infty} \left[\sum k^{-i} p_i(\tilde{\partial}_y) \tau(x + y) \tau(x - y) \right] \left[\sum k^j p_j(-2y) \right] dk \\ &= \sum_{j=0}^{\infty} p_j(-2y) p_{j+1}(\tilde{\partial}_y) \tau(x + y) \tau(x - y) \\ &= \sum_{j=0}^{\infty} p_j(-2y) e^{\sum y_i \frac{\partial}{\partial z_i}} p_{j+1}(\tilde{\partial}_z) \tau(x + z) \tau(x - z) \Big|_{z=0} \end{aligned}$$

where we introduce auxiliary variables $z = (z_1, z_2, \dots)$ and put, as above,

$$\tilde{\partial}_{z_n} = \frac{1}{n} \frac{\partial}{\partial z_n}.$$

We want now to introduce the following notation (the so-called Hirota bilinear operators). Given a function $f = f(x)$ and a linear differential operator $P(\partial_x)$ define the bilinear operator $P(D_x) f \cdot f$ as follows

$$P(D_x) f \cdot f := P(\partial_y) f(x+y) f(x-y)|_{y=0}. \quad (3.6.70)$$

Using this notation we can recast the result of the previous calculation into the form

$$\sum_{j=0}^{\infty} p_j (-2y) e^{\sum y_i D_i} p_{j+1}(\tilde{D}) \tau \cdot \tau = 0. \quad (3.6.71)$$

Here $D = (D_1, D_2, \dots)$ is the infinite vector of Hirota bilinear operators associated with $\partial/\partial t_1, \partial/\partial t_2$ etc.,

$$\tilde{D}_n = \frac{1}{n} D_n.$$

Expanding the left hand side of (3.6.71) in power series in y_1, y_2, \dots and equating to zero the coefficients of independent monomials one obtains the bilinear form of equations of KP hierarchy. The first two equations read

$$\begin{aligned} (D_1^4 + 3D_2^2 - 4D_1 D_3) \tau \cdot \tau &= 0 \\ (D_1^3 D_2 - 3D_1 D_4 + 2D_2 D_3) \tau \cdot \tau &= 0. \end{aligned}$$

3.7 “Infinite genus” extension: Wronskian solutions to KP, orthogonal polynomials and random matrices

We will now consider the limiting case $M \rightarrow \infty$ of the linear constraints (3.4.2) assuming all the multiplicities m_j to be equal to 0. The sums become integrals

$$\int_{\gamma_i} \alpha_i(k) \psi(k) dk = 0, \quad i = 1, \dots, N \quad (3.7.1)$$

over some curves on the complex plane; the functions $\alpha_i(k)$ are chosen in such a way to ensure existence of the integrals (3.7.1). The associated solution to KP can be written in a nice form in terms of the Wronskian of the following functions

$$\varphi_i(t) = \int_{\gamma_i} \alpha_i(k) e^{k t_1 + k^2 t_2 + \dots} dk, \quad i = 1, \dots, N. \quad (3.7.2)$$

We also denote the x -derivatives of these functions by

$$\varphi_i^{(m)}(t) := \partial_x^m \varphi_i(t), \quad x = t_1.$$

Theorem 3.7.1. *Let us assume that the Wronskian of the functions (3.7.2)*

$$W(t) := \det \begin{pmatrix} \varphi_1(t) & \varphi_2(t) & \dots & \varphi_N(t) \\ \varphi_1'(t) & \varphi_2'(t) & \dots & \varphi_N'(t) \\ \dots & \dots & \dots & \dots \\ \varphi_1^{(N-1)}(t) & \varphi_2^{(N-1)}(t) & \dots & \varphi_N^{(N-1)}(t) \end{pmatrix} \quad (3.7.3)$$

does not vanish identically in $t = (t_1, t_2, \dots)$. Then the functions

$$u(t) = 2 \partial_x^2 \log W(t), \quad w(t) = \frac{3}{2} \partial_x \partial_y \log W(t) \quad (3.7.4)$$

satisfies KP.

Proof. Since

$$\int_{\gamma_i} k^m \alpha_i(k) \psi(t; k) dk = \varphi_i^{(m)}(t),$$

the i -th equation of the linear system for the coefficients $a_1(t), \dots, a_N(t)$ of the degenerate BA function (3.5.1) reads

$$\varphi_i^{(N-1)} a_1 + \varphi_i^{(N-2)} a_2 + \dots + \varphi_i a_N = -\varphi_i^{(N)}, \quad i = 1, \dots, N.$$

Solving by Cramer rule this system we obtain for the coefficient $a_1(t)$ the expression

$$a_1(t) = -\partial_x \log W(t).$$

This proves the theorem.

Observe that the functions $\varphi_i(t)$ for every i satisfy the linear differential equations with constant coefficients of a very simple form

$$\frac{\partial \varphi_i(t)}{\partial t_m} = \frac{\partial^m \varphi_i(t)}{\partial x^m}, \quad m = 1, 2, \dots \quad (3.7.5)$$

In other words, the functions $\varphi_i(t)$ satisfy the linear differential equations of the KP theory

$$\frac{\partial \varphi_i(t)}{\partial t_m} = L_m^0 \phi_i(t) \quad (3.7.6)$$

with

$$L_m^0 = \partial_x^m, \quad m \geq 1. \quad (3.7.7)$$

Exercise 3.7.2: Given N arbitrary solutions to the system (3.7.5) with nonvanishing Wronskian (3.7.3), prove that the function (3.7.4) satisfies KP.

We will now apply the above trick to the following particular situation. Let us choose the system of linear constraints for the function

$$\psi_N(k) = (k^N + a_1 k^{N-1} + \dots + a_N) e^{k t_1 + k^2 t_2 + \dots} \quad (3.7.8)$$

in the form

$$\int_{-\infty}^{\infty} \psi(k)\mu(k) dk = 0, \quad \int_{-\infty}^{\infty} k \psi(k)\mu(k) dk = 0, \dots, \int_{-\infty}^{\infty} k^{N-1} \psi(k)\mu(k) dk = 0. \quad (3.7.9)$$

where the function $\mu(k)$ is chosen in such a way to ensure convergence of the integrals. More specific choice of the function $\mu(k)$ is given by

$$\mu(k) = e^{v(k)} \quad (3.7.10)$$

for a suitable polynomial $v(k)$.

The procedure can be repeated for various values of N keeping fixed the measure $\mu(k)dk$. The resulting solutions to KP satisfy the following interesting property.

Exercise 3.7.3: Representing

$$\psi_N(k) = P_N(k) e^{k t_1 + k^2 t_2 + \dots} \quad (3.7.11)$$

prove that the orthogonality relation

$$\int_{-\infty}^{\infty} P_i(k) P_j(k) e^{V(k)} dk = 0 \quad \text{for } i \neq j. \quad (3.7.12)$$

Here we denote

$$V(k) := k t_1 + k^2 t_2 + \dots + v(k). \quad (3.7.13)$$

In order to spell out the Wronskian formula (3.7.4) let us introduce the function

$$\phi(t) := \int_{-\infty}^{\infty} \mu(k) e^{k t_1 + k^2 t_2 + k^3 t_3 + \dots} dk = \int_{-\infty}^{\infty} e^{V(k)} dk. \quad (3.7.14)$$

Lemma 3.7.4. *The Wronskian (3.7.3) associated with the linear constraints (3.7.9) reads*

$$W_N(t) = \det \begin{pmatrix} \phi & \phi' & \dots & \phi^{(N-1)} \\ \phi' & \phi'' & \dots & \phi^{(N)} \\ \dots & \dots & \dots & \dots \\ \phi^{(N-1)} & \phi^{(N)} & \dots & \phi^{(2N-2)} \end{pmatrix}. \quad (3.7.15)$$

Proof. Indeed, the functions (3.7.2) read

$$\varphi_i(t) = \int_{-\infty}^{\infty} k^{i-1} e^{V(k)} dk = \partial_x^{i-1} \phi(t).$$

Exercise 3.7.5: Given an arbitrary function $\phi = \phi(x)$, prove that the Wronskians of the form (3.7.15) satisfy the following equation

$$\partial_x^2 \log W_N = \frac{W_{N+1} W_{N-1}}{W_N^2}. \quad (3.7.16)$$

We will return to (3.7.16) when considering the Toda lattice equations.

We will now show that the Wronskian solution to KP defined by the linear constraints (3.7.9) can be expressed in terms of Hermitean matrix integrals.

Denote

$$\mathcal{H}_N = \{H = (H_{ij}) \in \text{Mat}(N, \mathbb{C}) \mid H^* = H\}$$

the space of all $N \times N$ Hermitean matrices. It is a linear space of the dimension N^2 . Denote dH the Lebesgue measure on this space:

$$dH = \prod_{i=1}^N dH_{ii} \prod_{i<j} (d\text{Re}H_{ij} d\text{Im}H_{ij}) = \left(\frac{i}{2}\right)^{\frac{N(N-1)}{2}} \prod_{i=1}^N dH_{ii} \prod_{i<j} (dH_{ij} d\bar{H}_{ij}) \quad (3.7.17)$$

(the bar stands for the complex conjugation). Let us consider the matrix integral of the form

$$Z_N(t) = \int_{\mathcal{H}_N} e^{\text{tr} V(H)} dH. \quad (3.7.18)$$

Theorem 3.7.6. *The tau-function (3.7.14), (3.7.15) coincides, up to a t -independent factor, with the matrix integral (3.7.18):*

$$Z_N(t) = c_N W_N(t). \quad (3.7.19)$$

Corollary 3.7.7. *The functions*

$$u(t) = 2 \partial_x^2 \log Z_N(t), \quad w(t) = \frac{3}{2} \partial_x \partial_y \log Z_N(t) \quad (3.7.20)$$

satisfy KP.

Proof of the Theorem. The Wronskian formula (3.7.15) reads

$$W_N(t) = \det \begin{pmatrix} \int e^{V(k)} dk & \int k e^{V(k)} dk & \dots & \int k^{N-1} e^{V(k)} dk \\ \int k e^{V(k)} dk & \int k^2 e^{V(k)} dk & \dots & \int k^N e^{V(k)} dk \\ \dots & \dots & \dots & \dots \\ \int k^{N-1} e^{V(k)} dk & \int k^N e^{V(k)} dk & \dots & \int k^{2N-2} e^{V(k)} dk \end{pmatrix}$$

(all integrals over real line). We first rename the integration variables in different columns of the above matrix to rewrite the Wronskian as a multiple integral

$$\begin{aligned} W_N(t) &= \int \det \begin{pmatrix} e^{V(k_1)} & k_2 e^{V(k_2)} & \dots & k_N^{N-1} e^{V(k_N)} \\ k_1 e^{V(k_1)} & k_2^2 e^{V(k_2)} & \dots & k_N^N e^{V(k_N)} \\ \dots & \dots & \dots & \dots \\ k_1^{N-1} e^{V(k_1)} & k_2^N e^{V(k_2)} & \dots & k_N^{2N-2} e^{V(k_N)} \end{pmatrix} dk_1 dk_2 \dots dk_N \\ &= \int k_2 k_3^2 \dots k_N^{N-1} e^{V(k_1) + \dots + V(k_N)} \prod_{i<j} (k_i - k_j) dk_1 dk_2 \dots dk_N. \end{aligned} \quad (3.7.21)$$

Here the integral is over \mathbb{R}^N . Since the original formula does not depend on the order of k_1, k_2, \dots, k_N , we can symmetrize over all permutations of these variables.

Exercise 3.7.8:

$$\sum_{\text{permutations } i_1, \dots, i_N} k_{i_2} k_{i_3}^2 \dots k_{i_N}^{N-1} \prod_{m < n} (k_{i_m} - k_{i_n}) = (-1)^{\frac{N(N-1)}{2}} \prod_{i < j} (k_i - k_j)^2. \quad (3.7.22)$$

We arrive at the following formula for the Wronskian

$$W_N(t) = \pm \frac{1}{N!} \int dk_1 dk_2 \dots dk_N e^{V(k_1) + \dots + V(k_N)} \prod_{i < j} (k_i - k_j)^2. \quad (3.7.23)$$

It remains to identify the above formula with the matrix integral (3.7.18) (within a constant factor depending on N).

Exercise 3.7.9: Denote

$$h_N(t) := \int_{-\infty}^{\infty} P_N^2(k) e^{V(k)} dk \quad (3.7.24)$$

where the polynomials $P_N(k)$ are defined as in (3.7.11). Prove that

$$W_N(t) = h_0(t) h_1(t) \dots h_{N-1}(t). \quad (3.7.25)$$

Hint: Given an arbitrary system of monic polynomials $P_0(k), P_1(k), \dots, \deg P_n(k) = n$ prove first that

$$\det \begin{pmatrix} P_0(k_1) & P_0(k_2) & \dots & P_0(k_N) \\ P_1(k_1) & P_1(k_2) & \dots & P_1(k_N) \\ \dots & \dots & \dots & \dots \\ P_{N-1}(k_1) & P_{N-1}(k_2) & \dots & P_{N-1}(k_N) \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ k_1 & k_2 & \dots & k_N \\ \dots & \dots & \dots & \dots \\ k_1^{N-1} & k_2^{N-1} & \dots & k_N^{N-1} \end{pmatrix}.$$

Use then the orthogonality (3.7.12) for evaluation of the integral (3.7.23).

It remains to identify the integral (3.7.23) with the matrix integral (3.7.18) (within a factor depending on N). The basic idea is to use an appropriate change of variables in the matrix integral. We will refer to the following well known theorem from linear algebra: every Hermitean matrix H can be represented in the form

$$H = U^\dagger K U \quad (3.7.26)$$

for a unitary matrix U ,

$$U^\dagger U = 1$$

(the dagger stands for the Hermitean conjugation, $U^\dagger := \bar{U}^t$) and a real diagonal matrix K ,

$$K = \text{diag}(k_1, \dots, k_N).$$

The eigenvalues k_1, \dots, k_N of H are determined uniquely up to a permutation. For a generic H they are pairwise distinct and hence can be ordered

$$k_1 < k_2 < \dots < k_N.$$

The subset of non generic Hermitean matrices has the co-dimension 3 and thus it does not contribute into the integral.

We want to consider the transformation

$$H \mapsto (K, U), \quad K \in \mathbb{R}^N, \quad U \in U(N)$$

as a change of coordinates on the space of Hermitean matrices. However, the diagonalizing unitary matrix U is not determined uniquely. At the generic point, where all the eigenvalues k_1, \dots, k_N are pairwise distinct, the matrix U is determined up to a transformation

$$U \mapsto DU, \quad D = \text{diag} (e^{i\phi_1}, \dots, e^{i\phi_N})$$

for real phases ϕ_1, \dots, ϕ_N . Denote $Diag \subset U(N)$ the subgroup of unitary diagonal matrices and

$$Q(N) := U(N)/Diag$$

the quotient. We obtain a diffeomorphism of an open dense subset in the space of Hermitean $N \times N$ matrices with pairwise distinct eigenvalues to the direct product

$$(\mathbb{R}^N \setminus \text{diagonals}) / S_N \times Q(N), \quad H \mapsto (K, U).$$

Here S_N is the symmetric group acting by permutation of eigenvalues.

Our nearest goal is to rewrite the Lebesgue measure dH in the coordinates (K, U) . Denote dU the Haar measure on the unitary group; it can be obtained in the following way. Consider the embedding

$$U(N) \subset \mathbb{C}^{N^2} = Mat(N, \mathbb{C})$$

of the unitary group into the space of matrices. The latter is equipped with the standard Euclidean metric

$$(A, B) = \text{Re tr } A^\dagger B.$$

Rewriting it as a Riemannian metric

$$ds^2 = \sum_{i=1}^N |dA_{ii}|^2 + \sum_{i < j} |dA_{ij}|^2$$

and restricting it onto the submanifold of unitary matrices one obtains the Riemannian metric on $U(N)$ that is obviously biinvariant. Let dU be the volume element² with respect to *this* Riemannian metric. We will use the same symbol dU for the projection of the measure onto the quotient $Q(N)$.

Lemma 3.7.10.

$$dH = \prod_{i < j} (k_i - k_j)^2 dk_1 \dots dk_N dU. \quad (3.7.27)$$

Proof. We first prove the formula for the matrices U close to the identity,

$$U = 1 + iX, \quad \|X\| \ll 1.$$

²Let us recall that a Riemannian metric $ds^2 = g_{ij}(x)dx^i dx^j$ on a n -dimensional manifold defines a volume element on this manifold according to the formula $dV = \sqrt{g(x)} dx^1 \dots dx^n$, $g(x) := \det(g_{ij}(x))$. Isometries of the metric clearly preserve this volume element.

The matrix $X = (X_{ij})$ must be Hermitean. We can use the entries of X as the local coordinates on $U(N)$ in a small neighborhood of 1. The off-diagonal entries $(X_{ij})_{i < j}$ and $(\bar{X}_{ij})_{i < j}$ can serve as local coordinates on the quotient $Q(N)$ in the neighborhood of $Diag \cdot 1$. One has

$$H = U^\dagger K U = (1 - iX) K (1 + iX) = K + i[K, X] + O(\|X\|^2).$$

So,

$$H_{rr} \simeq k_r, \quad H_{rs} \simeq i(k_r - k_s) X_{rs}, \quad r \neq s.$$

Hence

$$dH_{rr} = dk_r$$

$$dH_{rs} d\bar{H}_{rs} = (k_r - k_s)^2 dX_{rs} d\bar{X}_{rs} + \dots \quad \text{for } r < s$$

where dots stand for terms containing dk_r or dk_s . We obtain

$$\prod_r dH_{rr} \prod_{r < s} dH_{rs} d\bar{H}_{rs} = \prod_{i < j} (k_i - k_j)^2 dk_1 \dots dk_N \prod_{i < j} dX_{ij} d\bar{X}_{ij}.$$

It remains to observe that the Riemannian metric on $U(N)$ in the neighborhood of $U = 1$ in the local coordinates $X_{ij}, \bar{X}_{ij}, i < j$, takes the form

$$ds^2 \simeq \sum_{i < j} dX_{ij} d\bar{X}_{ij}.$$

So, at the point $U = 1$ one has

$$dU = \prod_{i < j} dX_{ij} d\bar{X}_{ij}.$$

We have proved (3.7.27) at the point $U = 1$. To derive the same formula near a generic point $U_0 \in U(N)$ we will use invariance of the measure dU with respect to right shifts

$$U \mapsto U U_0^{-1}, \quad dU \mapsto dU.$$

The shift is a diffeomorphism $U(N) \rightarrow U(N)$ that maps a neighborhood of U_0 to a neighborhood of 1. Let us introduce a Hermitean matrix

$$H_0 := U_0 H U_0^{-1} = U_0 H U_0^\dagger. \quad (3.7.28)$$

Repeating the above arguments we prove the formula of Lemma with dH_0 instead of dH in the left hand side. It remains to prove that $dH_0 = dH$. Indeed, the transformation $H \mapsto H_0$ preserves the Euclidean structure

$$\text{tr } H^2 = \sum_i H_{ii}^2 + \sum_{i < j} H_{ij} \bar{H}_{ij}$$

on the space of Hermitean matrices:

$$\text{tr } (U_0 H U_0^{-1} U_0 H U_0^{-1}) = \text{tr } (U_0 H^2 U_0^{-1}) = \text{tr } H^2.$$

So, (3.7.28) is an isometry. Hence $dH_0 = dH$. The Lemma is proved.

Using Lemma one can rewrite the matrix integral (3.7.18) in the form

$$Z_N(t) = \int_{k_1 < k_2 < \dots < k_N} e^{V(k_1) + \dots + V(k_N)} \prod_{i < j} (k_i - k_j)^2 dk_1 \dots dk_N \cdot \int_{Q(N)} dU$$

since

$$\text{tr}(V(H)) = V(k_1) + \dots + V(k_N).$$

Due to symmetry of the integrand with respect to permutations of the variables k_1, k_N one can rewrite it as the integral over all values of these variables divided by the number of permutations

$$\begin{aligned} & \int_{k_1 < k_2 < \dots < k_N} e^{V(k_1) + \dots + V(k_N)} \prod_{i < j} (k_i - k_j)^2 dk_1 \dots dk_N \\ &= \frac{1}{N!} \int_{\mathbb{R}^N} e^{V(k_1) + \dots + V(k_N)} \prod_{i < j} (k_i - k_j)^2 dk_1 \dots dk_N. \end{aligned}$$

The last integral

$$c_N = \int_{Q(N)} dU$$

is a constant depending only on N (the volume of the compact manifold $Q(N)$). We arrive at the formula (3.7.23).

In order to compute the normalizing constant c_N let us evaluate the both sides of (3.7.23) for a particular choice of the potential $V(H) = -H^2$. The integral (3.7.23) becomes Gaussian and can be easily computed:

Exercise 3.7.11: Prove that

$$Z_N = \int e^{-\text{tr} H^2} dH = \frac{\pi^{\frac{N^2}{2}}}{2^{\frac{N(N-1)}{2}}}. \quad (3.7.29)$$

To evaluate the integral in the right hand side of the formula (3.7.23) with $V(H) = -H^2$ we will use the expression of Exercise 3.7.9 representing the integral W_N via the square norms of orthogonal polynomials:

$$\frac{1}{N!} \int e^{-k_1^2 - \dots - k_N^2} \prod_{i < j} (k_i - k_j)^2 dk_1 \dots dk_N = h_0 h_1 \dots h_{N-1} \quad (3.7.30)$$

$$h_n = \int P_n^2(k) e^{-k^2} dk$$

$$P_n(k) = k^n + \text{terms of degree} < n$$

$$\int P_i(k) P_j(k) e^{-k^2} dk = h_i \delta_{ij}.$$

Exercise 3.7.12: Prove that the polynomials $P_n(k)$ in (3.7.30) are proportional to the classical Hermite polynomials

$$P_n(k) = 2^{-n} H_n(k), \quad H_n(k) := (-1)^n e^{k^2} \partial_k^n e^{-k^2}.$$

Use this representation for evaluating the square norms of the polynomials $P_n(k)$:

$$h_n = \int P_n^2(k) e^{-k^2} dk = \sqrt{\pi} 2^{-n} n!.$$

Comparing the result of this exercise with the Gaussian integral (3.7.29) we obtain the value for the constant c_N in the formula (3.7.23):

$$c_N = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{n=1}^{N-1} n!}. \quad (3.7.31)$$

3.8 Real theta-functions and solutions to KP2

KP-2:

$$\frac{3}{4} u_{yy} = \frac{\partial}{\partial x} \left[u_t - \frac{1}{4} (6u u_x + u_{xxx}) \right]. \quad (3.8.1)$$

In the physics literature more often is written in the form

$$u_{yy} + u_{xt} + (u u_x)_x + u_{xxxx} = 0. \quad (3.8.2)$$

The KP-1 equation is obtained after the substitution

$$x \mapsto i x, \quad y \mapsto i y, \quad t \mapsto i t,$$

that yields

$$u_{yy} + u_{xt} + (u u_x)_x - u_{xxxx} = 0. \quad (3.8.3)$$

Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(\phi + 2\pi n) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \hat{f}(m) e^{i m \phi} \quad (3.8.4)$$

where

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x) e^{-i p x} dx \quad (3.8.5)$$

is the Fourier transform of the function f .

Applying to

$$f(x) = \frac{1}{2} \sqrt{\frac{2\pi}{b}} e^{-\frac{x^2}{2b}}, \quad \hat{f}(p) = e^{-\frac{1}{2} b p^2}$$

for a real positive number b one obtains

$$\frac{1}{2} \sqrt{\frac{2\pi}{b}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2b}(\phi + 2\pi n)^2} = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-\frac{1}{2} b m^2 + i m \phi} = \theta(\phi; b). \quad (3.8.6)$$

Hence

$$\theta(\phi; b) > 0 \quad \text{for all } \phi, b \in \mathbb{R}, \quad b > 0.$$

In similar way one proves the following

Theorem 3.8.1. *Given a real positive definite symmetric matrix $B = (B_{ij})_{1 \leq i, j \leq g}$, then the theta-function*

$$\theta(\phi) = \sum_{m \in \mathbb{Z}^g} e^{-\frac{1}{2} \langle m, B m \rangle + i \langle m, \phi \rangle} \quad (3.8.7)$$

is real and positive for all real $\phi = (\phi_1, \dots, \phi_g)$.

Observe that, since the function (3.8.7) is 2π -periodic with respect to every ϕ_1, \dots, ϕ_g , one has

$$\min_{\phi \in \mathbb{R}^g} \theta(\phi) > 0. \quad (3.8.8)$$

We will now describe the algebro-geometric data (Γ, Q, k, D) providing the conditions of reality and smoothness of the theta-functional solutions to KP.

The first condition will be imposed onto the Riemann surface Γ : it must carry a *real structure*. The simplest way to describe the idea of a real structure is to assume that Γ is the Riemann surface of an algebraic function $w(z)$ defined by a polynomial equation

$$F(z, w) = \sum_{i, j} a_{ij} z^i w^j = 0$$

with all *real* coefficients. The complex conjugation map

$$\sigma : (z, w) \mapsto (\bar{z}, \bar{w})$$

leaves invariant the algebraic curve $F(z, w) = 0$. Hence an *antiholomorphic involution* σ acts on the Riemann surface Γ

$$\sigma : \Gamma \rightarrow \Gamma, \quad \sigma^2 = \text{id}, \quad d\sigma/dz = 0. \quad (3.8.9)$$

Definition 3.8.2. *A pair (Γ, σ) , where Γ is a Riemann surface and σ is an antiholomorphic involution on Γ , is called a real Riemann surface. Two real Riemann surfaces (Γ, σ) and (Γ', σ') are called equivalent if there exists a biholomorphic equivalence $f : \Gamma \rightarrow \Gamma'$ such that*

$$f \circ \sigma = \sigma' \circ f.$$

The next condition is for the marked point $Q \in \Gamma$ and for the (inverse) local parameter k defined near Q : Q must be stable under σ ,

$$\sigma(Q) = Q \quad (3.8.10)$$

and the local parameter k must behave as follows

$$\sigma^* k = \bar{k} \quad (3.8.11)$$

(for brevity, we will say that k is σ -invariant).

Exercise 3.8.3: Given an antiholomorphic involution $\sigma : U \rightarrow U$ on the disc in the complex z -plane such that the point $Q = \{z = 0\}$ is stable under the involution, prove that there exists a holomorphic change of coordinates $\zeta = \zeta(z)$, $\zeta(0) = 0$, $d\zeta/dz \neq 0$ defined on a smaller disc $Q \in V \subset U$ such that the involution acts on ζ by complex conjugation

$$\sigma^*\zeta = \bar{\zeta}.$$

The above conditions for (Γ, Q, k) ensure reality of the solutions to KP equation (3.8.1) if the nonspecial divisor D is invariant with respect to the involution σ :

$$\sigma(D) = D. \quad (3.8.12)$$

Theorem 3.8.4. *Let the point $Q \in \Gamma$, the local parameter k and the divisor D for the real Riemann surface (Γ, σ) satisfy the conditions (3.8.10) - (3.8.12). Then the BA function $\psi = \psi(\mathbf{t}; P)$ associated with the data (Γ, Q, k, D) for all real values of $\mathbf{t} = (t_1, t_2, \dots)$ satisfy*

$$\sigma^*\psi = \bar{\psi}. \quad (3.8.13)$$

Proof. The function

$$\tilde{\psi} := \overline{\sigma^*\psi}$$

is again a BA function with the same data (Γ, Q, k, D) . Due to uniqueness it must coincide with ψ . The Theorem is proved.

Corollary 3.8.5. *The coefficients $\xi_1(\mathbf{t})$, $\xi_2(\mathbf{t})$, \dots of the BA function described in Theorem 3.8.4 take real values for real \mathbf{t} . The same is true for the solutions to the KP equation (3.8.1).*

We arrive at the main condition providing smoothness of the theta-functional solutions. It will be formulated in terms of the set Γ^σ of stable points of the involution σ . According to Exercise 3.8.3 near every point $P \in \Gamma^\sigma$ the stable set is defined by equation

$$\bar{\zeta} = \zeta$$

for a suitably chosen local parameter ζ . Therefore the set of stable points is a collection of smooth closed real curves in Γ . They are called *ovals* of the real Riemann surface (Γ, σ) .

Exercise 3.8.6: Consider the hyperelliptic curve

$$w^2 = \prod_{i=1}^{2g+1} (z - z_i)$$

where the roots z_1, \dots, z_{2g+1} are all real; assume

$$z_1 < z_2 < \dots < z_{2g+1}.$$

Prove that the involution

$$\sigma : (z, w) \mapsto (\bar{z}, \bar{w})$$

has g finite real ovals

$$\Gamma_k^\sigma = (z, \pm w(z)), \quad z \in \mathbb{R}, \quad z_{2k-1} \leq z \leq z_{2k}, \quad k = 1, \dots, g$$

and one infinite oval

$$\Gamma_\infty^\sigma = (z, \pm w(z)), \quad z \in \mathbb{R}, \quad z < z_1 \quad \text{or} \quad z_{2g+1} < z.$$

Determine the ovals of the same Riemann surface with respect to the involution

$$\tilde{\sigma} : (z, w) \mapsto (\bar{z}, -\bar{w}).$$

Exercise 3.8.7: Consider the genus g hyperelliptic real Riemann surface of the form

$$w^2 = \prod_{i=1}^{g+1} (z - z_i)(z - \bar{z}_i), \quad \sigma(z, w) = (\bar{z}, \bar{w})$$

$$z_i \neq \bar{z}_j \quad \text{for any } i, j,$$

$$z_i \neq z_j \quad \text{for } i \neq j.$$

Prove that the Riemann surface has 1 real oval for odd g and two real ovals for even g .

It is clear that the number of ovals is an invariant of a real Riemann surface (Γ, σ) . The following classical result gives an upper estimate for this number.

Theorem 3.8.8. (Harnack inequality) (i) *The number of ovals of a real Riemann surface of genus g cannot exceed $g+1$.* (ii) *If the number of ovals is equal to $g+1$ then the complement $\Gamma \setminus \Gamma^\sigma$ consists of two components*

$$\Gamma \setminus \Gamma^\sigma = \Gamma^+ \sqcup \Gamma^- \tag{3.8.14}$$

each of them is homeomorphic to the disc with g holes.

Exercise 3.8.9: Prove that any real structure on the Riemann sphere $\Gamma = \mathbf{P}^1$ is equivalent to one of the following two:

$$\sigma(z) = \bar{z}$$

or

$$\sigma(z) = -\frac{1}{\bar{z}}.$$

Prove that the latter has no ovals.

We are now ready to formulate the last condition ensuring smoothness of real solutions to KP-2.

Theorem 3.8.10. *Let (Γ, σ) be a real Riemann surface of genus g with the maximal number of real ovals. Denote Γ_∞ the oval containing the marked point Q ; let $\Gamma_1, \dots, \Gamma_g$ be the other ovals. Assume the divisor D has exactly one point on every oval $\Gamma_1, \dots, \Gamma_g$. Then the function $\psi(\mathbf{t}; P)$ is smooth for all real values of \mathbf{t} . The associated solutions to KP hierarchy are real smooth quasiperiodic functions for all real \mathbf{t} .*

Proof. Denote $D_1 = D(\mathbf{t})$ the divisor of zeroes of the BA function. Clearly $\sigma(D_1) = D_1$ for any \mathbf{t} . The BA function ψ has a pole for a given value of \mathbf{t} iff there exists a linearly equivalent divisor \tilde{D}_1 containing the marked point Q . One can assume the divisor \tilde{D}_1 to be σ -invariant, due to σ -invariance of the space $L(D_1)$. The statement of the theorem will follow from the following

Lemma 3.8.11. *Given a σ -invariant divisor D of degree g , denote*

$$n_i(D) := \# \text{ points of } D \text{ on the } i\text{-th oval } \Gamma_i, \quad i = 1, \dots, g. \quad (3.8.15)$$

Then for any other divisor D' linearly equivalent to D one has

$$n_i(D) \equiv n_i(D') \pmod{2}, \quad i = 1, \dots, g. \quad (3.8.16)$$

Proof. Let $f : \Gamma \rightarrow \mathbf{P}^1$ be the meromorphic function with poles in D and zeroes in D' . Without loss of generality one may assume $\sigma^* f = \bar{f}$. Hence f takes real values on Γ^σ . Restricting f onto a real oval Γ_i one obtains a smooth map of a circle to the circle. The degree of this map reduced modulo 2 coincides with $n_i(D)$. The theorem about invariance of the degree with respect to deformations [13] completes the proof of the Lemma.

To complete the proof of the Theorem it remains to observe that $n_i(D) = 1$ for $i = 1, \dots, g$. Therefore the $n_i(D(\mathbf{t})) = 1$ for any $i = 1, \dots, g$. Therefore any representative of the class of the divisor $D(\mathbf{t})$ cannot contain Q . The Theorem is proved.

An analogue of this theorem for the degenerate case. Consider

$$\psi = (k^N + a_1 k^{N-1} + \dots + a_N) e^{k t_1 + k^2 t_2 + \dots} \quad (3.8.17)$$

where the coefficients $a_i = a_i(\mathbf{t})$ are determined from the system of linear constraints

$$\sum_{j=1}^M \alpha_{ij} \psi(k = \kappa_j) = 0, \quad i = 1, \dots, N \quad (3.8.18)$$

where $M \geq N$. Suppose that all numbers κ_j are real and distinct; assume that

$$\kappa_1 < \kappa_2 < \dots < \kappa_M.$$

Moreover, assume that the $M \times N$ matrix

$$\alpha = (\alpha_{ij})_{1 \leq i \leq N, 1 \leq j \leq M}$$

is real and satisfies the following condition: all $N \times N$ minors are positive:

$$A_{j_1 \dots j_N} = \det \begin{pmatrix} \alpha_{1 j_1} & \alpha_{1 j_2} & \dots & \alpha_{1 j_N} \\ \alpha_{2 j_1} & \alpha_{2 j_2} & \dots & \alpha_{2 j_N} \\ \dots & \dots & \dots & \dots \\ \alpha_{N j_1} & \alpha_{N j_2} & \dots & \alpha_{N j_N} \end{pmatrix} > 0, \quad \text{for all } j_1 < j_2 < \dots < j_N. \quad (3.8.19)$$

The associated solutions to KP are expressed via the following tau-function (cf. (3.4.7))

$$\tau(\mathbf{t}) = \det (A_{ik}(\mathbf{t}))_{1 \leq i, k \leq N}, \quad A_{ik}(\mathbf{t}) = \sum_{j=1}^M \alpha_{ij} \kappa_j^{N-k} e^{\omega_j} \quad (3.8.20)$$

where

$$\omega_j = \kappa_j t_1 + \kappa_j^2 t_2 + \kappa_j^3 t_3 + \dots$$

Exercise 3.8.12: Under assumption (3.8.19) prove that the tau-function (3.8.20) is positive for all real values of $\mathbf{t} = (t_1, t_2, t_3, \dots)$.

Hint: Prove that under above assumptions the polynomial $k^N + a_1 k^{N-1} + \dots + a_N$ must have N zeroes on the interval (κ_1, κ_N) .

We will now establish a connection between the Theorems 3.8.1 and 3.8.10. Let us choose a particular basis of cycles on Γ . Choose

$$a_i = \Gamma_i, \quad i = 1, \dots, g$$

with some orientation. In order to construct the second part b_1, \dots, b_g of the basis choose points $P_i \in \Gamma_i$ and $P_\infty \in \Gamma_\infty$; connect P_∞ with P_i by a segment $l_i^+ \in \Gamma_+$ in such a way that the segments l_i^+, \dots, l_i^+ have no other intersection points but P_∞ . Choosing in a suitable way the orientation of these segments define

$$b_i = l_i^+ \cup (-\sigma(l_i^+)), \quad i = 1, \dots, g.$$

By construction

$$a_i \circ b_j = \delta_{ij} \quad \text{in } H_1(\Gamma; \mathbb{Z}).$$

Moreover, the action of the involution σ on the basis has the form

$$\begin{aligned} \sigma_* a_i &= a_i \\ \sigma_* b_i &= -b_i \\ i &= 1, \dots, g \end{aligned} \tag{3.8.21}$$

Denote $\omega_1, \dots, \omega_g$ the basis of holomorphic differentials on Γ normalized by the usual condition

$$\oint_{a_j} \omega_k = 2\pi i \delta_{jk}.$$

Let

$$\beta_{ij} = - \oint_{b_i} \omega_j$$

be the matrix of periods of the Riemann surface with the opposite sign. Like in (3.6.48), denote Ω_i the normalized second kind differential; let U_i be the vector of its b -periods (see (3.6.50)).

Lemma 3.8.13. *Let (Γ, σ, Q, k) be as above. Then the basic holomorphic differentials satisfy*

$$\sigma^* \omega_k = -\bar{\omega}_k, \quad k = 1, \dots, g. \tag{3.8.22}$$

The normalized second kind differentials Ω_i satisfy

$$\sigma^* \Omega_i = \bar{\Omega}_i. \tag{3.8.23}$$

The period matrix β_{ij} is real and positive definite. The vectors of b -periods U_i are purely imaginary,

$$\bar{U}_i = -U_i. \tag{3.8.24}$$

Proof. The differential $\sigma^*(\omega_k)$ must be antiholomorphic. Decompose it

$$\sigma^*(\omega_k) = \sum_{l=1}^g c_{kl} \bar{\omega}_l$$

for some complex coefficients c_{kl} . Using

$$\begin{aligned} \oint_{a_j} \sigma^*(\omega_k) &= \sum c_{kl} \oint_{a_j} \bar{\omega}_l = -2\pi i c_{kj} \\ &= \oint_{\sigma_*(a_j)} \omega_k = 2\pi i \delta_{jk} \end{aligned}$$

one obtains

$$c_{jk} = -\delta_{jk}.$$

The reality of the period matrix follows from a similar computation:

$$\bar{\beta}_{ij} = - \oint_{b_i} \bar{\omega}_j = \oint_{b_i} \sigma^*(\omega_j) = \oint_{\sigma_*(b_i)} \omega_j = - \oint_{b_i} \omega_j = \beta_{ij}.$$

Positive definiteness of β_{ij} follows from the negative definiteness of the real part of the period matrix.

Finally, to prove invariance (3.8.23) of the second kind differentials one has to use that the meromorphic differential

$$\tilde{\Omega}_i := \overline{\sigma^* \Omega_i}$$

has the same singularity

$$\tilde{\Omega}_i = dk^i + \text{regular terms}$$

due to (3.8.11). It is normalized with respect to the same basis of a -cycles. Hence $\tilde{\Omega}_i = \Omega_i$. The equation (3.8.24) readily follows from (3.8.23). The Lemma is proved.

3.9 Dual BA function, vanishing lemma and smooth solutions to KP-1

We proceed now to the theory of real smooth solutions to the KP-1 equation

$$\frac{3}{4} u_{yy} = \frac{\partial}{\partial x} \left[u_t - \frac{1}{4} (6u u_x - u_{xxx}) \right]. \quad (3.9.1)$$

It can be represented as the compatibility conditions of the following linear problem

$$\begin{aligned} \frac{1}{i} \frac{\partial \psi}{\partial y} &= -\psi_{xx} + u \psi \\ \frac{\partial \psi}{\partial t} &= -\psi_{xxx} + \frac{3}{4} (u \partial_x + \partial_x u) \psi + i w \psi. \end{aligned} \quad (3.9.2)$$

In order to select real smooth solutions $u(x, y, t)$, $w(x, y, t)$ to this equation we will first present an algebro-geometric realization of the dual BA function introduced in the Section 3.6.

Let $\psi = \psi(\mathbf{t}; P)$ be a BA function

$$\psi = \left(1 + \frac{\xi_1(\mathbf{t})}{k} + \frac{\xi_2(\mathbf{t})}{k^2} + \dots \right) e^{k t_1 + k^2 t_2 + \dots}$$

associated with the data (Γ, Q, k, D) .

Definition 3.9.1. *The dual BA function $\psi^\dagger = \psi^\dagger(\mathbf{t}; P)$ is a function meromorphic on $\Gamma \setminus Q$ with an essential singularity at Q of the form*

$$\psi^\dagger = \left(1 + \frac{\xi_1^\dagger(\mathbf{t})}{k} + \frac{\xi_2^\dagger(\mathbf{t})}{k^2} + \dots \right) e^{-k t_1 - k^2 t_2 - \dots} \quad (3.9.3)$$

with poles at the divisor D^\dagger such that

$$D + D^\dagger = K_\Gamma + 2Q. \quad (3.9.4)$$

Here K_Γ is the canonical class of Γ . Observe that the degree of D^\dagger is equal to g . The equation 3.9.4 can be reformulated as follows: there exists an Abelian differential Ω_D with double pole at Q and with zeroes at D and D^\dagger :

$$(\Omega_D) = D + D^\dagger - 2Q. \quad (3.9.5)$$

Such a differential is determined uniquely, within a constant factor, for a nonspecial divisor D .

Lemma 3.9.2. *Denote $L = \partial_x + \sum_{i \geq 1} u_i(\mathbf{t}) \partial_x^{-i}$, $A_n = [L^n]_+$ the operators of the KP hierarchy associated with ψ , i.e.,*

$$\frac{\partial \psi}{\partial t_n} = A_n \psi, \quad n = 1, 2, \dots \quad (3.9.6)$$

Then ψ^\dagger satisfies

$$-\frac{\partial \psi^\dagger}{\partial t_n} = A_n^\dagger \psi^\dagger, \quad n = 1, 2, \dots \quad (3.9.7)$$

where

$$A_n^\dagger = [(L^\dagger)^n]_+, \quad L^\dagger = (-\partial_x) + \sum_{i \geq 1} (-\partial_x)^{-i} u_i(\mathbf{t})$$

is the (formally) adjoint operator.

Proof. Consider the differential

$$\psi(\mathbf{t}; P) \psi^\dagger(\mathbf{t}'; P) \Omega_D(P)$$

depending on two sets of times \mathbf{t} and \mathbf{t}' . By construction it has no other singularities on Γ but the one at $P = Q$. Hence

$$\operatorname{Res}_{P=Q} \psi(\mathbf{t}; P) \psi^\dagger(\mathbf{t}'; P) \Omega_D(P) = 0$$

for any \mathbf{t}, \mathbf{t}' . Comparing with the Sato bilinear formulation of the KP hierarchy (see Theorem 3.6.19) one obtains the needed result.

Exercise 3.9.3: Let the differential Ω_D be normalised by the condition

$$\Omega_D(P) = \left(1 + O\left(\frac{1}{k^2}\right) \right) dk, \quad P \rightarrow Q.$$

Prove that the meromorphic differential $\psi(\mathbf{t}; P) \psi^\dagger(\mathbf{t}; P) \Omega_D(P)$ has the following expansion near $P = Q$:

$$\psi(\mathbf{t}; P) \psi^\dagger(\mathbf{t}; P) \Omega_D(P) = \left(1 - \frac{u(\mathbf{t}) + u_0}{2k^2} + \frac{v(\mathbf{t})}{k^3} + \dots \right) dk \quad (3.9.8)$$

where u_0 is a constant and the function $v(\mathbf{t})$ satisfies

$$v_x = -\frac{1}{2} u_y. \quad (3.9.9)$$

Remark 3.9.4. Recall that the divisor D enters into the theta-functional formula via the point $\zeta \in J(\Gamma)$ of the form

$$\zeta = A_Q(D) + \mathcal{K}_Q = D - \Delta - Q \in J(\Gamma)$$

(here Δ is the Riemann divisor, see Section 2.12). Denote

$$\zeta^\dagger := A_Q(D^\dagger) + \mathcal{K}_Q = D^\dagger - \Delta - Q.$$

Since $2\Delta = K_\Gamma$ we obtain

$$\zeta + \zeta^\dagger = D + D^\dagger - 2\Delta - 2Q = 0 \quad \text{on } J(\Gamma). \quad (3.9.10)$$

So, the duality is the involution $\zeta \mapsto -\zeta$ on the Jacobian.

We are ready to formulate the conditions of reality of solutions to KP-1.

Theorem 3.9.5. *Let the data $(\Gamma, \sigma, Q, k, D)$, where $\sigma : \Gamma \rightarrow \Gamma$ being an antiholomorphic involution, satisfy the following conditions:*

$$\begin{aligned} \sigma(Q) &= Q \\ \sigma^* k &= \bar{k} \\ \sigma(D) &= D^\dagger. \end{aligned}$$

Then the BA function with the essential singularity at Q of the form

$$\psi = \left(1 + \frac{\xi_1(\mathbf{t})}{k} + \frac{\xi_2(\mathbf{t})}{k^2} + \dots \right) e^{i k t_1 + i k^2 t_2 + \dots}$$

for real \mathbf{t} satisfies

$$\overline{\sigma^* \psi} = \psi^\dagger; \quad (3.9.11)$$

the linear equations of the KP hierarchy for ψ take the form

$$\frac{1}{i} \frac{\partial \psi}{\partial t_n} = A_n \psi \quad (3.9.12)$$

where the operators A_n satisfy

$$A_n^\dagger = \bar{A}_n \quad (3.9.13)$$

for all real $\mathbf{t} = (t_1, t_2, \dots)$. In particular, the solution

$$u(\mathbf{t}) = 2i \partial_x \xi_1$$

to the KP equation is real for real \mathbf{t} .

Proof. For real \mathbf{t} the function

$$\overline{\sigma^* \psi} = \overline{\psi(\mathbf{t}; \sigma(P))} = \left(1 + \frac{\bar{\xi}_1}{k} + \frac{\bar{\xi}_2}{k} + \dots \right) e^{-i k t_1 - i k^2 t_2 - \dots}$$

has the asymptotics like ψ^\dagger ; by assumption $\sigma(D) = D^\dagger$ it has the poles at D^\dagger . Hence it coincides with $\psi^\dagger(\mathbf{t}; P)$. Applying complex conjugation to (3.9.12) and replacing P by $\sigma(P)$ one obtains

$$-\frac{1}{i} \frac{\partial \psi^\dagger}{\partial t_n} = \bar{A}_n \psi^\dagger.$$

Comparison with (3.9.7) yields (3.9.13). The theorem is proved.

We will now illustrate the ideas about reality and smoothness of solutions to KP-1 on the level of degenerate BA functions. Let us look for the degenerate BA function of the form

$$\psi(\mathbf{t}; k) = \left(1 + \sum_{j=1}^N \frac{r_j(\mathbf{t})}{k - r_j} \right) e^{i k t_1 + i k^2 t_2 + \dots} \quad (3.9.14)$$

where the coefficients $r_1(\mathbf{t}), \dots, r_N(\mathbf{t})$ are determined from the following linear system

$$\psi(\mathbf{t}; \bar{\kappa}_i) + i \sum_{j=1}^N c_{ij} \operatorname{Res}_{k=\kappa_j} \psi(\mathbf{t}; k) dk = 0, \quad i = 1, \dots, N \quad (3.9.15)$$

for some pairwise distinct complex numbers $\kappa_1, \dots, \kappa_N$,

$$\kappa_i \neq \kappa_j \quad \text{for } i \neq j, \quad \kappa_i \neq \bar{\kappa}_j \quad \text{for all } i, j$$

and a given square matrix of complex numbers c_{ij} . The coefficient before the summation sign in (3.9.15) is $i = \sqrt{-1}$. Observe that the numbers κ_i can never be real.

The linear system (3.9.15) can be considered as a particular case of the defining relation (3.4.2) of the theory of degenerate BA function. Let us describe the reality conditions for the associated solutions to KP-1.

Exercise 3.9.6: (i) Let the matrix $C = (c_{ij})_{1 \leq i, j \leq N}$ be Hermitean,

$$C^* = C.$$

Prove that the corresponding solution to KP-1, when defined, takes real value for real \mathbf{t} .

(ii) Assuming

$$\Im \kappa_j > 0, \quad j = 1, \dots, p, \quad \Im \kappa_j < 0, \quad j = p + 1, \dots, N.$$

suppose that the Hermitean $p \times p$ and $(N - p) \times (N - p)$ matrices

$$C_{\text{up}} := (c_{ij})_{1 \leq i, j \leq p}, \quad C_{\text{down}} := -(c_{ij})_{p+1 \leq i, j \leq N} \quad (3.9.16)$$

are positive definite. Prove that then the solution to the linear system (3.9.15) exists and is smooth for any real \mathbf{t} .

Hint. (i) Consider the meromorphic 1-form

$$\Omega = \psi(\mathbf{t}; k) \overline{\psi(\mathbf{t}; \bar{k})} dk.$$

Prove that

$$\begin{aligned} \operatorname{Res}_{k=\kappa_i} \Omega &= i \sum_j c_{ji} \bar{\psi}_j \psi_i \\ \operatorname{Res}_{k=\bar{\kappa}_i} \Omega &= -i \sum_j c_{ij} \bar{\psi}_i \psi_j \end{aligned}$$

where

$$\psi_i := \operatorname{Res}_{k=\kappa_i} \psi(\mathbf{t}; k) dk.$$

Applying the residue theorem derive reality of u .

(ii) The determinant of the linear system (3.9.15) vanishes at a given value \mathbf{t} iff the associated homogeneous linear system has a nontrivial solution. The latter claim boils down to existence of a function

$$\tilde{\psi}(\mathbf{t}; k) = \left(\sum_{j=1}^N \frac{\tilde{r}_j(\mathbf{t})}{k - r_j} \right) e^{i k t_1 + i k^2 t_2 + \dots} \quad (3.9.17)$$

satisfying the same linear constraints (3.9.15). The differential

$$\tilde{\Omega} = \tilde{\psi}(\mathbf{t}; k) \overline{\tilde{\psi}(\mathbf{t}; \bar{k})} dk.$$

is positive for real k , hence it satisfies

$$I := \int_{-\infty}^{\infty} \tilde{\Omega} > 0.$$

On another side, computing the integral via the sum of residues of this differential over the poles in the upper half plane yields

$$I = -2\pi \left[\sum_{i,j=1}^p \overline{\tilde{\psi}_i} c_{ij} \tilde{\psi}_j - \sum_{i,j=p+1}^N \overline{\tilde{\psi}_i} c_{ij} \tilde{\psi}_j \right] < 0$$

where, as above,

$$\tilde{\psi}_i := \operatorname{Res}_{k=\kappa_i} \tilde{\psi}(\mathbf{t}; k) dk.$$

The contradiction obtained proves that $\tilde{\psi} = 0$.

3.10 Symmetries and reductions of KP

Adding a condition $w = \text{const}$ reduces the KP system (3.2.14) to one equation

$$u_t = \frac{1}{4} (6u u_x + u_{xxx}) \quad (3.10.1)$$

where the function $u = u(x, t)$ does not depend on y due to the first of the equations (3.2.14). This is the celebrated Korteweg - de Vries (KdV) equation. The modern theory of integrable systems began with KdV.

We will now obtain conditions for the triple (Γ, Q, k) that yield the reduction of KP to KdV.

Let the Riemann surface Γ and the point $Q \in \Gamma$ be such that a meromorphic function $\lambda(P)$ exists with only one pole of multiplicity two at $P = Q$. Then Γ must be a hyperelliptic curve and Q must be a Weierstrass point on it - see Exercise 2.10.9. Let us choose the local parameter near $Q \in \Gamma$ as follows

$$k^{-1}(P) := [\lambda(P)]^{-1/2}. \quad (3.10.2)$$

Lemma 3.10.1. *For a triple (Γ, Q, k) of the above form and for an arbitrary nonspecial divisor D of degree g on Γ the corresponding BA function $\psi(x, y, t; P)$ has the form*

$$\psi(x, y, t; P) = e^{y\lambda(P)} \varphi(x, t; P) \quad (3.10.3)$$

where $\varphi(x, t; P)$ is the BA function for the same triple (Γ, Q, k) with the same poles D and with essential singularity at Q of the form

$$\varphi(x, t; P) = e^{kx+k^3t} \left(1 + \frac{\xi_1}{k} + O(k^{-2}) \right). \quad (3.10.4)$$

Proof. Due to the choice of the local parameter k one has $k^2(P) = \lambda(P)$ for any point $P \in \Gamma$ sufficiently close to Q . So the product

$$\varphi := \psi(x, y, t; P) e^{-y\lambda(P)}$$

is a BA function on Γ with poles at D and essential singularity of the form (3.10.4). Since the exponential term in (3.10.4) is y -independent, the BA function φ depends only on x, t . Lemma is proved.

Corollary 3.10.2. *Let Γ be a hyperelliptic curve of genus g of the form*

$$w^2 = P_{2g+1}(z),$$

the point $Q = \infty$ be the infinite point of Γ , $k = \sqrt{z}$, $D = P_1 + \dots + P_g$ be a nonspecial divisor of the degree g (i.e., $z(P_i) \neq z(P_j)$ for $i \neq j$). Then: (i) for those values of x, t for which the corresponding BA function $\varphi(x, t; P)$ with essential singularity (3.10.4) exists it satisfies the “eigenfunction” equation

$$L \varphi(x, t; P) = \lambda(P) \varphi(x, t; P) \quad (3.10.5)$$

where

$$L = \partial_x^2 + u(x, t), \quad u = -2\partial_x \xi_1. \quad (3.10.6)$$

Observe that

$$\lambda(P) = z \quad \text{for } P = (z, w) \in \Gamma.$$

(ii) The t -dependence of φ obeys the equation

$$\frac{\partial \varphi}{\partial t} = A \varphi, \quad A = \partial_x^3 + \frac{3}{4} (u \partial_x + \partial_x u). \quad (3.10.7)$$

(iii) The function $u(x, t)$ of the form (3.10.6) satisfies KdV equation (3.10.1).

(iv) This solution can be expressed via the theta-function of the hyperelliptic curve

$$u(x, t) = 2 \partial_x^2 \log \theta(xU + tW + z_0) + c \quad (3.10.8)$$

where the vectors U, W and the constant c are constructed for (Γ, Q, k) using the formulae (3.2.21), z_0 is an arbitrary phase shift.

Proof. The BA function $\psi(x, y, t; P)$ for the triple (Γ, Q, k) and for a nonspecial divisor D must be of the form (3.10.3) (the assumptions of the Corollary is just a reformulation of those of Lemma 3.10.1). Plugging

$$\psi(x, y, t; P) = \varphi(x, t; P) e^{y \lambda(P)}$$

into equation $\partial \psi / \partial y = L \psi$ yields (3.10.5). Furthermore, the coefficients of the expansion of $\psi(x, y, t; P) e^{-kx - k^3 t - y \lambda(P)}$ at $P = Q$ do not depend on y . Hence $w = \text{const}$. So, equation (3.2.13) takes the form (3.10.7). Finally, (3.10.8) comes from (3.2.21) since $V = 0$. Indeed, this is the vector of b -periods of the exact differential

$$-2\Omega_Q^{(2)} = dz.$$

Corollary is proved.

Remark 1. The zero-curvature representation (3.2.18) of KP in the case under consideration rewrites in the form of Lax representation of KdV

$$\frac{\partial L}{\partial t} = [A, L] \quad \Leftrightarrow \quad u_t = \frac{1}{4} (6u u_x + u_{xxx}). \quad (3.10.9)$$

Clearly this is the condition of compatibility of linear equations (3.10.5) and (3.10.7) (cf. the proof of Corollary 3.2.3).

Exercise 3.10.3: Let

$$Q_1 = (\gamma_1, \sqrt{P_{2g+1}(\gamma_1)}), \dots, Q_g = (\gamma_g, \sqrt{P_{2g+1}(\gamma_g)})$$

be the zeroes of the BA function $\varphi(x, t; P)$ described in Corollary 3.10.2. Derive the following ODEs for the dependence of these zeroes on x and t

$$\frac{\partial \gamma_k}{\partial x} = -\frac{2i \sqrt{P_{2g+1}(\gamma_k)}}{\prod_{j \neq k} (\gamma_k - \gamma_j)}, \quad k = 1, \dots, g \quad (3.10.10)$$

$$\frac{\partial \gamma_k}{\partial t} = \frac{8i \left(\sum_{j \neq k} \gamma_j - \frac{1}{2} \bar{z} \right) \sqrt{P_{2g+1}(\gamma_k)}}{\prod_{j \neq k} (\gamma_k - \gamma_j)}, \quad k = 1, \dots, g \quad (3.10.11)$$

where

$$\bar{z} := \sum_{j=1}^{2g+1} z_j, \quad P_{2g+1}(z) = \prod_{j=1}^{2g+1} (z - z_j).$$

Let us now consider the Riemann surfaces Γ with a marked point Q such that a meromorphic function $\lambda(P)$ exists having a triple pole at the point Q and no other poles. Choosing the local parameter

$$k^{-1}(P) := \lambda^{-1/3}(P)$$

near Q we obtain the following

Lemma 3.10.4. *Under the above conditions for the triple (Γ, Q, k) and for an arbitrary nonspecial divisor of the degree g the corresponding BA function $\psi(x, y, t; P)$ has the form*

$$\psi(x, y, t; P) = e^{t \lambda(P)} \varphi(x, y; P) \quad (3.10.12)$$

where $\varphi(x, y, t; P)$ is the BA function with the same data (Γ, Q, k, D) with the expansion of the form

$$\varphi(x, y; P) = e^{kx + k^2y} (1 + O(k^{-1})) \quad \text{for } P \rightarrow Q. \quad (3.10.13)$$

The function φ satisfies the equations

$$\frac{\partial \varphi}{\partial y} = L \varphi \quad (3.10.14)$$

$$A \varphi(x, y; P) = \lambda(P) \varphi(x, y; P) \quad (3.10.15)$$

for the operators L, A of the form (3.2.10), (3.2.11). The coefficients u, w of these operators are solutions to the Boussinesq equations

$$\frac{3}{4} u_y = w_x \quad (3.10.16)$$

$$w_y + \frac{1}{4} (6u u_x + u_{xxx}) = 0.$$

Theta-functional formulae for these solutions can be obtained from (3.2.21), (3.2.22) dropping the t -dependence.

Proof is completely analogous to that of Lemma 3.10.1 and Corollary 3.10.2. From compatibility of (3.10.14) and (3.10.15) we obtain, like we did before, a Lax-type representation for the Boussinesq equation

$$\frac{\partial A}{\partial y} = [L, A]. \quad (3.10.17)$$

We will list now the simplest examples of Riemann surfaces carrying meromorphic functions with a single third order pole.

Clearly, such a function exists for any point of an arbitrary elliptic curve. We leave as a simple exercise for the reader to construct the corresponding elliptic solutions of Boussinesq equation. Let us consider less obvious examples.

Example 3.10.5. Let Γ be a Riemann surface of genus 2 (recall that any such surface is hyperelliptic - see Exercise 2.10.16). Due to Riemann - Roch theorem for any non-Weierstrass point $Q \in \Gamma$ (i.e., Q does not coincide with any of the 6 branch points of the hyperelliptic Riemann surface) a function exists having the only third order pole at Q (construct such a function explicitly for the Riemann surface of the form $w^2 = P_5(z)$ for a given degree 5 polynomial $P_5(z)$ and a given point $(z_0, w_0) \in \Gamma$, $w_0 \neq 0$).

Example 3.10.6. Let Γ be a non-hyperelliptic curve of genus 3 (see Exercise 2.10.20). According to the results of Lecture 2.10 there exists a Weierstrass point Q on Γ (i.e., such a point that $l(3Q) \geq 2$). Since Γ is not a hyperelliptic curve, there are no functions with the only pole of the order ≤ 2 at Q . Hence, there exists a function with the only triple pole at Q .

Exercise 3.10.7: Prove that any genus 3 non-hyperelliptic curve carries 24 Weierstrass points. Prove that the Weierstrass points on the smooth quartic $R(z, w) = \sum_{i+j \leq 4} a_{ij} z^i w^j = 0$ can be determined from the system

$$\left. \begin{aligned} R(z, w) &= 0 \\ R_{zz}R_w^2 - 2R_{zw}R_zR_w + R_{ww}R_z^2 &= 0 \end{aligned} \right\}.$$

Hint: Prove that there exist polynomials $A(z, w)$, $B(z, w)$ of the total degrees 4 and 6 respectively such that

$$R_{zz}R_w^2 - 2R_{zw}R_zR_w + R_{ww}R_z^2 = A(z, w)R(z, w) + B(z, w).$$

Generalizing the reduction procedure of KP explained in the beginning of this Lecture we arrive at

Theorem 3.10.8. (i) Let $\lambda(P)$ be a meromorphic function on Γ with the only pole at $Q \in \Gamma$ of the multiplicity n . We put

$$k(P) := \lambda^{1/n}(P) \quad (3.10.18)$$

for P sufficiently close to Q . Consider the BA function $\psi = \psi(x, t_2, \dots, \hat{t}_n, \dots; P)$ with the data (Γ, Q, k, D) for some non-special degree g divisor D and with the expansion of the form (3.3.1) for $P \rightarrow Q$ with the t_n -dependence omitted. Then for any $P \in \Gamma$ the function $\psi = \psi(x, t_2, \dots, \hat{t}_n, \dots; P)$ satisfies the equations

$$L\psi = \lambda(P)\psi, \quad L := A_n \quad (3.10.19)$$

$$\frac{\partial \psi}{\partial t_m} = A_m \psi, \quad m = 1, 2, \dots, \quad m \neq n. \quad (3.10.20)$$

The coefficients of the operators A_2, A_3, \dots satisfy a system of equations admitting a Lax-type representation

$$\frac{\partial L}{\partial t_m} = [A_m, L], \quad m = 1, 2, \dots, \quad (3.10.21)$$

(ii) Let us assume that another meromorphic function $\mu(P)$ exists on Γ with the only pole at Q of the order m . Denote c_0, c_1, \dots the coefficients of the Laurent expansion of $\mu(P)$ near $P = Q$ with respect to the local parameter (3.10.18),

$$\mu(P) = c_0 k^m + c_1 k^{m-1} + \dots + c_{m-1} k + c_m + O(k^{-1}). \quad (3.10.22)$$

Introduce the differential operator

$$A := \sum_{i=0}^m c_i A_{m-i} \quad (3.10.23)$$

where we denote $A_0 = 1$ the identity operator. Then the BA function ψ for any t_2, t_3, \dots satisfies

$$A\psi(x, \dots; P) = \mu(P)\psi(x, \dots; P) \quad (3.10.24)$$

and the ordinary linear differential operators $L = A_n$ and A commute,

$$[L, A] = 0. \quad (3.10.25)$$

We leave the proof of this theorem as an exercise for the reader.

Observe that a pair of meromorphic functions $\lambda(P), \mu(P)$ with the only poles at $Q \in \Gamma$ of the orders n, m resp. exists on an arbitrary Riemann surface Γ of the given genus g for sufficiently large n and m . For example, for a generic (i.e., a non-Weierstrass) point Q one can take $n = g + 1, m = g + 2$. Choosing such a pair (λ, μ) on (Γ, Q) one obtains a family of commuting ordinary differential operators L_λ, L_μ of the orders

$$\text{ord } L_\lambda = \text{deg } \lambda, \quad \text{ord } L_\mu = \text{deg } \mu$$

such that

$$L_\lambda \psi(x, \dots; P) = \lambda(P)\psi(x, \dots; P), \quad L_\mu \psi(x, \dots; P) = \mu(P)\psi(x, \dots; P).$$

It turns out that any such a pair of commuting ordinary differential operators of the orders m, n can be obtained by the above construction if the numbers m and n are coprime. For the general case of commuting ordinary differential operators of not coprime orders the classification theory is more complicated. It involves the technique of multidimensional vector bundles on Riemann surfaces. Appearance of Riemann surfaces in the problem of classification of commuting ordinary linear differential operators becomes clear due to the following statement.

Theorem 3.10.9. *Let L, A be two commuting linear ordinary differential operators of the orders n and m resp. Then there exists a polynomial*

$$F(z, w) = \sum a_{ij} z^i w^j$$

of the degree m in z and degree n in w such that

$$F(L, A) = 0. \quad (3.10.26)$$

The common eigenfunctions

$$L\psi = z\psi, \quad A\psi = w\psi \quad (3.10.27)$$

are meromorphic on the Riemann surface $F(z, w) = 0$.

We recommend to the reader to prove this theorem after studying the Lecture ?? where the particular case of commuting ordinary linear differential operators of the orders $n = 2$ and $m = 2k + 1$ will be studied. The results of this investigation will be applied to the theory of KdV equation and of the higher analogues of it (the so-called *KdV hierarchy*). In this case the equations of commutativity $[L, A] = 0$ can be reduced to an ordinary differential equation for the potential $u(x)$ of the differential operator $L = \partial_x^2 + u(x)$. The related non-stationary equations

$$\frac{\partial L}{\partial t} = [L, B]$$

can also be written as nonlinear evolutionary PDEs for the function u . These are the equations of the KdV hierarchy. The explicit form of these equations can be obtained by expressing recursively the coefficients of the operator A via $u(x)$ and its x -derivatives using the equations $[L, A] = 0$ and $\partial L/\partial t = [L, B]$ resp.

Let us illustrate again the basic idea of reducing KP to nKdV working with degenerate BA functions. Let us consider the BA function of the form (??) constraint by the system of linear equations (??) where the functions $g_i(k)$ have the form (??). As usual denote

$$L = \partial_x + \sum_{i \geq 1} u_i(\mathbf{t}) \partial_x^{-i} = W \partial_x W^{-1}, \quad W = 1 + \sum_{i \geq 1} \xi_i(\mathbf{t}) \partial_x^{-i}$$

the associated pseudodifferential operator.

Theorem 3.10.10. *Let the subspace*

$$U = \text{span}(g_1(k), g_2(k), \dots)$$

be invariant with respect to multiplication by k^n :

$$k^n U \subset U. \quad (3.10.28)$$

Then the BA function $\psi(\mathbf{t}; k)$ can be represented in the form

$$\psi(\mathbf{t}; k) = e^{\sum_{m \geq 1} k^m t_m} \varphi(\mathbf{t}'; k) \quad (3.10.29)$$

and the function $\varphi(\mathbf{t}', k)$ satisfies the equations

$$A_n \varphi = k^n \varphi \quad (3.10.30)$$

$$\frac{\partial \varphi}{\partial t_j} = A_j \varphi, \quad \text{for all } j \neq m n.$$

Here, as usual,

$$A_i = [L^i]_+, \quad i \geq 0,$$

moreover, the n -th power of L is a differential operator

$$\mathcal{L} := L^n = A_n = \partial_x^n + a_1(\mathbf{t}) \partial_x^{n-2} + \dots + a_{n-1}(\mathbf{t}). \quad (3.10.31)$$

The operators \mathcal{L} and A_j satisfy the equations of the n KdV hierarchy

$$\frac{\partial \mathcal{L}}{\partial t_j} = [A_j, \mathcal{L}], \quad j \neq n \quad (3.10.32)$$

written in the Lax form.

Proof. Let us consider the function

$$\varphi(\mathbf{t}; k) := e^{-\sum_{m \geq 1} k^{m n} t_m n} \psi(\mathbf{t}; k) = \left(1 + \frac{\xi_1}{k} + \frac{\xi_2}{k^2} + \dots\right) e^{\sum_{j \neq n} k^j t_j}.$$

Let us derive from (3.10.28) that this function satisfies the same linear constraints (??). Indeed, due to (3.10.28) there exists a constant matrix a_{ij} such that

$$k^n g_i(k) = g_{n+i}(k) + \sum_{j=1}^{n+i-1} a_{ij} g_{n+i-j}(k), \quad i = 1, 2, \dots$$

So the system of orthogonality constraints

$$\frac{1}{2\pi i} \oint g_i(k) \psi(\mathbf{t}; k) \frac{dk}{k} = 0, \quad i \geq 1$$

implies

$$\frac{1}{2\pi i} \oint k^{m n} g_i(k) \psi(\mathbf{t}; k) \frac{dk}{k} = 0 \quad \text{for all } i \geq 1, \quad m \geq 1.$$

Hence

$$\frac{1}{2\pi i} \oint e^{-\sum_{m \geq 1} k^{m n} t_m n} g_i(k) \psi(\mathbf{t}; k) \frac{dk}{k} = 0, \quad i \geq 1.$$

By assumption this system of linear constraints for φ has a unique solution. As the exponential term in φ does not depend on the variables $t_m n$, hence φ does not depend on these time variables as well. This proves (3.10.29). The proof of the remaining statements of the theorem reproduces the proof explained in the algebro-geometric situation. The theorem is proved.

Exercise 3.10.11: Let ψ be a degenerate BA function of the form (3.4.1) constrained by the conditions of the form

$$\psi(\kappa_i) = c_i \psi(-\kappa_i), \quad i = 1, \dots, N \quad (3.10.33)$$

for some numbers $\kappa_1, \dots, \kappa_N$ such that $\kappa_i \neq \kappa_j$ for $i \neq j$ and $\kappa_i \neq -\kappa_j$ for all i, j , and arbitrary nonzero numbers c_1, \dots, c_N . Show that the function ψ for any y, t is an eigenfunction³ of the differential operator $L = \partial_x^2 + u$

$$L \psi(x, y, t; k) = k^2 \psi(x, y, t; k) \quad (3.10.34)$$

where the potential

$$u = 2\partial_x^2 \log \det (A_{jk}(x, t)) \quad (3.10.35)$$

$$A_{jk}(x, t) = \delta_{jk} + \tilde{c}_j \frac{\exp[-(\kappa_j + \kappa_k)x - (\kappa_j^3 + \kappa_k^3)t]}{\kappa_j + \kappa_k}, \quad j, k = 1, \dots, N \quad (3.10.36)$$

does not depend on y . Here $\tilde{c}_1, \dots, \tilde{c}_N$ are some constants expressed in terms of c and κ . Prove that $u(x, t)$ is a solution to the KdV equation (3.10.1).

[*Hint*: rewrite the linear system (3.4.6) introducing new unknowns r_1, \dots, r_N instead of the unknowns a_1, \dots, a_N , where

$$\tilde{\psi} := \frac{\psi(x, y, t; k)}{\prod_{i=1}^N (k - \kappa_i)} = \left(1 + \sum_{i=1}^N \frac{r_i(x, t)}{k - \kappa_i} \right) e^{kx + k^2y + k^3t}.$$

Observe that the constraints (3.10.33) in terms of the function $\tilde{\psi}$ recast in the form

$$\operatorname{Res}_{k=\kappa_i} \tilde{\psi} = \tilde{c}_i \tilde{\psi}(-\kappa_i), \quad i = 1, \dots, N$$

(this defines the constants $\tilde{c}_1, \dots, \tilde{c}_N$ in (3.10.36)).]

For real positive numbers $\tilde{c}_1, \dots, \tilde{c}_N, \kappa_1, \dots, \kappa_N$ the function $u(x, t)$ of the form (3.10.35), (3.10.36) are smooth real solutions of KdV. They are celebrated multisoliton solutions of KdV. In particular, for $N = 1$ one obtains a soliton, i.e., a localized solution of the form

$$u(x, t) = \frac{2\kappa^2}{\cosh^2 \kappa[x - x_0 + \kappa^2 t]}. \quad (3.10.37)$$

3.11 Additional symmetries of KP and Virasoro algebra

We start from the following elementary observation: the trivial BA function

$$\psi_0 = e^{\sum t_m k^m}$$

satisfies

$$\frac{\partial \psi_0}{\partial k} = M_0 \psi_0, \quad M_0 = \sum m t_m \partial_x^{m-1}.$$

In general one can derive a similar equation for the k -dependence of the BA function by applying the dressing procedure. Introduce the operator

$$M = W M_0 W^{-1}, \quad W = 1 + \sum_{i \geq 1} \xi_i(\mathbf{t}) \partial_x^{-i}. \quad (3.11.1)$$

³Here we understand the notion of eigenfunction of the operator L in the formal sense as just a solution of the differential equation (3.10.34).

Then the following equation holds true

$$\frac{\partial \psi}{\partial k} = M \psi. \quad (3.11.2)$$

The operator M does not commute with

$$L = W \partial_x W^{-1}$$

but it satisfies the Heisenberg commutation relation

$$[L, M] = 1. \quad (3.11.3)$$

Exercise 3.11.1: Prove (3.11.3). Moreover, derive the following commutation relations

$$[L^i, M] = i L^{i-1} \quad \text{for any } i \in \mathbb{Z} \quad (3.11.4)$$

$$[L, M^j L^i] = j M^{j-1} L^i \quad \text{for all } i \in \mathbb{Z}, \quad j = 1, 2, \dots \quad (3.11.5)$$

Let us now introduce *additional symmetries* of the KP hierarchy as the vector fields $\partial/\partial s_{ij}$, $i \in \mathbb{Z}$, $j = 1, 2, \dots$ by the following formula

$$\frac{\partial W}{\partial s_{ij}} = - (M^j L^i)_- W. \quad (3.11.6)$$

For the Lax operator L the equation (3.11.6) yields

$$\frac{\partial L}{\partial s_{ij}} = - [(M^j L^i)_-, L]. \quad (3.11.7)$$

Lemma 3.11.2. *The additional flows (3.11.6) (or (3.11.7)) commute with equations of the KP hierarchy:*

$$\left[\frac{\partial}{\partial t_n}, \frac{\partial}{\partial s_{ij}} \right] = 0. \quad (3.11.8)$$

Proof. From the obvious commutativity

$$\left[\frac{\partial}{\partial t_n} - \partial_x^n, M_0 \right] = 0$$

it follows by dressing that

$$\left[\frac{\partial}{\partial t_n} - A_n, M \right] = 0, \quad A = (L^n)_+.$$

Hence

$$\left[\frac{\partial}{\partial t_n} - A_n, M^j L^i \right] = 0.$$

The commutativity (3.11.8) easily follows from the last equation. The lemma is proved.

The additional symmetries (3.11.6) *do not* commute between themselves.

Exercise 3.11.3: Prove that the mapping

$$M^j L^i \mapsto \frac{\partial}{\partial s_{ij}}$$

is a Lie algebra homomorphism. Derive the following commutation relations

$$\left[\frac{\partial}{\partial s_{ij}}, \frac{\partial}{\partial s_{kl}} \right] = \sum_{s \geq 1} s! \left[\binom{i}{s} \binom{l}{s} - \binom{j}{s} \binom{k}{s} \right] \frac{\partial}{\partial s_{i+k-s, j+l-s}}. \quad (3.11.9)$$

We will now look more carefully at the particular class of the additional symmetries corresponding to $j = 1$. We will *change the notations* defining

$$\frac{\partial}{\partial s_i} := \frac{\partial}{\partial s_{i+1,1}}. \quad (3.11.10)$$

So

$$\frac{\partial L}{\partial s_i} = - \left[(M L^{i+1})_-, L \right]. \quad (3.11.11)$$

From (3.11.9) one derives the commutation relations of the *Virasoro algebra* (with zero central charge) for these vector fields

$$\left[\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j} \right] = (i - j) \frac{\partial}{\partial s_{i+j}}. \quad (3.11.12)$$

Exercise 3.11.4: Prove that the flows

$$\frac{\partial L}{\partial \bar{s}_i} = - \left[(L^{i+1} M)_-, L \right]$$

coincide with (3.11.11).

We will now consider the subalgebra of the Virasoro algebra (3.11.12) of additional symmetries preserving the n -reduction of KP.

Lemma 3.11.5. *The n -reduction of KP given by the constraint*

$$\mathcal{L} := L^n = a \text{ differential operator} \quad (3.11.13)$$

is invariant with respect to the additional symmetries of the form $\partial/\partial s_{m n}$ for any $m \geq -1$. The dependence of the differential operator \mathcal{L} on the parameters of the additional symmetries is determined from the equations

$$\frac{\partial}{\partial s_{m n}} \mathcal{L} = [\mathcal{M}_m, \mathcal{L}] + n \mathcal{L}^{m+1}, \quad \mathcal{M}_m := (M L^{m n+1})_+. \quad (3.11.14)$$

Proof. Let us first prove (3.11.14). From

$$[L, M L^{m n+1}] = L^{m n+1}$$

(see (3.11.5)) it follows

$$[L^k, M L^{m n+1}] = k L^{m n+k}$$

for any $k \geq 1$. For $k = n$ it follows that

$$[\mathcal{L}, M L^{m n+1}] = n \mathcal{L}^{m+1}.$$

Hence

$$\frac{\partial \mathcal{L}}{\partial s_{m n}} = - \left[(M L^{m n+1})_-, \mathcal{L} \right] = \left[(M L^{m n+1})_+, \mathcal{L} \right] - [M L^{m n+1}, \mathcal{L}] = [\mathcal{M}_m, \mathcal{L}] + n \mathcal{L}^{m+1}.$$

Observe now that for $m \geq -1$ all the operators in the right hand side of (3.11.14) are purely differential⁴. This proves invariance of (3.11.13). The theorem is proved.

3.12 NLS equation and KP

We will now explain how to construct solutions to the *nonlinear Schrödinger equation* (NLS) using the dual BA functions.

Let us assume that there exists a rational map

$$\lambda : \Gamma \rightarrow \mathbf{P}^1$$

such that

$$\lambda^{-1}(\infty) = Q_+ + Q_-, \quad Q_+ = Q. \quad (3.12.1)$$

Clearly, in that case Γ must be hyperelliptic. Denoting P_1, \dots, P_{2g+2} the zeroes of the differential $d\lambda$ we obtain the equation of the Riemann surface in the form

$$\mu^2 = R(\lambda) \quad \text{where} \quad R(\lambda) = \prod_{i=1}^{2g+2} (\lambda - \lambda_i), \quad \lambda_i := \lambda(P_i). \quad (3.12.2)$$

The points Q_+ and Q_- are the two infinite points of (3.12.2). We choose

$$k(P) = \lambda(P), \quad P \rightarrow Q_+ \quad (3.12.3)$$

as the (inverse) local parameter near Q_+ and construct the BA function

$$\psi(x, t; P) = \left(1 + \frac{\xi_1(x, t)}{k} + \frac{\xi_2(x, t)}{k^2} + \dots \right) e^{k x + k^2 t} \quad (3.12.4)$$

with the essential singularity at $P = Q_+$ and a nonspecial degree g divisor D . As usual, put

$$u(x, t) = -2\partial_x \xi_1(x, t). \quad (3.12.5)$$

Let

$$\psi^\dagger(x, t; P) = \left(1 + \frac{\xi_1^\dagger(x, t)}{k} + \frac{\xi_2^\dagger(x, t)}{k^2} + \dots \right) e^{-k x - k^2 t} \quad (3.12.6)$$

⁴Actually, \mathcal{M}_m is a finite order differential operator when only finite number of time variables are nonzero.

be the dual BA function. Let Ω_D be the second kind differential on Γ with zeroes at D and double pole at Q_+ normalized as

$$\Omega_D = \left(1 + O\left(\frac{1}{k^2}\right)\right) dk.$$

Denote

$$\varphi(x, t) := \psi(x, t; Q_-), \quad \varphi^\dagger(x, t) := \psi^\dagger(x, t; Q_-). \quad (3.12.7)$$

Lemma 3.12.1. *The functions $\varphi = \varphi(x, t)$, $\varphi^\dagger = \varphi^\dagger(x, t)$, $u = u(x, t)$ satisfy the system*

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \varphi_{xx} + u \varphi \\ -\frac{\partial \varphi^\dagger}{\partial t} &= \varphi^\dagger_{xx} + u \varphi^\dagger \\ u &= -2\sigma \varphi \varphi^\dagger - u_0 \end{aligned} \quad (3.12.8)$$

where u_0 is some constant,

$$\sigma = -\operatorname{Res}_{P=Q_-} \lambda(P) \Omega_D(P). \quad (3.12.9)$$

Proof. The first two equations are already known (see Lemma 3.9.2 above). To prove the last equation let us consider the Abelian differential

$$\Omega(x, t; P) := \psi(x, t; P) \psi^\dagger(x, t; P) \Omega_D(P). \quad (3.12.10)$$

The differential has a double pole at $P = Q_+$ and no other poles. Near Q_+ it has the expansion

$$\Omega(x, t; P) = \left(1 - \frac{u(x, t) + u_0}{2k^2} + \dots\right) dk$$

(see exercise 3.9.3) with some constant u_0 . Applying the residue theorem to the meromorphic differential $\lambda(P) \Omega(x, t; P)$ one obtains the third equation of (3.12.8). The Lemma is proved.

Exercise 3.12.2: Let Q_+ be the infinite point on Γ defined by

$$Q_+ : \quad \lambda \rightarrow \infty, \quad \frac{\mu}{\lambda^{g+1}} \rightarrow +1.$$

Another infinite point Q_- then is defined by

$$Q_- : \quad \lambda \rightarrow \infty, \quad \frac{\mu}{\lambda^{g+1}} \rightarrow -1.$$

Prove that the differential Ω_D must have the form

$$\Omega_D = \frac{1}{2} \left(1 + \frac{P(\lambda)}{\mu}\right) d\lambda$$

where $P(\lambda)$ is a polynomial of degree $g + 1$ having the form

$$P(\lambda) = \lambda^{g+1} - a \lambda^g + \dots, \quad a = \frac{1}{2} \sum_{j=1}^{2g+2} \lambda_j.$$

The $2g$ points (γ_i, μ_i) , $i = 1, \dots, 2g$ of the divisor $D + D^\dagger$ are determined from the equations

$$R(\gamma_i) = P^2(\gamma_i), \quad \mu_i = -P(\gamma_i), \quad i = 1, \dots, 2g.$$

Let us assume now existence of an antiholomorphic involution $\sigma : \Gamma \rightarrow \Gamma$ such that

$$\sigma^* \lambda = \bar{\lambda}.$$

The points Q_\pm then are stable under σ . Let us assume that the real Riemann surface (Γ, σ) and the divisor D satisfy all the conditions of Theorem ??.

Theorem 3.12.3. *Let $\psi = \psi(x, t; P)$ be the BA function with the essential singularity at $P = Q_+$ of the form*

$$\psi = \left(1 + \frac{\xi_1(x, t)}{k} + \dots \right) e^{i k x + i k^2 t}$$

and with poles at the divisor D . Then the function

$$\varphi(x, t) := \psi(x, t; Q_-)$$

satisfy the NLS equation

$$i \varphi_t = \varphi_{xx} + (2\sigma |\varphi|^2 + u_0) \varphi. \quad (3.12.11)$$

It is clear that the constant σ defined in (3.12.9) is real. The case $\sigma > 0$ corresponds to *focusing* NLS; the case $\sigma < 0$ gives *defocusing* NLS.

Exercise 3.12.4: Prove that for $\sigma < 0$ all the branch points λ_i of the hyperelliptic curve (3.12.2) are real. For the focusing case $\sigma > 0$ prove that all them are non real.

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