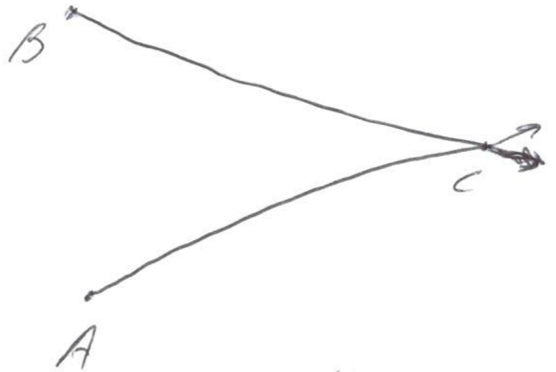
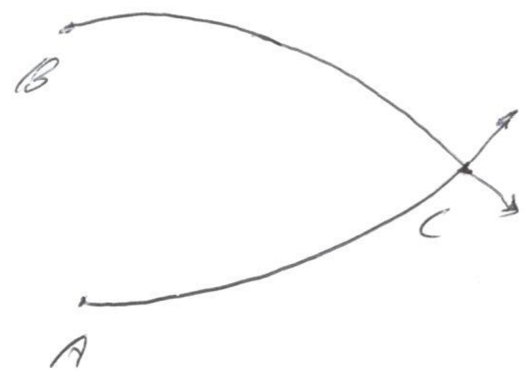
Basic idea:

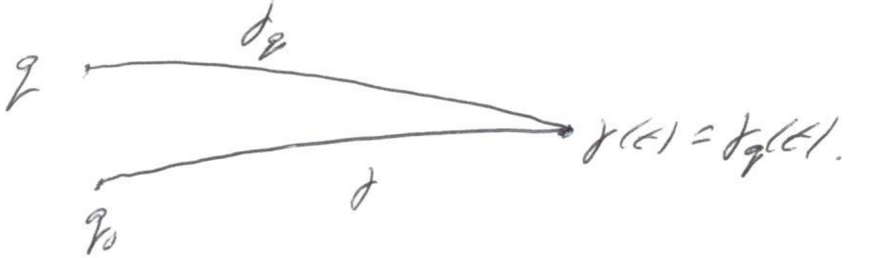




negative curvature positive curvature

Bigger the curvature - bigger the difference between velocities in the intersection point.

Do it infinitesimaly:



$$b_t(q) \doteq \frac{1}{2}|\dot\gamma_q(t)-\dot\gamma(t)|^2.$$

In particular $b_t(\gamma(s)) = \frac{1}{2}\left|\frac{t}{t-s}\dot\gamma(t)-\dot\gamma(t)\right|^2 = \frac{s^2}{2(t-s)^2}|\dot\gamma(t)|^2$

Riemannian case:

$$D^2_{q_0} b_t(v) = \frac{1}{t^2}|v|^2 + \frac{1}{3}\langle R(v,\dot\gamma)\dot\gamma, v\rangle + O(t).$$

General SR case (ample geodesic):

$$D^2_{q_0} b_t(v) = \frac{1}{t^2} Q(v) + \frac{1}{3} R_\gamma(v) + O(t), \quad v \in \Delta_{q_0};$$

$Q(v) \geq |v|^2$. Curvature $\doteq R_\gamma$.
(along γ)

We have: $Q(v) = |v|^2, \ \forall v \iff \Delta_{q_0} = T_{q_0}M$.

Quadratic form Q measures "nonholonomic orders" and precedes the curvature.

Let $\varphi^t : M \to M$ be a horizontal flow

s.t. $\varphi^t(q_0) = \gamma(t)$, $t \in \mathbb{R}$.

We set $\Delta^t = \varphi_*^{-t} \Delta_{\gamma(t)}$,

$\Delta^t \subset T_{q_0} M$.

Geodesic flag : $\Delta_{q_0} = \Delta^{(1)} \subset \Delta^{(2)} \subset \dots \subset \Delta^{(i)} \subset \dots$

is defined as follows: $\Delta^{(i)} = \frac{d^{i-1}}{dt^{i-1}} \Delta^t \Big|_{t=0}$;

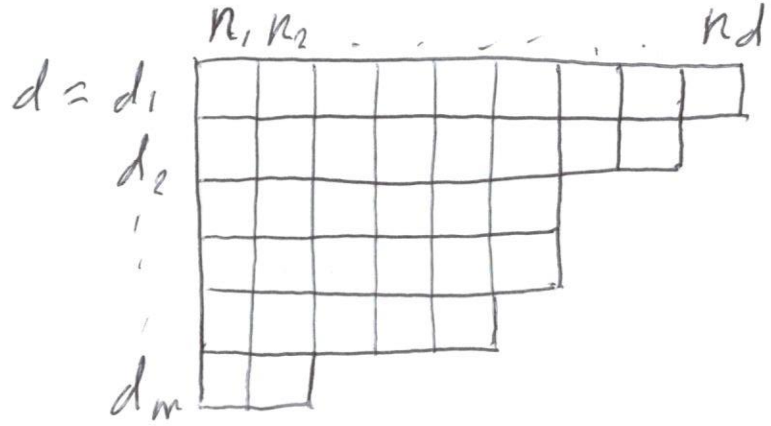
it depends only on γ and not on the choice of φ^t.

γ is ample if $\Delta^{(m)} = T_{q_0} M$ for some $m > 0$.

$\boxed{\text{III-4}}$

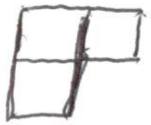
Let $\dim \Delta_{q_0} = d$, $\dim M = n$, γ is equiregular

Young diagram: $d_1 = d$, $d_i = \dim \Delta^{(i)} - \dim \Delta^{(i-1)}$.



$\operatorname{Spec} Q = \{n_1^2, \dots, n_d^2\}$.

<u>If $n=3$, $d=2$, then:</u>



$\operatorname{spec} Q = \{1, 4\}$, R_γ is a quadratic form on $\Delta/\mathbb{R}\dot{\gamma}(t)$

$\operatorname{spec} R_\gamma = \{0, r_\gamma\}$

Parameterized proportionally to the length extremals are trajectories of the Hamiltonian system with a quadratic on fibers Hamiltonian

$$\frac{1}{2}h^2 : T^*M \longrightarrow \mathbb{R}.$$

Given $\lambda \in T^*_{q_0}M$, let γ_λ be the projection to M of the started at λ trajectory. We set $r(\lambda) \doteq r_{\gamma_\lambda}$.

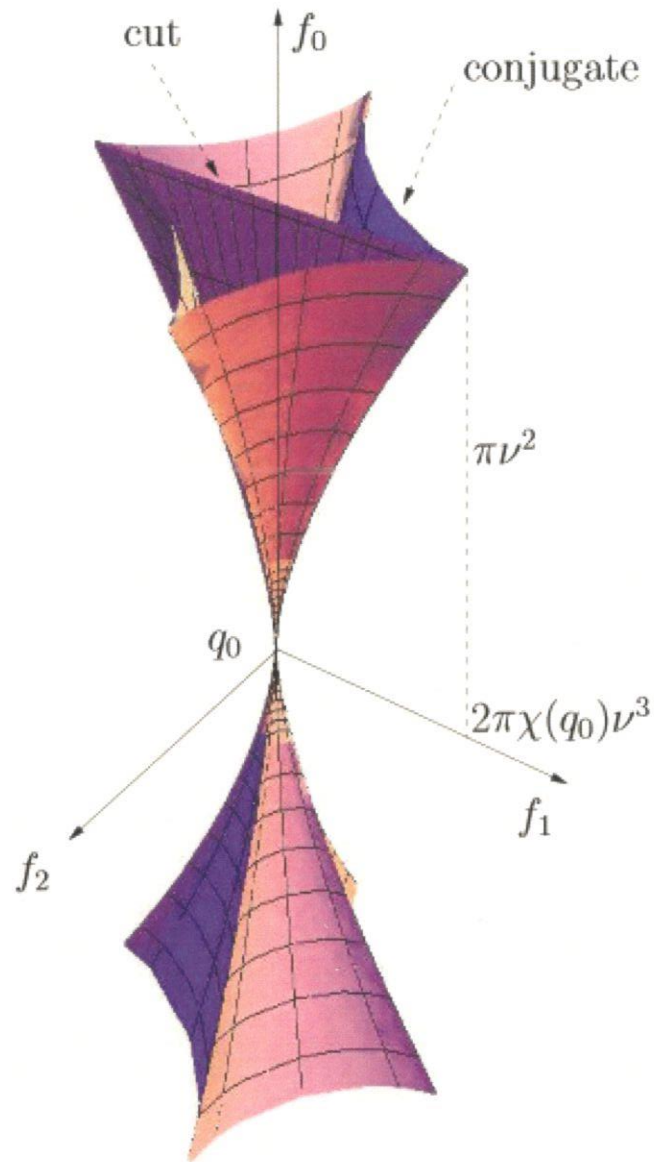
<u>Theorem</u> $r : T^*_{q_0}M \to \mathbb{R}$ is a quadratic form, $r\big|_{\Delta^\perp_{q_0}} > 0$. We have: $\frac{5}{2} r(\lambda) = \langle \lambda, f_0 \rangle^2 + \alpha_1 \langle \lambda, f_1 \rangle^2 + \alpha_2 \langle \lambda, f_2 \rangle^2$,

where f_1, f_2 is an orthonormal basis of Δ_{q_0}

$$\varkappa = \frac{1}{2}(\varkappa_1 + \varkappa_2),$$

$$\chi = |\varkappa_1 - \varkappa_2|.$$

$$\text{length}_{\text{conj}}(\nu) = 2\pi\nu - \pi\varkappa\nu^3 + O(\nu^4)$$



An SR-structure on $\mathbb{R}^3$ is locally isometric to the Dido isoperimetric problem on the surface with Gaussian curvature $\mathcal{K}$ $\iff$ $\chi \equiv 0$.

It is isometric to its own tangent metric space $\iff$ $\mathcal{K} = \chi \equiv 0$ $\iff$ η is a rank 1 quadratic form.