Hamiltonian Systems of Negative Curvature are Hyperbolic

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Abstract

The curvature and the reduced curvature are basic differential invariants of the pair: \langle Hamiltonian system, Lagrange distribution \rangle on the symplectic manifold. We show that negativity of the curvature implies that any bounded semi-trajectory of the Hamiltonian system tends to a hyperbolic equilibrium, while negativity of the reduced curvature implies the hyperbolicity of any compact invariant set of the Hamiltonian flow restricted to a prescribed energy level. Last statement generalizes a well-known property of the geodesic flows of Riemannian manifolds with negative sectional curvatures.

1 Regularity and Monotonicity

Smooth objects are supposed to be \( C^\infty \) in this note; the results remain valid for the class \( C^k \) with a finite and not large \( k \) but we prefer not to specify the minimal possible \( k \).

Let \( M \) be a \( 2n \)-dimensional symplectic manifold endowed with a symplectic form \( \sigma \). A Lagrange distribution \( \Delta \subset TM \) is a smooth vector subbundle of \( TM \) such that each fiber \( \Delta_x = \Delta \cap T_x M \), \( x \in M \), is a Lagrange subspace of the symplectic space \( T_x M \); in other words, \( \dim \Delta_x = n \) and \( \sigma_x(\xi, \eta) = 0 \ \forall \xi, \eta \in \Delta_x \).

Basic examples are cotangent bundles endowed with the standard symplectic structure and the “vertical” distribution:

\[
M = T^*N, \quad \Delta_x = T_x(T^*_qN), \quad \forall x \in T^*_qN, \quad q \in N. \tag{1}
\]

Let \( h \in C^\infty(M) \); then \( \vec{h} \in \text{Vec}M \) is the associated to \( h \) Hamiltonian vector field: \( dh = \sigma(\cdot, \vec{h}) \). We assume that \( \vec{h} \) is a complete vector field,

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i.e. solutions of the Hamiltonian system $\dot{x} = \vec{h}(x)$ are defined on the whole time axis. We may assume that without a lack of generality since we are going to study dynamics of the Hamiltonian system on compact subsets of $M$ and may reduce the general case to the complete one by the usual cut-off procedure.

The generated by $\vec{h}$ Hamiltonian flow is denoted by $e^{t\vec{h}}$, $t \in \mathbb{R}$. Other notations: $\bar{\Delta} \subset \text{Vec}M$ is the space of sections of the Lagrange distribution $\Delta$: $[v_1, v_2] \in \text{Vec}M$ is the Lie bracket (the commutator) of the fields $v_1, v_2 \in \text{Vec}M$, $[v_1, v_2] = v_1 \circ v_2 - v_2 \circ v_1$.

**Definition 1** We say that $\vec{h}$ is regular at $x \in M$ with respect to the Lagrange distribution $\Delta$ if $\{[\vec{h}, v](x) : v \in \bar{\Delta}\} = T_x M$.

An effective version of Definition 1 is as follows: Let $v_i \in \bar{\Delta}$, $i = 1, \ldots, n$ be such that the vectors $v_1(x), \ldots, v_n(x)$ form a basis of $\Delta_x$; then $\vec{h}$ is regular at $x$ with respect to $\Delta$ if and only if the vectors $v_1(x), \ldots, v_n(x), [\vec{h}, v_1](x), \ldots, [\vec{h}, v_n](x)$ form a basis of $T_x M$.

We define a bilinear mapping $g^h : \bar{\Delta} \times \bar{\Delta} \to C^\infty(M)$ by the formula:

$$ g^h(v_1, v_2) = \sigma([\vec{h}, v_1], v_2). $$

**Lemma 1** $g^h(v_2, v_1) = g^h(v_1, v_2)$, $\forall v_1, v_2 \in \bar{\Delta}$ and $g^h(v_1, v_2)(x)$ depends only on $v_1(x), v_2(x)$.

**Proof.** Hamiltonian flows preserve $\sigma$ and $\sigma$ vanishes on $\bar{\Delta}$. Using these facts, we obtain:

$$ 0 = \sigma(v_1, v_2) = \left( e^{t\vec{h}} \sigma \right)(v_1, v_2) = \sigma(e^{t\vec{h}} v_1, e^{t\vec{h}} v_2). $$

Differentiation of the identity $0 = \sigma(e^{t\vec{h}} v_1, e^{t\vec{h}} v_2)$ with respect to $t$ at $t = 0$ gives: $0 = \sigma([\vec{h}, v_1], v_2) + \sigma(v_1, [\vec{h}, v_2])$. Now the anti-symmetry of $\sigma$ implies the symmetry of $g^h$. Moreover, $g^h$ is $C^\infty(M)$-linear with respect to each argument, hence $g^h(v_1, v_2)(x)$ depends only on $v_1(x), v_2(x)$. □

Let $x \in M$, $\xi_i \in \Delta_x$, $\xi_i = v_i(x)$, $v_i \in \Delta$, $i = 1, 2$. We set $g^h_x(\xi_1, \xi_2) = g^h(v_1, v_2)(x)$. According to Lemma 1, $g^h_x$ is a well-defined symmetric bilinear form on $\Delta_x$. It is easy to see that the regularity of $h$ at $x$ is equivalent to the nondegeneracy of $g^h_x$. 2
If $M = T^*N$ and $\Delta$ is the vertical distribution (see (1)), then $g^h_x = D^2_x(h|_{T^*_qN})$, where $x \in T^*_qN$. The last equation can be easily checked in local coordinates. Indeed, local coordinates defined on a neighborhood $O \subset N$ provide the identification of $T^*N|_O$ with $\mathbb{R}^n \times \mathbb{R}^n = \{(p, q) : p, q \in \mathbb{R}^n\}$ such that $T^*_qN$ is identified with $\mathbb{R}^n \times \{q\}$, the form $\sigma$ is identified with $\sum_{i=1}^n dp_i \wedge dq_i$ and the field $\vec{h}$ with $\sum_{i=1}^n \left( \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i} \frac{\partial}{\partial p_i} \right)$.

The fields $\frac{\partial}{\partial p_i}$ form a basis of the vertical distribution and $g^h_x \left( \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right) = -\left\langle dq_j, \left[ \sum_{i=1}^n \left( \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i} \frac{\partial}{\partial p_i} \right), \frac{\partial}{\partial p_i} \right] \right\rangle = \frac{\partial^2 h}{\partial p_i \partial p_j}$.

**Definition 2** We say that a regular Hamiltonian field $\vec{h}$ is monotone at $x \in M$ with respect to $\Delta$ if $g^h_x$ is a sign-definite form.

2 The Curvature

Let $X_1, X_2$ be a pair of transversal $n$-dimensional subspaces of $T_xM$, then $T_xM = X_1 \oplus X_2$. We denote by $\pi_x(X_1, X_2)$ the projector of $T_xM$ on $X_2$ parallel to $X_1$. In other words, $\pi_x(X_1, X_2)$ is a linear operator characterized by the relations $\pi_x(X_1, X_2)|_{X_1} = 0$, $\pi_x(X_1, X_2)|_{X_2} = 1$.

Now consider the family of subspaces $J_x(t) = e^{-t\vec{h}}_x \Delta_e^{x_j(x)} \subset T_xM$, where $\vec{h}$ is a regular Hamiltonian field; in particular, $J_x(0) = \Delta_x$. It is easy to check that the regularity of $\vec{h}$ implies the transversality of $J_x(t)$ and $J_x(\tau)$ for $t \neq \tau$, if $t$ and $\tau$ are close enough to 0. Hence $\pi_x(J_x(t), J_x(\tau))$ is well-defined and smooth with respect to $(t, \tau)$ in a neighborhood of $(0, 0)$ with the removed diagonal $t = \tau$. The mapping $(t, \tau) \mapsto \pi_x(J_x(t), J_x(\tau))$ has a singularity at the diagonal, but this singularity can be controlled. In particular, the following statement is valid:

**Lemma 2** (see [1]). For any regular field $\vec{h}$,

$$\frac{\partial^2}{\partial t \partial \tau} \left( \pi_x(J_x(t), J_x(\tau)) |_{\Delta_x} \right) \bigg|_{\tau=0} = t^{-2} 1 + R^h_x + O(t) \quad \text{as} \ t \to 0,$$

where $R^h_x \in \mathfrak{gl}(\Delta_x)$ is a self-adjoint operator with respect to the scalar product $g^h_x$, i.e. $g^h_x(R^h_x \xi_1, \xi_2) = g^h_x(\xi_1, R^h_x \xi_2), \ \forall \xi_1, \xi_2 \in \Delta_x$.

We set $r^h_x(\xi) = g^h_x(R^h_x \xi, \xi)$.

**Definition 3** Operator $R^h_x$ and quadratic form $r^h_x$ are called the curvature operator and the curvature form of $\vec{h}$ at $x$ with respect to $\Delta$. We say that
Riemannian geodesic flow

Mechanical system on a Riemannian manifold

We set

\[ \bar{h} \text{ has a negative (positive) curvature at } x \text{ if } r^h_x(\xi)g^h_x(\xi, \xi) < 0 \text{ (} > 0, \forall \xi \in \Delta_x \setminus \{0\}. \]

It follows from the definition that only monotone fields may have negative or positive curvature. If \( \bar{h} \) is monotone at \( x \), then \( R^h_x \) has only real eigenvalues and negativity (positivity) of the curvature is equivalent to the negativity (positivity) of all eigenvalues of \( R^h_x \).

Let us give a coordinate presentation of \( R^h_x \). Fix local coordinates \((p, q)\), \( p, q \in \mathbb{R}^n \) in a neighborhood of \( x \) in \( M \) in such a way that \( \Delta_x \cong \{(p, 0) : p \in \mathbb{R}^n \}. \) Let \((p(t; p_0), q(t; p_0)) \) be the trajectory of the field \( \bar{h} \) with the initial conditions \( p(0; q_0) = p_0, \ q(0; p_0) = 0 \). We set \( S_t = \left. \frac{\partial q(t; p_0)}{\partial p_0} \right|_{p_0 = 0} \); regularity of \( \bar{h} \) is equivalent to the nondegeneracy of the \( n \times n \)-matrix \( \dot{S}_0 = \frac{dS_t}{dt}|_{t=0} \). The curvature operator is presented by the matrix Schwartzian derivative:

\[ R^h_{(p, q)} = 1/2\dot{S}^{-1}_{0} - 3/4(\dot{S}^{-1}_{0}\ddot{S}_0)^2. \]

Examples:

1. **Natural mechanical system**, \( M = \mathbb{R}^n \times \mathbb{R}^n, \text{ } \sigma = \sum_{i=1}^{n} dp_i \wedge dq_i, \text{ } \Delta_{(p, q)} = (\mathbb{R}^n, 0), \text{ } h(p, q) = 1/2||p||^2 + U(q); \text{ then } R^h_{(p, q)} = d^2U/dq^2. \)

2. **Riemannian geodesic flow**, \( M = T^*N \text{ and } h|_{T^*_qN} \text{ is a positive quadratic form } \forall q \in N; \text{ then } \bar{h} \text{ is actually a Riemannian structure on } N \text{ which identifies the tangent and cotangent bundles and we have: } R^h_{x} = \mathcal{R}(x', \xi)x', \text{ where } \mathcal{R} \text{ is the Riemannian curvature tensor and } x', \xi' \in T_qM \text{ are obtained from } x, \xi \in T^*_qM \text{ by the "raising of the indices".} \)

3. **Mechanical system on a Riemannian manifold**, \( M = T^*N \text{ and } h \) is the sum of the Riemannian Hamiltonian from Example 2 and the function \( U \circ \pi, \text{ where } \pi : T^*N \to N \text{ is standard projection and } U \text{ is a smooth function on } N \). Then \( R^h_{x} = \mathcal{R}(x, \xi)x + \nabla_{\xi}(\nabla U), \text{ where } \nabla_{\xi} \text{ is the Riemannian covariant derivative.} \)

Now we introduce a reduced curvature form \( \hat{r}^h_x \) defined on \( \Delta_x \cap \ker d_x h \) and related to the restriction of the Hamiltonian system on the prescribed energy level. To do that, we need some notations. Symplectic form \( \sigma_x \) on \( T_xM \) induces a nondegenerate pairing of \( \Delta_x \) and \( T_xM/\Delta_x \). Hence there exists a unique linear mapping \( G_x : \Delta_x \to T_xM/\Delta_x \text{ such that } g_x(\xi_1, \xi_2) = \sigma_x(G_x\xi_1, \xi_2), \text{ } \forall \xi_1, \xi_2 \in \Delta_x. \) The mapping \( G_x \) is invertible since the form \( g_x \) is nondegenerate. Let \( \Pi_x : T_xM \to T_xM/\Delta_x \) be the canonical projection.

We set \( v(x) = G^{-1}_x \Pi_x \bar{h}(x); \text{ then } v \text{ is a smooth section of } \Delta, \text{ i.e. } v \in \Delta. \)
Assume that $\vec{h}$ is a monotone field and $\vec{h}(x) \notin \Delta_x$; the reduced curvature form is defined by the formula:

$$\hat{r}^h_x(\xi) = r^h_x(\xi) + \frac{3\sigma_x([\vec{h}, [\vec{h}, v]](x), \xi)^2}{4g_x(v(x), v(x))}, \quad \xi \in \Delta_x \cap \ker d_xh.$$ 

In Ex. 1, we obtain: $\hat{r}^h_{(p,q)}(\xi) = r^h_{(p,q)}(\xi) + \frac{3|p|^2}{\|p\|^2} (\frac{dU}{dq}, \xi)^2$. In Ex. 2, $\hat{r}^h_x(\xi) = r^h_x(\xi)$. Finally, in Ex. 3 (which includes both Ex. 1 and Ex. 2) we have:

$$\hat{r}^h_x(\xi) = r^h_x(\xi) + \frac{3g_x(q_x, \xi)^2}{2(h(x) - U(q))}, \quad \text{where } q = \pi(x).$$ 

We say that $\vec{h}$ has a negative (positive) reduced curvature at $x$ if $\hat{r}^h_x(\xi)_{q^h_x(\xi)} < 0$ $(> 0)$, $\forall \xi \in \Delta_x \cap \ker d_x h \setminus \{0\}$.

### 3 Main Results

**Theorem 1** Let $\vec{h}$ be a monotone field and $x_0 \in M$. Assume that the semi-trajectory $\{e^{\vec{h}}(x_0) : t \geq 0\}$ has a compact closure and $\vec{h}$ has a negative curvature at each point of its closure. Then there exists $x_{\infty} = \lim_{t \to +\infty} e^{\vec{h}}(x_0)$, where $\vec{h}(x_{\infty}) = 0$ and $D_x h$ is hyperbolic (i.e. $D_x h$ has no eigenvalues on the imaginary axis).

**Remark.** Monotonicity of $\vec{h}$ is equivalent to the monotonicity of $-\vec{h}$ and $R^{-h}_x = R^h_x$, hence Theorem 1 can be applied to the negative time semi-trajectories of the field $\vec{h}$ as well.

Example. Consider a natural mechanical system (Ex. 1 in Sec. 2) where $U(q)$ is a strongly concave function, then any bounded semi-trajectory of $\vec{h}$ satisfies conditions of Theorem 1.

**Theorem 2** Let $\vec{h}$ be a monotone field, $S$ be a compact invariant subset of the flow $e^{\vec{h}}$ contained in a fix level set of $h$, $S \subset h^{-1}(c)$, and $\vec{h}(x) \notin \Delta_x \forall x \in S$. If $\vec{h}$ has a negative reduced curvature at each point of $S$, then $S$ is a hyperbolic set of the flow $e^{\vec{h}}\big|_{h^{-1}(c)}$ (see [2, Sec. 17.4] for the definition of a hyperbolic set).

Example. Mechanical system on a Riemannian manifold (Ex. 3 in Sec. 2). Let $\kappa_q$ be the maximal sectional curvature of the Riemannian manifold $N$ at $q \in N$. Then any compact invariant set $S$ of the flow $e^{\vec{h}}\big|_{h^{-1}(c)}$ such that the projection of $S$ to $N$ is contained in the domain

$$\{q \in N : \kappa_q < 0, 2 \max\{\|\nabla^2 U\|/|\kappa_q|, |\nabla U|^2\} < c - U(q)\}$$


is hyperbolic. In particular, if $N$ is a compact Riemannian manifold of a negative sectional curvature, then $e^{\tilde{E}(\cdot\cdot\cdot)}$ is an Anosov flow for any big enough $c$. Last statement generalizes a classical result on geodesic flows.

Both theorems are based on the structural equations derived in [1]. These equations are similar to the standard linear differential equation for Jacobi vector fields in Riemannian Geometry with the curvature operators $R^h_x$ playing the same role as the Riemannian curvature. In particular, the proof of Theorem 2 simply simulates the proof of the correspondent classical result on geodesic flows. Theorem 1 describes a new phenomenon, which is not performed by geodesic flows. Indeed, if the curvature is negative, then the operators $R^h_x$ are nondegenerate, while in the Riemannian case (Ex. 2 in Sec. 2) we have $R^h_xe(x) = 0$, where $e$ is the Euler field (i.e. the field generating homothety of the fibers $T^*_qN$).

References
