Optimal Transport and Geometric Inequalities

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Plan of the talk

Aim

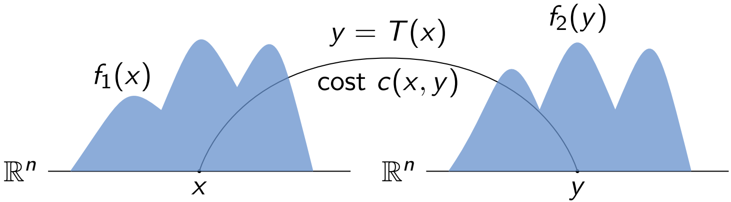
Present some applications of Optimal Transport to geometric inequalities for smooth/non-smooth manifolds.

Plan

- General overview on Optimal Transport
- Theory of Curvature-Dimension condition
- Functional Inequalities: Levy-Gromov isoperimetric inequality

Optimal Transport: Formulation

How to minimise total transport cost? (Monge 1781, Kantorovich 1942)



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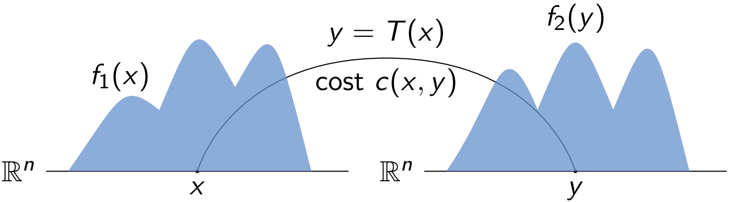

If $\int f_1 = \int f_2$, T is a *transport map* from f_1 to f_2 iff for any $A \subset \mathbb{R}^n$

$$\int_A f_2(x)\,dx = \int_{T^{-1}(A)} f_1(x)\,dx, \quad i.e.\ T_\sharp(f_1\,dx) = f_2\,dx,$$

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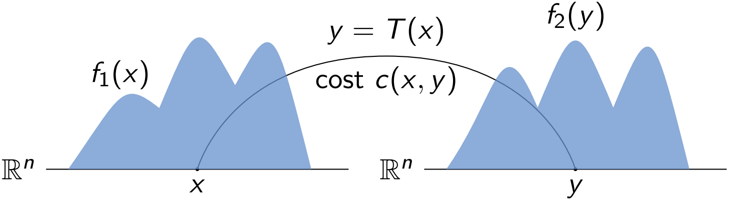

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Given a cost fuction $c(x, y)$, the *total transport cost* of T is

$$C(T) = \int_{\mathbb{R}^n} c(x, T(x)) f_1(x)\, dx.$$

Optimal Transport: Monge problem

Monge Optimal transport problem minimize

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Main issues with the minimization problem

- T is (smooth) transport map iff $f_2(T(x))|\det DT(x)| = f_1(x)$. Highly non-linear constrain.
- The set of transport maps is not closed in any reasonable topology.
- Replace f_1, f_2 with any $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^n)$ to obtain the general Monge problem: the set of transport maps can be empty.

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$\rightsquigarrow$ Kantorovich relaxation rewrite the total transportation cost

$$\int_{\mathbb{R}^n} c(x, T(x)) \mu_1(dx) = \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) ((id, T)_\sharp \mu_1)(dxdy)$$

Optimal Transport: Monge-Kantorovich problem

A transport map T seen as a measure on its graph $(id, T)_\sharp \mu_1$ becomes a *transport plan*

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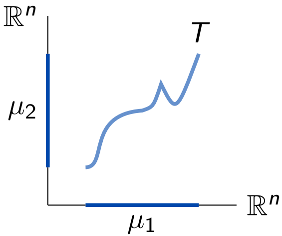

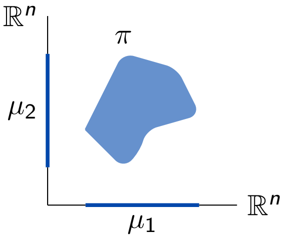

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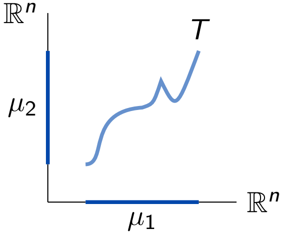

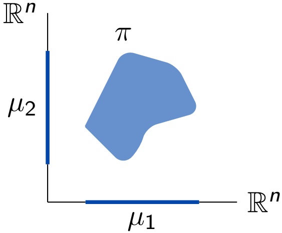

Set of transport plans is weakly closed and convex.
Monge-Kantorovich problem minimize the linear functional

$$\Pi(\mu_1, \mu_2) \ni \pi \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) \pi(dxdy).$$

If $c : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is l.s.c., existence of a solution.

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$$T(x) = \exp_x(-\nabla\psi(x)), \quad \psi : M^n \to \mathbb{R},\ d_g^2 - concave,$$

$\psi^{cc} = \psi$ where $\psi^c(y) = \inf_{x\in M} \frac{d_g^2(x,y)}{2} - \psi(x)$.

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- $X = M^n$ Riem. mfld $c = d_g$ (Feldman, McCann).

Given μ_1, μ_2 there exists $u : X \to \mathbb{R}$ 1-Lipschitz function so that

$$\pi \text{ optimal } \iff \pi(\{(x,y) \colon u(x) - u(y) = d_g(x,y)\}) = 1.$$

Optimal path for $c = d_g$ are along steepest descent of u.

Optimal Transport: cost function c

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- Analysis and PDEs: Gradient flows, JKO scheme. Monge-Ampere equation ($c(x, y) = |x - y|^2$).
- Physics: Random matching problem (sqrd Eucl. dist.), Density Functional Theory (Coulomb cost), Einstein equation of general relativity (Lorentzian cost function).
- Geometry of metric spaces: new class of metric spaces by Lott-Sturm-Villani verifying $Ric \geq K$ and $dim \leq N$ in a synthetic sense, called $\mathrm{CD}(K, N)$.
- Data science and Economy: Entropic regularisation (Sinkhorn, Shrodinger problem), mixed problems (Hellinger-Kantorovich).

Geometry of metric spaces: basics

Let (M^n, g) be an n-dimensional Riemannian manifold. Denote Sec the sectional curvature and Ric the Ricci curvature.

- For $K \in \mathbb{R}$ we write $Sec \geq K$ (resp.$\leq K$) if for every $p \in M$ and every 2-dim plane $\Pi \subset T_pM$ it holds $Sec_p(\Pi) \geq K$ (resp. $\leq K$).
- $Ric_p : T_pM \times T_pM \to \mathbb{R}$ is a quadratic form. We write $Ric \geq K$ (resp. $\leq K$) if the quadratic form $Ric_p - Kg_p$ is non-negative (resp. non-positive) definite at every $p \in M$.

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Examples

- n-dimensional euclidean space: $Sec \equiv 0, Ric \equiv 0$.
- n-dimensional round sphere of radius 1: $Sec \equiv 1, Ric \equiv n-1$.
- n-dimensional hyperbolic space: $Sec \equiv -1, Ric \equiv -(n-1)$.

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Natural question (M, g) smooth Riem. manifold. Assume some upper/lower bounds on Sec or on Ric; what can we say on (M, g)?

Basics on comparison geometry

- Upper/Lower bounds on the *Sec* are strong assumptions with strong implications (definition of Alexandrov spaces: non smooth spaces with upper/lower bounds on *Sec*).
- Upper bounds on the Ricci curvature are very (too) weak assumption for geometric conclusions. Lokhamp Theorem: any closed mfld of $dim \geq 3$ carries a metric with negative Ric.

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Lower bounds on the Ric natural framework for comparison geom.

- Bishop-Gromov volume comparison: If $Ric \geq 0$ then for all $x \in M$ $R \to vol_g(B_R(x))/\omega_N R^N$ is monotone non-increasing
- Laplacian comparison,
- Cheeger-Gromoll splitting,
- Levy-Gromov isoperimetric inequality
- ...

Origin of the topic: motivation

Gromov in the '80ies:

- notion of convergence for Riemannian manifolds: Gromov-Hausdorff convergence (for non-compact manifolds, more convenient a pointed version, called pointed Gromov-Hausdorff convergence $\sim$ GH-convergence of metric balls of every fixed radius).

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Big Question what about the compactification of the space of Riem. mfld with Ricci curvature bounded below (by, say, -1)?

Hope useful also to establish properties for smooth manifolds.

Origin of the topic: extrinsic Vs intrinsic

Cheeger-Colding 1997-2000: three fundamental works on the structure of Ricci limit spaces.
Non-intrinsic point of view consider the non-smooth space arising as limits of smooth objects. Dichotomy collapsing (loss of dim in the limit)-non collapsing. Very powerful for local struct. properties.

Analogy Define $W^{1,2}$ as completion of C^∞ endwed with $W^{1,2}$-dist.

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Analogy Define $W^{1,2}$ as completion of C^∞ endwed with $W^{1,2}$-dist.

$W^{1,2}$ can be defined also in completely intrinsic way without passing via approximations (very convenient for doing calculus of variations).

Role of OT define in an intrisic-axiomatic way a non-smooth space with Ricci curvature bounded below by K and dimension bounded above by N (containing ricci limits no matter if collapsed or not).

$\Longrightarrow$ Weak version of a Riemannian manifold with $Ric \geq K$.

Optimal Transport: Cornerstone

Interplay of Optimal Transport, entropy and curvature

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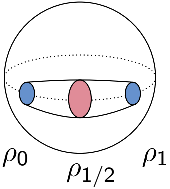
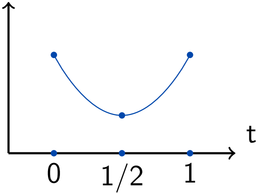
- **Ricci curvature** in terms of geodesic convexity of entropy along L^2 Optimal Transport, $c(x,y) = d_g^2(x,y)$ (Lott-Villani, Sturm '06)

$$Ent(\rho) = \int \rho(x) \log \rho(x) dx$$





Giving: $Ric \geq K$ **if and only if** $Hess\, Ent \geq K$.

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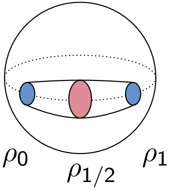

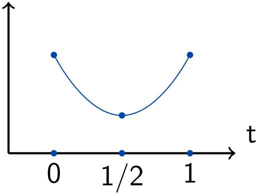

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- **LSV theory:** new approach to non-smooth metric spaces
  Examples: manifolds with $Ric \geq K$, Alexandrov spaces, normed and Finsler spaces, limits of those spaces