Universality of Causet d'Alembertians in curved spacetime and the mass operator problem

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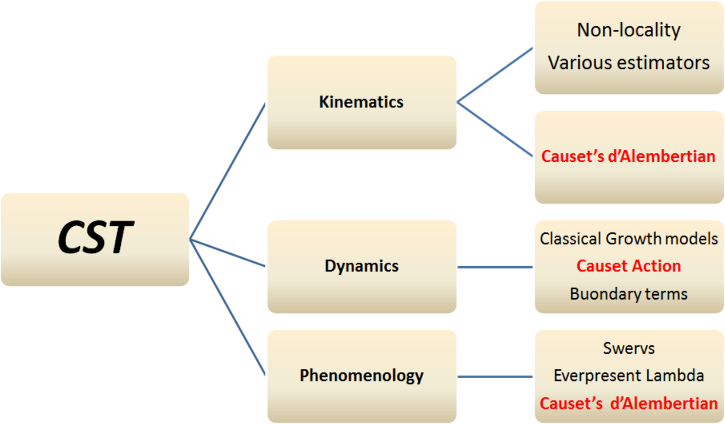
Overview

Casual–Set theory (CST) is a mathematically elegant proposal for QG taking seriously the various hints towards **discreteness** of spacetime and the increasing evidences of the fundamental nature of (local) **Lorentz Invariance**



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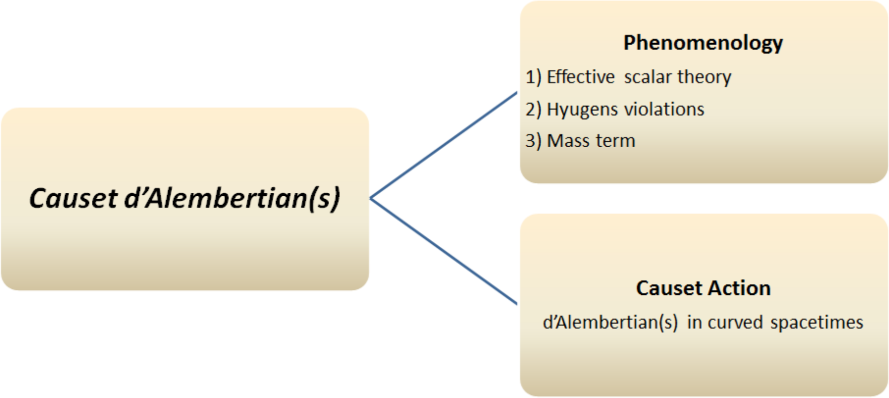

UNIVERSALITY

Generalized d'Alembertians [S. Aslanbeigi, M. Saravani, and R. D. Sorkin, *JHEP*, 1406:024, 2014]

Physical requirements

1. **Linearity**
2. **Retardedness**
3. **Label Invariance**
4. **Neighbourly democracy**

Generalized Causal–Set's d'Alembertians

$$(B_\rho^{(D)}\phi)(x) = \rho^{2/D}\left(a\,\phi(x) + \sum_{n=0}^{L_{max}} b_n \sum_{y\in I_n(x)} \phi(y)\right),$$

where $\rho = l_{nl}^{-D}$ and $L_{max} \geq L_{minimal}$

Generalized d'Alembertians [S. Aslanbeigi, M. Saravani, and R. D. Sorkin, *JHEP*, 1406:024, 2014]

We can now take the expectation value of the random variable just define over sprinklings of a generic spacetime $(\mathcal{M}, g)$, this leads to

Generalized Causal–Set's d'Alembertians: continuum version

$$\mathbb{E}(B_\rho^{(D)}\phi)(x) = \rho^{2/D} a\, \phi(x) + \rho^{(2+D)/D} \sum_{n=0}^{L_{max}} \frac{b_n}{n!} \int_{J^-(x)} \sqrt{-g} e^{-\rho V(x,y)} [\rho V(x,y)]^n \phi(y) d^D y,$$

where $J^-(x)$ is the causal past of x and $V(x,y)$ is the spacetime volume of the causal interval between x and y.

Flat spacetime: $\Box_\rho^{(D)} e^{ip\cdot x} = g_\rho^{(D)}(p) e^{ip\cdot x}$

General eigenvalue in integral form

$$g(p) = a\rho^{2/D} + 2(2\pi)^{D/2-1} \rho^{\frac{2+D}{2D}} p^{2\frac{2-D}{4}} \sum_{n=0}^{L_{max}} \frac{b_n}{n!} C_D^n \int_0^\infty s^{D(n+1/2)} e^{-C_D s^D} K_{\frac{D}{2}-1}\left(s\sqrt{\rho^{\frac{-2}{D}} p^2}\right) ds$$

Generalized d'Alembertians [S. Aslanbeigi, M. Saravani, and R. D. Sorkin, *JHEP*, 1406:024, 2014]

We can now take the expectation value of the random variable just define over sprinklings of a generic spacetime $(\mathcal{M}, g)$, this leads to

Generalized Causal–Set's d'Alembertians: continuum version

$$(\Box^{(D)}_{(g,\rho)}\phi)(x) = \rho^{2/D} a\, \phi(x) + \rho^{(2+D)/D}\hat{O} \int_{J^-(x)} \sqrt{-g} e^{-\rho V(x,y)} \phi(y) d^D y,$$

where $J^-(x)$ is the causal past of x and $V(x,y)$ is the spacetime volume of the causal interval between x and y.

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Generalized d'Alembertians: requiring the local limit

$$\hat{O} = \sum_{n=0}^{L_{max}} \frac{b_n}{n!} (-1)^n H_n, \qquad \text{where } H_n \equiv \rho^n \frac{\partial}{\partial \rho^n}.$$

$$\lim_{\rho\to\infty} \Box_\rho^{(D)} \phi(x) = \Box \phi(x)$$

Even Dimensions

$$\sum_{n=0}^{L_{max}} \frac{b_n}{n!} \Gamma(n + \frac{k+1}{N+1}) = 0, \quad k = 0, 1, \ldots, N+1$$

$$a + \frac{2(-1)^{N+1}\pi^N}{N! D^2 C_D} \sum_{n=0}^{L_{max}} b_n \psi(n+1) = 0$$

$$\sum_{n=0}^{L_{max}} \frac{b_n}{n!} \Gamma(n + \frac{N+2}{N+1}) \psi(n + \frac{N+2}{N+1}) = \frac{2(-1)^N (N+1)!}{\pi^N} D^2 C_D^{\frac{N+2}{N+1}},$$

where $D = 2N+2$ and $L_{minimal} = \frac{D+2}{2}$

Odd Dimensions

$$\sum_{n=0}^{L_{max}} \frac{b_n}{n!} \Gamma(n + \frac{2k+2}{2N+1}) = 0, \quad k = 0, 1, \ldots, N$$

$$a + \frac{(-1)^N \pi^{N+\frac{1}{2}}}{D C_D \Gamma(N+\frac{1}{2})} \sum_{n=0}^{L_{max}} b_n = 0$$

$$\sum_{n=0}^{L_{max}} \frac{b_n}{n!} \Gamma(n + \frac{2N+3}{2N+1}) = \frac{4(-1)^{N-1} \Gamma(N+\frac{3}{2})}{\pi^{N+\frac{1}{2}}} D C_D^{\frac{2N+3}{2N+1}}.$$

where $D = 2N+1$ and $L_{minimal} = \frac{D+1}{2}$

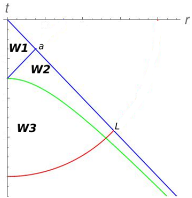
Universality in even dimension I

Starting Motivation:
Is it possible to find, in the family of Generalized Causet d'Alembertians, an operator that gives (say in 4D) a non-minimally coupled field with $\zeta = 1/6$ as required by EEP?

Universal behavior: $\lim_{\rho \to \infty} \Box^{(D)}_{(g,\rho)} \phi(x) = \Box^{(D)}_{(g)} \phi(x) - \frac{1}{2} R(x) \phi(x)$

- Already proved for the minimal operator in every dimension in [Lisa Glaser, *Class. Quantum Grav.* **31**, (2014)]
- We have obtained a new proof for the whole family of *Generalized Causal–Set d'Alembertian*
- **Key point:** the requirements that lead to the flat limit are sufficient to uniquely determine the curved spacetime limit
- Kind of *trivial a posteriori result* but nevertheless interesting since bridging between two formalisms

Universality in even dimension II



Let us concentrate on the **near region** *W1*:

$$\sqrt{-g} = 1 - \frac{1}{6} R_{\mu\nu}(0) y^\mu y^\nu + \mathcal{O}(R^2) \equiv 1 + \delta\sqrt{-g} + \mathcal{O}(R^2)$$

$$V(y) = V_0^D \left(1 - \frac{D}{24(D+1)(D+2)} R(0)\tau^2 + \frac{D}{24(D+1)} R_{\mu\nu}(0) y^\mu y^\nu + \mathcal{O}(R^2)\right)$$

$$\phi(y) = \phi(0) + y^\mu \phi_{,\mu}(0) + \frac{1}{2} y^\nu y^\mu \phi_{,\nu\mu}(0) + y^\mu y^\nu y^\sigma \Phi_{\mu\nu\sigma}(y),$$

$$\begin{aligned} I^{(D)} = & \int_0^{\tilde{a}} dv \int_0^v du \, \frac{(v-u)^{D-2}}{2^{(D-2)/2}} \\ & \Big[(D-1)\omega_{D-1}\phi(0) + (D-1)\omega_{D-1} y^0 \phi_{,0}(0) \\ & + \frac{(u+v)^2}{2} \omega_{D-1} (D-1) \left(\frac{1}{2}\phi_{,00}(0) - \frac{1}{6} R_{00}(0)\phi(0) - \rho\,\phi(0) c_D (uv)^{D/2} \frac{D}{24(D+1)} R_{00}(0) \right) \\ & + \frac{(v-u)^2}{2} \omega_{D-1} \left(\frac{1}{2}\phi_{,ii}(0) - \frac{1}{6} R_{ii}(0)\phi(0) - \rho\,\phi(0) c_D (uv)^{D/2} \frac{D}{24(D+1)} R_{ii}(0) \right) \\ & + \omega_{D-1}(D-1)\rho\,\phi(0) c_D (uv)^{1+D/2} \frac{2D}{24(D+2)(D+1)} R(0) \Big] \, e^{-\rho V_0}, \end{aligned}$$

Universality in even dimension III

$\hat{O}$ *killing* properties:

$$\boxed{\hat{O}\rho^{-\frac{2(k+1)}{D}} = 0, \quad k = 0, 1, 2, \ldots, \frac{D}{2}} \quad \Leftrightarrow$$

$$\sum_{n=0}^{L_{max}} \frac{b_n}{n!}\Gamma(n + \frac{k+1}{N+1}) = 0, \quad k = 0, 1, \ldots, N+1$$

$$a + \frac{2(-1)^{N+1}\pi^N}{N!D^2C_D}\sum_{n=0}^{L_{max}} b_n\psi(n+1) = 0$$

+ eliminating the constant

Requiring the **local limit** in flat spacetime specify (obviously) the last equation. Until here we have (only) bridged between two formalism. From the local limit in flat spacetime we have that

$$\rho^{\frac{D+2}{D}}\hat{O}\frac{\Gamma[\frac{D+2}{D}]}{2(\frac{D}{2})^2(c_D)^{\frac{D+2}{D}}\rho^{\frac{D+2}{D}}}\log(\tilde{a}^D c_D\rho) = \frac{2^{\frac{D+2}{2}}}{\omega_{D-1}A_2},$$

Universality in even dimension IV

$$\frac{1}{2^{\frac{D-2}{2}}}\Bigg\{(D-1)\omega_{D-1}\phi(0)A_0 I_{\frac{D-2}{2},\frac{D-2}{2}} + A_1\left[\frac{1}{2}\omega_{D-1}\left(\frac{D-1}{2}\phi_{,00}(0) - \frac{1}{6}\phi(0)(D-1)R_{00}\right)\right] I_{D/2,D/2}$$
$$+A_2\left[\frac{1}{2}\omega_{D-1}\left(\frac{1}{2}\phi_{,ii} - \frac{1}{6}\phi(0)R_{ii}\right)\right] I_{D/2,D/2} + A_1\left[\frac{\omega_{D-1}}{2}\left(-\frac{D}{24(D+1)}c_D\phi(0)(D-1)R_{00}\right)\right]\rho I_{D,D}$$
$$+A_2\left[\frac{\omega_{D-1}}{2}\left(-\frac{D}{24(D+1)}c_D\phi(0)R_{ii}\right)\right]\rho I_{D,D} + A_5\left[\frac{2D}{24(D+1)(D+2)}\omega_{D-1}(D-1)c_D\phi(0)R\right]\rho I_{D,D}\Bigg\}.$$

Red terms: they are determined by the previous requirements and give in the limit

$$-\frac{1}{3}R\,\phi$$

Green terms: they are determined by the action of $\hat{\mathcal{O}}$ on the integral I_{DD}. It can be shown that again also this result is determined by the previous requirements. They give

$$-\frac{1}{6}R\,\phi$$

Universality in even dimension: Conclusion and Discussion

In the end we have our universality result:

$$\Box_{(g,\rho)^D}\phi(x) \to \Box_g\phi(x) - \frac{1}{2}R(x)\phi(x) \quad \forall\, D \text{ and } \forall\, \textit{Generalized Causal-set d'Alembertians}$$

- **Basic physical requirements + consistency with local physics in flat spacetime** directly imply the local limit result in **curved spacetimes**
- What about the EEP?

Outlook:

- Which physical condition can be changed in order to find another local limit in CS?
- or reversing the argument which are the consequences of this non-minimal coupling?
- What happen if we add mass?

MASS

The "mass problem" [AB, Benincasa, Liberati, *JHEP 1503 (2015) 036*; Saravani, Aslanbeigi, *arXiv:1502.01655*]

Let's go back to **flat spacetime** and consider the massive extension of the non-local d'Alembertians, i.e. **non-local KG equation**

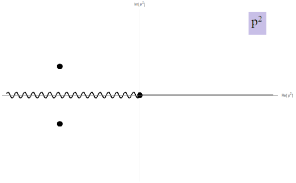
Naïve extension[1]:

$$f_\rho(\Box)\phi = 0 \rightarrow (f_\rho(\Box) - m^2)\phi = 0$$

This extension cannot work due to the fact that there are **no (on-shell) modes** satisfying this equation. Indeed we know that

$$g_\rho^D(p) = 0 \text{ for real momenta}$$

$$\text{iff } p^2 = 0$$



[1](-,+,+,+) signature is used.

The "mass problem" [AB, Benincasa, Liberati, *JHEP 1503 (2015) 036*; Saravani, Aslanbeigi, *arXiv:1502.01655*]

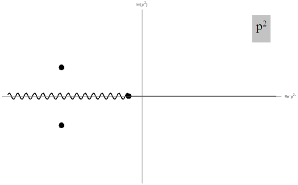
Smarter extension:

$$f_\rho(\Box)\phi = 0 \rightarrow f_\rho(\Box - m^2)\phi = 0$$

In this way we retain all the properties of the operators by simply shifting the physical pole from $p^2 = 0$ to $p^2 = -m^2$

$$g^D_\rho(-p^2 - m^2) = 0 \text{ for real momenta}$$

$$\text{iff } p^2 = -m^2$$



BUT:

at present we do **not know** which is (if any) the discrete version of the massive operator and so the **meaning of** a scalar field **mass** in the Causet

Rise and fall of a mass operator proposal: Rise I

Strategy: Find a continuum expression for a non-local KG operator similar to the d'Alembertian one in such a way that **it gives the right local limit** and it is possible to discretize it. This could give us an understanding of the mass term in Causet starting from continuum physics.

2D case:

Averaged proposal:

$$\mathbb{E}(B_m\phi(x)) = \left[\alpha\rho\phi(x) + 4\rho^2\hat{O}\int_{y\in J^-(x)} d^2y\phi(y)e^{-\rho\frac{1}{2}\tau^2}(1+\frac{1}{4}m^2\tau^2)\right]$$

Discrete version:

$$B_m\phi(x) = -2\rho\phi(x) + 4\rho\left(\sum_{y\in L_1} -(2-\frac{1}{2}\frac{m^2}{\rho})\sum_{y\in L_2} +(1-2\frac{m^2}{\rho})\sum_{y\in L_3} +\frac{3}{2}\frac{m^2}{\rho}\sum_{y\in L_4}\right)\phi(y)$$

Rise and fall of a mass operator proposal: Rise II

4D case:

Averaged proposal:

$$\mathbb{E}(B\phi) = \alpha\sqrt{\rho}\phi(x) + \rho^{3/2}\sum_{n=0}^{L'_{max}} \frac{c_n}{n!}\int_{J^-(x)} d^4y e^{-\rho V}(\rho V)^n \phi(y)$$

$$+ m^2\rho \sum_{n=0}^{L_{max}} \frac{b_n}{n!}\int_{J^-(x)} d^4y e^{-\rho V}(\rho V)^{n+1}\phi(y)$$

$$= \alpha\sqrt{\rho}\phi(x) + \rho^{3/2}\hat{O}\int_{J^-(x)} d^4y e^{-\rho V}\phi(y) + m^2\rho\tilde{O}\int_{J^-(x)} d^4y e^{-\rho V}\phi(y),$$

Discrete version:

$$B\phi(x) = \sqrt{\rho}\alpha\phi(x) + \sqrt{\rho}\sum_{n=0}^{L'_{max}} c_n \sum_{y\in L_n(x)}\phi(y) + m^2 \sum_{n=1}^{L_{max}+1} n b_{n-1}\sum_{y\in L_n(x)}\phi(y)$$

$$= \sqrt{\rho}\alpha\phi(x) + \sqrt{\rho}\sum_{n=0}^{L''_{max}}\left(c_n + \frac{m^2}{\sqrt{\rho}} n\, b_{n-1}(1-\delta_{n,0})\right)\sum_{y\in L_n(x)}\phi(y).$$

Rise and fall of a mass operator proposal: Observations

- In **2D** there is no contribution to the mass operator coming from the first layer (links are the nearest thing to null rays)
- In **2D** we need an extra layer in order to have the massive operator
- In **4D** again no contribution from links
- In **4D** again we need at least 5 layers when for the wave operators they were at least 4

Rise and fall of a mass operator proposal: Fall (?)

Hint to a failure in 2D:
From a brute force calculation in 2D it ***seems*** that the massive operator does not present the physical pole at

$$p^2 = -m^2$$

so not solving the "mass problem"

A more refined analysis has to be done both in 2D and 4D (using result for Laplace transform of LI functions). This leave open the possibility that, at least the 4D operator, can be a good candidate for a massive operator.

Rise and fall of a mass operator proposal: Conclusion and Discussion

- Massive extension of Causet's d'Alembertian
- Need to verify their pole structure and address the "mass problem"
- They could give some interesting insight on a possible Causet meaning of a scalar field mass
- What is the local limit of this operator in curved spacetime?
- Note that the latter will be different from the massless case!! Can we reach accordance with EEP? If no what effects may arise?

Thank you!

Backup Slides

Backup I: EP and couplings

$$(\eta^{ab}\partial_a\partial_b - m^2)\phi = 0 \quad \rightarrow \quad (g^{ab}\nabla_a\nabla_b - m^2 - \xi R)\phi = 0$$

- $\xi = 0$ **minimal coupling**, requiring the same formal structure
- $\xi = 1/6$ **conformal coupling**, conformal invariance and **EEP**

EP:
WEP: Universality of free fall
EEP: WEP+LLI+LPI
SEP, GWEP,.....

Local physics should be described by SR according to EEP⇒ Physical features of the solution should be locally the same

$$G_R(x', x) = \Sigma(x', x)\delta_R(\Gamma(x', x)) + V(x', x)\Theta_R(-\Gamma(x', x))$$

$$lim_{x' \rightarrow x}\Sigma(x', x) = lim_{x' \rightarrow x}\Sigma_M(x', x)$$
$$lim_{x' \rightarrow x}V(x', x) = lim_{x' \rightarrow x}V_M(x', x)$$

Back up II: EP and couplings

$$\Sigma(x',x) = \frac{1}{4\pi} + O_1(x',x)$$

$$\Sigma_M(x',x) = \frac{1}{4\pi}$$

$$V(x',x) = -\frac{1}{8\pi}\left[m^2 + \left(\xi - \frac{1}{6}R(x)\right)\right] + O_2(x',x)$$

$$V_M(x',x) = -\frac{m^2}{8\pi}$$

$$(g^{ab}\nabla_a\nabla_b - m^2 - \xi R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{abcd}R^{abcd} + \ldots.)\phi + \lambda F(\phi) = 0$$

EP require:

$$\xi = \frac{1}{6},\ \alpha = \beta = \gamma = \ldots.. = 0$$

Back up III: EP and couplings

A non vanishing $V(x', x)$ implies wave tails inside the light cone.

Flat spacetime:

Type **a**: due to $m \neq 0$, $lim_{x' \to x} V_M(x', x) \neq 0$

Type **b**: due to backscattering off a potential, $lim_{x' \to x} V_M(x', x) = 0$

In **curved spacetimes** we have the same classification.

If $\xi = 1/6$:

- Tails of type **a** iff $m \neq 0$

If $\xi \neq 1/6$:

- If $m = 0$ there could be type **a** tails (HP violations)
- When $m \neq 0$ if $m^2 + (\xi - 1/6)R(x) = 0$ then **no tails of type a**$\Rightarrow$ Propagation of massive particles on the light cone, unacceptable if local physics has to be described by SR according to EEP