Докл. Акад. Наук СССР
Том 284 (1985), № 4

Soviet Math. Dokl.
Vol. 32 (1985), No. 2

THE INDEX OF EXTREMALITY AND QUASIEXTREMAL CONTROLS

UDC 517.97

A. A. AGRACHEV AND R. V. GAMKRELIDZE

1. We begin with an informal description of index of extremality; for precise definitions see §2.

Consider the extremal problem for a functional $\varphi_0\colon Z \to \mathbf{R}$ under constraints $\varphi_i(z) = 0$ for $i = 1, \ldots, m$ (with Z to be specified later). Let $z_0 \in Z$ and $l < 0$; we assume that the index of extremality at z_0 is greater than l if the point z_0 can be made extremal on adding $(-l)$ new constraints in a "stable manner" (stability here meaning that if the new constraints are changed slightly, z_0 remains extremal). Second, suppose that $z_0 \in Z$ is an extremal point, and $0 \le k \le m$; we assume that the index of extremality at z_0 is greater than k if k of the constraints can be omitted in a "stable manner" while retaining extremality of z_0.

We shall actually use a more geometric approach, in which the functional is not considered separately from the constraints: instead of treating a functional φ_0 and constraints $\varphi_1, \ldots, \varphi_m$ we shall consider the vector-valued function $\Phi = (\varphi_0, \varphi_1, \ldots, \varphi_m)^{\mathrm{T}}$, and extremal values will be the boundary points of the image im Φ. The concept of extremality index is then modified appropriately. Further, we shall not treat quite arbitrary mappings Φ, but restrict ourselves to control systems. The quasiextremality index of a given control is the largest extremality index at the corresponding "point" that can be achieved by an arbitrarily small change of the system.

2. Let M be an n-manifold, and U an r-manifold, both of class C^∞, embedded as closed submanifolds in $\mathbf{R}^d$. Consider the controlled system

$$\dot{x} = f_t(x, u), \quad x \in M, u \in U, t \in [0, 1], \qquad x(0) = x_0; \tag{1}$$

here $f_t(x, u)$ is infinitely differentiable with respect to (x, u) and measurable in t, with

$$\int_0^1 \|f_t(\cdot, \cdot)\|_{K,\alpha}\, dt < +\infty \quad \text{for all } K \Subset M \times U, \alpha \ge 0,$$

where $\|\cdot\|_{K,\alpha}$ denotes the maximum of all derivatives to order α over the compact set K. The admissible controls are arbitrary bounded measurable mappings $u\colon [0, 1] \to U \subset \mathbf{R}^d$; clearly the collection $L_\infty([0, 1]; U)$ of admissible controls is a smooth Banach submanifold of $L^d_\infty[0, 1]$. The collection of seminorms $\int_0^1 \|\cdot\|_{K,\alpha}\, dt$, $K \Subset M \times U$, $\alpha \ge 0$, turns the linear space of controlled systems of the form (1) into a Fréchet space that will be denoted by $CS(M, x_0; U)$.

Fix an admissible control $\tilde{u}(t)$, $t \in [0, 1]$, and assume that the corresponding trajectory $\tilde{x}(t)$, which satisfies $\dot{\tilde{x}}(t) = f_t(\tilde{x}(t), \tilde{u}(t))$ and $\tilde{x}(0) = x_0$, is defined over the entire interval $[0, 1]$. Then for all controls $u(\cdot)$ in some neighborhood $\mathcal{U}$ of the "point" $\tilde{u}(\cdot)$ in the space $L_\infty([0, 1]; U)$ there is defined a mapping $F\colon u(\cdot) \mapsto x(1) \in M$, where $\dot{x}(t) = f_t(x(t), u(t))$ for $t \in [0, 1]$, and $x(0) = x_0$. It is not hard to show that $F\colon \mathcal{U} \mapsto M$ is infinitely differentiable. Before proceeding further let us describe several pertinent local invariants of smooth mappings.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 49A99; Secondary 34H05.

Let $\mathcal{A}$ be a Banach manifold of class C^∞, and $a \in \mathcal{A}$. Denote by $C^\infty_a(\mathcal{A}, M)$ the space of germs, at a, of smooth mappings from $\mathcal{A}$ to M, with the topology of convergence of all derivatives at a. In the subsequent definitions, the phrase "for almost every germ" means "for any germ in some open dense subset of the space of germs".

DEFINITION 1. A germ $\mathcal{H} \in C^\infty_a(\mathcal{A}, M)$ is said to be *extremal* if there exist a neighborhood $\mathcal{O}$ of a in $\mathcal{A}$ and a representative $H: \mathcal{O} \to M$ of $\mathcal{H}$ such that $H(a) \in \partial H(\mathcal{O})$ (i.e., the point $H(a)$ is on the boundary of $H(\mathcal{O})$).

DEFINITION 2. Again, take $\mathcal{H} \in C^\infty_a(\mathcal{A}, M)$.

(i) Let $\mathcal{H}$ be extremal. We say that $\mathcal{H}$ has *extremality index* $k > 0$ if k is the least integer such that, for almost every germ $\Phi \in C^\infty_{\mathcal{H}(a)}(M, \mathbf{R}^{n-k})$, the germ $\Phi \circ \mathcal{H} \in C^\infty_a(\mathcal{A}, \mathbf{R}^{n-k})$ is not extremal.

(ii) Suppose $\mathcal{H}$ is not extremal. We say that $\mathcal{H}$ has *extremality index* $l \leq 0$ if l is the smallest integer such that, for almost every germ $\Psi \in C^\infty_a(\mathcal{A}, \mathbf{R}^{-l})$, the germ $(\mathcal{H}, \Psi) \in C^\infty_a(\mathcal{A}, M \times \mathbf{R}^{-l})$ is not extremal. If a least l does not exist, the index of extremality is $-\infty$.

Thus each germ $\mathcal{H} \in C^\infty_a(\mathcal{A}, M)$ has an index of extremality, lying within some interval $[-\infty, n]$. A germ is *extremal* if its extremality index is positive.

Let us now return to the controlled system (1).

DEFINITION 3. The *index of local extremality* of a control $\tilde{u}(\cdot)$, relative to the system (1), is defined to be the extremality index of the germ of F at the "point" $\tilde{u}(\cdot)$. If the local extremality index is positive, the control $\tilde{u}(\cdot)$ is said to be *locally extremal* relative to system (1).

DEFINITION 4. The *index of quasiextremality* of a control $\tilde{u}(\cdot)$, relative to system (1), is defined to be the largest $k \in [-\infty, n]$ with the following property: in the space $CS(M, x_0; U)$, arbitrarily close to $f_t(x, u)$ there exists a controlled system $g_t(x, u)$ relative to which the control $\tilde{u}(\cdot)$ has local extremality index k. If the quasiextremality index is positive, the control $\tilde{u}(\cdot)$ is said to be *quasiextremal* relative to the system (1).

Thus the quasiextremality index of a control relative to a given system $f_t(x, u)$ is the limit superior of the local extremality indices of $\tilde{u}(\cdot)$ relative to systems $g \in CS(M, x_0; U)$ as g tends to f (it is easy to see that the corresponding limit inferior is always $-\infty$). In particular, a given control has its quasiextremality index depending upper semicontinuously on the system.

3. It turns out that the index of quasiextremality of a control $\tilde{u}(\cdot)$ can be computed on the basis of only the differential and the Hessian of F at the "point" $\tilde{u}(\cdot)$. To describe these we shall need some further notation. For any y_1 in a neighborhood $O_1 \subset M$ of the point $\tilde{x}(1)$ in M, the solution of the equation $\dot{y}(\tau) = f_\tau(y(\tau), \tilde{u}(\tau))$, $y(1) = y_1$, is defined for all $\tau \in [0, 1]$; moreover, for each $t \in [0, 1]$ the mapping $p_t: y(t) \mapsto y(1)$ is a diffeomorphism of a neighborhood O_t of $\tilde{x}(t)$ onto O_1. In the usual manner denote the differential of p_t by p_{t*}, and the codifferential by p_t^* (p_{t*} takes vector field on O_t to vector fields on O_1, while p_t^* takes differential forms on O_1 to differential forms on O_t). The tangent and cotangent spaces of M at x are T_xM and T_x^*M and T_uM is the tangent space of U at u. Now define

$$(\alpha) \qquad \tilde{f}_t'(x) = (\partial f_t/\partial u)(x, \tilde{u}(t)), \qquad \tilde{f}_t''(x) = (\partial^2 f_t/\partial u_1 \partial u_2)(x, \tilde{u}(t)).$$

Then $\tilde{f}'(x): T_{\tilde{u}(t)}U \to T_xM$ is a linear mapping, and for each $v \in T_{\tilde{u}(t)}U$ the correspondence $x \mapsto \tilde{f}_t'(x)v$ defines a vector field $\tilde{f}_t'v$ on M. Analogously, $\tilde{f}_t''(x): T_{\tilde{u}(t)}U \times T_{\tilde{u}(t)}U \to \operatorname{coker} \tilde{f}_t'(x)$ is a symmetric bilinear mapping (we are using the standard notation $\operatorname{coker} \tilde{f}_t'(x) = T_xM/\operatorname{im} \tilde{f}_t'(x)$; the values of the second derivatives $\partial^2 f/\partial u_1 \partial u_2$ are well-defined only modulo image $\partial f/\partial u$). Finally we note that the tangent space $T_{\tilde{u}(\cdot)}L_\infty([0, 1]; U)$ of the Banach manifold $L_\infty([0, 1]; U)$ at the "point" $\tilde{u}(\cdot)$ consists of the bounded measurable mappings $t \mapsto v(t)$, $0 \le t \le 1$, for which $v(t) \in T_{\tilde u(t)}U$ for all $t \in [0,1]$.

PROPOSITION 1. *Let $\tilde F' : T_{\tilde u(\cdot)}L_\infty([0,1];U) \to T_{\tilde x(1)}M$ be the differential of a mapping F at the "point" $\tilde u(\cdot)$, and let* $\ker \tilde F'$ *be its kernel,* $\operatorname{im} \tilde F'$ *its image,* $\operatorname{coker} \tilde F' = T_{\tilde x(1)}M/\operatorname{im}\tilde F'$ *its cokernel, and* $\tilde F''$: $\ker \tilde F' \times \ker \tilde F' \to \operatorname{coker} \tilde F'$ *the Hessian of F at the "point" $\tilde u(\cdot)$. Then the following equalities are true:*

$$\tilde F' v(\cdot) = \int_0^1 p_{t*}\tilde f_t'(\tilde x(t))v(t)\,dt,$$

$$\operatorname{im} \tilde F' = \operatorname{span}\{p_{t*}\tilde f_t'(\tilde x(t))v \mid v \in T_{\tilde u(t)}U, \text{ where } t \text{ is a Lebesgue point of the mapping } \tau \mapsto p_{t*}\tilde f_\tau'(\tilde x(\tau))\}$$

$$\tilde F''(v_1(\cdot), v_2(\cdot)) = \int_0^1 \left\{ (p_{t*}\tilde f_t''(\tilde x(t))(v_1(t), v_2(t))) + \left[\int_0^t p_{\tau*}\tilde f_\tau' v_1(\tau)\,d\tau, p_{t*}\tilde f_t' v_2(t)\right](\tilde x(1)) \right\} dt + \operatorname{im} \tilde F',$$

for all $v_i(\cdot) \in \ker \tilde F'$, $i = 1,2$. The brackets $[\ ,\ ]$ denote the commutator of vector fields on M.

The orthogonal complement of the image is $\tilde F'$ is

$$(\operatorname{im} \tilde F')^\perp = \{\psi \in T^*_{\tilde x(1)}M \mid (p_t^*\psi)\tilde f_t(\tilde x(t))v = 0 \text{ for every } v \in T_{\tilde u(t)}U \text{ and almost every } t \in [0,1]\}.$$

For any $\psi \in (\operatorname{im} \tilde F')^\perp$, the mapping $v(\cdot) \mapsto \psi F''(v(\cdot), v(\cdot))$ is a scalar quadratic form on $\ker \tilde F'$, to be denoted by $\psi \tilde F''$. We recall that the Morse index of a quadratic form Q is defined to be the maximal dimension, possibly $+\infty$, of subspaces on which Q is negative definite; the standard notation is $\operatorname{ind} Q$. By convention, $\min \varnothing = +\infty$.

THEOREM 1. *The quasiextremality index of a control $\tilde u(\cdot)$ relative to (1) is*

$$\dim \operatorname{coker} \tilde F' - \min\{\operatorname{ind}(\psi \tilde F'') \mid \psi \in (\operatorname{im} \tilde F')^\perp, \psi \ne 0\}.$$

4. The generalised Legendre conditions take their definitive form as an estimate of the quasiextremality index (see [1]–[3]). For the remainder of this section we assume that $f_t(x,u)$ and $\tilde u(t)$ are piecewise smooth and left-continuous in t; t-derivatives at points of discontinuity are to be interpreted as limits from the left of the corresponding derivatives.

If we set $\tilde f_t(x) = f_t(x, \tilde u(t))$, the correspondence $x \mapsto \tilde f_t(x)$ defines a vector field $\tilde f_t$ on M. In the customary manner we define the operator $\operatorname{ad} \tilde f_t$, mapping the set of vector fields on M into itself: namely, $(\operatorname{ad} \tilde f_t)g = [\tilde f_t, g]$ for any vector field g.

DEFINITION 5. Let $t \in (0,1]$, and let $k \ge 0$ be an integer. The bilinear mapping

$$L_t^k : T_{\tilde u(t)}U \times T_{\tilde u(t)}U \to T_{\tilde x(t)}M$$

taking (v_1, v_2) to $[\tilde f_t' v_1, (\partial/\partial t + \operatorname{ad} \tilde f_t)^k \tilde f_t' v_2](\tilde x(t))$ is called the *Legendre form of order k at t.*

The following notation will also be convenient:

$$L_t^{-1}(v_1, v_2) = \tilde f_t''(\tilde x(t))(v_1, v_2).$$

Let $\psi \in T^*_{\tilde x(1)}M$ and set $\psi_t = p_t^*\psi$; then the covector ψ belongs to $(\operatorname{im} \tilde F')^\perp$ if and only if $\psi_t \tilde f_t'(\tilde x(t)) = 0$ for all $t \in (0,1]$; furthermore, the products $\psi_t L_t^k$ for $k = -1, 0, 1, \ldots$ are scalar bilinear forms on $T_{\tilde u(t)}U$, $t \in (0,1]$. Let $k_t(\psi)$ be the least $k \ge -1$ such that $\psi_\tau L_\tau^k$ does not vanish identically on any interval of the form $[\bar t, t]$ with $0 < \bar t < t$.

PROPOSITION 2. *Assume that the family of covectors* $\psi_t = p_t^*\psi$ *satisfies* $\psi_t f_t'(\tilde{x}(t)) \equiv 0$ *for* $t \in (0,1]$. *For each* $t \in (0,1]$ *the following assertions are true*:

(i) *If* $k_t(\psi) \geq 2 \dim \operatorname{span}\{p_{\tau *}\tilde{f}_\tau'(\tilde{x}(\tau))v | v \in T_{\tilde{u}(\tau)}U, 0 < \tau \leq t\}$, *then* $k_t(\psi) = +\infty$.

(ii) *The bilinear form* $\psi_t L_t^{k_t(\psi)}(v_1, v_2)$ *is symmetric if* $k_t(\psi)$ *is odd, and skew-symmetric if* $k_t(\psi)$ *is even.*

For odd $k_t(\psi)$ the quadratic form $v \mapsto \psi_t L_t^{k_t(\psi)}(v,v)$ will be denoted by $\psi_t L_t^{k_t(\psi)}$; I_t denotes the quadratic form $v \mapsto |v|^2$ on $T_{\tilde{u}(t)}U$.

THEOREM 2. *Assume that the control* $\tilde{u}(\cdot)$ *has finite quasiextremality index relative to system* (1). *Then there exists* $\psi \in T^*_{\tilde{x}(1)}M\backslash\{0\}$ *such that for* $t \in (0,1]$ *and* $\psi_t = p_t^*\psi$ *the following relations hold*:

(a) $\psi_t \tilde{f}_t'(\tilde{x}(t)) = 0$.

(b) *If* $k_t(\psi)$ *is finite, then it is odd, and* $(-1)^{(k_t(\psi)+1)/2} L_t^{k_t(\psi)} \geq 0$. *Conversely, if for some family* $\psi_t = p_t^*\psi \neq 0$, $t \in (0,1]$, *relations* a) *and* b) *hold, and if, in additon,* $k_t(\psi) < \infty$ *and* $(-1)^{(k_t(\psi)+1)/2} L_t^{k_t(\psi)} \geq \varepsilon I_t$ *with* $\varepsilon > 0$ *for all* $t \in (0,1]$, *then the control* $\tilde{u}(\cdot)$ *has finite quasiextremality index relative to* (1).

REMARK. The main concepts and results of this paper extend to the case where the set U of control parameters is a "curvilinear polyhedron" rather than a smooth manifold. We expect to give the precise definitions and proofs in a more extensive paper.

All-Union Institute for Scientific and Technical Information
Moscow

Received 22/NOV/84

REFERENCES

1. A. A. Agrachev and R. V. Gamkrelidze, Mat. Sb. **100(142)** (1976), 610–643; English transl. in Math. USSR Sb. **29** (1976).

2. A. A. Agrachev, Mat. Sb. **102(144)** (1977), 551–568; English transl. in Math. USSR Sb. **31** (1977).

3. Arthur J. Krener, SIAM J. Control Optim. **15** (1977), 256–293.

4. A. A. Agrachev, Internat. Congr. Math. (Warsaw, 1982 [1983]): Short Communications (Abstracts). XII, Internat. Congr. Math., Warsaw, 1983, p. 29. (Russian)

Translated by O. HAJEK