Low-Dimensional Control of the 2D Navier–Stokes and Euler Equations

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We consider the Navier–Stokes and Euler equations on the 2-dimensional Riemannian surface $M$ homeomorphic to the sphere, torus or disc. In the last case we assume that $\partial M$ is a piecewise smooth curve and impose Lions boundary condition. The equations written in terms of the vorticity $w$ and the stream functions $\psi$ read:

\[
\frac{\partial w}{\partial t} + \{\psi, w\} - \nu \Delta w = f(t, x), \quad \Delta \psi = w, \tag{1}
\]

\[
0 \leq t \leq T, \ x \in M, \quad \psi|_{\partial M} = w|_{\partial M} = 0,
\]

where $\{\cdot, \cdot\}$ is the Poisson bracket, $\Delta$ the Laplace–Beltrami operator, $\nu$ a nonnegative real number, and the right-hand side $f$ is the vorticity of the external force. We assume that the right-hand side has the form:

\[
f(t, x) = f_0(x) + \sum_{i=1}^{k} v_i(t) f_i(x),
\]

where $f_0, f_1, \ldots, f_k$ are fixed smooth functions and $v_1(\cdot), \ldots, v_k(\cdot)$ are control functions at our
disposal. We assume that $v_i(\cdot)$ belong to the space of admissible controls $V \subset L_\infty[0,T]$ and that $V$ is an everywhere dense vector subspace of $L_1[0,T]$.

Given $\varphi_0 \in H_2(M)$, we say that $\varphi_T \in H_2(M)$ is reachable from $\varphi_0$ if there exist admissible control functions $v_1(\cdot), \ldots, v_k(\cdot)$ such that the solution of system (1) with the initial condition $w(0, \cdot) = \varphi_0$ satisfies the equation $w(T, \cdot) = \varphi_T$. Let $\mathcal{R}(\varphi_0) \subset H_2(M)$ be the set of all reachable functions. We say that the system is $L_2$-approximately controllable if $\mathcal{R}_T(\varphi_0)$ is everywhere dense in $L_2(M)$ for any $\varphi_0 \in H_2(M)$. The system is controllable in finite dimensional projections if the $L_2$-orthogonal projection of $\mathcal{R}_T(\varphi_0)$ on any finite dimensional subspace of $H_2(M)$ is surjective.

The input–state map $S_{\varphi_0} : V^k \to H_2(M)$ sends a control vector-function $(v_1, \ldots, v_k)$ to $w(T, \cdot)$. 
In particular, $R(\varphi) = S_{\varphi_0}(V^k)$. Given a finite dimensional subspace $E$ of $H^2(M)$ we denote by $P_E : L^2(M) \to E$ the orthogonal projector. The system is controllable in finite dimensional projections iff the mapping $P_E \circ S_{\varphi_0}$ is surjective for any $E$ and $\varphi_0$.

Solid controllability in finite dimensional projections is a robust version of the usual one. We say that the mapping $P_E \circ S_{\varphi_0}$ is robustly surjective if for any ball $B$ in $E$ there exists a finite dimensional ball $B$ in $V^k$ such that $\Phi(B) \supset B$ for any sufficiently close to $P_E \circ S_{\varphi_0}|_B$ in $C^0$-topology continuous mapping $\Phi : B \to E$. The system is solidly controllable in finite dimensional projections if $P_E \circ S_{\varphi_0}$ is robustly surjective for any $E$ and $\varphi_0$.

Assume that $f_1, \ldots, f_l$ are steady states of the Euler equation:

$$\{\Delta^{-1}f_i, f_i\} = 0, \quad i = 1, \ldots, l, \quad l \leq k.$$
We denote $D_{f_i} = \{\Delta^{-1} \cdot, f_i\} + \{\Delta^{-1} f_i, \cdot\}$, the operator obtained by the linearization of the Euler equation at the steady state $f_i$.

**Theorem 1.** Let $\mathcal{F}$ be the minimal common invariant subspace of the operators $D_{f_1}, \ldots, D_{f_l}$ which contains $f_1, \ldots, f_k$. If $\mathcal{F}$ is everywhere dense in $L_2(M)$, then the system is $L_2$-approximately controllable and solidly controllable in finite dimensional projections.

In all applications below $f_1, \ldots, f_k$ are eigenfunctions of $\Delta$ and $l = k$.

**Examples.**

1. Torus $S^1 \times S^1$. Eigenfunctions of $\Delta$:

   $$\sin(n_1 x_1 + n_2 x_2), \cos(n_1 x_1 + n_2 x_2),$$

   $n_1, n_2 \in \mathbb{Z}_+$. Take $k = 4$, $\{f_1, \ldots, f_4\} =$

   $\{\sin x_1, \cos x_1, \sin(x_1 + x_2), \cos(x_1 + x_2)\}$. 
2. Square $[0, \pi] \times [0, \pi]$. Eigenfunctions of $\Delta$:

$$\sin(n_1 x_1) \sin(n_2 x_2), \ n_1, n_2 \in \mathbb{Z}_+.$$  

Take $k = 8$,

$$\{f_1, \ldots, f_8\} = \{\sin(n_1 x_1) \sin(n_2 x_2) : n_1, n_2 \leq 3, (n_1, n_2) \neq (3, 3)\}.$$

3. Sphere $S^2$. Eigenfunctions of $\Delta$ are homogeneous harmonic polynomials of 3 variables. Take $k = 5$ and the set $\{f_1, \ldots, f_5\}$ containing three linear, one quadratic and one cubic polynomials.

**Proposition.** Given $k > 0$ assume that for some Riemannian structure on $M \exists$ eigenfunctions $f_1, \ldots, f_k$ of $\Delta$ which satisfy conditions of Theorem 1. Then the eigenfunctions of $\Delta$ with such a property do exist for generic Riemannian structure on $M$. 
Sketch of the proof:

The set of appropriate Riemannian structures is the intersection of a countable number of open subsets in the space of all Riemannian structures. It remains to prove that this is a everywhere dense subset.

Given Riemannian structures \( \mu_0, \mu_1 \), connect them by a continuous family \( \mu_t, \quad 0 \leq t \leq 1 \) that is analytic w. r. t. \( t \) on the interval \((0, 1)\). Then any eigenfunction \( f^0 \) of the Laplace–Beltrami operator \( \Delta_{\mu_0} \) is included in the continuous family \( f^t \) of the eigenfunctions of \( \Delta_{\mu_t}, \quad 0 \leq t \leq 1 \), and the family \( f^t \) is analytic on the interval \((0, 1)\). Let \( f^0_1, \ldots, f^0_k \) be eigenfunctions of \( \Delta_{\mu_0} \); it is not hard to show that the set

\[
\{ t \in [0, 1] : (f^t_1, \ldots, f^t_k) \text{ satisfies Th. 1} \}
\]

is either empty or the complement of a countable subset of \([0, 1]\).
Any homeomorphic to the disc Riemannian surface is isometric to the disc endowed with a Riemannian structure of the form

\[ e^{a(x_1,x_2)}(dx_1^2 + dx_2^2). \]

This Riemannian disc is isometric to a simply connected domain in \( \mathbb{R}^2 \) iff \( \Delta a = 0 \).

The specification of the above proof: take \( \mu_t = e^{at}(dx_1^2 + dx_2^2), \Delta a_t = 0 \). We obtain:

**Proposition.** Given \( k \geq 0 \), assume that for some bounded simply connected domain \( M \subset \mathbb{R}^2 \) there exist eigenfunctions \( f_1, \ldots, f_k \) of \( \Delta \) which satisfy conditions of Theorem 1. Then the eigenfunctions of \( \Delta \) with such a property do exist for generic domain.
Outline of the proof of Th. 1.

The control system: \( \frac{\partial w}{\partial t} + \{\Delta^{-1}w, w\} - \nu \Delta w = \)
\[ = f_0 + \sum_{i=1}^{k} v_i(t) f_i, \quad w(0, \cdot) = \varphi_0. \]

We use fast oscillating control functions \( v_i(t) \). Our method is based on the continuity of the input–state map \( S_{\varphi_0} : V^k \to H_2(M) \) w. r. t. controls endowed with the ‘relaxation norm’
\[
\|v(\cdot)\|_{rx} \overset{\text{def}}{=} \max_{t \in [0, T]} | \int_0^t v(\tau) d\tau |.
\]

We show that controllability of the extended system \( \frac{\partial w}{\partial t} + \{\Delta^{-1}w, w\} - \nu \Delta w = \)
\[ = f_0 + \sum_{i=1}^{k} \left( v_i(t) f_i + \sum_{j=1}^{l} v_{ij}(t) D_{f_j} f_i \right) \]
implies controllability of the original system and then iterate the procedure: substitute \( \{f_i : 1 \leq i \leq k\} \) by \( \{f_i, D_{f_j} f_i : 1 \leq i \leq k, 1 \leq j \leq l\} \) e. t. c.
Induction step.

To simplify notations, we make calculations for the case $l = 1$, $k = 2$.

1. Take Lipschitzian functions $\tilde{v}_1(t), \tilde{v}_2(t)$ and substitute $v_1, v_2$ by $\frac{d\tilde{v}_1}{dt} + v_1$ and $\frac{d\tilde{v}_2}{dt} + v_2$. Let $q = w - \tilde{v}_1 f_1 - \tilde{v}_2 f_2$; then:

$$\frac{\partial q}{\partial t} + \{\Delta^{-1}(q + \tilde{v}_1 f_1 + \tilde{v}_2 f_2), q + \tilde{v}_1 f_1 + \tilde{v}_2 f_2\} - \nu \Delta (q + \tilde{v}_1 f_1 + \tilde{v}_2 f_2) = f_0 + v_1 f_1 + v_2 f_2.$$

Write it slightly differently:

$$\frac{\partial q}{\partial t} + \{\Delta^{-1} q, q\} - \nu \Delta q$$

$$+ \tilde{v}_1 (Df_1 q - \nu \Delta f_1) + \tilde{v}_2 (Df_2 q - \nu \Delta f_2)$$

$$= f_0 + v_1 f_1 + v_2 f_2 - \tilde{v}_1 \tilde{v}_2 Df_1 f_2 - \frac{\tilde{v}_2^2}{2} Df_2 f_2.$$

If $\tilde{v}_1(T) = \tilde{v}_2(T) = 0$, then

$$q_T = S_{\varphi_0} \left(\frac{d\tilde{v}_1}{dt} + v_1, \frac{d\tilde{v}_2}{dt} + v_2\right).$$
2. Substitute \( \hat{v}_i(t) \) by \( \text{sgn}(\sin(t/\varepsilon))\hat{v}_i(t) \), \( \varepsilon \to 0 \); this kills linear terms \( \hat{v}(2D_{f_i}q - \nu \Delta f_i) \) without affecting quadratic terms. We arrive to the system:

\[
\frac{\partial q}{\partial t} + \{\Delta^{-1}q, q\} - \nu \Delta q = 
\]

\[
f_0 + v_1f_1 + v_2f_2 - \hat{v}_1\hat{v}_2D_{f_1}f_2 - \frac{\hat{v}_2^2}{2}D_{f_2}f_2.
\]

Solid controllability of this system implies solid controllability of the original one.

3. Substitute \( \hat{v}_1 \) and \( \hat{v}_2 \) by \( \frac{\hat{v}_1}{\varepsilon} \) and \( \varepsilon \hat{v}_2 \), and set \( v_{12} = -\hat{v}_1\hat{v}_2 \). We obtain:

\[
\frac{\partial q}{\partial t} + \{\Delta^{-1}q, q\} - \nu \Delta q = 
\]

\[
f_0 + v_1f_1 + v_2f_2 + v_{12}D_{f_1}f_2 + O(\varepsilon^2).
\]

Go to the limit as \( \varepsilon \to 0 \). Solid controllability of the limit system implies solid controllability of the original one.