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Local controllability for families of diffeomorphisms

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Abstract: We study groups and semigroups which are generated by analytic families of diffeomorphisms. The central notion is that of local controllability of a family of diffeomorphisms at a given point of the state manifold, which generalizes the familiar notion of local controllability of control systems with continuous, as well as discrete time. Lie theory methods are used. We systematically exploit the so called fast switching variations and properties of the jet spaces of curves on the state manifold.

Keywords: Semigroup of diffeomorphisms; Lie algebra; local controllability; fast switching variations; jet of curve.

1. Introductory remarks. Suppose a real analytic manifold M is fixed and

$$Q(u) \in \mathrm{Diff}(M), \quad u \in O \subset R^r(O \text{ open}), \, Q(0) = \mathrm{id}, \tag{1}$$

is a family of diffeomorphisms of M, analytically depending on $u \in O$. Analyticity means that the mapping

$$R^r \times M \to M : (u, x) \mapsto Q(u)(x)$$

is analytic. Furthermore, let $U \subset O$ be a starshaped (not necessarily open) set with respect to the origin of R^r, containing interior points. Denote by $\mathrm{Gr}(Q(U))$ the subgroup in $\mathrm{Diff}(M)$ generated by the diffeomorphisms $Q(u)$, $u \in U$, and by $\mathrm{SGr}(Q(U))$ the semigroup in $\mathrm{Diff}(M)$ generated by the same family of the diffeomorphisms. Thus

$$\mathrm{SGr}(Q(U)) = \{Q(u_1) \circ \cdots \circ Q(u_k) \mid u_i \in U, \, k \geq 1\},$$

where '$\circ$' denotes the composition of diffeomorphisms.

When dealing with families of diffeomorphisms and vector fields it is convenient to use operator notations. In these notations an arbitrary diffeomorphism Q of M is identified with the corresponding automorphism of the algebra of smooth functions $C^\infty(M)$ on M, acting according to the formula $\phi(-) \mapsto \phi(Q(-))$, $\forall \phi \in C^\infty(M)$, and an arbitrary point $x \in M$ is identified with the homomorphism $C^\infty(M) \to R$: $\phi \mapsto \phi(x)$. Thus, in operator notations the value of the diffeomorphism Q at x is denoted as the composition $x \circ Q$ of the automorphism Q followed by the homomorphism x: $(x \circ Q)\phi = \phi(Q(x))$. There is no need to introduce different letters for a diffeomorphism as such and the corresponding automorphism, or the point and the corresponding homomorphism, since the meaning will always be clear from the context.

Smooth vector fields on M are identified with the derivations of the algebra $C^\infty(M)$, i.e. with the linear operators $f: C^\infty(M) \to C^\infty(M)$, which satisfy the Leibniz rule for the product differentiation: $f(\phi_1\phi_2) = (f\phi_1)\phi_2 + \phi_1(f\phi_2)$. The Lie bracket of a pair of vector fields is the commutator of the corresponding operators: $[f, g] = f \circ g - g \circ f$. Tangent vectors at the point $x \in M$ are identified with

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linear functionals $\xi : C^\infty(M) \to R$, which satisfy the condition $\xi(\phi_1\phi_2) = (\xi\phi_1)\phi_2(x) + \phi_1(x)(\xi\phi)$. In particular, the value of a vector field f at the point x is a tangent vector to M at x and is represented as a composition $x \circ f$ of the operator f and the functional $x : \phi \mapsto \phi(x)$.

In the sequel we shall be constantly concerned with families of diffeomorphisms and families of vector fields depending on parameters with values in R^n. Continuity, differentiability, summability, etc. of such families is always considered in a weak operator sense: the family obtained by applying the operator family to an arbitrary function from $C^\infty(M)$ should enjoy the corresponding property. For a detailed exposition of this viewpoint, see [1,3,5].

2. Locally controllable families of diffeomorphisms. For an arbitrary $x_0 \in M$, the expression

$$x_0 \circ \mathrm{Gr}(Q(U)) = \{x_0 \circ P \mid P \in \mathrm{Gr}(Q(U))\} \subset M$$

is the orbit of the group $\mathrm{Gr}(Q(U))$ through the point x_0, and

$$x_0 \circ \mathrm{SGr}(Q(U)) = \{x_0 \circ Q(u_1) \circ \cdots \circ Q(u_k) \mid u_i \in U,\ k \geq 1\} \tag{2}$$

is the orbit of the semigroup.

For every $\varepsilon > 0$ we put $U_\varepsilon = \{u \in U \mid |u| \leqslant \varepsilon\}$, and let Gr $Q(U_\varepsilon)$, SGr $Q(U_\varepsilon)$ be the group and the semigroup, generated by the diffeomorphisms $Q(u)$, $u \in U_\varepsilon$.

Definition. The family of diffeomorphisms $Q(u)$, $u \in U$, is called *locally controllable at* x_0 if for every $\varepsilon > 0$ we have $x_0 \in \mathrm{int}(x_0 \circ \mathrm{SGr}(Q(U_\varepsilon)))$; in other words, if x_0 is an interior point of the corresponding orbit of the semigroup $\mathrm{SGr}(Q(U_\varepsilon))$.

Consider a control system with the discrete time

$$x_{(i+1)} = Q(u)(x_{(i)}), \quad u \in U,\ x_0 \text{ fixed}.$$

It is obvious that the local controllability of the family $Q(u)$, $u \in U$, at x_0 is equivalent to the usual local controllability of this system. Furthermore, consider a control system with continuous time

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, v), \quad v \in V \subset R^{r-1},\ x(0) = x_0. \tag{3}$$

Let $U = \{(\alpha, \alpha v) \in R^r \mid 0 \leq \alpha \leq 1,\ v \in V\}$ be a cone with base V and the vertex at the origin, and let $Q((\alpha, \alpha v)) = \mathrm{e}^{\alpha f(-, v)}$.

It is easily verified that the local controllability of the family $Q((\alpha, \alpha v))$ at x_0 is equivalent to the fact that x_0 is an interior point of the attainable set of the system (3) for an arbitrary time, if piecewise constant controls are used. For analytic systems this is equivalent to the fact that x_0 is an interior point of the attainable set for arbitrary measurable controls with values from a compact set.

3. Volterra series expansions of families of diffeomorphisms. Now we return to an arbitrary family $Q(u)$. For every multiindex $i = (i^1, \ldots, i^r)$, $|i| = i^1 + \cdots + i^r$, we have the corresponding differential operator

$$D_i : C^\infty(M) \to C^\infty(M), \quad D_i\phi = \frac{1}{i!}\frac{\partial^{|i|}}{\partial u_{i_1} \ldots \partial u_{i_r}} Q(u)\phi \bigg|_{u=0} \quad \forall \phi \in C^\infty(M).$$

It is clear that the order of the operator D_i does not exceed $|i|$.

The Taylor series expansion of $Q(u)$ at zero has the form $Q(u) \approx \mathrm{id} + \Sigma_{|i|>0} u^i D_i$. The operator $Q^{-1}(u) \circ (\mathrm{d}/\mathrm{d}t)Q(tu)|_{t=1}$ is a vector field, analytically depending on $u \in R^r$ and zero for $u = 0$. Let

$$Q^{-1}(u) \circ \frac{\mathrm{d}}{\mathrm{d}t} Q(tu)|_{t=1} \approx \vec{\omega}(u) = \sum_{|i|>0} u^i \vec{\omega}_i$$

be its Taylor series expansion; the coefficients $\vec{\omega}_i$ are vector fields.

For every series $\vartheta_t(u) = \sum_{|i|>0} \vartheta_t^i u^i$, having as coefficients nonstationary vector fields, summable over $t \in [0, 1]$, we shall denote by $\overrightarrow{\exp}\int_0^1 \vartheta_t(u)\,\mathrm{d}t$ the Volterra series

$$\overrightarrow{\exp}\int_0^1 \vartheta_t(u)\,\mathrm{d}t = \mathrm{id} + \sum_{n=1}^{\infty} \int \underset{\Delta_n}{\cdots} \int \vartheta_{t_n}(u) \circ \cdots \circ \vartheta_{t_1}(u)\,\mathrm{d}t_1 \cdots \mathrm{d}t_n, \tag{4}$$

where

$$\Delta_n = \{(t_1, \ldots, t_n) \mid \geq t_1 \geq \cdots \geq t_n \geq 0\}.$$

The Taylor series expansion of an arbitrary family of diffeomorphisms can be easily represented as a Volterra series. Indeed

$$\mathrm{id} + \sum_i (tu)^i D_i = \mathrm{id} + \int_0^t \Big(\mathrm{id} + \sum_i (\tau u)^i D_i\Big) \circ \vec{\omega}(\tau u) \frac{1}{\tau}\,\mathrm{d}\tau, \quad t \in R,$$

from which it easily follows that

$$\mathrm{id} + \sum_{|i|>0} u^i D_i = \overrightarrow{\exp}\int_0^1 \frac{1}{t} \vec{\omega}(tu)\,\mathrm{d}t. \tag{5}$$

Finally, we denote

$$\Lambda(u) = \sum_{|i|>0} u^i \Lambda_i = \ln\Big(\mathrm{id} + \sum_i u^i D_i\Big) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \Big(\sum_i u^i D_i\Big)^n.$$

4. Proposition. *For every multiindex i the differential operator Λ_i is a vector field: $\Lambda_i \in \mathrm{Vect}(M)$. Furthermore, the following relations hold:*

$$\mathrm{Lie}(\{\Lambda_i \| i| > 0\}) = \mathrm{Lie}(\{\vec{\omega}_i \| i| > 0\}),$$
$$\mathrm{Ass}(\{D_i \| i| > 0\}) = \mathrm{Ass}(\{\Lambda_i \| i| > 0\}) = \mathrm{Ass}(\{\vec{\omega}_i \| i| > 0\}).$$

Here Lie($\cdot$) denotes the Lie subalgebra in Vect(M) generated by the family of vector fields in the parenthesis; Ass($\cdot$) denotes the associative algebra generated by the family of differential operators in parenthesis, with composition of operators '$\circ$' as algebra-multiplication.

Proof. From (5) we deduce $\Lambda(u) = \ln(\overrightarrow{\exp}\int_0^t t^{-1}\vec{\omega}(\mathrm{tu})\,\mathrm{d}t)$. At the same time we have

$$\ln\Big(\overrightarrow{\exp}\int_0^t \vartheta_t(u)\,\mathrm{d}t\Big) = \sum_{n=1}^{\infty} \int \underset{\Delta_n}{\cdots} \int \pi_n(\vartheta_{t_n}(u), \ldots, \vartheta_{t_1}(u))\,\mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

where π_n is a *commutator polynomial* of n variables, $n = 1, 2, \ldots$, which can be explicitly computed, cf. [1]; hence, $\Lambda_i \in \mathrm{Lie}(\{\vec{\omega}_i \| i| > 0\})$. On the other hand, $\mathrm{id} + \sum_i u^i D_i = \mathrm{e}^{\Lambda(u)}$, and therefore,

$$\vec{\omega}(u) = \mathrm{e}^{-\Lambda(u)} \circ \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\Lambda(tu)}\big|_{t=1} = \int_0^1 \mathrm{e}^{(\tau-1)\mathrm{ad}\Lambda(u)} \frac{\mathrm{d}}{\mathrm{d}t} \Lambda(tu)\Big|_{t=1} \mathrm{d}\tau.$$

Thus, we obtained the inclusion $\vec{\omega}_i \in \mathrm{Lie}(\{\Lambda_i \| i| > 0\})$, and therefore we have proved that the Lie algebras and the associative algebras generated by the families $\{\vec{\omega}_i \| i| > 0\}$ and $\{\Lambda_i \| i| > 0\}$ coincide respectively. The inclusion $\mathrm{Ass}(\{\vec{\omega}_i \| i| > 0\}) \subset \mathrm{Ass}(\{D_i \| i| > 0\})$ is evident; the opposite inclusion follows from (5).

5. The group of curves. The standard finite-dimensional (or Banach) Lie theory could not be applied to groups like Gr($Q(U)$). Nevertheless, some modification of this theory could be applied to our case. To avoid insignificant technical difficulties, we suppose that all vector fields encountered below are complete. It is convenient to have a simple sufficient condition for completeness. For such, we shall suppose that M is imbedded in R^N as a closed submanifold, and the diffeomorphisms $Q(u)$ satisfy the relation $|Q(u)(x)-x| = O(|x|^k)$, $|x| \to \infty$, $\forall k > 0$, uniformly in $u \in U$, where U is compact.

By $U(-)$ we denote the set of all smooth functions $\varepsilon \mapsto u(\varepsilon)$, $\varepsilon \geq 0$, with values in U and initial condition $u(0) = 0$. Every curve $u(-)$ defines the curve $\varepsilon \mapsto Q(u(\varepsilon))$ in the group of diffeomorphisms. We define a multiplication of two curves in $\mathrm{Diff}(M)$ pointwise by the formula:

$$(P_1 \circ P_2)(\varepsilon) := P_1(\varepsilon) \circ P_2(\varepsilon), \quad \varepsilon \mapsto P_i(\varepsilon),\ i = 1, 2, \text{ curves in } \mathrm{Diff}(M).$$

The subgroup generated by the set of curves $Q(U(-)) = \{Q(u(-)) \mid u(-) \in U(-)\}$ with the '$\circ$' multiplication we denote by $\mathrm{Gr}(Q(U(-)))$.

The *order* of a smooth curve $\varepsilon \mapsto P(\varepsilon) \in \mathrm{Diff}(M)$, $P(0) = 0$, is defined to be the number

$$\operatorname{ord} P(-) = \min\left\{k > 0 \mid \frac{\mathrm{d}^k}{\mathrm{d}\varepsilon^k} P(\varepsilon)|_{\varepsilon=0} \neq 0\right\}.$$

If $\operatorname{ord} P(-) = n$, then $(\mathrm{d}^n/\mathrm{d}\varepsilon^n)P(\varepsilon)|_{\varepsilon=0}$ is a vector field, and the expression $T_0P(-) = \{\alpha(\mathrm{d}^n/\mathrm{d}\varepsilon^n)P(\varepsilon)|_{\varepsilon=0} \mid \alpha > 0\}$ defines a ray, tangent to $P(-)$.

6. Theorem. (i) $\bigcup_{P(-)\in \mathrm{Gr}(Q(U(-)))} T_0P(-) = \mathrm{Lie}(\{A_i \| i| > 0\})$.
(ii) *The orbits of the groups* $\mathrm{Gr}(Q(U))$ *and* $\mathrm{Gr}(\{e^{tA_i} \mid t \in R, |i| > 0\})$ *in* M *coincide.*

The theorem is a generalization of Proposition 1 from [2] and is proved similarly.

Corollary. *The orbits of the group* $\mathrm{Gr}(Q(U_\varepsilon))$ *in* M *do not depend on* ε *and are immersed submanifolds in* M; *their tangent spaces are given by the expression*:

$$T_x(x \circ \mathrm{Gr}(Q(U_\varepsilon))) = x \circ \mathrm{Lie}(\{A_i \| i| > 0\}) \quad \forall x \in M.$$

The orbit description for the groups $\mathrm{Gr}(Q(U_\varepsilon))$ is fully satisfactory. The same problem for the semigroups $\mathrm{SGr}(Q(U_\varepsilon))$ is much more complicated. Generally, their orbits depend on ε and do not constitute submanifolds in M. Furthermore, Theorem 6 implies that the tangent space to the orbit $x \circ \mathrm{Gr}(Q(U(-)))$ coincides with the union of rays $x \circ T_0P(-)$, $P(-) \in \mathrm{Gr}(Q(U(-)))$. We can consider a semigroup $\mathrm{SGr}(Q(U(-)))$, which consists of the curves

$$\varepsilon \mapsto P(\varepsilon) = Q(u_1(\varepsilon)) \circ \cdots \circ Q(u_k(\varepsilon)), \quad u_i(-) \in U(-). \tag{6}$$

The tangent ray at the origin to the curve $\varepsilon \mapsto x \circ P(\varepsilon)$, where $P(-)$ has the form (6), is tangent to the orbits $x \circ \mathrm{SGr}(Q(Q(U_\varepsilon))$ $\forall \varepsilon > 0$. However, the tangents of this kind can not give a satisfactory approximation of the orbits. We can obtain a much richer stock of tangent directions if we increase the amount of factors in (6) indefinitely for $\varepsilon \to 0$. An interesting example of this phenomenon for control systems with continuous time was discovered by Kawski [6]. We shall show below that for general families of diffeomorphisms (including the systems with discrete time) this effect is a usual one.

To proceed further we must adopt a precise definition of a tangent direction to the family of orbits $x \circ \mathrm{SGr}(Q(U_\varepsilon))$, $\varepsilon > 0$. Not claiming the ultimate, we adopt the following:

Definition. A ray $l \subset T_xM$ is said to be a *tangent direction to the family of orbits* $x \circ \mathrm{SGr}(Q(U_\varepsilon))$, $\varepsilon > 0$, if l is tangent to the curve $\varepsilon \mapsto x \circ P(\varepsilon)$, $\varepsilon \geq 0$, where $P(-)$ is a smooth curve in $\mathrm{Diff}(M)$, $P(0) = \mathrm{id}$ and an integer $N > 0$ exists such that

$$P\left(\frac{1}{kN}\right) \in \mathrm{SGr}(Q(U_{\varepsilon_k})), \quad \forall k > 0,\ \varepsilon_k \to 0\ (k \to \infty).$$

The following proposition is easy to prove.

Proposition. *The closure of the union of all tangent directions to the family of orbits* $\mathrm{SGr}(Q(U_\varepsilon))$, $\varepsilon > 0$, *at* x *is a convex cone in* $T_x M$.

7. Fast switching variations. The Corollary of Theorem 6 and the inclusion $\mathrm{SGr}(Q(U_\varepsilon)) \subset \mathrm{Gr}(x(U_\varepsilon))$ imply that every vector tangent to the family of orbits $x_0 \circ \mathrm{SGr}(Q(U_\varepsilon))$, $\varepsilon > 0$, is the value at x_0 of some commutator polynomial of Λ_i, $|i| > 0$. Here we describe a procedure for obtaining a rich variety of such polynomials.

Let $v(t)$, $t \in [0, 1]$, be a piecewise constant function, with values in U and discontinuities only in rational points of the interval $[0, 1]$. Suppose that $v(t)$ is constant on the intervals $(i/N, (i+1)/N)$, $i = 0, 1, \ldots, N-1$:

$$v(t) = v_i \quad \text{for} \frac{i}{N} < t < \frac{i+1}{N}.$$

Consider the composition $Q(v_0) \circ \cdots \circ Q(v_{N-1}) \in \mathrm{SGr}(Q(U))$. The Taylor series expansion in the powers of $v_0, \ldots, v_{N-1}$ has the form

$$Q(v_0) \circ \cdots \circ Q(v_{N-1}) \approx \mathrm{e}^{\Lambda(v_0)} \circ \cdots \circ \mathrm{e}^{\Lambda(v_{N-1})} = \overrightarrow{\exp} \int_0^1 N\Lambda(v(t))\, \mathrm{d}t,$$

where, cf. (4),

$$\overrightarrow{\exp} \int_0^1 N\Lambda(v(t))\, \mathrm{d}t = \mathrm{id} + \sum_{l=1}^{\infty} \int \underset{\Delta_l}{\cdots} \int N^l \Lambda(v(t_l)) \circ \cdots \circ \Lambda(v(t_1))\, \mathrm{d}t_1 \cdots \mathrm{d}t_l.$$

Let $0 < m < n$ be natural numbers. Put $N = \varepsilon^{-m}$ and substitute $\varepsilon^n v(t)$ for $v(t)$. We obtain

$$Q(\varepsilon^n v_0) \circ \cdots \circ Q(\varepsilon^n v_{N-1}) \approx \overrightarrow{\exp} \int_0^1 \varepsilon^{-m} \Lambda(\varepsilon^n v(t))\, \mathrm{d}t, \tag{7}$$

where the right-hand side is a well defined power series of ε with coefficients expressed as polynomials of $\Lambda_i, |i| > 0$:

$$\begin{aligned} &\overrightarrow{\exp} \int_0^1 \varepsilon^{-m} \Lambda(\varepsilon^n v(t))\, \mathrm{d}t \\ &\quad = \mathrm{id} + \sum_{l=1}^{\infty} \sum_{(i_1, \ldots, i_l)} \varepsilon^{(|i_1| + \cdots + |i_l|)n - lm} \int \underset{\Delta_l}{\cdots} \int v^{i_l}(t_l) \cdots v^{i_1}(t_1)\, \mathrm{d}t_1 \cdots \mathrm{d}t_l \Lambda_{i_l} \circ \cdots \circ \Lambda_{i_1}. \end{aligned} \tag{8}$$

(We remind that i_j are the multiindices $i_j = (i_j^1, \ldots, i_j^r)$.)

The relation (7) implies the following:

8. Proposition. *Let* $0 < m < n$ *be natural numbers, and* $v(t)$, $t \in [0, 1]$, *is a piecewise constant vector function with values in* U, *with discontinuities only in rational points. Furthermore, let* $k > 0$ *be the least integer, such that the coefficient at* ε^k *of the series* $x_0 \circ \overrightarrow{\exp} \int_0^1 \varepsilon^{-m} \Lambda(\varepsilon^n v(t))\, \mathrm{d}t$ *is not zero*:

$$x_0 \circ \overrightarrow{\exp} \int_0^1 \varepsilon^{-m} \Lambda(\varepsilon^n v(t))\, \mathrm{d}t = x_0 + \varepsilon^k x_0 \circ D + \mathrm{O}(\varepsilon^{k+1}). \tag{9}$$

Then $x_0 \circ D \in T_{x_0} M$ *and the ray* $\{\alpha(x_0 \circ D) \mid \alpha > 0\}$ *is a tangent direction to the family of orbits* $x_0 \circ \mathrm{SGr}(Q(U\varepsilon))$, $\varepsilon > 0$.

Formula (8) gives an explicit expression for the coefficients of the series (9), but as values at x_0 of an *associative* rather than a *commutator* polynomial of the variables Λ_i. Generally speaking, an associative polynomial of the vector fields Λ_i is a differential operator of higher order and does not define a tangent vector at x_0. However, $x_0 \circ D$ is a tangent vector and the following proposition gives its representation as the value of a commutator polynomial at x_0.

9. Proposition. *Let $\vartheta_t(\varepsilon)$, $t \in [0, 1]$, be a power series in ε, with nonstationary vector fields as coefficients, summable over $[0, 1]$. Then for every $x \in M$ the first nonzero terms (coefficients as well as exponents) of the following three series in ε coincide:*

$$x \circ \overrightarrow{\exp} \int_0^1 \vartheta_t(\varepsilon)\, dt - x = \sum_{l=1}^{\infty} \int \underset{\Delta_l}{\cdots} \int x \circ \vartheta_{t_l}(\varepsilon) \circ \cdots \circ \vartheta_{t_1}(\varepsilon)\, dt_1 \cdots dt_l, \tag{a}$$

$$\sum_{l=1}^{\infty} \frac{1}{l} \int \underset{\Delta_l}{\cdots} \int x \circ \left[\vartheta_{t_l}, \left[\vartheta_{t_{l-1}}, \ldots, \left[\vartheta_{t_2}, \vartheta_{t_1}\right] \ldots\right]\right] dt_1 \cdots dt_l, \tag{b}$$

$$\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \int \underset{\Delta_l}{\cdots} \int x \circ \left[\vartheta_{1-t_l}, \left[\vartheta_{1-t_{l-1}}, \ldots, \left[\vartheta_{1-t_2}, \vartheta_{1-t_1}\right] \ldots\right]\right] dt_1 \cdots dt_l. \tag{c}$$

The proof is based on the following formulas of chronological calculus, cf. [1,3,5]:

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} \overrightarrow{\exp} \int_0^1 \vartheta_t(\varepsilon)\, dt &= \int_0^1 \left(\overrightarrow{\exp} \int_0^t \operatorname{ad} \vartheta_\tau(\varepsilon)\, d\tau \right) \frac{\partial}{\partial \varepsilon} \vartheta_t(\varepsilon)\, dt \circ \overrightarrow{\exp} \int_0^1 \vartheta_t(\varepsilon)\, dt \\ &= \overrightarrow{\exp} \int_0^1 \vartheta_t(\varepsilon)\, dt \circ \int_0^1 \left(\overrightarrow{\exp} \int_0^t - \operatorname{ad} \vartheta_{1-\tau}(\varepsilon)\, d\tau \right) \frac{\partial}{\partial \varepsilon} \vartheta_{1-t}(\varepsilon)\, dt. \end{aligned}$$

10. Generalization. The procedure described above can be generalized essentially. Let $v(t_0, t_1)$, $(t_0, t_1) \in [0, 1] \times [0, 1]$, be a vector function with values in U, piecewise constant in the following sense: $\exists N_0, N_1$ such that $v(t_0, t_1) = v_{i_0 i_1}$ for $i_0/N_0 < t_0 < (i_0+1)/N_0$, $i_1/N_1 < t_1 < (i_1+1)/N_1$, $i_0 = 0, 1, \ldots, N_0 - 1$; $i_1 = 0, 1, \ldots, N_1 - 1$. Then

$$\begin{aligned} & Q(v_{00}) \circ \cdots \circ Q(v_{N_0-1,0}) \circ Q(v_{01}) \circ \cdots \circ Q(v_{N_0-1,1}) \circ \cdots \circ Q(v_{0,N_1-1}) \\ & \quad \circ \cdots \circ Q(v_{N_0-1,N_1-1}) \\ & \approx \overrightarrow{\exp} \int_0^1 N_0 A(v(t_0, 0))\, dt_0 \circ \cdots \circ \overrightarrow{\exp} \int_0^1 N_0 A\left(v\left(t_0, \frac{N_1 - 1}{N_1}\right)\right) dt_0 \\ & = \exp\left\{\ln \overrightarrow{\exp} \int_0^1 N_0 A(v(t_0, 0))\, dt_0\right\} \circ \cdots \circ \exp\left\{\ln \overrightarrow{\exp} \int_0^1 N_0 A\left(v\left(t_0, \frac{N_1-1}{N_1}\right)\right) dt_0\right\} \\ & = \overrightarrow{\exp} \int_0^1 \left(N_1 \ln \overrightarrow{\exp} \int_0^1 N_0 A(v(t_0, t_1))\, dt_0 \right) dt_1. \end{aligned}$$

Let n, m_0, m_1 be integers, $0 < m_0 + m_1 < n$. Put $N_0 = \varepsilon^{-m_0}$, $N_1 = \varepsilon^{-m_1}$ and substitute $v(t_0, t_1)$ by $\varepsilon^n v(t_0, t_1)$. We obtain

$$\begin{aligned} & Q(v_{00}) \circ \cdots \circ Q(v_{N_0-1,0}) \circ \cdots \circ Q(v_{0,N_1-1}) \circ \cdots \circ Q(v_{N_0-1,N_1-1}) \\ & \approx \overrightarrow{\exp} \int_0^1 \left(\varepsilon^{-m_1} \ln \overrightarrow{\exp} \int_0^1 \varepsilon^{-m_0} A(\varepsilon^n v(t_0, t_1))\, dt_0 \right) dt_1, \end{aligned} \tag{10}$$

where the right-hand side in (10) is a well defined power series in ε.

Proceeding in the same manner and considering instead of $v(t_0, t_1)$ a vector function of arbitrarily many variables we come to the following generalization of Proposition 8.

11. Proposition. *Let $v(t_0, t_1, \dots, t_d)$, $t_i \in [0, 1]$, $i = 0, \dots, d$, be a vector function with values in U, locally constant on some rectangle net in $[0, 1]^{d+1}$, with rational vertices. Let $m_0, \dots, m_d$, n be integers such that $0 < m_0 + \cdots + m_d < n$. Furthermore, suppose that*

$$x_0 \circ \overrightarrow{\exp}\int_0^1\left(\varepsilon^{-m_d} \ln \cdots \overrightarrow{\exp}\int_0^1\left(\varepsilon^{-m_1} \ln \overrightarrow{\exp}\int_0^1 \varepsilon^{-m_0}\Lambda(\varepsilon^n v(t_0, \dots, t_d))\, \mathrm{d}t_0\right) \mathrm{d}t_1 \dots\right) \mathrm{d}t_d$$

$$= x_0 + \varepsilon^k x_0 \circ D + \mathrm{O}(\varepsilon^{k+1}), \quad x_0 \circ D \neq 0. \tag{11}$$

Then $x_0 \circ D \in T_{x_0}M$ and the ray generated by the vector $x_o \circ D$ is a tangent direction to the family of orbits $x_0 \circ \mathrm{SGr}(Q(U_\varepsilon))$, $\varepsilon > 0$.

12. Jet bundles. Asymptotic expansions given in Propositions 8 and 11 and the commutator representation of the main term, Proposition 9, determine the tangent vectors to the orbits of families of semigroups $\mathrm{SGr}(Q(U_\varepsilon))$, $\varepsilon > 0$, as values of some commutator polynomials of Λ_i at the initial point. However, the problem is complicated by the fact that the set of the commutator polynomials in consideration essentially depends on the initial point; more strictly speaking, which of the polynomials vanish at the initial point. To take into account this information we must consider the action of the curves in the group of diffeomorphisms on the jet space of curves in M, with x_0 as initial point.

Denote by $C^n_{x_0}$ the manifold of n-jets at zero of smooth curves $\gamma : R \to M$ with initial condition $\gamma(0) = x_0$. By $J^n_0\gamma$ we shall denote the n-jet at zero of a given curve γ, and by $\mathrm{pr}^n : C^n_{x_0} \to C^{n-1}_{x_0}$ the canonical projection of the manifold of the n-jets onto the manifold of the $(n-1)$-jets.

The inverse image $(\mathrm{pr})^{-1}J^{n-1}_0\gamma$ under this mapping of a given $(n-1)$-jet has a natural affine space structure over the linear space $T_{x_0}M$. Indeed, suppose that the curves $\varepsilon \mapsto \gamma_i(\varepsilon)$, $i = 1, 2$, have identical $(n-1)$-jets, $J^{n-1}_0\gamma_1 = J^{n-1}_0\gamma_2$. Then, in the group of diffeomorphisms, there exists a curve $\varepsilon \mapsto Q(\varepsilon)$, such that $Q(0) = \mathrm{id}$, $\gamma_1(\varepsilon) \circ Q(\varepsilon) = \gamma_2(\varepsilon)$ for all sufficiently small $\varepsilon \geq 0$ and $x_0 \circ Q(\varepsilon) = x_0 + \varepsilon^n\xi + \mathrm{O}(\varepsilon^{n+1})$.

It is not difficult to show that the tangent vector $\xi \in T_{x_0}M$ depends only on the n-jets $J^n_0\gamma_1$, $J^n_0\gamma_2$ and defines the 'difference' of these n-jets. We can even formulate a stronger result:

13. Proposition. *For every $n > 0$ the mapping $\mathrm{pr}^n : C^n_{x_0} \to C^{n-1}_{x_0}$ is a projection onto the base space of the affine bundle with the affine space over the linear space $T_{x_0}M$ as a fibre.*

14. The group action on jets. Let S be a set and $\varepsilon \mapsto P_\varepsilon(s)$, $\varepsilon \geq 0$, $s \in S$, $P_0(s) = \mathrm{id}\ \forall s \in S$, be a family of smooth curves in $\mathrm{Diff}(M)$, indexed with the elements of S. The group of curves in $\mathrm{Diff}(M)$, starting at id is acting in a standard way on the space of curves on M. For example, $P_\_(s)$ transforms the curve γ into the curve $\gamma \circ P_\_(s) : \varepsilon \mapsto \gamma(\varepsilon) \circ P_\varepsilon(s)$. It is evident that $J^n_{x_0}(\gamma \circ P_\_(s))$ depends only on $J^n_{x_0}\gamma$, rather than on the whole curve γ, and we obtain an action of curves in $\mathrm{Diff}(M)$ on the space of jets $C^n_{x_0}\gamma$: the curve $P_\_$ transforms the jet $J^n_{x_0}\gamma$ into the jet $(J^n_{x_0}\gamma) \circ P_\_ : = J^n_{x_0}(\gamma \circ P_\_)$.

Denote by $\mathrm{Gr}(P_\_(S))$ the subgroup in the group of curves in $\mathrm{Diff}(M)$, generated by the curves $\varepsilon \mapsto P_{\phi(\varepsilon)}(s)$, $s \in S$, where $\phi : R \to R$ is a smooth mapping (the variable substitution), subject to the conditions $\phi(0) = 0$, $\phi(\varepsilon) \geq 0$ for $\varepsilon \geq 0$. The space $C^n_{x_0}$ contains an exceptional element: $J^n x_0$, the jet of the constant curve $\gamma(\varepsilon) \equiv x_0$. It is our immediate aim to describe the orbit of the group $\mathrm{Gr}(P_\_(S))$ in $C^n_{x_0}$ through $J^n x_0$.

Let $P_\varepsilon(s) \approx \mathrm{id} + \sum_{n=1}^\infty \varepsilon^n D_n(s)$ be the Taylor series expansion in the powers of ε, and

$$\ln\left(\mathrm{id} + + \sum_{n=1}^\infty \varepsilon^n D_n(s)\right) = \sum_{n=1}^\infty \varepsilon^n \Delta_n(s).$$

Proposition 4 implies that $\Delta_n(\alpha) \in \mathrm{Vect}(M)$, $n > 0$, $\alpha \in \mathscr{A}$. Finally, put

$$L_n = \mathrm{span}\left\{\left[\Delta_{k_1}(s_1), \left[\Delta_{k_2}(s_2), \ldots, \left[\Delta_{k_{l-1}}(s_{l-1}), \Delta_{k_l}(s_l)\right]\ldots\right]\right] \Big| \sum_{i=1}^{l} k_i \leq n,\ s_i \in S,\ l > 0\right\},$$

$$L_{n-1} \subset L_n \subset \mathrm{Vect}(M).$$

15. Theorem. *For every $n > 0$ the orbit $(J^n x_0) \circ \mathrm{Gr}(P\_(S))$ of the group $\mathrm{Gr}(P\_(S))$ in $C^n_{x_0}$ is an affine subbundle of the bundle $\mathrm{pr}^n : C^n_{x_0} \to C^{n-1}_{x_0}$, restricted to the orbit $(J^{n-1}x_0) \circ \mathrm{Gr}(P(S)) \subset C^{n-1}_{x_0}$. The fibre of this subbundle is an affine space over $x_0 \circ L_n$.*

16. Normally accessible jets. Denote by $\mathrm{SGr}(P\_(S))$ the semigroup generated by the curves $\varepsilon \mapsto P_{\phi(\varepsilon)}(s)$, $s \in S$, $\phi : R \to R$ (smooth), $\phi(0) = 0$, $\phi(\varepsilon) \geq 0$ for $\varepsilon \geq 0$. Thus, $\mathrm{SGr}(P\_(S)) \subset \mathrm{Gr}(P\_(S))$. Consider the orbit $J^n x_0 \circ \mathrm{SGr}(P\_(S))$ of this semigroup in $C^n_{x_0}$.

Definition. The point $q \in (J^n x_0 \circ \mathrm{SGr}(P\_(S)))$ is *normally accessible for a family of curves $P\_(s)$, $s \in S$*, if $m, k_1, \ldots, k_m > 0$ exist such that q is a regular value of the mapping

$$(\phi_1, \ldots, \phi_m) \mapsto (J^n x_0) \circ P_{\varepsilon^{k_1}\phi_1(\varepsilon)} \circ \cdots \circ P_{\varepsilon^{k_m}\phi_m(\varepsilon)},$$

with the domain consisting of the sequences of polynomials $\phi_1, \ldots, \phi_m$, of degree $\leq n$ and satisfying the conditions $\phi(0) > 0$, $i = 1, \ldots, m$, and with range the manifold $J^n x_0 \circ \mathrm{Gr}(P\_(S))$.

The set of normally accessible points is everywhere dense in the orbit $J^n x_0 \circ \mathrm{SGr}(P\_(S))$, and every normally accessible point has a neighborhood in $J^n x_0 \circ \mathrm{Gr}(P\_(S))$, completely contained in $J^n x_0 \circ \mathrm{SGr}(P\_(S))$. This remark, together with Theorem 15, implies:

17. Proposition. *If $J^n x_0$ is a normally accessible point then*

$$x_0 \circ L_n \subset \bigcup_{R(-) \in \mathrm{SGr}(P\_(S))} T_{x_0}(x_0 \circ R(-)),$$

where $T_{x_0}\gamma(-) \subset T_{x_0}M$ denotes the tangent ray to the curve $\varepsilon \mapsto \gamma(\varepsilon)$, $\epsilon \geq 0$, $\gamma(0) = x_0$.

Recall that the level sets of the mapping pr^n are affine spaces over $T_{x_0}M$. Let $q^n \in (J^n x_0) \circ \mathrm{Gr}(P\_(S))$, $q^{n-1} = \mathrm{pr}^n q^n$. Theorem 15 implies that the intersection of the level set $(\mathrm{pr}^n)^{-1}(q^{n-1})$ with the orbit $(J^n x_0) \circ \mathrm{Gr}(P\_(S))$ is an affine space over $x_0 \circ L_n$. We can write down this assertion in the following way: $(\mathrm{pr}^n)^{-1}(q^{n-1}) \cap (J^n x_0) \circ \mathrm{Gr}(P\_(S)) = q^n + x_0 \circ L_n$.

The following proposition indicates how to get many normally accessible points and, in particular, it gives a handy sufficient condition for $J^n x_0$ to be a normally accessible point.

18. Proposition. *If at least one point of the affine space $J^n x_0 + x_0 \circ L_{n-1}$ is normally accessible then every point of this space, including $J^n x_0$, is normally accessible.*

19. Sufficient conditions of local controllability. We shall give here a short description of a scheme for obtaining sufficient conditions of local controllability based on the techniques developed above. An evident necessary condition is given by the relation $x_0 \circ \mathrm{Lie}(\{\Lambda_i \,\|\, |i| > 0\}) = T_{x_0}M$. Suppose this condition to be satisfied. The fast switching variations from part 7 supply us with a vast set of curves P_ε in $\mathrm{Diff}(M)$, such that the tangent directions to the curves $\varepsilon \mapsto x_0 \circ P_\varepsilon$ are at the same time tangent to the orbits $x_0 \circ \mathrm{SGr}(Q(U_\varepsilon))$, $\varepsilon > 0$. Moreover, if for every $n > 0$ the jet $J^n x_0$ is normally accessible for this family of curves in $\mathrm{Diff}(M)$, then the family $Q(U)$, $u \in U$, is locally controllable at the point x_0.

Hence, it remains to find appropriate relations which should be satisfied by the values of the commutator polynomials of Λ_i at x_0 in order to guarantee the normal accessibility of $J^n x_0$, provided at least one normally accessible point exists. To this end we have to make use of Proposition 18 and a method of 'noncommutative averaging' over the symmetric group of the problem used by Sussmann [7,8]. It is even possible to extend this method and exploit not only the symmetries of the set U of the control parameters, but also the 'internal symmetries' of the variable substitutions in the functions $v(t_0, t_1, \dots, t_d)$ in (11).

For the time being we can only survey a small part of the opening possibilities, but already this suffices to obtain a whole range of pretty strong sufficient conditions. The theorem formulated below exploits only the simplest fast-switching variations, which are defined by a vector function $v(t)$, $t \in [0, 1]$, of one real argument, (cf. Proposition 8), and a single 'internal' symmetry $v(t) \mapsto v(1-t)$, (cf. Proposition 9).

Let Γ be a finite group of linear transformations of R^r, preserving the set $U \subset R^r$. For every $g \in \Gamma$ we have $(g^{-1}u)^i = \sum_j c_j^i(g) u^j$, $u \in R^r$, $i = (i^1, \dots, i^r)$. Consider a free Lie algebra over R, generated by the elements s_i, where $i = (i^1, \dots, i^r)$ is an arbitrary multiindex of the length $r, |i| > 0$.

Define an action of the group Γ by the automorphisms of the Lie algebra $\mathrm{Lie}(\{s_i \| i| > 0\})$, defining it on the generators by $g(s_i) = \sum c_j^i(g) s_j$, $\forall g \in \Gamma$.

Furthermore, for any integers $n > m \geq 0$ put

$$\mathscr{L}(n, m) = \mathrm{span}\Big\{\big[s_{i_m}, \big[s_{i_{m-1}}, \dots, [s_{i_1}, s_{i_0}] \dots\big]\big] \Big| |i_0| + \cdots + |i_m| = n\Big\}.$$

It is easy to see that the subspaces $\mathscr{L}(n, m)$ are invariant under the action of Γ. Let

$$\mathrm{Inv}_\Gamma(\mathscr{L}(n, m)) = \{\sigma \in \mathscr{L}(n, m) \mid g(\sigma) = \sigma \forall g \in \Gamma\}$$

be the subspace of invariant elements in $\mathscr{L}(n, m)$. Finally, let $\boldsymbol{\Lambda}(n, m)$ be the image of $\mathscr{L}(n, m)$ under the canonical homomorphism of the algebra $\mathrm{Lie}(\{s_i \| i| > 0\})$ onto the algebra $\mathrm{Lie}(\{\Lambda_i \| i| > 0\})$, transforming the variable s_i into Λ_i, and let $\mathrm{Inv}_\Gamma(\boldsymbol{\Lambda}(n, m))$ be the image of the subspace $\mathrm{Inv}_\Gamma(\mathscr{L}(n, m))$ under this homomorphism. Thus, $\mathrm{Inv}_\Gamma(\boldsymbol{\Lambda}(n, m)) \subset \mathrm{Lie}(\{\Lambda_i \| i| > 0\})$.

20. Theorem. *Suppose that* $x_0 \circ \mathrm{Lie}(\{\Lambda_i \| i| > 0\}) = T_{x_0}M$, *and* $\exists \theta \in [0, 1)$ *such that for arbitrary integers* $n > 2m \geq 0$,

$$x_0 \circ \mathrm{Inv}_\Gamma(\boldsymbol{\Lambda}(n, 2m)) \subset \mathrm{span}\{x_0 \circ \boldsymbol{\Lambda}(k, l) \mid k - \theta l < n - 2\theta m\}$$

for a certain finite group $\Gamma \subset \mathrm{GL}(R^r)$ *which preserves* U. *Then the family of diffeomorphisms* $Q(u)$, $u \in U$, *is locally controllable at* x_0.

The proof of the theorem is fairly long and technical, and could not be given here. Therefore, we shall restrict ourselves to the following comments. The essential meaning of the theorem consists in the statement that the family of diffeomorphisms is locally controllable if the value of every, in the given sense, invariant Lie polynomial at the initial point could be neutralized by the values of the Lie polynomials of 'lesser orders'. It should be emphasized that the formulation of our theorem only partly follows a standard scheme, used in controllability conditions for families of vector fields, cf. [8]. The essential difference from this scheme consists in the way the orders of the Lie polynomials are counted. We assign to usual variations the value $\theta = 0$ and the order of the polynomials from $\Lambda(k, l)$ turns out to be equal to k. But to the fast-switching variations of the type (7)–(8) we assign $\theta = m/n$, and the order of the polynomials from $\Lambda(k, l)$ is equal to $k - \theta(l + 1)$.

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