METHODS OF CONTROL THEORY IN NONHOLONOMIC GEOMETRY

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1.Introduction. Let M be a C^∞ -manifold and TM the total space of the tangent bundle. A control system is a subset $V \subset TM$. Fix an initial point $q_0 \in M$ and a segment $[0,t] \subset \mathbb{R}$. Admissible trajectories are Lipschitzian curves $q(\tau), 0 \le \tau \le t,\ q(0) = q_0,$ satisfying a differential equation of the form

$$\dot{q} = v_\tau(q), \tag{1}$$

where $v_\tau(q) \in V \cap T_qM,\ \forall q \in M,\ v_\tau(q)$ is smooth in q, bounded and measurable in τ. The mapping $q(\cdot) \mapsto q(t)$ which maps admissible trajectories in their end points is called an end-point mapping.

Control Theory is in a sense a theory of end-point mappings. This point of view is rather restrictive but sufficient for our purposes. For instance, attainable sets are just images of end-point mappings. Geometric Control Theory tends to characterize properties of these mappings in terms of iterated Lie brackets of smooth vector fields on M with values in V. A number of researchers have shown a remarkable ingenuity in this regard leading to encouraging results. See, for instance, books [1],[2],[3] to get an idea of various periods in the development of this domain and for other references. A complete list of references would probably run to thousands of items.

A great part of the theory is devoted to the case of nonsmooth V such that $V \cap T_qM$ are polytopes or worse. There is a wide-spread view that such a nonsmoothness is the essence of Control Theory. This is not my opinion, and I am making the following radical assumption.

Let us assume that V forms a smooth locally trivial bundle over M with fibers V_q — smooth closed convex submanifolds in T_qM of positive dimension, symmetric with respect to the origin. So we consider a very special class of control systems.

Examples. 1) V_q is an ellipsoid centered at the origin. This is the case of Riemannian Geometry.

2) V_q is a proper linear subspace of T_qM. This case includes Nonholonomic Geometry.

3) V_q is the intersection of an ellipsoid and a subspace. This is sub-Riemannian Geometry.

This paper essentially deals with cases 2) and 3).

2.Extremals. Denote by $\Omega_{q_0}(t)$ the space of all admissible trajectories on $[0,t]$ equipped with $W_{1,\infty}$ -topology, i.e. the topology of uniform convergence for curves and their velocities. Under our assumptions for V, the space $\Omega_{q_0}(t)$ possesses the natural structure of a smooth Banach manifold, and the end-point mapping

$$f_t : \Omega_{q_0}(t) \to M, \ f_t(q(\cdot)) = q(t)$$

is a smooth mapping. We will denote by $D_q f_t : T_q\Omega_{q_0}(t) \to T_{f_t(q)}M$ the differential of f_t at $q(\cdot)$.

A trajectory $q(\cdot)$ is a critical point for f_t iff $\exists \lambda \in T^*_{f_t(q)}M$, $\lambda \neq 0$, such that $\lambda D_q f_t = 0$, i.e. λ is orthogonal to the image of the linear mapping $D_q f_t$. It is a natural thing that critical points of f_t are the main object of our investigation. We study critical levels of f_t and restrictions of f_t on the sets of their critical points.

The cotangent bundle T^*M possesses the canonical symplectic structure. We will denote by $\overrightarrow{\phi}$ the Hamiltonian vector field on T^*M associated to the Hamiltonian $\phi \in C^\infty(T^*M)$. Let v be a smooth vector field on M, then $v^* : \lambda \mapsto \langle \lambda, v(q)\rangle, \lambda \in T^*_q M, q \in M$ is a Hamiltonian on T^*M, which is linear on fibers, and $\overrightarrow{v^*}$ is a lift of the vector field v on T^*M.

Set $\Omega_{q_0} = \Omega_{q_0}(1), \ f = f_1$. Let $q(\cdot) \in \Omega_{q_0}$ be a critical point of f. Then $q|_{[0,t]}$ is obviously a critical point of $f_t, \ \forall t \in (0,1]$. Moreover, let $q(\cdot)$ satisfy the equation (1). If $\lambda_t, 0 \leq t \leq 1$, is a solution of the nonstationary Hamiltonian system

$$\dot{\lambda} = \overrightarrow{v_t^*}(\lambda) \quad \text{and} \quad \lambda_1 D_q f = 0, \quad \lambda_1 \neq 0, \tag{2}$$

then $\lambda_t D_{q|_{[0,t]}} f_t = 0, \lambda_t \neq 0, \ \forall t \in [0,1]$.

The curves in T^*M which satisfy (2) for some v_t are called extremals associated with $q(\cdot)$. Let $q \in M, \ \lambda \in T^*_q M, \ \lambda \neq 0$. Set $h(\lambda) = \max\limits_{v \in V_q} \langle \lambda, v \rangle$ if the maximum exists. The function h is defined on a subset of T^*M. It is convex and positively homogeneous on fibers. We call h the Hamiltonian of the control system.

Let σ be the canonical symplectic structure on T^*M. The following proposition is a corollary of the Pontryagin Maximum Principle.

Proposition 1. *a) $h(\lambda_t) = const, \ 0 \leq t \leq 1$, for arbitrary extremal λ_t.*

b) Let a level set $h^{-1}(c)$ be a smooth submanifold of T^*M. Then any extremal $\lambda_t \in h^{-1}(c), \ 0 \leq t \leq 1$ is a characteristic of the differential form $\sigma|_{h^{-1}(c)}$ (i.e. $\dot{\lambda} \rfloor \sigma|_{h^{-1}(c)} = 0$), and any properly parametrized characteristic of this form started at $T^*_{q_0}M$ is the extremal.

Note, that level sets of h are smooth in the above examples 1-3.

3.Distributions. Distributions are just smooth vector subbundles of the tangent bundle. Let Δ be the space of smooth sections of a distribution, and $\Delta_q \subset T_q M$ is the fiber at $q \in M$ of the corresponding subbundle.

Set $\Delta^1 = \Delta, \ \Delta^n = [\Delta, \Delta^{n-1}], \ n = 2, 3, \ldots$ where the Lie bracket of spaces of vector fields is, by definition, the linear hull of the pairwise brackets of their elements. The distribution is called bracket generating if a number n_q exists for $\forall q \in M$ such that $\Delta_q^{n_q} = T_qM$. We will consider only bracket generating distributions in this paper. The distribution defines a control system $V = \bigcup_{q\in M} \Delta_q$. The well known Rashevskij—Chow theorem asserts that the end-point mapping $f : \Omega_{q_0} \to M$ is a surjective one. It is not a submersion, however, if $\Delta_q \neq T_qM$. Critical points of f are called singular or abnormal geodesics for Δ.

Let $\Delta_q^\perp \subset T_q^*M$ be the set of all nonzero covectors which are orthogonal to Δ_q, $\Delta^\perp = \bigcup_{q\in M} \Delta_q^\perp$. The manifold $\Delta^\perp$ is the domain of the Hamiltonian of the control system V. This Hamiltonian is identical to zero in its domain. Singular geodesics are exactly projections on M of the characteristics of the form $\sigma|_{\Delta^\perp}$, started at $\Delta_{q_0}^\perp$.

4. Rigidity. Let $\Omega_{q_0,q_1} = f^{-1}(q_1)$ be the set of admissible trajectories which connect q_0 with q_1. An admissible trajectory $q(\cdot)$ is called rigid if there exists a neighborhood of $q(\cdot)$ in $\Omega_{q_0,q(1)}$ which contains only reparametrizations of $q(\cdot)$. It is called locally rigid if its small enough pieces are rigid, cf.[8].

Theorem 2. *(see[6]) Let $q(\cdot) \in \Omega_{q_0}$, $\dot q = v(q)$, $v \in \Delta$.*

a) If $q(\cdot)$ is a locally rigid trajectory, then there exists an extremal λ_t associated with $q(\cdot)$ such that

$$\lambda_t \perp \Delta^2_{q(t)},\ \langle \lambda_t, [[v,w],w](q(t))\rangle \geqslant 0,\ \forall w \in \Delta, \quad 0 \le t \le 1 \tag{3}$$

b) Let there exist an extremal λ_t which satisfies (3) and

$$\langle \lambda_t, [[v,w],w](q(t))\rangle > 0,\ \forall w \in \Delta,\ w(q(t)) \nparallel \dot q(t), \quad 0 \le t \le 1. \tag{4}$$

Then $q(\cdot)$ is indeed a locally rigid trajectory.

We call $q(\cdot)$ the singular geodesic of the first order if there exists a unique up to a positive multiplier extremal λ_t associated with $q(\cdot)$ which satisfies (3), (4).

5. Jacobi curves. Let λ_t be an extremal which satisfies (3), and $Q_\tau : M \to M$ — the flow generated by v. Set $\Gamma_t = \{\overrightarrow{w^*}|_{\lambda_t} \ : \ w \in \Delta\}$ —an isotropic subspace of the symplectic space $T_{\lambda_t}(T^*M)$. Let $0 = t_0 < t_1 < ... < t_{k+1} = 1$ be a subdivision of the segment $[0,1]$, $I = \{t_1, ..., t_k\}$. Let us identify $T_\lambda(T_q^*M) = T_q^*M$ for $\lambda \in T_q^*M$ and set

$$\Lambda_0(I) = T^*_{q_0}M,\ \Lambda_t(I) = Q^*_{t_i - t}(\Lambda_{t_i}(I)^{\Gamma_{t_i}}) \text{ for } t_i < t \le t_{i+1}, i = 0, ..., k,$$

where Λ^Γ denotes the intersection of $\Lambda+\Gamma$ with the skew-orthogonal complement to Γ. Then $\Lambda_t(I) \subset T_{\lambda_t}(T^*M)$ is a piecewise smooth family of Lagrangian subspaces.

Proposition 3. *Let $q(\cdot)$ be a locally rigid trajectory, $\dot q = v(q)$, $v \in \Delta$. Then there exists an extremal λ_t which satisfies (3) and such that*

$$\exists\ \mathcal{I}\text{-}\lim \Lambda_t(I) = \Lambda_t,\ \forall t \in [0,1],\ \text{where } \mathcal{I} = \{I \subset (0,1) : \#I < \infty\}.$$

We call Λ_t the Jacobi curve associated with λ_t. The Jacobi curve is smooth in $t \in (0,1]$ and satisfies a simple Hamiltonian equation if $q(\cdot)$ is a singular geodesic of the first order. See details in [4],[5],[6].

Theorem 4. *Let $q(\cdot)$ be a singular geodesic of the first order, and Λ_t be the Jacobi curve associated with the corresponding extremal λ_t. Suppose that $\Lambda_1 \cap T^*_{q(1)}M = \mathbb{R}\lambda_1$. Then there exists an integer $d \geq 0$ and a neighborhood $\mathcal{O}_q$ of q in $\Omega_{q_0,q(1)}$ such that $\mathcal{O}_q \setminus \{q\}$ is homotopy equivalent to the sphere S^{d-1}. If $\Lambda_t \cap T^*_{q(t)}M = \mathbb{R}\lambda_t,\ \forall t \in (0,1]$, then $d = 0$.*

We write $d = \mathrm{ind}q(\cdot)$. This index has an explicit expression in terms of the Maslov cocycle on T^*M, cf.[5].

6. Low dimensions. Let Δ be a rank 2 distribution and $\dim M = 3$. Then $N = \{q \in M : \Delta_q = \Delta^2_q \neq \Delta^3_q\}$ is a smooth 2-dimensional submanifold in M (maybe empty), and $\Delta_q \pitchfork N\ \forall q \in N$. Integral curves of the rank 1 distribution $\Delta_q \cap T_qN$ on N are singular geodesics of the first order and all of them are *rigid*.

One may say more about generic distributions using local normal forms, see [13],[14]. The closure $\bar{N}$ is a smooth submanifold in M for generic Δ, and $\bar{N} \setminus N$ consists of isolated points. These points are singularities of the foliation on N generated by rank 1 distribution $\Delta_q \cap T_qN$. They may be saddles or focuses. We obtain a nonsmooth rigid trajectory pasting together two neighboring separatrixes of the saddle, and a smooth but not a rigid singular geodesic if we paste together separatrixes lying opposite each other. One more interesting phenomenon: any neighborhood of the focus contains rigid trajectories of arbitrary length! It happens since the foliation is never generated by a linearizable vector field in a neighborhood of our focus.

Let $\mathrm{rank}\Delta = 2, \dim M = 4$ and $\Delta_q \neq \Delta^2_q \neq \Delta^3_q,\ \forall q \in M$. Such a distribution is called the Engel one. A characteristic rank1 subdistribution $K \subset \Delta$ is defined by the relation $[K, \Delta^2] \subset \Delta^2$. Singular geodesics for Δ are exactly parametrizations of integral curves of K. These integral curves are singular geodesics of the first order. Let $q(\cdot) \in \Omega_{q_0}$ be a piece of one of them without self-intersections, and $\mathcal{K}$ be the foliation generated by K. Replace M by a neighborhood M_0 of $\{q(t) : 0 \leq t \leq 1\}$ such that a factor-manifold $M_0/\mathcal{K}$ is well defined. Let $\kappa : M_0 \to M_0/\mathcal{K}$ be the canonical projection. Then $\kappa_*\Delta^2_{q(t)}$ is a two-dimensional subspace in $T_q(M_0/\mathcal{K})$ which does not depend on t. Let $\bar{\kappa}_*$ denote the composition of the κ_* and the projectivization of $T_q(M_0/\mathcal{K})$. Hence $\bar{\kappa}_*\Delta^2_{q(t)}$ is a projective line.

Proposition 5. *The singular geodesic $q(\cdot)$ satisfies conditions of the theorem 4 iff $\bar{\kappa}_*\Delta_{q(1)} \neq \bar{\kappa}_*\Delta_{q_0}$, and*

$$\mathit{ind}q(\cdot) = \#\{t \in (0,1) : \bar{\kappa}_*\Delta_{q(t)} = \bar{\kappa}_*\Delta_{q_0}\}.$$

See also [6],[8].

Let $\mathrm{rank}\Delta = 2,\ \dim M$ is arbitrary. If $\Delta^3_{q_0} \neq \Delta^2_{q_0}$, then Ω_{q_0} contains a smooth rigid trajectory. If $\dim(\Delta^3_{q_0}/\Delta^2_{q_0}) = 2$, then there exists a smooth rigid trajectory $q(\cdot) \in \Omega_{q_0}$ such that $\dot{q}(0) = \xi,$ for $\forall \xi \in \Delta_{q_0} \setminus \{0\}$, see [6], [8]. Note that 2 is the maximum possible dimension for $\Delta^3_{q_0}/\Delta^2_{q_0}$.

Remark. Remind that the spaces Ω_{q_0,q_1} have the $W_{1,\infty}$ -topology. Homotopy types may change dramatically and become independent on the distribution if we replace this topology by a weaker $W_{1,s}$ -topology (i.e. the L_s -topology for velocities), $1 \leq s < \infty$. The embedding of Ω_{q_0,q_1} in the space of *all* Lipschitzian curves in M connecting q_0 with q_1 is a homotopy equivalence in the $W_{1,s}$ -topology , $1 \leq s < \infty$, see [10].

7. Sub-Riemannian geodesics. Let V_q^1 be the intersection of Δ_q with an ellipsoid in T_qM centered at the origin and smoothly depending on $q \in M$. Set $V_q^l = lV_q^1,\ V^l = \bigcup_{q\in M} V_q^l,\ l > 0$. The family of control systems V^l is called the sub-Riemannian structure on M coordinated with Δ. We will denote by $\Omega_{q_0}^l$ the space of admissible trajectories for V^l on $[0,1]$ equipped with the $W_{1,1}$ -topology. Note that all $W_{1,s}$ -topologies, $1 \leq s < \infty$ are equivalent in the sub-Riemannian case since V_q^l are compact. The number l is, by definition, the length of any curve in $\Omega_{q_0}^l$. Set $\Omega_{q_0,q_1}^l = \{q(\cdot) \in \Omega_{q_0}^l : q(1) = q_1\}$ —a subspace in $\Omega_{q_0}^l$.

We call $q(\cdot) \in \Omega_{q_0}^l$ the strong length minimizer if it is a $W_{1,1}$ -isolated point in $\bigcup_{l'\leqslant l} \Omega_{q_0,q(1)}^{l'}$. We call $q(\cdot)$ the global length minimizer if $\Omega_{q_0,q(1)}^{l'} = \emptyset,\ \forall l' < l$.

Proposition 6. *Let $q(\cdot)$ be an isolated point in $\Omega_{q_0,q(1)}^l$. Then $q(\cdot)$ is a strong length minimizer and its small enough pieces (reparametrized in the obvious way) are global length minimizers.*

Critical points of the end-point mappings for Control Systems V^l are called sub-Riemannian geodesics. Let h^l be the Hamiltonian of V^l. The function $h^l = lh_1$ is smooth outside its zero level set, which is equal to $\Delta^\perp$.

Let λ_t be an extremal associated with a sub-Riemannian geodesic. The extremal is called normal if $h^l(\lambda_t) \neq 0$, otherwise it is called abnormal. Normal extremals are exactly trajectories of the Hamiltonian system $\dot\lambda = \overrightarrow{h^l}(\lambda),\ h(\lambda) \neq 0$, started at $T_{q_0}^*M$. Abnormal extremals are just extremals associated with properly parametrized singular geodesics for Δ.

A sub-Riemannian geodesic $q(\cdot)$ is called regular if there exists a unique up to a positive multiplier normal extremal associated with $q(\cdot)$, otherwise it is called singular or abnormal. It is easy to show that an abnormal extremal is associated with any singular geodesic $q(\cdot)$. If all extremals associated with $q(\cdot)$ are abnormal, then $q(\cdot)$ is called strictly abnormal.

Let λ_t be a normal sub-Riemannian extremal and $H_t^l : T^*M \to T^*M$ —the Hamiltonian flow generated by the vector field $\overrightarrow{h^l}$. Set

$$\Lambda_0^l = T_{\lambda_0}(T_{q_0}^*M),\ \Lambda_t^l = H_{t*}^l(\Lambda_0^l),\ 0 \leq t \leq 1.$$

Then $\Lambda_t^l \subset T_{\lambda_t}(T^*M)$ is a smooth family of Lagrangian subspaces. We call Λ_t^l the Jacobi curve associated with λ_t.

Theorem 4^l**.** *Let $q(\cdot)$ be a regular sub-Riemannian geodesic. The statement of Theorem 4 remains true if symbols Λ and Ω are replaced by Λ^l and Ω^l everywhere in its formulation.*

Theorem 7. *Let rank*$\Delta = 2$ *and* $q(\cdot) \in \Omega^l_{q_0}$ *be a singular geodesic meeting conditions of Theorem 4.*

a) If $q(\cdot)$ *is rigid, then it is a strong length minimizer.*

b) If $q(\cdot)$ *is strictly abnormal, then* $\mathcal{O}^l_q \setminus \{q\}$ *is homotopy equivalent to* $\mathcal{O}_q \setminus \{q\}$ *for some neighborhoods* $\mathcal{O}_q \subset \Omega_{q_0,q(1)}$, $\mathcal{O}^l_q \subset \Omega^l_{q_0,q(1)}$.

In particular, smooth rigid trajectories described in the previous section are strong length minimizers for an arbitrary sub-Riemannian structure coordinated with Δ. It turns out however that nonsmooth rigid curves constructed there for typical rank 2 distributions on the three-dimensional manifold are never strong length minimizers. Recall, that a strong minimum is a local minimum in the $W_{1,1}$ -topology (see the remark in the end of the previous section.) See also [7], [9], [11].

8. The Lie group case. In this section we consider examples of sub-Riemannian geodesics which are neither regular nor strictly abnormal. While most likely nongeneric, these geodesics are common in symmetric situations.

Let $M = G$ be a compact semisimple Lie group with the Lie algebra g of left-invariant vector fields and a bi-invariant Riemannian structure $(v_1|v_2), v_1, v_2 \in T_qG,\ q \in G$. Any left-invariant corank 1 distribution on G has a form $\Delta(a)$, where $a \in g, (a|a) = 1,\ \Delta_q(a) = \{v \in T_qG : (v|a(q)) = 0\}$. Consider a sub-Riemannian structure

$$V^l = \{v \in \Delta_q(a) : (v|v) = l^2,\ q \in G\}.$$

Sub-Riemannian geodesics for V^l which are not strictly abnormal, are exactly the curves

$$q(t) = q_0 e^{tb} e^{-t(b|a)a},\ b \in g,\ (b|b) - (b|a)^2 = l^2. \tag{5}$$

Let a be a regular element of g. The geodesic (5) is regular iff $[b, a] \neq 0$, otherwise it is neither regular nor strictly abnormal. Let $A = \{v \in g : [v, a] = 0\}$ be a Cartan subalgebra in g. Fix $c \in A,\ (c|a) = 0, (c|c) = 1$. Let $\pm\rho_i \in A^*, i = 1, ..., m$, be all roots of g (relative to A), $\langle \rho_i, c\rangle \geq 0$.

Let $q^l(t) = q_0 e^{tlc}$. Homology groups of the pair $(\Omega^l_{q_0,q_1}, \Omega^l_{q_0,q_1} \setminus \{q^l\})$ are determined by the disposition of the affine line $lc + \mathbb{R}a$ with respect to the Stifel diagram, i.e. the maximal triangulation of the complex $\{v \in A : \exists i \text{ s.t.} \langle \rho_i, v\rangle \in \mathbb{Z}\}$.

Proposition 8. *Suppose that* $lc + \mathbb{R}a$ *is transversal to the Stifel diagram and* c *belongs to the interior of a Weyl chamber. Let* E *be the intersection of this Weyl chamber with* $lc + \mathbb{R}a$,

$$E_k = \{e \in E : 2\sum_{i=1}^{m} [\langle \rho_i, e\rangle] \leq k\},\ k = 0, 1, 2, ...,$$

where $[\cdot]$ *is the integral part of the number in brackets. Then*

$$H_n(\Omega^l_{q_0}, \Omega^l_{q_0} \setminus \{q^l\}) = H_0(E_n, E_{n-1}) \oplus H_1(E_{n+1}, E_n),\ n \geq 0.$$

Example. Let $G = SU(3)$, then $\dim A = 2$, $m = 3$. Let $0 < \langle \rho_1, c\rangle < \langle \rho_2, c\rangle < \langle \rho_3, c\rangle$. A possible disposition of $lc + \mathbb{R}a$ is shown in fig.1.

$fig.1$

There is a rather involved explicit expression for the Betti numbers of the pair $(\Omega^l_{q_0}, \Omega^l_{q_0} \setminus \{q^l\})$ via $\langle \rho_i, lc\rangle$, but some asymptotic relations for $l \to \infty$ are transparent. Let $d(lc) = \min\{n : H_n(\Omega^l_{q_0}, \Omega^l_{q_0}\setminus\{q^l\}) \neq 0\}$, $D(lc) = \max\{n : H_n(\Omega^l_{q_0}, \Omega^l_{q_0}\setminus\{q^l\}) \neq 0\}$. Then

$$\lim_{l\to\infty} \frac{d(lc)}{D(lc)} = \frac{\langle \rho_1 + \rho_3, c\rangle}{\langle \rho_2 + \rho_3, c\rangle}.$$

This limit is a rough homological invariant of the sub-Riemannian structure and it is a rational function of a!

9. Contact structures. Our next topic is exponential mappings, i.e. the restrictions of the end-point mappings on the sets of sub-Riemannian geodesics. Following the philosophy of this paper, we deal with the most "smooth" case.

Let Δ be a contact structure, i.e. a corank 1 distribution such that $[v, \Delta]_q = T_qM$, $\forall v \in \Delta$, $v(q) \neq 0$, $q \in M$. Hence the dimension of M is odd, $\dim M = 2m + 1$. We will consider a sub-Riemannian structure V^l, $l > 0$, coordinated with Δ. All geodesics for such a structure are regular except the constant trajectory $q(t) \equiv q_0$. While all nontrivial geodesics are regular, they form a smooth manifold $\mathcal{Q} = \bigcup\limits_{l>0} \mathcal{Q}^l$ naturally diffeomorphic to an open subset of $T^*_{q_0}M \setminus \Delta^\perp_{q_0}$. We obtain a desired diffeomorphism just by identifying a geodesic for V^l with the initial point of the extremal λ_t associated with this geodesic and normalized by the relation $h^l(\lambda_t) = 1$. Then $\mathcal{Q}^l$ is identified with $(h^l_{q_0})^{-1}(1)$ for all l small enough and with an open subset in $(h^l_{q_0})^{-1}(1)$ for the arbitrary $l > 0$, where $h^l_{q_0} = h^l|_{T^*_{q_0}M}$.

A dilation $\delta_\tau : \mathcal{Q} \to \mathcal{Q}$, $0 < \tau \leq 1$, is defined by the relation $(\delta_\tau q)(t) = q(\tau t)$, $q(\cdot) \in \mathcal{Q}$, $t \in [0, 1]$. Then $\delta_\tau(\mathcal{Q}^l) = \mathcal{Q}^{\tau l}$ for $l > 0$ small enough. In other words, $\mathcal{Q}^{l'}$ consists of reparametrized pieces of curves from $\mathcal{Q}^l$ if $l' < l$.

Consider "the exponential mapping" $ex : \ q(\cdot) \mapsto q(1)$, $q(\cdot) \in \mathcal{Q}$. Let us denote by $\mathcal{C}$ the set of critical points of ex.

Proposition 9. *a)* $\#\{\tau \in (0,1) : \delta_\tau q \in \mathcal{C}\} < \infty$, $\forall q \in \mathcal{Q}$.

b) Let $q \in \mathcal{Q}\setminus\mathcal{C}$, *then*
q *is a strong length minimizer* $\Longleftrightarrow \delta_\tau q \notin \mathcal{C}$, $\forall \tau \in (0,1)$.

c) For any $K \Subset \mathcal{Q}$ *there exists* $\tau_K > 0$ *such that* $\delta_\tau(K) \cap \mathcal{C} = \emptyset$ $\forall \tau \leq \tau_K$.

d) $\mathcal{Q}^l \cap \mathcal{C} \neq \emptyset$ *for any small enough* $l > 0$.

Properties a) - c) of the exponential mapping are similar to the case of Riemannian Geometry but d) is the exact opposite of the Riemannian case. It follows from a), b), d) that there exist arbitrarily short geodesics started at q_0 which are not strong length minimizers. A formal reason is the noncompactness of

$\mathcal{Q}^l \approx (h^l_{q_0})^{-1}(1)$, as opposed to the Riemannian Geometry. Actually, this phenomenon is easily predictable since arbitrarily short geodesics cover a neighborhood of q_0, although all of them are tangent to the hyperplane Δ_{q_0}.

The set

$$\mathbf{C} = \{q(1) : q(\cdot) \in \mathcal{C},\ \delta_\alpha q \notin \mathcal{C},\ \forall \alpha \in (0,1)\} \subset M$$

is called the sub-Riemannian caustic. It is an "envelope" of the family of geodesics. Initial point q_0 belongs to the closure of $\mathbf{C}$. We need more notations to say more.

The sub-Riemannian structure $V^l,\ l > 0$, induces a Euclidean structure on $\Delta_q,\ q \in M$, such that the Euclidean length of $\forall v \in V^l_q$ is equal to l. Let ω be a differential one-form which is orthogonal to Δ and normalized by the following condition: $2m$ -form $(d_q\omega)^m|_{\Delta_q}$ is the volume form for the Euclidean structure induced by V^l. The form ω is defined up to a sign in a neighborhood of q_0, it is defined globally iff contact structure Δ is coorientable. Our considerations are local and we fix a sign of ω.

Set $h = \frac{1}{2}(h^1)^2 = \frac{1}{2l^2}(h^l)^2$ a Hamiltonian which is quadratic on the fibers of T^*M. Relations $e\rfloor\omega = 1, \quad e\rfloor d\omega = 0$ define a vector field e and a Hamiltonian $e^* : \lambda \mapsto \langle \lambda, e(q)\rangle,\ \lambda \in T_qM$, which is linear on fibers. Let us consider the Poisson bracket $\{e^*, h\}$. It is a one more Hamiltonian which is quadratic on fibers. It is possible to show that $\Delta_q^\perp$ is contained in the kernel of the quadratic form $\{e^*, h\}_q = \{e^*, h\}|_{T_q^*M}$. Hence we may consider $\{e^*, h\}_q$ as a quadratic form on $\Delta_q^* = T_q^*M/\Delta_q^\perp$. Moreover, the Euclidean structure on Δ_q permits us to identify Δ_q^* with Δ_q and to consider $\{e^*, h\}_q$ as a quadratic form on Δ_q or, in other words, as a symmetric operator on the Euclidean space Δ_q. In particular, the trace and the determinant of $\{e^*, h\}_q$ are well defined. It turns out that $\mathrm{tr}\{e^*, h\}_q = 0$ but the determinant doesn't vanish, generally speaking.

If M is the total space of a principle bundle with one-dimensional fibers transversal to Δ, and V is invariant under the action of structure group (so that Δ is just a connection on the principle bundle), then e is a "vertical' vector field and $\{e^*, h\} = 0$. Conversly, if $\{e^*, h\} = 0$ for a contact sub-Riemannian structure, then the structure is invariant under the one-parametric group generated by e.

We have $T^*_{q_0}M = \mathbb{R}\omega_{q_0} + \Delta^*_{q_0}$. Let $\nu \in \mathbb{R}$ and $\eta \in \Delta^*_{q_0},\ \eta \neq 0$. We will denote by $q(\cdot;\nu,\eta)$ the geodesic which is the projection on M of the extremal, starting at $(\nu\omega_{q_0} + \eta) \in T^*_{q_0}M$. It turns out that the mapping $\nu \mapsto q(\frac{1}{\nu};\nu,\eta)$ possesses an asymptotic expansion for $\nu \to \infty$ in the power series in $\frac{1}{\nu}$ with coefficients which are elementary functions of η. It was the study of this expansion that made it possible to obtain fundamental invariants of the contact sub-Riemannian structurs and to understand the form of the caustic near q_0 in the generic situation for $m = 1$.

Dimension 3. Let $\mathrm{dim}M = 3$. Interesting calculations were made by various authors in this minimal possible dimension for a symmetric (Lie group) case where geodesics have a simple explicit expression (see especially [12]). We'll see, however, that principal invariants vanish in that symmetric case.

The fig.2 shows the form of the caustic $\mathbf{C}$ near q_0 if $\{e^*, h\}_{q_0} \neq 0$. "Horizontal" sections have 4 cusps.

fig.2

We don't use below a special notation for the standard identification of Δ_{q_0} and $\Delta^*_{q_0}$, and just put elements of Δ_{q_0} instead of $\Delta^*_{q_0}$ in formulas. Thus $q(\cdot;\nu,v),\ v\in\Delta_{q_0}$ is a geodesic whose velocity equals v at the starting point. The form $d_{q_0}\omega$ induces an orientation of Δ_{q_0} and of $V^1_{q_0}$ which is the unit circle in the Euclidean plane Δ_{q_0}. We will denote by $d\theta_\xi,\ \xi\in V^1_{q_0}$, the angle differential form on the oriented circle.

Let $\nu\in\mathbb{R},\ v\in V^1_{q_0}$; set

$$l_c(\nu,v)=\min\{l>0: q(\cdot;\nu,lv)\in\mathcal{C}\},\quad q_c(\nu,v)=q(l_c(\nu,v);\nu,v).$$

Then $l_c(\nu,v)$ is the supremum of the length of strong length minimizing pieces of the geodesic $q(\cdot;\nu,v)$, and $q_c(\nu,v)$ is the point of the caustic $\mathbf{C}$ where this geodesic ceases to be a strong length minimizer.

Theorem 10. *The following asymptotic expansions holds for $\nu\to\pm\infty,\ v\in V^1_{q_0}$:*

$$q_c(\nu,v)=\pm\nu^{-2}\pi e(q_0)-\nu^{-3}\frac{3\pi}{2}\int_v^{-v}\{e^*,h\}_{q_0}(\xi)\xi\,d\theta_\xi+O(\nu^{-4})$$

$$l_c(\nu,v)=|\nu|^{-1}2\pi-|\nu|^{-3}\pi\rho(q_0)+O(\nu^{-4}),$$

where $\rho(q_0)$ is a constant.

The curve $v\mapsto\frac{3}{2}\int_v^{-v}\{e^*,h\}_{q_0}\xi\,d\theta_\xi,\ v\in V^1_{q_0}$, is a symmetric astroid in Δ_{q_0}; its radius equals $(-\det\{e^*,h\}_{q_0})^{\frac{1}{2}}$, and cuspidal points belong to the isotropic lines of the form $\{e^*,h\}_{q_0}$.

The invariant $\rho(q),\ q\in M$, is, in fact, a nonholonomic analog of the Gaussian curvature of a surface. Let $v_1(q),v_2(q)$ be an orthonormal frame in Δ_q with a right orientation. Then $[v_1,v_2]=\alpha_1v_1+\alpha_2v_2-e,\quad[e,v_j]\in\Delta,\ j=1,2$, where α_j are smooth functions, and

$$\rho=v_1\alpha_2-v_2\alpha_1-\alpha_1^2-\alpha_2^2+\frac{1}{2}(\langle[e,v_2],v_1\rangle-\langle[e,v_1],v_2\rangle).$$

A simple count of parameters shows that sub-Riemannian structures on a three-dimensional manifold should have two "functional invariants". We already have two: $\det\{e^*,h\}$ and ρ.

Theorem 11. *Let $d\rho=0,\ det\{e^*,h\}_q=0\ \forall q\in M$, where M is a parallelizable manifold, and $H^1(M;\mathbb{R})=0$.*

Then there exists an orthonormal frame v_1,v_2 in Δ such that

$$[v_2,v_1]=e,\ [v_1,e]=\rho v_2,\ [v_2,e]=-\rho v_1.$$

So a contact sub-Riemannian structure on a three-dimensional manifold, with the identically vanishing $\{e^*, h\}$ and ρ, is locally equivalent to the Heisenberg group with a left-invariant sub-Riemmanian structure— the most popular example in Nonholonomic Geometry. We obtain a model of the sub-Riemannian manifold with the identically vanishing $\{e^*, h\}$ and constant positive (negative) ρ if we consider the group $SU(2)$ ($\tilde{SL}(2;\mathbb{R})$) with the sub-Riemannian structure wich is defined by the restriction on a left-invariant distribution of the bi-invariant (pseudo-)Riemannian structure on $SU(2)$ ($\tilde{SL}(2;\mathbb{R})$).

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