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ENSEMBLE CONTROLLABILITY BY LIE ALGEBRAIC METHODS

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Abstract. We study possibilities to control an ensemble (a parameterized family) of nonlinear control systems by a single parameter-independent control. Proceeding by Lie algebraic methods we establish genericity of exact controllability property for finite ensembles, prove sufficient approximate controllability condition for a model problem in $\mathbb{R}^3$, and provide a variant of Rashevsky-Chow theorem for approximate controllability of control-linear ensembles.

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1. Introduction

1.1. Motivation

Over the last decade there has been a rise of interest regarding controllability of ensembles - parameterized families - of nonlinear control systems

$$\dot{x} = f^{\theta}(x, u), \ \theta \in \Theta,$$

by a single θ-independent control $u(\cdot)$. Such problems arise for example, from a necessity to control a system with a "structured uncertainty", when some parameters of the system are subject to "dispersion".

The problems of designing a control, which compensates the dispersion, appear for example in NMR spectroscopy. Study of the control of Bloch equation under various types of dispersion has been initiated by S. Li and N. Khaneja ([11–13]). The state space of Bloch equation is a (matrix) Lie group, and therefore Lie algebraic notions and tools, such as e.g. Campbell-Hausdorff formula, appeared in its study naturally. The core of the approach of S. Li and N. Khaneja is "generating higher order Lie brackets by use of the control vector fields which carry higher order powers of the dispersion parameters to investigating ensemble controllability". More recent publication by K. Beauchard-J.-M. Coron-P. Rouchon ([6]), also dedicated to the Bloch equations with dispersed parameter, invoked *analytic* methods to obtain finer results on ensemble controllability.

In the current presentation we search for an extension of the Lie algebraic approach of geometric control theory onto ensembles of nonlinear systems.

Continual ensemble is an infinite-dimensional control system with finite-dimensional space of control parameters. Therefore exact controllability would in general fail, a mechanism for such failure is explained in

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another context in [5]. We concentrate on *approximate ensemble controllability* by means of controls of *fixed* finite dimension.

On the contrast to the above mentioned publications we do not use any expansion in the parameter θ, nor do we assume any smoothness of ensembles in θ. Instead we advocate an approach, which combines use of iterated Lie brackets and hence Taylor series in state variables, and Fourier-type series in the parameter θ.

We start with *finite* ensembles. For such ensembles the Lie rank criteria of *exact* controllability of a single system can be reformulated in a rather direct way. We prove in Section 3 that the property of global controllability for a *finite ensemble of control-linear systems* is generic (Theorem A). In Section 4 we establish (Theorem B) global controllability by means of a single scalar control for a *finite ensemble of rigid bodies* with generic inertial parameters.

Two examples of *continual ensembles* are studied in Sections 5,6.

The first one is a model example of an ensemble in $\mathbb{R}^3$. We seek for a control, which generates a loop in $\mathbb{R}^2$ and makes the third coordinate to trace approximately a prescribed target (say, a curve or a surface). Theorem C provides sufficient and necessary condition for the approximate controllability.

In Section 6 we study general ensemble of control-linear systems on a manifold. Theorem D provides sufficient approximate controllability criterion; it is an ensemble version of Rashevsky-Chow theorem.

Both criteria are formulated in terms of *Lie algebraic span*.

A number of publications (see [9, 10, 16]) contain variants of Rashevsky-Chow theorem in infinite dimension. We explain in Section 6 the difference between our criteria and the results of the publications cited.

1.2. **Definitions of ensemble controllability**

Let M be a C^ℓ-manifold[1]; $U \subset \mathbb{R}^r$; Θ - compact subset of a Lebesgue measure space.

We consider *ensembles* of control systems parameterized by $\theta \in \Theta$

$$\frac{dx^\theta}{dt} = f^\theta(x, u), x^\theta \in M, \ u \in U, \ \theta \in \Theta. \tag{1}$$

Ensemble is *finite*, whenever Θ is finite and is *infinite* otherwise. Note that the control $u(\cdot)$ in (1) is assumed to be θ-independent, i.e. *all the systems of the ensemble are driven by the same control.*

We are going to study approximate controllability of ensembles (1).

Definition 1.1 (cf. [11])**.** *Let $\alpha(\theta)$ be an ensemble of initial data*

$$x^\theta(0) = \alpha(\theta), \tag{2}$$

and $\omega(\theta)$ be a target ensemble.

We say that ensemble (1) is L_p-approximately steerable from $\alpha(\theta)$ to $\omega(\theta)$ in time $T > 0$, if for any $\delta > 0$ there exists a θ-independent control $\bar{u}(t), \ t \in [0, T]$ (depending on δ) such that for the trajectories of the ensemble

$$\frac{dx^\theta}{dt} = f^\theta(x^\theta, \bar{u}(t))$$

with the initial data (2), there holds:

$$\|x^\theta(T) - \omega(\theta)\|_{L_p(\Theta)} < \delta.$$

Ensemble (1) is L_p-approximately controllable (in time T) if for each pair of measurable bounded maps $\alpha(\theta), \omega(\theta)$ it is L_p-approximately steerable from $\alpha(\theta)$ to $\omega(\theta)$ (in time T). □

Another definition of controllability, which in some cases is slightly stronger, than approximate controllability, can in our view be useful.

[1]In our presentation we consider analytic case $\ell = \omega$; infinitely smooth case $\ell = \infty$ is commented in Remark 6.5

For the ensemble (1) we define a *moment* corresponding to a probability density $p(\theta)$ on the space of parameters Θ:

$$\langle p, x^\theta(t)\rangle = \int_\Theta \langle p(\theta), x^\theta(t)\rangle d\theta.$$

Definition 1.2. *The ensemble (1) is controllable in momenta if for any finite ensemble of probability densities $p_1(\theta), \ldots, p_m(\theta)$, for any initial data (2) and each m-ple $(\pi_1, \ldots, \pi_m) \in \mathbb{R}^m$ there exists a control $\bar{u}(t),\ t \in [0,T]$, which steers the ensemble of initial data (2) to a terminal ensemble $x^\theta(T)$, for which $\langle p_j, x^\theta(T)\rangle = \pi_j,\ j = 1, \ldots, m$.* □

The criteria of controllability in the momenta can be obtained by the methods, introduced below. Still the technicalities differ and we leave the presentation for another occasion.

Also for continual ensembles we restrict our attention to the case, where the parameter θ enters the right hand side of (1), while the initial data does not depend on $\theta:\ \alpha(\theta) \equiv \tilde{x}$. An interesting question of controllability of continual ensembles of initial data (interpreted as controllability in the spaces of surfaces/curves) will be treated elsewhere.

1.3. On Lie algebraic or geometric control approach

The *geometric control theory* approaches controllability, observability and optimality properties of nonlinear control systems investigating the structure of the Lie algebra, generated by the set of vector fields, which "constitute" a control system. Nagano's theorem puts it in strict terms, stating that two control systems, satisfying the same *Lie relations*, are equivalent up to a coordinate change. Identification of the complete set of Lie relations is in general not possible, but often a finite subset of this set suffices for establishing controllability (see [2] for details).

It is rather straightforward to extend the geometric control approach to controllability of a single system onto the case of *finite ensembles*: $|\Theta| = N$. One can just see the ensemble as a single system on a carthesian product of N copies[2] of the state space M and apply Lie algebraic (Lie rank) methods to establish controllability of this system. Two observations are due: i) for finite ensembles approximate controllability "usually" implies exact controllability; ii) the Lie rank and hence the number of iterated Lie brackets, needed for the verification of controllability, grows and tends to infinity with $N \to \infty$.

When dealing with infinite and in particular with continual ensembles tempting is the idea, firt, to discretize Θ, then to establish (when possible) *exact* controllability of the discretized *finite* ensemble and finally refine the discretization (increasing the number of "nodes") and conclude the approximate controllability of the continual ensemble.

Unfortunately this artless idea seems to fail. The reason is that with the refinement of the discretization the number of the iterated Lie brackets, involved, and hence the *complexity* of the corresponding controls grow unboundedly. The 'nodal' systems are driven by the control of high complexity to the target, but one looses control of what happens with the systems "between the nodes".

Leaving this idea out we instead view the ensemble (1) as a system in an infinite-dimensional space of functions, defined on Θ, and seek for an infinite-dimensional variant of the *method of Lie extensions*. In the next few paragraphs we describe the idea informally.

The classical Lie extensions method deals with the vector fields, which are the sections of the tangent bundle TM. Below we consider instead the fiber bundles over the base M with the infinite-dimensional fibers $L_p(\Theta, T_xM)$ over each $x \in M$. Analogues of vector fields are the sections of the fiber bundle. We introduce kind of Lie structure for these sections by taking Lie brackets on M for each $\theta \in \Theta$. We define the Lie extensions and iterating them seek for an analogue of *Lie rank condition*.

Note that if Θ is finite then the fiber is just a Carthesian product of a finite number of copies of T_xM and we come back to the above described approach to finite ensembles.

[2] Minor modification of the approach allows to deal with control systems defined on different C^∞ manifolds $M^1, \ldots, M^N$.

Infinite dimensionality intervenes in two ways. First, since we take a large but finite number of iterated Lie brackets, we end up with approximate controllability. Second, the usual notions of rank, dimension and linear independence should be treated with more care in the infinite-dimensional situation. For model example in Section 5 we invoke Fourier series in θ; in general it is useful in constructing appropriate controls.

A natural extension of the notion of ensemble controllability would be the study of controllability in the space of curves or surfaces, or more generally, on the group of diffeomorphisms Diff M. A criterion of *exact* controllability, presented in [1], required the set of controls to be rich enough to allow for multiplying vector fields of the system by any smooth functional multiplier. We look forward to obtaining results on *approximate* controllability on Diff M by means of finite-dimensional control.

2. Basic assumptions

The following two assumptions for the dynamics of (1) hold for the continual ensembles, treated in Sections 5,6.

Let M be a real analytic (C^ω) manifold.

Assumption 2.1 (Uniform analyticity in x)**.** *For each $\theta \in \Theta$ the vector fields $X^\theta(x),\ x \in M$ can be extended to (complex)-analytic vector fields $X^\theta(z),\ z \in B_\rho(M)$ - a complex ρ-neighborhood of the manifold M.* □

Assumption 2.2 (Dependence on parameter θ)**.** *The set of parameters Θ is a separable compact Hausdorff space equipped with a Borel measure. For each $z \in B_\rho(M)$ the map $\theta \to X^\theta(z)$ is continuous.* □

3. Elementary case I: control of a finite ensemble of control-linear systems

For finite ensembles the controllability in momenta and approximate controllability are equivalent[3] to exact controllability.

We consider finite ensembles of *control-linear* systems and prove that exact controllability property is *generic*.

Finite ensemble of N control-linear systems on a C^∞ manifold M is:

$$\dot{x}^\theta = \sum_{j=1}^{r} X^{\theta j}(x^\theta)u_j(t),\ x^\theta \in M,\ (u_1, \ldots, u_r) \in \mathbb{R},\ \theta = 1, \ldots, N. \tag{3}$$

Once again the control $u(t) = (u_1(t), \ldots, u_r(t))$ is θ-independent.

For a *single system* of ensemble (3), defined by a r-tuple ($r \geq 2$) of vector fields $\left(X^{\theta 1}, \ldots, X^{\theta r}\right)$, classical result by C.Lobry [14] states, that global controllability property is generic, i.e. holds for each $\left(X^{\theta 1}, \ldots, X^{\theta r}\right)$ from a subset of $(\mathrm{Vect}^\infty(M))^r$, which is open and dense in C^ν-metric with ν sufficiently large.

We extend this result to the case of ensembles.

Theorem A. *There exists a natural number ν and a subset $\mathcal{C} \subset \left((\mathrm{Vect}^\infty(M))^N\right)^r$, which is open and dense in $C^\nu(M)$-metric and such that for each rN-tuple of vector fields $\left(X^{\theta j}\right), \theta = 1, \ldots N,\ j = 1, \ldots, r$ from $\mathcal{C}$ the ensemble* (3) *is globally exactly controllable.* □

Proof. It suffices to provide a proof for $r = 2$ in (3), or the same, for the ensemble

$$\dot{x}^\theta = X^\theta(x^\theta)u(t) + Y^\theta(x^\theta)v(t),\ x^\theta \in M,\ \theta = 1, \ldots, N. \tag{4}$$

We make an obvious step considering on M^N the cartesian product of the systems of the ensemble (4). Obviously the vector fields $X = (X^1, \ldots, X^N)$, $Y = (Y^1, \ldots, Y^N)$, belong to $\left((\mathrm{Vect}^\infty(M))^N\right)^2 \subset \left(\mathrm{Vect}^\infty(M^N)\right)^2$.

The following technical lemma is immediate consequence of Rashevsky-Chow theorem.

[3] under full Lie rank condition

Lemma 3.1. *If the pair (X,Y) is bracket generating at each point of M^N, then ensemble (4) is globally controllable.* $\square$

It rests to prove that the bracket generating property is generic in $\left((\mathrm{Vect}^\infty(M))^N\right)^2$ in C^ν-metric for some ν.

C.Lobry's theorem, applied to the couple (X,Y), guarantees existence and density of globally controllable couples of vector fields from $\left(\mathrm{Vect}^\infty(M^N)\right)^2$, while we need them to belong to a smaller set $\left((\mathrm{Vect}^\infty(M))^N\right)^2$. Still one can modify the original idea of C.Lobry in order to cover this case. This is done in Appendix. $\square$

4. Elementary case II: Controllability of a finite ensemble of rigid bodies

Consider an ensemble of N rigid bodies, with the evolution of the momenta, described by an ensemble of Euler equations

$$\dot{K}^\theta = K^\theta \times J^\theta K^\theta + Lu,\ \theta = 1, \ldots, N,\ u \in U \subset \mathbb{R},\ \mathrm{int\ conv}(U) \ni 0; \tag{5}$$

with the *scalar* control torque $u(t)$ (*in body*), applied along one and the same direction L to all bodies.

Here $J^\theta \in \mathbf{J}$ are the (inverses of the) inertia tensors of the bodies; $\mathbf{J}$ is a closed subset with nonempty interior of the set of symmetric positive definite (3×3)-matrices.

We restrict ourselves to an open subset of dynamically asymmetric bodies, or equivalently, the matrices J^θ with distinct positive eigenvalues in int $\mathbf{J}$.

Without loss of generality we may take as $U = [-\beta, \beta],\ \beta > 0$. Admissible controls $u(t)$ are arbitrary measurable functions, but piecewise-continuous and piecewise constant $u(t)$ suffice for controllability.

Consider the cartesian product of the systems (5) defined on $(\mathbb{R}^3)^N$. We provide $(\mathbb{R}^3)^N$ with the euclidean structure and with the standard volume measure of the cartesian product.

Putting $K = (K^1, \ldots, K^N)$, $b_L = (L, \ldots, L)$, $J = \mathrm{diag}(J^1, \ldots J^N)$ the $(3N \times 3N)$ block diagonal matrix, we get a control-affine system on $(\mathbb{R}^3)^N$:

$$\dot{K} = K \times JK + b_L u,$$

(the cross product is applied componentwise).

We denote by $\mathcal{E}_J(K)$ the Euler term: $\mathcal{E}_J(K) = K \times JK$.

4.1. Recollection: controllability result for a single rigid body

For (5) with $N = 1$ controllability result has been established by two different methods in [3, 8], see also [2, Ch.6,8]).

Proposition 4.1. *For an asymmetric J^1 and generic L, the single body is globally controllable.* $\square$

Global controllability of single equation (5) can be derived from the bracket generating property (see Proposition 4.4), satisfied by the pair of vector fields $(\mathcal{E}_{J^1}(K), b_L(K))$ and the *recurrence property* of $\mathcal{E}_{J^1}(K)$.

In the next Subsection we establish controllability of a generic ensemble of N rigid bodies. Proposition 4.1 is a special case of this result.

4.2. Controllability of ensemble of rigid bodies

Theorem B. *Given $L \in \mathbb{R}^3 \setminus 0$, and integer $N \geq 1$ there exists an open dense subset $\mathcal{D} \subset \mathbf{J}^N$, such that for each $(J^1, \ldots, J^N) \in \mathcal{D}$ the finite ensemble of rigid bodies (5) is globally exactly controllable by a torque along L. Besides for each compact subset $\mathcal{C} \subset \mathcal{D}$ there exists an upper bound $T_{\mathcal{C}} > 0$ for minimal attainability times.*
$\square$

Remark 4.2. *The set $\mathbf{J}^N = \mathbf{J} \times \cdots \times \mathbf{J}$ is an open subset of the linear space $\left(Sym(\mathbb{R}^3)\right)^N$* $\square$.

A more interesting question to be answered is

Question 1. *Given an N-tuple $(J^1, \ldots, J^N)$, with J^θ pairwise distinct (or belonging to a sphere in* $\mathbf{J}$ *and pairwise distinct, or a generic N-tuple) and possessing simple eigenvalues, does there exist an open dense subset* $\mathbf{L} \subset \mathbb{R}^3$*, such that* $\forall L \in \mathbf{L}$ *the finite ensemble of rigid bodies* (5) *is globally controllable?* □

We start proving Theorem B.

A vector field on a manifold M is *recurrent* if $\forall x \in M$, each neighborhood W_x of x and each $t > 0$, there exists a point $\hat{x} \in W_x$ and time $\hat{t} > t$ such that $e^{\hat{t}f}(\hat{x}) \in W_x$.

Poincare recurrence theorem establishes this property for a broad class of vector fields, which includes $\mathcal{E}_{J^1}(K)$.

Proposition 4.3 (Poincare recurrence theorem)**.** *If a complete vector field f on a manifold M preserves the volume form (that is, divergence free) and leaves a set A of finite volume invariant. Then the restriction of f to A is recurrent.* □

It is immediate to see, that the drift vector field $\mathcal{E}_J(K) = K \times JK$ is divergence free and preserves $\|K\|^2 = \sum_{i=1}^r \|K_i\|^2$. Hence the sets $B_r = \{K| \ \|K\|^2 = \sum_{i=1}^r \|K_i\|^2 \leq r^2\}$ of finite volume are invariant for the volume preserving vector field $\mathcal{E}_J(K)$, wherefrom the recurrence property follows.

The following result states that bracket generating property of a system of vector fields plus the recurrence property of one of them suffices for controllability of the respective control-affine system

Proposition 4.4 ([7], [15])**.** *Let f_0 possess recurrence property on M and the bracket generating condition $Lie(f_0, \ldots, f_r)(x) = T_xM$ hold at each point $x \in M$. Then the control-affine system $\dot{x} = f_0(x) + \sum_{i=1}^r f^i(x)u_i$, $(u_1, \ldots, u_r) \in U$, (where int conv$(U) \ni 0$) is globally controllable on M.* □

Given the recurrence property of $\mathcal{E}_J(K)$ it remains only to verify the bracket generating property for the pair of vector fields $\{\mathcal{E}_J(K), b_L\}$ on $\left(\mathbb{R}^3\right)^N$.

Given that the vector field $\mathcal{E}_J(K)$ is a polynomial of second degree in K and the vector field b_L is constant, it is convenient to take into account those iterated Lie brackets, which result in constant vector fields. These are for example

$$V^0_{J,L} = b_L(K) = L, V^1_{J,L} = \left[V^0_{J,L}, [\mathcal{E}_J, V^0_{J,L}]\right] = 2L \times JL, \ V^{m+1}_{J,L} = \left[V^m_{J,L}, [\mathcal{E}_J, V^0_{J,L}]\right], \ m \geq 1. \tag{6}$$

We form $(3N \times 3N)$-matrix

$$R_N\left(J^1, \ldots, J^N; L\right) = \begin{pmatrix} V^0_{J^1,L} & V^1_{J^1,L} & \cdots & V^{3N-1}_{J^1,L} \\ V^0_{J^2,L} & V^1_{J,L} & \cdots & V^{3N-1}_{J^2,L} \\ \cdots & \cdots & \cdots & \cdots \\ V^0_{J^N,L} & V^1_{J^N,L} & \cdots & V^{3N-1}_{J^N,L} \end{pmatrix}. \tag{7}$$

Bracket generating property for fixed J and L would be implied by the non-nullity of the determinant $\det R_N\left(J^1, \ldots, J^N; L\right)$.

For fixed L (7) is a polynomial with respect to (the elements of) J, its nullity determines an algebraic variety in $\left(Sym(\mathbb{R}^3)\right)^N$. If the determinant does not vanish *identically* on $\left(Sym(\mathbb{R}^3)\right)^N$, i.e. the variety is proper; then its complement, intersected with $\mathbf{J}^N$, contains an open dense subset of $\mathbf{J}^N$.

Lemma 4.5. *For each L from an open dense subset of $\mathbb{R}^3$ and for each $N \geq 1$ the determinant* $\det R_N(J^1, \ldots, J^N; L)$ *does not vanish identically on $\left(Sym(\mathbb{R}^3)\right)^N$.* □

The proof of the Lemma goes by induction in N. We provide the initial inductive step; `the rest of the proof can be found in Appendix`.

The set of zeros of the determinant

$$R_1\left(J^1; L\right) = \det\left(V^0_{J^1,L}\middle| V^1_{J^1,L}\middle|V^1_{J^1,L}\right)$$

can be characterized. Direct computation shows (see [2]) that for a dynamically asymmetric J^1 and for L, lying in a complement to the union $\tilde{\mathbf{L}}$ of 3 straight lines (principal axes) and two (separatrix) planes, the determinant $R_1\left(J^1; L\right) \neq 0$.

This fact together with Proposition 4.4, implies the statement of Proposition 4.1. □

Remark 4.6. *One should be selective in choosing iterated Lie brackets, when establishing bracket generating property. For example the constant vector field $\tilde{V}^2_{J,L} = \left[V^1_{J,L}, [\mathcal{E}_J, V^1_{J,L}]\right]$ is collinear to $V^0_L = L$ for any J.* □

4.3. A remark on the bounds for the attainability time

We will provide the following reinforcement of the previous statement.

Proposition 4.7. *Let $\mathcal{C} \subset \left(\mathbb{R}^3\right)^N$ be compact. Under the conditions of Theorem B there exists uniform upper bound $T_{\mathcal{C}} > 0$, such that $\forall \tilde{K}, \hat{K} \in \mathcal{C}$ ensemble* (5) *can be steered from $\tilde{K}$ to $\hat{K}$ in time $T \leq T_{\mathcal{C}}$.* □

Proof. Existence of minimal attainability $T(\tilde{K}, \hat{K})$ time for each couple $\tilde{K}, \hat{K}$ is part of classical Filippov theorem. Being the system globally controllable one can conclude that $\hat{K}$ is *normally attainable* ([17]) from $\tilde{K}$ in the greater time $T(\tilde{K}, \hat{K}) + 1$. Then each point of a small neighborhood of $\hat{K}$ is attainable from any point in small neighborhood of $\tilde{K}$, or, equivalently, that for any (x, y) in a small neighborhood of $(\tilde{K}, \hat{K})$, y can be reached from x in time $T(\tilde{K}, \hat{K}) + 1$. By compactness, one can choose a finite cover by such neighborhoods of $\mathcal{C} \times \mathcal{C}$, implying the proposition. □

5. Continual ensemble of control-linear system: model example

We elaborate our approach to approximate controllability of continual ensembles on a simple model with 2 controls:

$$\dot{x} = u, \ \dot{y} = v, \ \dot{z}^\theta = f^\theta(x)v, \tag{8}$$

$$x(0) = y(0) = z^\theta(0) = 0. \tag{9}$$

The ensemble is constituted by control-linear systems whose right-hand side is spanned by the vector fields

$$X = \frac{\partial}{\partial x}, \ Y^\theta = \frac{\partial}{\partial y} + f^\theta(x)\frac{\partial}{\partial z^\theta}, \ \theta \in \Theta. \tag{10}$$

One proceeds under Assumptions 2.1,2.2, in particular $f^\theta(x)$ is analytic in x. We set a slightly modified `Approximate ensemble controllability problem.` *Given $T > 0$ and a target function $\hat{z}(p) \in L_\infty(\Theta)$ and $\varepsilon > 0$ does there exist θ-independent controls $u(\cdot), v(\cdot) \in L_2[0, T]$, such that for the trajectory, driven by $u(\cdot), v(\cdot)$ there holds:*

$$x(T) = y(T) = 0, \ \int_\Theta \|z^\theta(T) - \hat{z}(\theta)\|^2 d\theta \leq \varepsilon. \tag{11}$$

Note that, on the contrast to the previous problem setting, we ask for exact controllability in coordinates x, y.

For simple model (8)-(9) the trajectory can be computed explicitly:

$$x(t) = U(t) = \int_0^t u(\tau)d\tau, \ y(t) = V(t) = \int_0^t v(\tau)d\tau, \ z^\theta(t) = \int_0^t f^\theta(U(\tau))v(\tau)d\tau = \int_0^t f^\theta(U(\tau))dV(\tau). \tag{12}$$

Consider Taylor expansion for f^θ in x at 0:

$$f^\theta(x) = \sum_{m=1}^{\infty} a_m(\theta)x^m, \ a_m(\theta) = \frac{1}{m!}\left.\frac{\partial^m f^\theta}{\partial x^m}\right|_{x=0}. \tag{13}$$

The following condition is central for the controllability of the ensemble (8)-(9).

Definition 5.1 (*Lie algebraic span condition*)**.** *The functions* $(a_m(\theta))_1^\infty$*, defined by* (13)*, span dense subspace of* $L_2(\Theta)$*:*

$$\overline{span}\{a_m(\theta),\ m=1,\ldots\}=L_2(\Theta). \quad \Box \tag{14}$$

Remark 5.2. *We talk about Lie algebraic condition, since the functions* $a_m(\theta)$ *are* z^θ*-components of the evaluations at* $x=0$ *of the iterated Lie brackets* $\frac{1}{m!}\left((\mathrm{ad}X)^m Y^\theta\right)$ *of the vector fields* (10)*.* $\Box$

Theorem C. *Ensemble* (8) *is time-T approximately controllable for each* $T>0$ *if and only if the Lie algebraic span condition* (14) *holds.* $\Box$

Rescaling of the time and control $t\to k^{-1}t,\ (u,v)\to(ku,kv),\ k\in\mathbb{R}_+$, leaves (8) invariant, therefore we can assume $T=1$.

By (12):

$$z^\theta(1)=\int_0^1 f^\theta(U(t))v(t)dt. \tag{15}$$

One needs to construct functions $U(t),v(t)$ such that $U(1)=x(1)=V(1)=y(1)=0$ and $z^\theta(1)$, defined by (15), would satisfy the inequality (11) for $T=1$.

To accomplish this we proceed by a variant of moments method.

From now on assume the magnitude of the function $U(t)$ to be small, so that the series

$$f^\theta(U(t))=\sum_{m=1}^{\infty}a_m(\theta)(U(t))^m \tag{16}$$

will be converging.

We will seek $v(t)$ as a linear combination: $v(t)=\sum_{r=1}^{R}y_r v_r(t)$; integer parameter R depends on the rate of approximation and will be specified in a moment.

For the controls defined one derives from the expansion (16):

$$z^\theta(1)=\sum_{m=1}^{\infty}a_m(\theta)\sum_{r=1}^{R}\gamma_{mr}y_r. \tag{17}$$

where

$$\gamma_{mr}=\int_0^1 (U(t))^m\, v_r(t)dt. \tag{18}$$

If Lie algebraic span condition is satisfied, then for each $\varepsilon_1>0$ one can find a finite linear combination $\sum_{r=1}^{R}c_m a_m(\theta)$, such that

$$\|\hat{z}(\theta)-\sum_{r=1}^{R}c_r a_r(\theta)\|_{L_2(\Theta)}<\varepsilon_1. \tag{19}$$

This sets the number R, which depends on the rate of approximation $\varepsilon_1:\ R=R(\varepsilon_1)$.

Our goal is to choose $U(t),v_r(t)$ in such a way that the equation

$$\sum_{m=1}^{\infty}\left(\sum_{r=1}^{R}\gamma_{mr}y_r\right)a_m(\theta)=\sum_{r=1}^{R}c_r a_r(\theta)$$

with the coefficients γ_{mr}, defined by (18), would be *approximately solvable* with respect to y_r.

This fact, `proved in Appendix`, completes the proof of sufficiency part of the Theorem C.

Now we prove the *necessity.* If the closure in (14) is a proper subspace in $L_2(\Theta)$ take an element $\nu(\theta)$ orthogonal to the closure: $\int_\Theta \nu(\theta)a_m(\theta)d\theta=0$. By (17) $\int_\Theta \nu(\theta)z^\theta(1)d\theta=0$ and hence the system can not be approximately steered to any target function $\hat{z}(\cdot)$, which is not orthogonal to $\nu(\cdot)$. $\Box$

6. Controllability of ensembles of driftless (control-linear) systems. Ensemble version of Rashevsky-Chow theorem

6.1. Formulation of the result

Consider the ensemble of control-linear systems

$$\frac{d}{dt}x^{\theta}(t) = \sum_{j=1}^{r} f_j^{\theta}(x^{\theta})u_j(t). \tag{20}$$

We study controllability of the ensemble for the case where the parameter θ enters the dynamics, while the initial data $\tilde{x}$ and the target $\hat{x}$ are θ-independent. Let $d(x, y)$ be a Riemannian distance on M.

Definition 6.1. *The ensemble* (20) *is time-T L_1-approximately steerable from $\tilde{x}$ to $\hat{x}$, if $\forall \varepsilon > 0$ there exists a control $u(\cdot)$, which steers in time T the ensemble* (20) *from $\tilde{x}$ to $x^{\theta}(T)$, and:*

$$\int_{\Theta} d(x^{\theta}(T), \hat{x})d\theta < \varepsilon. \quad \square. \tag{21}$$

Remark 6.2. *For technical reasons we opt here for $L_1(\Theta)$-approximations of the target on the contrast to $L_2(\Theta)$-approximations invoked in Section 5.* $\square$

Let assumptions 2.1,2.2 hold.

Definition 6.3. *Lie algebraic span condition holds for* (20)*, if $\forall x \in M$ the evaluations at x of the iterated Lie brackets of the vector fields $f_{\alpha}^{\theta}(x)$*

$$X_{\alpha}^{\theta}(x) = [f_{\alpha_1}^{\theta}, [f_{\alpha_2}^{\theta}, [\ldots, f_{\alpha_N}^{\theta}]\ldots]](x), \ \theta \in \Theta, \tag{22}$$

span dense subspace of the Banach space $L_1(\Theta, T_x M)$. $\square$

Theorem D (ensemble controllability criterion)**.** *Let the assumptions 2.1,2.2 and the Lie algebraic span condition hold for* (20)*. Then for each couple $(\tilde{x}, \hat{x})$ and each $T > 0$ the ensemble* (20) *is time-T L_1-approximately steerable from $\tilde{x}$ to $\hat{x}$.* $\square$

Remark 6.4. *For Θ being finite ($|\Theta| = N$), the space $L_1(\Theta, T_x M)$ becomes finite-dimensional, isomorphic to $T_x M^N$, and the Lie algebraic span condition is equivalent to the bracket generating condition on M^N. In this case stronger result on exact ensemble controllability holds (see Section 3).*

For $|\Theta| = 1$, i.e. for single system, one gets Rashevsky-Chow theorem. $\square$

Remark 6.5. *While in this paper we mainly work with the analytic systems, the results of Theorem D remain true for the vector fields f_j^{θ}, which are merely C^{∞}-smooth in x. This can be achieved by modification of some technical details of the arguments below.*

We believe that the requirement of continuity of f_j^{θ} in θ can also be loosened. $\square$

Remark 6.6. *There was a number of publications ([9, 10, 16]) which presented variants of approximate Rashevsky-Chow theorem in infinite dimension. All those results regard control-linear systems $\dot{y} = \sum_{i=1}^{r} g^i(y)u_i(t)$, $y \in E$ in infinite-dimensional vector space E and, roughly speaking, state that whenever approximate bracket generating property holds, i.e. the iterated Lie brackets of the vector fields $g^1, \ldots, g^r$ evaluated at each point of the infinite-dimensional space span a dense subspace of E, then the system is approximately controllable.*

The controlled ensemble (20) *can be seen as a control-linear system in a space E of the functions $x(\theta) = x^{\theta}$. One can introduce vector fields on this space, define the Lie brackets in standard way, and apply the results, just mentioned, to get a controllability criterion.*

This would require verification of the approximate bracket generating property, or the same, density of the span of the iterated Lie brackets (22)*, evaluated at each "point" $x(\theta)$ of the functional space E. This is a vast set of conditions, "indexed" by the elements of the functional space E.*

On contrast to it Theorem D just requires verification of the approximate bracket generating property at the points of finite-dimensional manifold M, or , in other words, at the constant functions $x(\theta) \equiv \bar{x}$. $\square$

The rest of the Section is dedicated to the *proof of Theorem* D, which is based on an infinite-dimensional version of the method of Lie extensions.

According to the method we first establish the possibility to steer an *extended ensemble*

$$\frac{d}{dt}x^{\theta}(t) = \sum_{\alpha \in A} X_{\alpha}^{\theta}(x)v_{\alpha}(t), \tag{23}$$

which involves the vector fields $X_{\alpha}^{\theta}(x)$, defined by (22), and a high-dimensional *extended control* (v_{α}). Then we demonstrate how the action of the extended control can be approximated by the action of a small-dimensional original control.

Remark 6.7. *Without lack of generality we assume that all the extended ensembles, we invoke, contain the vector fields $f_j^{\theta}(x)$, which define the dynamics of the original ensemble* (20). $\square$

6.2. Steering an extended ensemble

Proposition 6.8. *Under the assumptions of the Theorem for each $\tilde{x}, \hat{x} \in M$, and each $\varepsilon > 0, T > 0$ there exists a finite set of multi-indices $A_{\varepsilon} = \{(\alpha_1, \ldots, \alpha_N)\}$, and an extended control $(v_{\alpha}(t))_{\alpha \in A}$, which steers in time T the extended ensemble* (23) *from $\tilde{x}$ to $x^{\theta}(T)$, so that* (21) *holds.* $\square$

Time and control rescaling $t \to k^{-1}t,\ v_{\alpha} \to kv_{\alpha}, \alpha \in A,\ k \in \mathbb{R}_{+}$ leaves the control-linear ensemble (23) invariant; therefore whenever controllability property holds for some $T_0 > 0$, then it holds for any $T > 0$.

Now let us choose any C^{∞}-smooth vector field $Y(x)$ on M with a trajectory $\bar{x}(t)$, which satisfies the boundary conditions

$$\bar{x}(0) = \tilde{x},\ \bar{x}(1) = \hat{x}.$$

Denote $\bar{\gamma} = \{\bar{x}(t)|\ t \in [0,1]\} \subset M$.

We prove the following technical Lemma.

Lemma 6.9. *Under the assumptions of Theorem D there exists a pair of compact neighborhoods $\tilde{V}, V$ of $\bar{\gamma}$ ($\tilde{V} \supset \bar{V}$) and for each $\varepsilon > 0$ a finite set of smooth functions $(v_{\alpha}(x))$, $\alpha \in A_{\varepsilon}$ with supports, contained in $\tilde{V}$ and such that $\forall x \in \bar{V}$:*

$$\left\| Y(x) - \sum_{\alpha \in A_{\varepsilon}} v_{\alpha}(x)X_{\alpha}^{\theta}(x) \right\|_{L_1(\Theta)} < \varepsilon. \tag{24}$$

Remark 6.10. *The vector $Y(x)$ in* (24) *is seen as constant vector-function of $\theta \in \Theta$.*

To prove Lemma 6.9 we fix a compact neighborhood $\bar{V}$ of γ such that at each point $x \in \bar{V}$ the Lie algebraic span condition (6.3) is satisfied. Then for each $\varepsilon > 0$ and each $x^0 \in \bar{V}$ there exists a neighborhood $U_{x^0} \ni x^0$ such that inequality (24) remains valid for each point $x \in U_{x^0}$. We can arrange a finite covering of $\bar{V}$ by the neighborhoods $U_i = U_{x_i},\ i = 1, \ldots, N$, such that

$$\forall x \in U_i: \ \left\| Y(x) - \sum_{\alpha_i \in A_i} v_{i\alpha_i}(x_i)X_{\alpha_i}^{\theta}(x) \right\|_{L_1(\Theta)} < \varepsilon,\ i = 1, \ldots, N.$$

Choose a smooth partition of unity $\{\lambda_i(x)\}$ subject to the covering $\{U_i\}$ of $\bar{V}$. Take the union $\tilde{V}$ of the supports of $\lambda_i,\ i = 1, \ldots, N$ and $A_{\varepsilon} = \bigcup_{i=1}^{N} A_i$. Put for each $\alpha \in A_{\varepsilon}:\ v_{\alpha}(x) = \sum_{[i,\alpha_i=\alpha]} \lambda_i(x)v_{i\alpha_i}(x_i)$. $\square$

Coming back to the proof of Proposition 6.8 we consider the trajectory $\bar{x}(\cdot)$ of the vector field $Y(x)$, which joins $\tilde{x}$ and $\hat{x}$. Denote $\bar{v}_\alpha(t) = v_\alpha(\bar{x}(t)),\ \alpha \in A_\varepsilon$ and consider the time-variant differential equation

$$\dot{x}^\theta = X_t^\theta(x) = \sum_{\alpha \in A_\varepsilon} \bar{v}_\alpha(t) X_\alpha^\theta(x)$$

Note that $\bar{v}_\alpha(t)$ are smooth and the vector field $X_t^\theta(x)$ is locally Lipschitzian in x.

Let $x^\theta(t)$ be the trajectory of this equation starting at $\tilde{x}$.

Without lack of generality we may act as if M were a bounded connected subset of R^n.

To find a bound for $\|x^\theta(T) - \hat{x}\|$ we compute

$$x^\theta(T) - \hat{x} = x^\theta(T) - \bar{x}(T) = \int_0^T \left(X^\theta(x^\theta(\tau)) - Y(\bar{x}(\tau)) \right) d\tau,$$

and proceed with the estimates for the norms in $\mathbb{R}^n$.

$$\|x^\theta(t) - \bar{x}(t)\| = \left\| \int_0^t \left(X_t^\theta(x^\theta(\tau)) - Y(\bar{x}(\tau)) \right) d\tau \right\| \leq \int_0^t \left\| X_t^\theta(x^\theta(\tau)) - X_t^\theta(\bar{x}(\tau)) \right\| d\tau +$$
$$+ \int_0^t \left\| X_t^\theta(\bar{x}(\tau)) - Y(\bar{x}(\tau)) \right\| d\tau \leq L_X \int_0^t \left\| x^\theta(\tau) - \bar{x}(\tau) \right\| + \int_0^t \left\| X_t^\theta(\bar{x}(\tau)) - Y(\bar{x}(\tau)) \right\| d\tau d\tau,$$

as long as $x^\theta(\cdot)$ does not leave $\bar{V}$. Here L_X is Lipschitz constant for $X_t^\theta(x)$ on $\bar{V}$.

Then integrating with respect to θ and applying Fubini theorem we get

$$\int_\Theta \|x^\theta(t) - \bar{x}(t)\| d\theta \leq \int_0^t \int_\Theta \left\| X_t^\theta(\bar{x}(\tau)) - Y(\bar{x}(\tau)) \right\| d\theta d\tau + L_X \int_0^t \int_\Theta \left\| x^\theta(\tau) - \bar{x}(\tau) \right\| d\theta d\tau.$$

By virtue of (24) the last inequality becomes

$$\int_\Theta \|x^\theta(t) - \bar{x}(t)\| d\theta \leq \varepsilon t + L_X \int_0^t \int_\Theta \left\| x^\theta(\tau) - \bar{x}(\tau) \right\| d\theta d\tau,$$

and by virtue of Gronwall lemma

$$\int_\Theta \|x^\theta(T) - \bar{x}(T)\| d\theta \leq \frac{\varepsilon}{L_X} \left(e^{L_X T} - 1 \right),$$

wherefrom the claim of Proposition 6.8 follows. □

6.3. **Lie extension**

We have just proved approximate controllability for an extended ensemble by means of a high-dimensional *extended* control. Now we have to prove, that the same goal is achievable by means of lower-dimensional control. This is done in iterative way via so-called *Lie extensions*.

The following result shows, that the control-linear 2-input ensemble

$$\frac{d}{dt} x^\theta(t) = X^\theta(x) u(t) + Y^\theta(x) v(t), \tag{25}$$

and the extended 3-input ensemble

$$\frac{d}{dt} x^\theta(t) = X^\theta(x) u_e(t) + Y^\theta(x) v_e(t) + [X^\theta, Y^\theta](x) w_e(t), \tag{26}$$

have (approximately) the same steering capacities, according to Definition 6.1.

Proposition 6.11. *If the ensemble* (26) *can be steered in time T from $\tilde{x}$ to $\hat{x}$ approximately, then the same is valid for the ensemble* (25). $\square$

Using the statement one can easily complete the proof of Theorem D. Proposition 6.8 demonstrates that an extended ensemble (23) can be steered from $\tilde{x}$ to $x^{\theta}(T)$ with (21) satisfied. By Proposition 6.11 the same result can be achieved with a diminished (by 1) dimension of controls. Proceeding by (inverse) induction we prove, that the original ensemble (20) can be steered approximately from $\tilde{x}$ to $\hat{x}$. $\square$

6.4. **Proof of Proposition 6.11**

The construction is based on fast-oscillating functions and on techniques adopted for relaxed controls; see [4] for an application of these ideas to the control of Navier-Stokes equation.

Let $u_e(t), v_e(t), w_e(t)$ be the controls, which steer the system (26) approximately to $\hat{x}$, acording to Definition 6.1. It suffices to establish the statement for smooth $w_e(t)$, as far as smooth functions are dense in the space of measurable functions in L_1-metric.

We will use the formula, which is a nonlinear version of the 'variation of constants' method. Its more general form - variational formula for time variant vector fields can be found in [2, Ch.2].

Let $\overrightarrow{\exp}\int_0^t X_\tau d\tau$ denote the flow generated by a time variant vector field X_t, $F_0 = Id$, while e^{tY} be the flow, generated by a time-invariant vector field Y.

Lemma 6.12. *Let $f_\tau(x), g(x)$ be real analytic in x, f_τ integrable in τ. The flow $P_t = \overrightarrow{\exp}\int_0^t f_\tau(x)+g(x)u(\tau)d\tau$, corresponding to the differential equation*

$$\dot{x} = f_t(x) + g(x)u(t),\ U(0) = 0, \tag{27}$$

can be represented as a composition of two flows

$$\overrightarrow{\exp}\int_0^t (f_\tau(x) + g(x)u(\tau))\, d\tau = \overrightarrow{\exp}\int_0^t e^{U(\tau)\mathrm{ad}_g} f_\tau d\tau \circ e^{gU(t)}, \tag{28}$$

where $U(t) = \int_0^t u(\tau)d\tau$. $\square$

The operator ad_Z, determined by the vector field Z, acts on vector fields as: $\mathrm{ad}_Z Z_1 = [Z, Z_1]$ - the Lie bracket of Z and Z_1, while the operator exponential $e^{\mathrm{ad}_Z} = \sum_{j=0}^{\infty} \frac{(\mathrm{ad}_Z)^j}{j!}$.

Note that $e^{U(t)g(x)}$ is time-$U(t)$ element of the flow of the time-invariant vector field g.

To relate the formula (28) to fast-oscillating functions we choose a 1-periodic measurable bounded function $v(t)$ with $\int_0^1 v(t)dt = 0$. Feeding into (27) a *fast-oscillating, possibly high-gain,* control $u_\varepsilon(t) = \varepsilon^{-\alpha} v(t/\varepsilon^\beta)$, $0 \leq \alpha < \beta$, we get by (28)

$$\overrightarrow{\exp}\int_0^t (f_\tau(x) + g(x)u_\varepsilon(\tau))\, d\tau = \overrightarrow{\exp}\int_0^t e^{\varepsilon^{\beta-\alpha} V(\tau/\varepsilon^\beta)\mathrm{ad}_g} f_\tau d\tau \circ e^{\varepsilon^{\beta-\alpha} V(t/\varepsilon^\beta)} g,$$

where $V(t) = \int_0^t v(\tau)d\tau$ is 1-periodic Lipschitzian function.

Expanding the exponential $e^{\varepsilon^{\beta-\alpha} V(\tau/\varepsilon^\beta)\mathrm{ad}_g}$ we get

$$\begin{aligned}\overrightarrow{\exp}\int_0^t (f_\tau(x) + g(x)u_\varepsilon(\tau))\, d\tau =& \overrightarrow{\exp}\int_0^t \left(f_\tau(x) + O(\varepsilon^{\beta-\alpha})\right) d\tau \circ \left(I + O(\varepsilon^{\beta-\alpha})\right) = \\ =& \overrightarrow{\exp}\int_0^t f_\tau(x) d\tau \circ (I + O(\varepsilon^{\beta-\alpha})).\end{aligned}$$

This demonstrates that the effect of fast-oscillating perturbation $g(x)u_\varepsilon(\tau)$ tends to 0 as $\varepsilon \to 0$.

Remark 6.13. *The expression $O(\varepsilon^{\beta-\alpha})$ above regards each of the seminorms $\|X(x)\|_{s,K}, \|P\|_{s,K}$, which define the convergence of the derivatives of order $\leq s$ on a compact K $\square$.*

Remark 6.14. *Similar conclusion holds if one takes $u_\varepsilon(t) = w(t)\varepsilon^{-\alpha}v(t/\varepsilon^\beta)$, where $w(\cdot)$ is, say, Lipschitzian function. The conclusion is achieved by similar reasoning, given the fact that the primitive of $u_\varepsilon(t)$ in this case is $\varepsilon^{\beta-\alpha}\left(w(t)V(t/\varepsilon^\beta) - \int_0^t V(\tau/\varepsilon^\beta)\dot{w}\tau d\tau\right) = O(\varepsilon^{\beta-\alpha})$, as $\varepsilon \to +0$.*

Coming back to the 2-input system (25) we choose the controls $u_\varepsilon(t), v_\varepsilon(t)$ of the form

$$u_\varepsilon(t) = u_e(t) + \varepsilon\dot{U}_\varepsilon(t),\ v_\varepsilon(t) = v_e(t) + \varepsilon^{-1}\hat{v}_\varepsilon(t), \tag{29}$$

where $U_\varepsilon(t)$ is function, $U_\varepsilon(0) = 0$. Both $U_\varepsilon(t)$ and $\hat{v}_\varepsilon(t)$ will be specified in a moment.

Feeding the controls (29) into the system (25) we get

$$\frac{d}{dt}x^\theta(t) = X^\theta(x)u_e(t) + Y^\theta(x)\left(v_e(t) + \varepsilon^{-1}\hat{v}_\varepsilon(t)\right) + X^\theta(x)\varepsilon\dot{U}_\varepsilon(t). \tag{30}$$

Applying (28) we represent the flow of (30) as a composition of flows

$$\overrightarrow{\exp}\int_0^t X^\theta(x)u_e(t) + e^{\varepsilon U_\varepsilon(t)\mathrm{ad}X^\theta}Y^\theta(x)(v_e(t) + \varepsilon^{-1}\hat{v}_\varepsilon(t))dt \circ e^{\varepsilon U_\varepsilon(t)X^\theta(x)}. \tag{31}$$

We impose the condition $U_\varepsilon(T) = 0$, so that $e^{\varepsilon U_\varepsilon(T)X^\theta(x)} = I$ and we can restrict our attention to the first factor of the composition (31).

Proceeding with the expansion of the exponential $e^{\varepsilon U_\varepsilon(t)\mathrm{ad}X^\theta(x)}$ we represent (31) as

$$\overrightarrow{\exp}\int_0^t (X^\theta(x)u_e(t) + Y^\theta(x)v_e(t) + Y^\theta(x)\varepsilon^{-1}\hat{v}_\varepsilon(t) + [X^\theta, Y^\theta](x)U_\varepsilon(t)\hat{v}_\varepsilon(t) + O(\varepsilon))dt. \tag{32}$$

We wish the flow (32) to approximate the flow generated by the equation (26). To achieve this we take the functions

$$U_\varepsilon(t) = 2\sin(t/\varepsilon^2)w_e(t),\ \hat{v}_\varepsilon(t) = \sin(t/\varepsilon^2); \tag{33}$$

we choose ε from the sequence

$$\varepsilon_n = (T/\pi n)^{1/2}, n = 1, 2, \ldots, \tag{34}$$

so that $U_\varepsilon(T) = 0$. Then

$$U_\varepsilon(t)\hat{v}_\varepsilon(t) = w_e(t) - w_e(t)\cos(2t/\varepsilon^2),$$

so that feeding $U_\varepsilon(t), \hat{v}_\varepsilon(t)$ into (32) gives us

$$\begin{aligned}\overrightarrow{\exp}\int_0^t \big(X^\theta(x)u_e(t) + Y^\theta(x)v_e(t) + [X^\theta, Y^\theta](x)w_e(t) + Y^\theta(x)\varepsilon^{-1}\sin(t/\varepsilon^2) - \\ - [X^\theta, Y^\theta](x)w_e(t)\cos(2t/\varepsilon^2) + O(\varepsilon)\big)\, dt.\end{aligned} \tag{35}$$

One can apply formula (28) to the flow taking $Y^\theta(x)\varepsilon^{-1}\sin(t/\varepsilon^2)$ as $gu(t)$ and the rest of the vector field in the exponential (35) as f_t . Then we represent the flow (35) as a composition

$$\overrightarrow{\exp}\int_0^t \big(X^\theta(x)u_e(t) + Y^\theta(x)v_e(t) + [X^\theta, Y^\theta](x)w_e(t) - [X^\theta, Y^\theta](x)w_e(t)\cos(2t/\varepsilon^2) + O(\varepsilon)\big)\, dt \circ e^{-\varepsilon\cos(t/\varepsilon^2)}.$$

According to the Remark 6.14 we conclude that the flow of the equation (30) can be represented as

$$\overrightarrow{\exp}\int_0^t (\ X^\theta(x)u_e(t)+Y^\theta(x)v_e(t)+[X^\theta,Y^\theta](x)w_e(t){+}O(\varepsilon)\ )dt\circ(I+O(\varepsilon))=$$
$$=\overrightarrow{\exp}\int_0^t(\ X^\theta(x)u_e(t)+Y^\theta(x)v_e(t)+[X^\theta,Y^\theta](x)w_e(t)\ )dt\circ(I+O(\varepsilon)).$$

Denote by $x_e^\theta(t)$ the trajectory of the 3-input ensemble (26). We have proved that for the trajectories $x_{\varepsilon_n}(t)$ of the 2-input ensemble (25), driven by the controls $u_{\varepsilon_n}(t), v_{\varepsilon_n}(t)$, defined by (29)-(33)-(34), there holds for each $\theta\in\Theta$:

$$\lim_{n\to\infty}\|x_e^\theta(T)-x_{\varepsilon_n}^\theta(T)\|=0. \tag{36}$$

Recall that the real-analytic vector fields f_j^θ of the ensemble (20) depend continuosly on θ; the same holds for the Lie brackets of the vector fields. Then, since Θ is compact, one can easily check that $x_e^\theta(T)$ and $x_{\varepsilon_n}^\theta(T)$ are equibounded for all θ and n. Then by Lebesgue theorem we get from (36)

$$\lim_{n\to\infty}\int_\Theta\|x_e^\theta(T)-x_{\varepsilon_n}^\theta(T)\|d\theta=0.\ \square$$

7. Appendix

7.1. **Proof of Theorem A**

We fix $\dim M=n$.

It is enough to show that the linear ensemble control is generically bracket generating already for two families of vector fields

$$X=(X^1,X^2,\dots,X^N),\ Y=(Y^1,Y^2,\dots,Y^N).$$

In the trivial ensemble ($N=1$) case, it is easy to see that for a generic pair X,Y, the linear span of these fields has rank at least one everywhere on M, hence we can always assume that either X or Y are non-vanishing in a vicinity of a point of M. After that, the generic generating property follows almost immediately.

If, say $X\neq 0$ at a point $p\in M$, then the codimension of the subset of $(S-1)$-jets of vector fields Y in M at p such that S vectors $\mathrm{ad}_X^kY(p),\ k=1,\dots,S-1$, do *not* span T_pM, is $S-(n-1)$. Hence, by R.Thom's transversality theorem, for $S\geq 2n$ this subset is avoided by an open dense set of vector fields Y in M.

In the nontrivial ensemble (finite $|\Theta|>1$) case, a similar approach works, but requires some modifications. We still will choose a control generating a vector field X, and will argue that differentiating a vector field corresponding to a different (constant) control iteratively will produce enough vectors to span $TM^N=\bigoplus_\theta TM_\theta$.

There is a small wrinkle here: as the vector fields, tangent to different components M_θ of the ensemble, do not interact, we will need to have all of $X^\theta(p_\theta)\neq 0$. We cannot however claim that that is true for either X or Y: consider, as an example, a point $(x_1,x_2,\dots,x_N)\in M^N$, where $X^\theta(x_\theta)=0$ are non-degenerate zeros. Then for any perturbation of X^{θ_i}'s they will have a point in M^N where all of the vector fields will vanish on one of the ensemble factors.

To overcome this difficulty, we fix a generic collection of controls

$$\{u_1,\dots,u_{N+1}\},u_k=(u_{k,1},u_{k,2})\in\mathbf{R}^2,$$

such that any 2 of them are linearly independent. Then generically, at least for one of the indices k, $L_k^\theta(p_\theta)=u_{k,1}X^\theta(p_\theta)+u_{k,2}Y^\theta(p_\theta)\neq 0$ for all $\theta\in\Theta$. Indeed, otherwise, by pigeonhole principle, there will be at least one of the factors M^θ, two of the vector fields L_k^θ,L_l^θ will vanish at $p_\theta\in M^\theta$, and, by assumptions on $u_{\cdot,\cdot}$'s, both X^θ and Y^θ would vanish at p_θ. This cannot happen generically, as discussed above.

Hence we can assume that for any point $p = (p_1, \dots, p_N) \in M^\Theta$, generically, $X^\theta := L_k^\theta$ is non null at all factors p_θ (for a $1 \le k \le N+1$). Setting $Y^\theta := L^\theta_{(k+1) \mod N}$, and forming S Lie derivatives

$$Y(p), \mathrm{ad}_X Y(p), \dots, \mathrm{ad}_X^{S-1} Y(p),$$

we conclude that generically for $S > 2Nn$, these derivatives generate the whole tangent space $T_p M = \bigoplus_\theta T_{p_\theta} M$, at p and in its vicinity. Compactness of M implies that generically, X, Y are bracket generating, and hence the system is fully controllable. □

7.2. Proof of Lemma 4.5

Continuing with the induction on N we introduce a linear map $\Lambda_{J,L} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$K \mapsto [\mathcal{E}_J, V_L^0](K), \quad [\mathcal{E}_J, V_L^0](K) = L \times JK + K \times JL.$$

Its matrix in the basis of principal axes of the body has form

$$\Lambda_{J,L} = D_J \hat{L},\ D_J = \mathrm{diag}\left(J_3 - J_2, J_1 - J_3, J_2 - J_1\right),\ \hat{L} = \begin{pmatrix} 0 & L_3 & L_2 \\ L_3 & 0 & L_1 \\ L_2 & L_1 & 0 \end{pmatrix}.$$

Elements of $\Lambda_{J,L}$ are homogeneous of first order in J. The constant vector fields in (6) can be represented as

$$V_{J,L}^{m+1} = \Lambda_{J,L} V_{J,L}^m = \Lambda_{J,L}^m V_{J,L}^1.$$

By direct computation $\det \hat{L} = 2L_1L_2L_3$ and hence given the asymmetry of the body, we conclude thet $\det \Lambda_L \neq 0$, provided that L does not belong to any of coordinate planes $P_i : \ L_i = 0; i = 1,2,3$.

Let $\mathbf{L}$ be the complement of the union of three principal axes of the body and the three planes P_1, P_2, P_3.

According to the aforesaid

$$\forall L \in \mathbf{L},\ m \ge 1 : \ \textit{the vectors } V_{J,L}^m, V_{J,L}^{m+1}, V_{J,L}^{m+2} \textit{ are linearly independent.} \tag{37}$$

The determinant R_N can be represented as [4]

$$R_N\left(J^1, \dots, J^N; L\right) = \det \left(\begin{array}{cccc|ccc} L & \Lambda_{J^1,L} L & \cdots & \Lambda_{J^1,L}^{3N-4} L & \Lambda_{J^1,L}^{3N-3} L & \Lambda_{J^1,L}^{3N-2} L & \Lambda_{J^1,L}^{3N-1} L \\ L & \Lambda_{J^2,L} L & \cdots & \Lambda_{J^2,L}^{3N-4} L & \Lambda_{J^2,L}^{3N-3} L & \Lambda_{J^2,L}^{3N-2} L & \Lambda_{J^2,L}^{3N-1} L \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline L & \Lambda_{J^N,L} L & \cdots & \Lambda_{J^N,L}^{3N-4} L & \Lambda_{J^N,L}^{3N-3} L & \Lambda_{J^N,L}^{3N-2} L & \Lambda_{J^N,L}^{3N-1} L \end{array}\right).$$

We will prove that for $L \in \mathbf{L}$, the determinant $R_N\left(J^1, \dots, J^N; L\right)$ defines a nontrivial polynomial in $J^1, \dots, J^N$. This is true for $N = 1$.

We write R_N in a block form

$$R_N = \left(\begin{array}{c|c} \tilde{R}_{N-1} & W_1 \\ \hline W_2 & \tilde{R} \end{array}\right),$$

where $\tilde{R}_{N-1}$ and $\tilde{R}$ are $3(N-1) \times 3(N-1)$ and 3×3 blocks correspondingly, and W_1, W_2 have appropriate dimensions.

By induction assumption $\det \tilde{R}_{N-1} \neq 0$, while $\det \tilde{R} \neq 0$ by (37).

[4] R_N is (non-commutative) matrix version of Vandermond determinant. Any explicit computations for such determinant are desirable, but we are not aware of any

To verify that $\det R_N$ does not vanish identically for all $J^1, \dots, J^N$, we substitute $\varepsilon J^1, \dots, \varepsilon J^{N-1}$ in place of $J^1, \dots, J^{N-1}$. This results in multiplication by ε^{k-1} of the k-th column *of the upper block* $(R_{N-1} \,|W_1\,)$, $k = 1, \dots 3N$.

Denote the resulting matrix by $R_N(\varepsilon)$:

$$R_N(\varepsilon) = \left(\begin{array}{c|c} \tilde{R}_{N-1}(\varepsilon) & W_1(\varepsilon) \\ \hline W_2 & \tilde{R} \end{array} \right),$$

with

$$(R_{N-1}(\varepsilon), W_1(\varepsilon)) = \left(\tilde{R}_{N-1}, W_1\right) D^\varepsilon, \;\; D^\varepsilon = \operatorname{diag}\left(1, \varepsilon, \dots, \varepsilon^{3N-1}\right).$$

Multiplying the matrix $R_N(\varepsilon)$ from the left by a *nonsingular* matrix

$$\left(\begin{array}{c|c} I & -W_1(\varepsilon)\tilde{R}^{-1} \\ \hline 0 & \tilde{R}^{-1} \end{array} \right)$$

we arrive to the matrix

$$\hat{R}_N(\varepsilon) = \left(\begin{array}{c|c} R_{N-1}(\varepsilon) - W_1(\varepsilon)\tilde{R}^{-1}W_2 & 0 \\ \hline \tilde{R}^{-1}W_2 & I \end{array} \right).$$

The elements of $W_1(\varepsilon)$ are $O(\varepsilon^{3N-3})$ as $\varepsilon \to 0$, so are the elements of $W_1(\varepsilon)\tilde{R}^{-1}W_2$.

Multiplying the matrix $\hat{R}_N(\varepsilon)$ from the right by a diagonal matrix

$$\operatorname{diag}(1, \varepsilon^{-1}, \dots, \varepsilon^{-(3N-4)}, 1, 1, 1)$$

we get the matrix

$$\bar{R}_N(\varepsilon) = \left(\begin{array}{c|c} R_{N-1} + Y_1(\varepsilon) & 0 \\ \hline Y_2(\varepsilon) & I \end{array} \right),$$

where $Y_1(\varepsilon) = O(\varepsilon)$. The determinant $\det \bar{R}_N(\varepsilon)$ is close to $\det \tilde{R}_{N-1} \neq 0$, and hence differs from 0, whenever ε is sufficiently small. Therefore $R_N(\varepsilon) = R_N\left(\varepsilon J^1, \dots, \varepsilon J^{n-1}, J^N; L\right)$ is nonsingular for sufficiently small $\varepsilon > 0$. $\square$

7.3. **Proof of Theorem C**

Introduce matrix $\Gamma = (\gamma_{mr}),\; m = 1, \dots, \infty; r = 1, \dots, R$, with γ_{mr}, defined by (18). Let $\hat{\Gamma}$ be the upper $(R \times R)$-block of the $(\infty \times r)$-matrix Γ and $\tilde{\Gamma}$ be the resting infinite block.

We will choose the controls $U(\cdot), v_1(\cdot), \dots, v_R(\cdot)$ in such a way that the matrix $\hat{\Gamma}$ would be (non singular) lower triangular matrix with nonvanishing diagonal elements. At the same time we will be able to guarantee smallness of $\|\tilde{\Gamma}y\|_{\ell_2}$.

We take Legendre polynomials: $P_k(t) = \frac{1}{k!}\frac{d^k}{dt^k}\left((t^2 - t)^k\right)$ orthogonal on $[0, 1]$ and put

$$U(t) = \int_0^t P_1(s)ds = (t^2 - t), \;\; v_r(t) = P_{2r}(t), \;\; r = 1, \dots, R.$$

Note that $U(1) = 0$ and $V(1) = \int_0^1 v_r(s)ds = 0, \;\; r = 1, \dots, R$, since $P_{2r}(t)$ is orthogonal to $1 = P_0(t)$.

Evidently $(U(t))^m$ is polynomial of degree $2m$, hence:

$$\gamma_{mr} = \int_0^1 (U(t))^m v_r(t)dt = 0, \;\text{ for } m < r,$$

while

$$\gamma_{rr} = \int_0^1 (U(t))^r v_r(t)dt = \int_0^1 t^{2r} P_{2r}(t)dt \neq 0, \ \forall r.$$

Note that $\gamma_{mr}, m = 1, \ldots, r = 1, \ldots$ admit a common upper bound:

$$|\tilde{\gamma}_{mr}| = \left| \int_0^1 (t-t^2)^m P_{2r}(t)dt \right| \leq \left(\int_0^1 (t^2-t)^{2m} dt \right)^{1/2} \left(\int_0^1 (P_{2r}(t))^2 dt \right)^{1/2}.$$

The first factor is $\leq 2^{-2}$ given that $|t - t^2| \leq 1/4$ on $[0, 1]$. The second factor equals $\frac{1}{\sqrt{4r+1}} < 2^{-1}$ for each $r \geq 1$. Hence $|\gamma_{mr}| < 2^{-3}$.

Now we introduce small parameter $\varepsilon > 0$ defining:

$$U^\varepsilon(t) = \varepsilon U(t) = \varepsilon(t - t^2), \ v_r^\varepsilon(t) = \varepsilon^{-r} v_r(t) = \varepsilon^{-r} P_{2r}(t).$$

Substituting $U^\varepsilon(t)$ and $v_r^\varepsilon(t)$ into (18) we get $\gamma_{mr}^\varepsilon = \varepsilon^{m-r}\gamma_{mr}$.

For chosen $U^\varepsilon(t), v^\varepsilon(t)$ the representation (17) for $z^\theta(1)$ takes form

$$z_\theta(1) = \sum_{r=1}^{R} a_r(\theta)\gamma_{rr} y_r + \varepsilon \sum_{r=1}^{R} \sum_{m=r+1}^{\infty} \varepsilon^{m-(r+1)} a_m(\theta)\gamma_{mr} y_r. \tag{38}$$

Let us take $y_r = c_r/\gamma_{rr}, \ r = 1, \ldots, R$, where c_r are the coefficients in (19), so that

$$\left\| \hat{z}(\theta) - \sum_{r=1}^{R(\varepsilon_1)} a_r(\theta)\gamma_{rr} y_r \right\|_{L_2(\Theta)} < \varepsilon_1.$$

Obviously the coefficients y_r depend on ε_1.

Now it rests to estimate the second addend at the right-hand side of (38). Given that $a_m(\theta) = \left(\int_{C_\rho} \frac{f^\theta(\zeta)}{\zeta^{m+1}} d\zeta \right)$, we get the estimate

$$|a_m(\theta)| = \left| \int_{C_\rho} \frac{f^\theta(\zeta)}{\zeta^{m+1}} d\zeta \right| \leq 2\pi\mu_f \rho^{-m}, \ \mu_f = \sup_{(\theta,\zeta)\in\Theta\times B_\rho} \left| f^\theta(\zeta) \right|.$$

Then

$$\left| \sum_{m=r+1}^{\infty} \varepsilon^{m-(r+1)} a_m(\theta)\gamma_{mr} \right| \leq 2^{-3} \left| \sum_{s=0}^{\infty} \varepsilon^s \frac{2\pi\mu_f}{\rho^{s+(r+1)}} \right| \leq \frac{\pi\mu_f}{4\rho^R(\rho-\epsilon)},$$

and the second addend in (38) admits an upper bound:

$$\frac{\varepsilon\pi\mu_f}{4\rho^{R(\varepsilon_1)}(\rho-\epsilon)} \sum_{r=1}^{R(\varepsilon_1)} |y_r|. \tag{39}$$

The term $\sum_{r=1}^{R(\varepsilon_1)} |y_r|$ admits an upper bound $b_y(\varepsilon_1) > 0$.

Now choose $\varepsilon > 0$ such that

$$\frac{\varepsilon\pi\mu_f}{2(\rho-\epsilon)} < \frac{\varepsilon_1\rho^{R(\varepsilon_1)}}{b_y(\varepsilon_1)}.$$

Then the estimate (39) of the "perturbation term" in (38) is $< \varepsilon_1$, and

$$\|\hat{z}(\theta) - z_\theta(1)\|_{L_2(\Theta)} < 2\varepsilon_1. \quad \square$$

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