Fast-oscillating Control and Combinatorics of Permutations

Andrei Agrachev

SISSA, Trieste and MIAN, Moscow
Smooth dynamical system:

\[ \dot{q}(t) = f(q(t)), \quad q \in M, \ t \in \mathbb{R}, \]

generates a flow

\[ P^t : M \to M, \quad P^t : q(0) \mapsto q(t), \quad t \in \mathbb{R}. \]
Control system:

\[ \dot{q} = f_u(q), \quad u \in U. \]

Control: \( t \mapsto u(t), \ t \geq 0. \)

Trajectory: \( t \mapsto q(t), \) where \( \dot{q}(t) = f_{u(t)}(q(t)). \)

Special case: \( U = \{1, 2\} \):
Trajectories:
Fast-oscillating controls allow to uniformly approximate flows generated by:

\[ \dot{q} = \nu f_1(q) + (1 - \nu)f_2(q), \quad 0 \leq \nu \leq 1. \]

Similarly, for \( U = \{1, 2, \ldots, k\} \) we approximate dynamics

\[ \dot{q} = \sum_{i=1}^{k} \nu_i f_i(q), \quad \nu_i \geq 0, \quad \sum_i \nu_i = 1. \]

If \( 0 \in \text{relint} \left( \text{conv} f_U(q) \right) \), then we can much more!
Consider the case
\[ f_u = \sum_{i=1}^{k} u^i f_i, \quad u = (u^1, \ldots, u^k) \in U, \]
where \( U \) is a neighborhood of \( 0 \in \mathbb{R}^k \).

Take a sample vector-function \( t \mapsto v(t), \supp\{v(\cdot)\} \subset [0, 1] \) and let
\[ \dot{q}_\varepsilon(t) = f_{v(t/\varepsilon)}(q_\varepsilon), \quad q_\varepsilon(0) = q_0. \]

Then, for any “observable” \( a : M \mapsto \mathbb{R} \), we have:
\[ a(q_{\varepsilon}(t)) \approx a(q_0) + \sum_{i=1}^{\infty} \varepsilon^i \int \int_{\Delta_i} f_{v(t_i)} \circ \cdots \circ f_{v(t_1)} a(q_0) \, dt_1 \cdots dt_i \]

\[ = a(q_0) + \sum_{i=1}^{\infty} \varepsilon^i \int \int_{\Delta_i} p_i(v(t_i)), \ldots, v(t_1) \, dt_1 \cdots dt_i, \]

where \( \Delta_i = \{(t_1, \ldots, t_i) : 0 \leq t_i \leq \cdots \leq t_1 \leq 1\} \) and \( p_i \) is a \( i \)-linear form, \( p_i(v_i, \ldots, v_1) = \langle \omega_i, v_i \otimes \cdots \otimes v_1 \rangle \).

We set \( \gamma(t) = \int_0^t u(\tau) \, d\tau \), the \( i \)-th order term takes the form:

\[ \varepsilon^i \left\langle \omega_i, \int \int_{\Delta_i} d\gamma(t_i) \otimes \cdots \otimes d\gamma(t_1) \right\rangle. \]
We set:

\[ D^n(\gamma) = \int \int_{\Delta_n} d\gamma(t_n) \otimes \cdots \otimes d\gamma(t_1), \]

where \( \gamma \) is a Lipschitz curve in \( \mathbb{R}^k \), \( \gamma(0) = 0 \).

In particular, \( D^1(\gamma) = \gamma(1) \). If \( \gamma \) is a closed curve then principal term is \( D^2(\gamma) \). Moreover,

\[ D^2(\gamma) = \int_0^1 \dot{\gamma}(t) \wedge \gamma(t) \, dt + \frac{1}{2} \gamma(1) \otimes \gamma(1). \]
Let \( \Omega_n = \{ \gamma : D^1(\gamma) = \cdots = D^{n-1}(\gamma) = 0 \} \).

We know that \( D^n(\Omega_n) = \text{Lie}^n(\mathbb{R}^k) \subset (\mathbb{R}^k)^\otimes n \).

If \( \gamma \in \Omega_n \) and \( D^n(\gamma) = \pi(e_1, \ldots, e_n) \), where \( \pi \) is a “Lie polynomial”, then
\[
q_\varepsilon(t) = q_0 + \varepsilon^n \pi(f_1, \ldots, f_n)(q_0) + O(\varepsilon^{n+1}).
\]

We are looking for symmetries of \( D^n|_{\Omega_n} \) in order to better understand the structure of \( \Omega_n \).

Let \( \Sigma_n \) be the symmetric group and \( \overline{\Sigma}_n = \{ \sum_i c_i \sigma_i : \sigma_i \in \Sigma_n \} \) its group algebra. We set:
\[
D^n_\sigma(\gamma) = \int \int_{\Delta_n} d\gamma(t_{\sigma(n)}) \otimes \cdots \otimes d\gamma(t_{\sigma(1)}), \quad D^n_{\sum c_i \sigma_i} = \sum c_i D^n_{\sigma_i}.
\]
Let $\sigma \in \Sigma_n$, the monotonicity type of $\sigma$ is a word $w_\sigma = s_1 \ldots s_{n-1}$ in the alphabet $\{\alpha, \beta\}$,

$$s_i = \begin{cases} 
\alpha, & \sigma(i) < \sigma(i+1); \\
\beta, & \sigma(i) > \sigma(i+1).
\end{cases}$$

Given a word $w$, we set $\bar{w} = \sum_{\{w_\sigma = w\}} \sigma$. The descent subalgebra of $\bar{\Sigma}_n$:

$$\mathcal{M}_n = \text{span} \{ \bar{w} : w = s_1 \ldots s_{n-1}, s_i \in \{\alpha, \beta\} \}.$$

It admits a homomorphism:

$$r : \mathcal{M}_n \to \mathbb{Z}, \quad r(s_1 \ldots s_{n-1}) = (-1)^{\#\{i : s_i = \beta\}}.$$
Example:

\[\alpha \cdots \alpha = 1, \quad \beta \cdots \beta = \begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}, \quad r(\beta \cdots \beta) = (-1)^{n-1}.\]

**Theorem.** A curve \(\gamma\) belongs to \(\Omega_n\) if and only if

\[D^n_m(\gamma) = r(m)D^n(\gamma), \quad \forall m \in \mathcal{M}.\]
Affine in control system:
\[ \dot{q} = h(q) + \sum_i u^i f_i(q), \quad u = (u^1, \ldots, u^k) \in \mathbb{R}^k. \] (*)

If \( h \in \text{Lie}\{f_1, \ldots, f_k\} \), then we can neutralize the drift \( h \), but this inclusion is violated for many important apparently controllable systems. Examples:

1. **Acceleration control:** \( \ddot{x} = \sum_i u^i g_i(x) \). We rewrite:
\[ \dot{x} = y, \quad \dot{y} = \sum_i u^i g_i(x); \quad q = (x, y), \]
\[ \dot{q} = h(q) + \sum_i u^i f_i(q), \quad [f_i, f_j] = 0, \quad i, j = 1, \ldots, k. \]

2. “**Fluid dynamics:**” \( \dot{y} = Ay + B(y, y) + \sum_i u^i g_i \).
Theorem. Assume that $[f_i, f_j] = 0$, $i, j = 1, \ldots, k$. If

$$\text{conv}\left\{ \sum_{i,j} u^i u^j [f_i, [f_j, h]] : u^i, u^j \in \mathbb{R} \right\}$$

is a subspace, then system

$$\dot{q} = h(q) + \sum_i u^i f_i(q) + \sum_{i,j} u^{ij} [f_i, [f_j, h]](q)$$

has "the same control properties" as system (*).

If the fields $[f_i, [f_j, h]], f_i$ are all commuting then we iterate the theorem etc.
Hint: Use a fast-oscillating control variation:
\[ u^i_\varepsilon(t) = \frac{1}{\varepsilon} \sin \left( \frac{t}{\varepsilon^2} \right), \quad u^j_\varepsilon(t) = \frac{1}{\varepsilon} \cos \left( \frac{t}{\varepsilon^2} \right) \]
to single out the desired bracket.

Indeed:
\[
\int_0^1 u^i_\varepsilon \, dt = O(\varepsilon), \quad \int\int_{\Delta_2} u^i_\varepsilon(t_1)u^j_\varepsilon(t_2) \, dt_1 \, dt_2 = O(1),
\]
as \( \varepsilon \to 0 \).