Towards robust Lie-algebraic stability conditions for switched linear systems

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Abstract— **This paper presents new sufficient conditions for exponential stability of switched linear systems under arbitrary switching, which involve the commutators (Lie brackets) among the given matrices generating the switched system. The main novel feature of these stability criteria is that, unlike their earlier counterparts, they are robust with respect to small perturbations of the system parameters. Two distinct approaches are investigated. For discrete-time switched linear systems, we formulate a stability condition in terms of an explicit upper bound on the norms of the Lie brackets. For continuous-time switched linear systems, we develop two stability criteria which capture proximity of the associated matrix Lie algebra to a solvable or a "solvable plus compact" Lie algebra, respectively.**

I. Introduction

A *switched system* is described by a family of systems and a rule that orchestrates the switching between them (see [1] for an overview). In the large body of literature devoted to stability analysis of switched systems, a specific research direction that has received a lot of attention is to develop stability criteria that take into account *commutation relations* among the constituent systems. In the simplest case when these systems pairwise commute, stability is preserved under arbitrary switching; this can be shown either by directly studying the solutions (which is straightforward for linear systems but takes more effort for nonlinear systems [2]) or by constructing a common Lyapunov function (which was done for linear systems in [3], for nonlinear exponentially stable systems in [4], and for general nonlinear asymptotically stable systems in [5]). To build on this observation, one can consider the *Lie algebra* generated by the constituent systems (a matrix Lie algebra in the linear case or a Lie algebra of vector fields in general) and ask whether the structure of this Lie algebra can be used to verify stability of the switched system. Provided that the constituent systems are linear and stable, it was shown that the switched system remains stable under arbitrary switching if the Lie algebra is nilpotent [6], solvable [7], [8], or has a compact semisimple part [9], [10]; each of these classes of Lie algebras strictly contains the previous one, and the existence of a quadratic common Lyapunov function is guaranteed for all of them. Moreover, it was shown in [10] that no further generalization is possible based solely on the properties of the Lie algebra.

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For nonlinear systems the story is much less complete, but recently some results connecting Lie brackets and stability of switched nonlinear systems (beyond the already mentioned commuting case) were established in [11] and [12].

While mathematically quite elegant, the available stability conditions based on commutation relations suffer from one serious drawback: they are not robust with respect to small perturbations of the system data. For example, if we take two matrices that commute with each other and perturb one of them slightly, they will cease to commute. If we take a family of matrices generating a nilpotent or solvable matrix Lie algebra and introduce arbitrarily small errors in their entries, the new Lie algebra will no longer possess any helpful structure (see [10] for a precise result along these lines). For this reason, the results mentioned in the previous paragraph have very limited applicability and serve primarily academic interests. It is important to note that stability itself, as well as the existence of a (quadratic) common Lyapunov function, are properties that do have inherent robustness to small perturbations; see [1, Section 2.2.4] for a detailed discussion of this issue. Thus the indicated lack of robustness is a shortcoming of the existing stability tests and is not an attribute of the problem itself.

To get a handle on robustness and obtain more satisfactory results, we must characterize "closeness" of a given collection of systems to one with "nice" commutation relations. Rather than searching for nearby systems to which known results can be applied (which is in general not feasible), we want to be able to verify such closeness directly from the given data. This is the basic task pursued here. A preliminary attempt to tackle this problem was reported by one of the authors in the recent paper [13], but the results were restricted to periodic switching. Approximate simultaneous triangularization, which is a robust version of the condition that the Lie algebra is solvable, is briefly examined in another recent work [14]. In the present paper we propose, for the first time, conditions formulated directly in terms of Lie brackets which guarantee stability under arbitrary switching and are robust to small perturbations of system parameters.

We develop two distinct approaches. In Section II we consider discrete-time switched linear systems and present a stability criterion which involves an upper bound on the norms of the Lie brackets; the proof technique, which is quite direct, relies on splitting matrix products into sums and using a counting argument (this is a different twist on the approach of [13] which used discrete commutators to rearrange terms within a product). In Section III we study continuous-time switched linear systems and formulate two stability criteria which utilize the structure of the associated Lie algebra (provided by its Levi and Cartan decompositions); these results directly extend the previous work in [9] and [10] but have built-in robustness.

II. Discrete time: bounds on commutators

Consider a finite collection of matrices $A_1, \ldots, A_N \in \mathbb{R}^{n\times n}$. For each matrix A_i, consider the corresponding discrete-time linear system $x(k+1) = A_i x(k)$ in $\mathbb{R}^n$. The discrete-time switched system generated by these systems is the system

$$x(k+1) = A_{\sigma(k)} x(k) \tag{1}$$

where $\sigma : \{0, 1, \ldots\} \to \{1, \ldots, N\}$ is an arbitrary switching function.

We say that the switched system (1) is *globally uniformly exponentially stable* (GUES) if there exist positive numbers c and λ such that the solutions of (1) satisfy

$$|x(k)| \le c e^{-\lambda k} |x(0)| \qquad \forall\, k \ge 0 \tag{2}$$

for arbitrary choices of the initial condition $x(0)$ and the switching function $\sigma(\cdot)$, where $|\cdot|$ denotes the Euclidean norm (or any other norm on $\mathbb{R}^n$). This is the property of interest to us in this section. The term "uniform" refers to the fact that the single bound (2) covers all switching functions.

For a matrix W given by a product of A_i's (a "word" in A_i's), let us denote by $|W|$ the number of terms in this product (the length of the word) and by $\|W\|$ the induced norm of W with respect to the chosen norm on $\mathbb{R}^n$. This notation is convenient, as it lets us restate the GUES property (2) equivalently as the requirement that for all W it should hold that

$$\|W\| \le c e^{-\lambda |W|}. \tag{3}$$

Since constant switching functions are allowed, for the switched system to be GUES it is necessary that each matrix A_i be Schur stable. We henceforth assume that this is the case. Consequently, there is a positive integer m such that

$$\|A_i^m\| \le \rho < 1, \qquad i = 1, \ldots, N. \tag{4}$$

We also let

$$M := \max\{\|A_i\| : 1 \le i \le N\}. \tag{5}$$

We define E_{ij} to be the commutator—or Lie bracket—of A_i and A_j:

$$E_{ij} := A_i A_j - A_j A_i, \qquad 1 \le i, j \le N. \tag{6}$$

The following result gives an upper bound on the induced norms of E_{ij}'s which guarantees the GUES property. This bound depends on ρ, m, M, and N.

Proposition 1 *Let the matrices A_i, $i = 1, \ldots, N$ satisfy* (4) *for some $m \ge 1$. Let M be defined by* (5)*. Let λ be an arbitrary positive number satisfying*

$$\rho e^{\lambda m} < 1. \tag{7}$$

Assume that the matrices E_{ij} defined by (6) *satisfy*

$$\|E_{ij}\| \le \varepsilon \qquad \forall\, i, j \tag{8}$$

with ε small enough so that

$$\rho e^{\lambda m} + m(N-1)(m-1)\varepsilon M^{N(m-1)-1} e^{\lambda(N(m-1)+1)} \le 1. \tag{9}$$

Then there exists a number $c > 0$ such that all products W of A_i's satisfy (3)*, and consequently the discrete-time switched system* (1) *is GUES.*

Proof. We will prove (3) by induction on the product length $|W|$. For the induction basis, select a value for c such that (3) holds for all products W with $|W| \le N(m-1)+1$. This is possible because the total number of products of length at most $N(m-1)+1$ is finite, hence we just need to pick c large enough. Now, suppose that $|W| \ge N(m-1)+2$ and (3) has been proved for all products of length less than $|W|$. Write $W = LR$, where the length of the prefix L is $|L| = N(m-1)+1 = (N-1)(m-1)+m$. Then there exists an index i such that L contains at least m A_i's. Assume for concreteness that $i = 1$. We can then represent L as $L = A_1^m L_1 + L_2$, where $|L_1| = (N-1)(m-1)$ and L_2 is the sum of at most $m(N-1)(m-1)$ terms of length $N(m-1)$, each containing one E_{1i} for some i and $N(m-1)-1$ A_i's. For example, for $N = 2$ and $m = 3$, letting $A := A_1$, $B := A_2$, and $E := E_{12}$ for better readability, we have $A\underline{BA}BA = AAB\underline{BA} - AEBA = AA\underline{BA}B - AABE - AEBA = AAABB - AAEB - AABE - AEBA$ where, at each step, the underlined term is transformed using (6). The bound $m(N-1)(m-1)$ on the number of terms in L_2 comes from the fact that, e.g., $L = A_2^{m-1} \cdots A_N^{m-1} A_1^m$ gives a worst-case scenario (the largest number of terms in L_2). We can now calculate, using (4), (5) and the triangle inequality, that

$$\begin{aligned} \|W\| &= \|A_1^m L_1 R + L_2 R\| \le \rho c e^{-\lambda(|W|-m)} \\ &+ m(N-1)(m-1)\varepsilon M^{N(m-1)-1} c \times \\ &\times e^{-\lambda(|W|-N(m-1)-1)} = c e^{-\lambda|W|}\Big(\rho e^{\lambda m} \\ &+ m(N-1)(m-1)\varepsilon M^{N(m-1)-1} e^{\lambda(N(m-1)+1)}\Big) \end{aligned} \tag{10}$$

where, to obtain the inequality on the first line, in the first summand we used the induction hypothesis for $\|L_1 R\|$ and in the second just for $\|R\|$ (plus the submultiplicativity of the norm). Applying (9) immediately leads to (3), which in turn implies that solutions of the switched system satisfy the GUES bound (1). □

The novelty of Proposition 1 is that stability of the switched system is deduced from properties of the commutators, yet these commutators do not need to vanish completely. Checking the hypotheses requires only elementary matrix computations. On the other hand, generalization to higher-order commutators does not appear to be straightforward.

Example 1 Consider the matrices $A_1 = \left(\begin{smallmatrix} 0.1 & -2 \\ \delta & 0.1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 0.2 & -1.5 \\ \delta & 0.2 \end{smallmatrix}\right)$ where $\delta > 0$ is a parameter. These matrices can be viewed as perturbations of the commuting matrices

$\begin{pmatrix} 0.1 & -2 \\ 0 & 0.1 \end{pmatrix}$ and $\begin{pmatrix} 0.2 & -1.5 \\ 0 & 0.2 \end{pmatrix}$ with the perturbation being given by δ. The same example was studied in [13], and the approach developed in that paper showed exponential stability under 1-periodic switching ($A_1A_2A_1A_2\cdots$) for δ up to about 0.352. (That bound is known to be conservative: it can be checked that A_1A_2 remains Schur until $\delta \approx 0.6$, guaranteeing stability under 1-periodic switching.) The current approach shows GUES—which is a stronger property since the switching is arbitrary—for δ up to about 0.16. For $\delta = 0.16$, we have $m = 2$, $\rho = 0.6556$, $M = 2.0046$, $\varepsilon = 0.08$, and (9) holds for λ close enough to 0 (the left-hand side equals 0.9763 for $\lambda = 0$). As already mentioned in the introduction, for systems with commuting matrices a quadratic common Lyapunov function can be used to show stability and also characterize robustness to small perturbations. In this example, the function $x^T Px$ with $P := \begin{pmatrix} 1 & 0 \\ 0 & 4.5 \end{pmatrix}$ serves as a quadratic common Lyapunov function for the two systems for δ up to about 0.46, guaranteeing GUES. On the other hand, the argument we used here to verify stability is direct from the given data. □

We can obtain a refined result if we suppose that for some $L \leq N$ the matrices $A_{L+1}, \dots, A_N$ can be represented as linear combinations of $A_1, \dots, A_L$. Our next result generalizes Proposition 1 in this direction (it reduces to Proposition 1 when $L = N$).

Proposition 2 *Let $A_1, \dots, A_L$, $L \leq N$ be a subset of the given set of matrices $A_1, \dots, A_N$ such that $\|A_i^m\| \leq \rho < 1$, $i = 1, \dots, L$ for some $m \geq 1$, and such that each of the remaining matrices $A_{L+1}, \dots, A_N$ can be written as*

$$A_i = \alpha_{i1}A_1 + \cdots + \alpha_{iL}A_L \tag{11}$$

with real coefficients α_{ij}, $L + 1 \leq i \leq N$, $1 \leq j \leq L$ satisfying

$$\sum_{j=1}^{L} |\alpha_{ij}| \leq 1, \qquad i = L+1, \dots, N. \tag{12}$$

Define M by $M := \max\{\|A_i\| : 1 \leq i \leq L\}$. Let λ be an arbitrary positive number fulfilling (7). *Assume that the commutators $E_{ij} := A_iA_j - A_jA_i$, $1 \leq i, j \leq L$ satisfy* (8) *with ε small enough so that*

$$\rho e^{\lambda m} + m(L-1)(m-1)\varepsilon M^{L(m-1)-1} e^{\lambda(L(m-1)+1)} \leq 1. \tag{13}$$

Then there exists a number $c > 0$ such that all products W of $A_1, \dots, A_N$ satisfy (3)*, and consequently the discrete-time switched system* (1) *is GUES.*

PROOF. By (11), each word W in A_i's can be written as

$$\begin{aligned} W = D_0(\alpha_{i_11}A_1 + \cdots + \alpha_{i_1L}A_L)D_1(\alpha_{i_21}A_1 + \dots \\ + \alpha_{i_2L}A_L)\cdots(\alpha_{i_K1}A_1 + \cdots + \alpha_{i_KL}A_L)D_K \end{aligned} \tag{14}$$

where $K \leq |W|$ and $D_0, \dots, D_K$ are (possibly empty) products of $A_1, \dots, A_L$. This product can be expanded into a sum of at most L^K products of length $|W|$ involving only $A_1, \dots, A_L$, each multiplied by a product of appropriate coefficients α_{ij}: $W = \sum_{1 \leq j_1, \dots, j_K \leq L} \alpha_{i_1j_1}\alpha_{i_2j_2}\cdots\alpha_{i_Kj_K} D_0 A_{j_1} \cdots A_{j_K} D_K$. Thus the norm of W satisfies, by the triangle inequality,

$$\begin{aligned} \|W\| \leq \sum_{1 \leq j_1, \dots, j_K \leq L} |\alpha_{i_1j_1}||\alpha_{i_2j_2}|\cdots|\alpha_{i_Kj_K}| \times \\ \times \|D_0 A_{j_1} \cdots A_{j_K} D_K\|. \end{aligned} \tag{15}$$

In view of (13), we can apply Proposition 1 with L instead of N to each induced norm $\|D_0 A_{j_1} \cdots A_{j_K} D_K\|$ appearing in (15), concluding that this norm does not exceed $ce^{-\lambda|W|}$. Pulling out this common bound and then returning the sum of products of $|\alpha_{ij}|$'s into the original factored form similar to (14), we arrive at

$$\begin{aligned} \|W\| \leq (|\alpha_{i_11}| + \cdots + |\alpha_{i_1L}|)(|\alpha_{i_21}| + \dots \\ + |\alpha_{i_2L}|)\cdots(|\alpha_{i_K1}| + \cdots + |\alpha_{i_KL}|)ce^{-\lambda|W|} \end{aligned} \tag{16}$$

and the result now follows from (12). □

The condition (13) is of the same form as (9) but has L in place of N. It gives a sharper bound on ε which in principle allows us to replace the set of all matrices by, for example, a subset of linearly independent ones. On the other hand, we must also check the condition (12) because otherwise the bound (16) in general grows with $|W|$. So, to apply Proposition 2 we can look for the smallest number L of matrices—not necessarily linearly independent—such that all the other matrices can be expressed as linear combinations of these L matrices with coefficients satisfying (12), i.e., lie in their "symmetric convex hull."

III. Continuous time: structure of the Lie algebra

In this section we consider a compact (with respect to the usual topology in $\mathbb{R}^{n \times n}$) set of real $n \times n$ matrices $\{A_p : p \in \mathcal{P}\}$, where $\mathcal{P}$ is an index set, and the corresponding continuous-time linear systems $\dot{x} = A_p x$ in $\mathbb{R}^n$. These systems generate the continuous-time switched system

$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{17}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant switching signal.

The GUES property for this switched system is the obvious continuous-time counterpart of the GUES property studied in the previous section. Namely, we say that the switched system (17) is GUES if there exist positive numbers c and λ such that the solutions of (17) satisfy

$$|x(t)| \leq ce^{-\lambda t}|x(0)| \qquad \forall\, t \geq 0 \tag{18}$$

for arbitrary choices of the initial condition $x(0)$ and the switching signal $\sigma(\cdot)$. Since this is the property we are seeking, we assume throughout that the matrices A_p, $p \in \mathcal{P}$ are all Hurwitz stable. (In fact, each of the stability conditions developed below will imply this.)

We will be working with the Lie algebra (over $\mathbb{R}$) generated by the matrices A_p, $p \in \mathcal{P}$, which we denote as $\mathfrak{g}$:

$$\mathfrak{g} := \{A_p : p \in \mathcal{P}\}_{LA}. \tag{19}$$

This is a linear vector space (of dimension at most n^2) spanned by the given matrices and all their iterated Lie brackets. We refer the reader to the appendices in [10] and [1] for a summary of necessary background on Lie algebras.

A. Levi decomposition

Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be the Levi decomposition of $\mathfrak{g}$, where $\mathfrak{r}$ is the radical (the maximal solvable ideal) and $\mathfrak{s}$ is a semisimple subalgebra. For each $p \in \mathcal{P}$, we can then write

$$A_p = R_p + S_p, \qquad R_p \in \mathfrak{r}, \quad S_p \in \mathfrak{s}. \tag{20}$$

The sets $\{R_p : p \in \mathcal{P}\}$ and $\{S_p : p \in \mathcal{P}\}$ are both compact as intersections of the compact set $\{A_p : p \in \mathcal{P}\}$ with the subspaces forming the Levi decomposition.

Denote by $\Phi(t, 0)$, or simply $\Phi(t)$, the transition matrix for the switched system, with initial time 0. It satisfies the matrix differential equation

$$\dot{\Phi}(t) = A_{\sigma(t)}\Phi(t) = (R_{\sigma(t)} + S_{\sigma(t)})\Phi(t), \quad \Phi(0) = I. \tag{21}$$

The following decomposition is well-known [15] and easily verified.

Lemma 1 *The matrix $\Phi(t)$ from* (21) *can be represented as*

$$\Phi(t) = \Phi_S(t)\Phi_R(t) \tag{22}$$

where

$$\dot{\Phi}_S(t) = S_{\sigma(t)}\Phi_S(t), \qquad \Phi_S(0) = I \tag{23}$$

and $\dot{\Phi}_R(t) = \big(\Phi_S^{-1}(t)R_{\sigma(t)}\Phi_S(t)\big)\Phi_R(t) =: C(t)\Phi_R(t)$, $\Phi_R(0) = I$.

Let

$$\bar{\lambda}_R := \max\{\operatorname{Re}\lambda : \lambda \in \operatorname{spec}(R_p),\, p \in \mathcal{P}\} \tag{24}$$

where $\operatorname{spec}(\cdot)$ is the set of eigenvalues of a matrix. Also, let

$$\lambda_S^* := \limsup_{t\to\infty} \frac{1}{t}\log\|\Phi_S(t)\| \tag{25}$$

which is the *characteristic exponent* of the system (23). We then have the following result.

Proposition 3 *Let each matrix A_p be written as in* (20) *in accordance with a Levi decomposition of the Lie algebra* (19)*. Assume that*

$$\bar{\lambda}_R + \lambda_S^* < 0 \tag{26}$$

where λ_R and λ_S^* are defined by (24) *and* (25)*, respectively. Then the continuous-time switched system* (17) *is GUES.*

Before we prove Proposition 3, we note that the condition (26) is difficult to verify because it involves estimating the characteristic exponent λ_S^* of the switched system (23). We can obtain a more useful stability condition by giving an upper bound on λ_S^*. As we show next, one such upper bound is

$$\hat{\lambda}_S := \max\{\|S_p\| : p \in \mathcal{P}\}. \tag{27}$$

Corollary 1 *Assume that*

$$\bar{\lambda}_R + \hat{\lambda}_S < 0 \tag{28}$$

where λ_R and $\hat{\lambda}_S$ are defined by (24) *and* (27)*, respectively. Then the continuous-time switched system* (17) *is GUES.*

PROOF OF COROLLARY 1. From (23) we have

$$\Phi_S(t) = e^{S_{\sigma(t_k)}(t-t_k)} \cdots e^{S_{\sigma(t_1)}(t_2-t_1)} e^{S_{\sigma(0)}t_1} \tag{29}$$

where $t_1, \dots, t_k$ are the discontinuities (switching times) of σ on the interval $(0, t)$. This yields the following (conservative) bound: $\|\Phi_S(t)\| \le e^{\|S_{\sigma(t_k)}\|(t-t_k)} \cdots e^{\|S_{\sigma(t_1)}\|(t_2-t_1)} e^{\|S_{\sigma(0)}\|t_1}$. In view of (27) we obtain $\|\Phi_S(t)\| \le e^{\hat{\lambda}_S t}$. Comparing this with the definition (25) of λ_S^*, we conclude that

$$\lambda_S^* \le \hat{\lambda}_S. \tag{30}$$

Thus (28) implies (26) and the result follows from Proposition 3. $\square$

PROOF OF PROPOSITION 3. We take advantage of Lemma 9.13 from [16] which provides additional structure for the matrices R_p and S_p (this structure has not been used in the previous works [9], [10]). That result says that in a suitable basis (over $\mathbb{C}$), all matrices $R_p \in \mathfrak{r}$ take the form

$$R_p = \begin{pmatrix} \lambda_{p,1} & \cdots & 0 & * & \cdots & * & * & \cdots & * & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & \cdots & \lambda_{p,1} & * & \cdots & * & * & \cdots & * & \cdots \\ 0 & \cdots & 0 & \lambda_{p,2} & \cdots & 0 & * & \cdots & * & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & \lambda_{p,2} & * & \cdots & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{31}$$

This structure is block-triangular, with each diagonal block being a multiple of the identity matrix. Moreover, in the same basis, all matrices $S_p \in \mathfrak{s}$ take the form

$$S_p = \begin{pmatrix} * & \cdots & * & * & \cdots & * & * & \cdots & * & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ * & \cdots & * & * & \cdots & * & * & \cdots & * & \cdots \\ 0 & \cdots & 0 & * & \cdots & * & * & \cdots & * & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & \cdots & 0 & * & \cdots & * & * & \cdots & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{32}$$

This means that if we denote by W the subspace spanned by eigenvectors of R_p with eigenvalue $\lambda_{p,1}$, then $\mathfrak{s}$ leaves this subspace invariant (this is the main content of Lemma 9.13 in [16]). Passing to the quotient space over W, the construction is repeated, resulting in the indicated block-triangular structure (see [16] for details). Let $S_{p,1}$, $S_{p,2}$, …, $S_{p,k}$ be the diagonal blocks in the decomposition (32) of S_p, and define $R_{p,i}$ similarly using (31). Let the dimensions of these blocks be n_1, n_2, …, n_k, with $n_1 + \cdots + n_k = n$. For each $i \in \{1, \dots, k\}$, let $\lambda_{S,i}^*$ be the characteristic exponent of the corresponding n_i-dimensional switched system $\dot{x}_i = S_{\sigma(t),i}x_i$. Also, let $\bar{\lambda}_{R,i} := \max\{\operatorname{Re}\lambda_{p,i} : p \in \mathcal{P}\}$. Then we have the following claims.

CLAIM 1: $\lambda_S^* = \max_i\{\lambda_{S,i}^*\}$. This follows from the block-triangular structure of S_p, by the same reasoning as in [7], [8] (the facts that here we have blocks instead of scalars and that the corresponding switched systems are not necessarily stable do not affect the argument).

CLAIM 2: For each i, the n_i-dimensional switched system generated by the matrices $A_{p,i} := R_{p,i} + S_{p,i}$ has characteristic exponent no larger than $\bar{\lambda}_{R,i} + \lambda_{S,i}^*$; in particular, it is GUES if $\bar{\lambda}_{R,i} + \lambda_{S,i}^* < 0$. This follows from the fact that the matrices $R_{p,i}$ are multiples of the identity and hence commute with the $S_{p,i}$'s, which means that we simply need to add the two individual characteristic exponents. This fact can also be seen from Lemma 1 applied to each diagonal block, noting that we have $R_{\sigma(t),i}$ in place of $C(t)$ because of the commutativity.

CLAIM 3: The characteristic exponent of the overall switched system is no larger than

$$\max_i\{\bar{\lambda}_{R,i} + \lambda_{S,i}^*\} \tag{33}$$

This again follows from the block-triangular structure and Claim 2 by the same arguments as in [7], [8]. (Another way of proving this is to rescale the spaces of the filtration, i.e., conjugate the matrices A_p by the matrices with $\varepsilon^i I_{i\times i}$ as the diagonal blocks and zeros elsewhere. Then for small ε the off-diagonal parts become small since the set $\{A_p : p \in \mathcal{P}\}$ is compact.)

We can upper-bound (33) by $\max_i \bar{\lambda}_{R,i} + \max_i \lambda_{S,i}^* = \bar{\lambda}_R + \lambda_S^*$ where we used the definition (24) of $\bar{\lambda}_R$ and Claim 1. The result now follows from Claim 3. □

Remark 1 Examining the proof of Proposition 3, we can also obtain an "intermediate" sufficient condition for stability, sharper than (28) but still more useful compared to (26). In the notation of the above proof, let $\hat{\lambda}_{S,i} := \max\{\|S_{p,i}\| : p \in \mathcal{P}\}$. We have $\lambda_{S,i}^* \le \hat{\lambda}_{S,i}$, as is readily shown by the same reasoning that led us to (30). This fact and Claim 3 in the above proof imply that the following condition guarantees GUES of (17): $\bar{\lambda}_{R,i} + \hat{\lambda}_{S,i} < 0\ \forall i$. Note that in the definition of $\hat{\lambda}_{S,i}$ we can actually use a different norm for different i, which gives extra flexibility. □

Corollary 1 provides a robust version of the result from [7], [8], in the sense that $\mathfrak{g}$ is allowed to have a semisimple part $\mathfrak{s}$ if the matrices in $\mathfrak{s}$ have sufficiently small norm (compared to the stability margin of the matrices in the solvable part $\mathfrak{r}$). On the other hand, it is also known from [9], [10] that the switched system is GUES if $\mathfrak{s}$ is a compact Lie algebra. Note that neither of these two conditions—the bound on the norm of the matrices $S_p \in \mathfrak{s}$ given by (27)–(28) and compactness of the Lie algebra $\mathfrak{s}$—implies the other. Thus, we also want to develop another robust stability condition in which closeness of $\mathfrak{s}$ to a compact Lie algebra would play a role.

Let us first look at the special case in which $\mathfrak{s}$ is compact. Then $\Phi_S(t)$ from (23) lives in the compact Lie group $\mathcal{S} = \{e^S : S \in \mathfrak{s}\}$, which implies that $\lambda_S^* = 0$ (see, e.g., [17, Theorem 6.4.6]). Note that for each $A_p = R_p + S_p$, the eigenvalues of A_p are pairwise sums of suitably ordered eigenvalues of R_p and S_p: $\operatorname{spec}(A_p) = \{\lambda_i = \mu_i + \nu_i : \mu_i \in \operatorname{spec}(R_p),\ \nu_i \in \operatorname{spec}(S_p),\ i = 1,\dots,n\}$. This is shown in the proof of Lemma 3 in [9], and also follows from the earlier block-triangular decomposition (31), (32). The eigenvalues of S_p have zero real parts, hence $\bar{\lambda}_R$ is simply the largest real part of the eigenvalues of A_p, $p \in \mathcal{P}$. When the matrices A_p are Hurwitz, the condition (26) is satisfied—even though (28) may be false—and we recover the result of [9], [10].

B. Cartan decomposition

In the general case, to better understand λ_S^*, we can go one step further and consider a Cartan decomposition $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{p}$. Here $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{s}$, $\mathfrak{p}$ is its orthogonal complement with respect to the Killing form, and we have the relations $[\mathfrak{k},\mathfrak{k}] \in \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \in \mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}] \in \mathfrak{k}$. For each $p \in \mathcal{P}$, we can write

$$S_p = K_p + P_p, \qquad K_p \in \mathfrak{k}, \quad \mathcal{P}_p \in \mathfrak{p}. \tag{34}$$

Consider the matrix Φ_S in (23). The following result is analogous to Lemma 1.

Lemma 2 *The matrix $\Phi_S(t)$ from* (23) *can be represented as*

$$\Phi_S(t) = \Phi_K(t)\Phi_P(t) \tag{35}$$

where

$$\dot{\Phi}_K(t) = K_{\sigma(t)}\Phi_K(t), \qquad \Phi_K(0) = I \tag{36}$$

and

$$\begin{aligned}\dot{\Phi}_P(t) &= \left(\Phi_K^{-1}(t)P_{\sigma(t)}\Phi_K(t)\right)\Phi_P(t)\\ &=: D(t)\Phi_P(t), \qquad \Phi_P(0) = I.\end{aligned} \tag{37}$$

Let

$$\hat{\lambda}_P := \max\left\{\left\|e^{-K}P_p e^K\right\| : K \in \mathfrak{k},\ p \in \mathcal{P}\right\} \tag{38}$$

which is well defined because $\mathfrak{k}$ is compact and the set $\{P_p : p \in \mathcal{P}\}$ is compact by the compactness of $\{A_p : p \in \mathcal{P}\}$. We can now state our next result.

Proposition 4 *Let each matrix A_p be written as in* (20) *in accordance with a Levi decomposition of the Lie algebra* (19)*, and let each matrix S_p be written as in* (34) *in accordance with a Cartan decomposition of $\mathfrak{s}$. Assume that*

$$\bar{\lambda}_R + \hat{\lambda}_P < 0 \tag{39}$$

where λ_R and $\hat{\lambda}_P$ are defined by (24) *and* (38)*, respectively. Then the continuous-time switched system* (17) *is GUES.*

PROOF. Since $\Phi_K(t)$ lives in the compact Lie group $\mathcal{K} = \{e^K : K \in \mathfrak{k}\}$, its norm is uniformly bounded. Thus the characteristic exponent of the system (36) equals 0. As for Φ_P, we have $\|\Phi_P(t)\| \le I + \int_0^t \|D(s)\|\|\Phi_P(s)\|ds$ hence (by the Bellman-Gronwall inequality) $\|\Phi_P(t)\| \le e^{t\max_{0\le s\le t}\|D(s)\|}$. The definition of $D(t)$ in (37) yields the bound $\|\Phi_P(t)\| \le e^{\hat{\lambda}_P t}$. We see that the characteristic exponent of the system (37) is no larger than $\hat{\lambda}_P$. Combining the previous two conclusions and using (35), we have $\lambda_S^* \le \hat{\lambda}_P$. The result now follows from Proposition 3. $\square$

Proposition 4 can be considered as an improvement over Corollary 1 because it singles out the noncompact part. Returning to the special case when $\mathfrak{s}$ is compact, we have $\mathfrak{p} = 0$ and the condition (39) holds if and only if the matrices A_p are Hurwitz. In general, Proposition 4 says that stability is preserved when noncompact perturbations are introduced, as long as they are small compared to the real parts of the eigenvalues of the matrices in the solvable part $\mathfrak{r}$. In contrast with Corollary 1, the norms of the matrices in the compact part $\mathfrak{k}$ are not restricted in any way.

Example 2 Suppose that the matrices A_p, $p \in \mathcal{P}$ take the form $A_p = \left(\begin{smallmatrix} \lambda_p & \alpha_p+\delta_p \\ -\alpha_p+\delta_p & \lambda_p \end{smallmatrix}\right)$ for some numbers $\lambda_p < 0$, $\alpha_p, \delta_p \in \mathbb{R}$, $p \in \mathcal{P}$. For generic values of these numbers, $\mathfrak{g} = gl(2,\mathbb{R})$ (the Lie algebra of all 2×2 matrices), $\mathfrak{r}$ consists of multiples of the identity matrix, $\mathfrak{s}$ consists of all traceless matrices, $\mathfrak{k}$ consists of skew-symmetric matrices, and $\mathfrak{p}$ consists of traceless symmetric matrices (see [18, p. 144] and [10]). We then have $R_p = \left(\begin{smallmatrix} \lambda_p & 0 \\ 0 & \lambda_p \end{smallmatrix}\right)$, $K_p = \left(\begin{smallmatrix} 0 & \alpha_p \\ -\alpha_p & 0 \end{smallmatrix}\right)$, $P_p = \left(\begin{smallmatrix} 0 & \delta_p \\ \delta_p & 0 \end{smallmatrix}\right)$. Clearly, $\bar{\lambda}_R = \max_{p\in\mathcal{P}} \lambda_p$. Since e^K, $K \in \mathfrak{k}$ are orthogonal matrices, $\hat{\lambda}_P = \max_{p\in\mathcal{P}} |\delta_p|$. By Proposition 4, the switched system is GUES if $\max_{p\in\mathcal{P}} \lambda_p + \max_{p\in\mathcal{P}} |\delta_p| < 0$. $\square$

Actually, the more general case when the matrices A_p, $p \in \mathcal{P}$ span the Lie algebra $gl(n,\mathbb{R})$ for an arbitrary n is not too different from Example 2. The components $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{k}$, and $\mathfrak{p}$ in the Levi and Cartan decompositions are described in the same way as in Example 2. For each p we have $R_p = \lambda_p I$ where $\lambda_p := \frac{1}{n}\mathrm{tr}(A_p)$, and it still holds that $\bar{\lambda}_R = \max_{p\in\mathcal{P}} \lambda_p$. Next, P_p is given by the formula $P_p = \frac{1}{2}(A_p + A_p^T) - \frac{1}{n}\mathrm{tr}(A_p)I$. Using the orthogonality of e^K, $K \in \mathfrak{k}$ and the fact that P_p is symmetric, we have $\hat{\lambda}_P = \max_{p\in\mathcal{P}} \sigma_{\max}(P_p) = \max_{p\in\mathcal{P}} \sqrt{\lambda_{\max}\Big(\Big(\frac{1}{2}(A_p + A_p^T) - \frac{1}{n}\mathrm{tr}(A_p)I\Big)^2\Big)}$ where $\sigma_{\max}(\cdot)$ stands for the largest singular value and $\lambda_{\max}(\cdot)$ stands for the largest eigenvalue of a symmetric matrix. Therefore, by Proposition 4 a sufficient condition for GUES is $\max_{p\in\mathcal{P}} \frac{1}{n}\mathrm{tr}(A_p) + \max_{p\in\mathcal{P}} \sqrt{\lambda_{\max}\Big(\Big(\frac{1}{2}(A_p + A_p^T) - \frac{1}{n}\mathrm{tr}(A_p)I\Big)^2\Big)} < 0$.

For comparison, let us now look at a case where the given matrices possess some additional structure. Namely, consider the matrices $A_p \in \mathbb{R}^{2n\times 2n}$ in the form $A_p = \lambda_p I + S_p$, where $\lambda_p = \frac{1}{2n}\mathrm{tr}(A_p)$ and S_p is a *symplectic* matrix, i.e., $S_p = \left(\begin{smallmatrix} V_p & W_p \\ U_p & -V_p^T \end{smallmatrix}\right)$ where U_p and W_p are symmetric. Such matrices appear in models of Hamiltonian mechanical systems (for quadratic Hamiltonians), with λ_p reflecting dissipation effects (for example, due to friction). The Cartan decomposition of the symplectic Lie algebra $sp(2n,\mathbb{R})$ is described as follows (see [19, Chapter X]). The compact part $\mathfrak{k}$ consists of matrices of the form $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$ with A skew-symmetric and B symmetric; so, matrices in $\mathfrak{k}$ are skew-symmetric with additional structure inherited from the symplectic Lie algebra (namely, off-diagonal blocks are symmetric and diagonal blocks are identical). The noncompact part $\mathfrak{p}$ consists of matrices of the form $\left(\begin{smallmatrix} C & D \\ D & -C \end{smallmatrix}\right)$ with C and D both symmetric; so, matrices in $\mathfrak{p}$ are symmetric but again with additional structure (off-diagonal blocks are symmetric and diagonal blocks are negatives of each other). Let us decompose each V_p into symmetric and anti-symmetric parts: $V_p = V_{p,s} + V_{p,a}$, so that $V_p^T = V_{p,s} - V_{p,a}$. The Cartan decomposition gives $S_p = K_p + P_p$ where $K_p = \left(\begin{smallmatrix} V_{p,a} & -\frac{1}{2}(U_p - W_p) \\ \frac{1}{2}(U_p - W_p) & V_{p,a} \end{smallmatrix}\right)$ and $P_p = \left(\begin{smallmatrix} V_{p,s} & \frac{1}{2}(U_p + W_p) \\ \frac{1}{2}(U_p + W_p) & -V_{p,s} \end{smallmatrix}\right)$. We can calculate P_p from $P_p = \frac{1}{2}(A_p + A_p^T) - \frac{1}{2n}\mathrm{tr}(A_p)I$ and a sufficient condition for GUES from Proposition 4 is $\max_{p\in\mathcal{P}} \frac{1}{2n}\mathrm{tr}(A_p) + \max_{p\in\mathcal{P}} \sqrt{\lambda_{\max}\Big(\Big(\frac{1}{2}(A_p + A_p^T) - \frac{1}{2n}\mathrm{tr}(A_p)I\Big)^2\Big)} < 0$, the same as for $gl(2n,\mathbb{R})$. We see that the above formula for P_p automatically accounts for symmetries in the system.

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