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One more condition for a conditional extremum

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The difficulties associated with the characterization of points of conditional extremum are well known in the case where the restrictions are not independent. In this note we will state a necessary and sufficient condition which takes into account, on the one hand, the method of Milyutin [4], and on the other hand, the results of [1], [3]. We only consider the finite-dimensional situation, where the essence of the problem is not overshadowed by details of a functional analytic character. The author is grateful to A.V. Arutyunov for calling his attention to the fact that it would be interesting to obtain a condition for conditional extremum.

1. Let V be a finite-dimensional real vector space. We denote by $\mathscr{P}(V)$ the space of real-valued symmetric bilinear forms on V and by $S^2(V)$ the symmetric product of V by itself. Any $q \in \mathscr{P}(V)$ can be regarded as a linear form on $S^2(V)$, so that $S^2(V) = \mathscr{P}(V)^*$. To each $q \in \mathscr{P}(V)$ there corresponds a number ind q, the maximal dimension of a subspace of V on which the quadratic form $v \mapsto q(v, v)$ is negative definite, and also the subspace

$$\ker q = \{v \in V \mid q(v, V) = 0\}.$$

Let $C \subset V$ be a convex polyhedral cone (with vertex at the origin) and let C° denote its polar, as usual: $C^\circ \subset V^*$.

Definition. A linear map φ: $V \to \mathscr{P}(M)$ is called *regular* on C if

$$\varphi(C)^c \cap S^2(\ker \varphi(v)) = 0 \quad \forall v \in C \setminus 0.$$

By using Sard's theorem, it is easy to prove that in the set of linear maps of V into $\mathscr{P}(M)$ the maps regular on a polyhedral cone form an open everywhere dense subset.

2. Let $f \in C^2(\mathbb{R}^N; \mathbb{R}^n)$, $f(0) = 0$, and let $K \subset \mathbb{R}^n$ be a convex polyhedral cone with vertex at the origin. We denote by f_0': $\mathbb{R}^N \to \mathbb{R}^n$ the derivative of f at the origin and by f_0'': $\ker f_0' \times \ker f_0' \to \mathbb{R}^n$ the restriction of the second derivative to the kernel of the first derivative. Therefore, f_0' is a linear and f_0'' a bilinear symmetric map. The space $\mathbb{R}^n$ consists of column vectors and $\mathbb{R}^{n*}$ of row vectors. If $\psi \in \mathbb{R}^{n*}$, then

$$\psi f_0'' \in \mathscr{P}(\ker f_0').$$

Let us introduce the notation

$$\Psi = K^\circ \cap (\operatorname{im} f_0')^\perp, \quad r = \dim \Psi, \ \Psi_r = \{\psi \in \Psi \setminus 0 \mid \operatorname{ind} \psi f_0'' < r\}.$$

Theorem. *Suppose that $r > 0$, $r(r-1) \leqslant N - n$, $K \cap \operatorname{im} f_0' = 0$, and that the map $\psi \mapsto \psi f_0''$, $\psi \in \mathbb{R}^{n*}$ is regular on Ψ. Then the following assertions are equivalent*: a) $f(O_0) \cap K = 0$ *for some neighbourhood O_0 of the origin in $\mathbb{R}^N$*; b) $f(O_0 \setminus 0) \cap K = \varnothing$ *for some neighbourhood O_0 of the origin in $\mathbb{R}^N$*; c) $\Psi_r \neq \varnothing$ *and* $\max\limits_{\psi \in \Psi_r} \psi f_0''(u, u) \geqslant 0$ *for all* $u \in \ker f_0'$.

3. *Proof.* We denote by Π the maximal subspace in the cone $\operatorname{im} f_0' + K$. Let π: $\mathbb{R}^n \to \mathbb{R}^n / \Pi$ be the canonical map onto the quotient space. Then $\pi(K)$ is an acute polyhedral cone. Let

$$S = \{u \in \ker f_0' \mid \|u\| = 1\}$$

be the unit sphere in $\ker f_0'$. We introduce the map p: $S \to \mathbb{R}^n / \Pi$ by putting $p(u) = \pi f_0''(u, u)$, $u \in S$. It is easy to deduce from the regularity of $\psi \mapsto \psi f_0''$ on $\Psi = \pi(K)^\circ$ that p is transversal to $\pi(K)$ (for the definition of transversality of a map to a convex set, see the appendix of [2]). In turn if p is transversal to $\pi(K)$, then, as is easy to show, each of the assertions a), b) is equivalent to the relation

(1) $$p(S) \cap \pi(K) = \varnothing.$$

At the same time, assertion c) is equivalent to the relation

(2) $$p(S) \cap \operatorname{int}(\Psi_r^\circ + \pi(K)) = \varnothing.$$

The remaining reasoning follows along the lines of the proof of Proposition 2.1 of [2]. First of all, it follows from the proof of Lemma 2.3 of [2] that under the hypotheses of the theorem $\operatorname{int} \Psi_r^\circ \neq \varnothing$. For the proof of the theorem it is sufficient to show that (1) implies (2).

Let us assume that (1) is satisfied but (2) is not. Then there is a $u_0 \in S$ such that

$$p(u_0) \in \operatorname{int}(\Psi_r^\circ + \pi(K))$$

and

$$(p(u) - \alpha p(u_0)) \notin \pi(K) \qquad \forall \alpha \in [0, 1), \quad u \in S.$$

In this case there exists $\psi \in \pi(K)^\circ \setminus 0$, $\psi p(u_0) \geqslant 0$, such that u_0 is a critical point of ψp and the Hessian of ψp at u_0 has index smaller than $r = \dim(\mathbb{R}^n/\Pi)$. Recalling that ψp is the restriction of the quadratic function $u \mapsto \psi f_0''(u, u)$ to S, we obtain $\operatorname{ind} \psi f_0'' < r$. Hence $\psi \in \Psi_r \cap \pi(K)^\circ$. Consequently, for any non-zero $y \in \operatorname{int}(\Psi_r^\circ + \pi(K))$ we have $\psi y < 0$. This contradicts the inequality $\psi p(u_0) \geqslant 0$. ∎

The convenience of the above theorem lies in the fact that both the statement and the proof rely only on elementary facts of convex analysis. The main shortcoming is obvious: for the verification of the condition of extremum, the maximum of a family of quadratic forms has to be considered; in the presence of the quadratic forms themselves, this is a very complex undertaking. However, a more basic study of the quadratic map $u \mapsto f_0''(u, u)$ requires much more powerful methods (see [2]).

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