Navier-Stokes Equations: Controllability by Means of Low Modes Forcing

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Abstract

We study controllability issues for 2D and 3D Navier-Stokes (NS) systems with periodic boundary conditions. The systems are controlled by a degenerate (applied to few low modes) forcing. Methods of differential geometric/Lie algebraic control theory are used to establish global controllability of finite-dimensional Galerkin approximations of 2D and 3D NS and Euler systems, global controllability in finite-dimensional projection of 2D NS system and L_2-approximate controllability for 2D NS system. Beyond these main goals we obtain results on boundedness and continuous dependence of trajectories of 2D NS system on degenerate forcing, when the space of forcings is endowed with so called relaxation metric.

Keywords: Navier-Stokes equation, controllability, geometric control

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1 Introduction

In the present paper we deal with 2- and 3- dimensional Navier-Stokes equations (2D and 3D NS systems) with periodic boundary conditions controlled by a nonrandom degenerate forcing

$$\partial u/\partial t + (u \cdot \nabla)u + \nabla p = \nu \Delta u + F(t,x), \tag{1}$$

$$\nabla \cdot u = 0. \tag{2}$$

The word "degenerate" means that $F(t,x)$ is a "low-order" trigonometric polynomial with respect to x, i.e. sum of a "small number" of harmonics: $F(t,x) = \sum_{k \in \mathcal{K}^1} v_k(t)e^{ik\cdot x}$, $\mathcal{K}^1$ is finite. The word "controlled" means that the components $v_k(t)$, $k \in \mathcal{K}^1$, $t \in [0,T]$ of the forcing are controls at our disposal; these are measurable essentially bounded functions. In fact along our presentation the controls are piecewise-continuous.

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Our goal is obtaining controllability results for the 2D and 3D systems (1)-(2) and for their finite-dimensional Galerkin approximations. More precisely we study global controllability for Galerkin approximations of 2D and 3D NS systems with periodic boundary conditions. Besides for the 2D NS system we study so called, controllability in finite-dimensional observed projection and L_2-approximate controllability. Exact definitions and detailed problems setting are provided in the Section 3.

There has been an extensive study of controllability of the Navier-Stokes and Euler equations in particular by means of boundary control. There are various results on exact local controllability of 2D and 3D Navier-Stokes equations obtained by A.Fursikov, O.Imanuilov, global exact controllability for 2D Euler equation obtained by J.-M. Coron, global exact controllability for 2D Navier-Stokes equation by A.Fursikov and J.-M. Coron. We refer the readers to the book [9] and to the surveys [10] and [7] for further references.

Our problem setting differs from the above results by the class of *degenerate distributed controls* which is involved.

The structure of the paper is the following. The problem setting in the Section 3 is preceded by the Section 2 which contains a necessary minimum of standard preliminary material on 2D and 3D NS systems.

Section 4 contains new (as far as we can judge) results on boundedness and continuity of solutions of 2D NS systems with respect to degenerate forcing. The difference with the classical results is in that we endow the space of forcings with rather weak topology determined by so called relaxation metric. This is an initial but important step towards a study of the NS and other classes of evolution PDE subject to relaxed (controlled) forcing. We will provide more comment on this subject elsewhere.

In the Section 5 we collect results and methods from geometric control theory which concern controllability of finite-dimensional nonlinear control systems; most of these results are known. In Sections 6,7 we proceed with application of these methods to finite-dimensional Galerkin approximations of the 2D and 3D NS systems. Global controllability results for the Galerkin approximations are formulated and proven in the section 8. These controllability results are also valid, when $\nu = 0$, i.e. hold for Galerkin approximations of 2D and 3D Euler systems which describe movement of incompressible ideal (inviscid) fluid.

The rest of the paper is devoted to controllability of 2D NS (with $\nu > 0$) system. We derive a sufficient condition (Theorem 9) for global controllability in finite-dimensional observed projection and sufficient condition for global approximate controllability (Theorem 10) for the 2D NS system. Section 11 contains descriptions of so-called "saturating sets" of controlled modes which suffice to guarantee the global approximate controllability. Sections 12 and 13 contain the proofs of the Theorems 9, 10.

Inside the sections the material is organized in relatively small subsections, each containing few results, definitions, notions etc. These latter are numbered by the numbers of corresponding subsection: e.g. Proposition 4.6 means Proposition from the Subsection 4.6 etc.

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2 Preliminaries on Navier-Stokes Systems

2.1 3D Navier-Stokes System: spectral method, Galerkin approximations

Consider $3D$ Navier-Stokes equation with a nonrandom forcing

$$\partial u/\partial t + (u \cdot \nabla)u + \nabla p = \nu \Delta u + v(t,x), \tag{3}$$

under incompressibility condition (2).

We assume the boundary conditions to be periodic, i.e. the domain of definition of the $\mathbb{R}^3$-valued function u to be 3-dimensional torus $\mathbb{T}^3$.

To reduce the equation (3) to an infinite-dimensional system of ODE we use "spectral algorithm" (see [11]). It invokes the Fourier expansion of solution $u(t,x)$ in a series with respect to the basis of eigenfunctions $e^{ik\cdot x}$ of the Laplacian operator on $\mathbb{T}^3$:

$$u(x,t) = \sum_k \underline{q}_k(t) e^{ik\cdot x}, \ k \in \mathbb{Z}^3.$$

Here $\underline{q}_k$ is vector-valued function. For u to satisfy the incompressibility condition the coefficients $\underline{q}_k(t)$ must be orthogonal to respective $k:\ \underline{q}_k \cdot k = 0$.

Similarly we introduce the expansions for the pressure and the forcing:

$$p(x,t) = \sum_k \underline{p}_k(t) e^{ik\cdot x}, \ v(x,t) = \sum_k \underline{v}_k(t) e^{ik\cdot x}, \ k \in \mathbb{Z}^3.$$

We assume that the forcing has zero average ($v_0 \equiv 0$). Then changing the reference frame to the one uniformly moving with the center of mass we may assume $\int u \ dx = 0$ and hence $\underline{q}_0 = 0$. It is known that the pressure term can be separated from equations for $\underline{q}_k$ and these latter can be written in the "ODE form":

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n) \Pi_k \underline{q}_n - \nu |k|^2 \underline{q}_k + \underline{v}_k, \ k,m,n \in \mathbb{Z}^3, \tag{4}$$

where Π_k stays for the orthogonal projection of $\mathbb{R}^3$ onto the plane $k^\perp$ orthogonal to k. Formally we also should take the projection $\Pi_k \underline{v}_k(t)$ of the forcing; instead the k-directed component of $\underline{v}_k$ is taken into account by the pressure term.

Since $u(x,t)$ is real-valued we have to assume: $\bar{\underline{q}}_k = \underline{q}_{-k}$.

Consider any subset $\mathcal{G} \subset \mathbb{Z}^3$ and introduce Galerkin $\mathcal{G}$-approximation of the system (4) by projecting this equation onto the linear space spanned by the harmonics $e^{ik\cdot x}$ with $k \in \mathcal{G}$. It corresponds to keeping in the system (4) only the equations for the variables $\underline{q}_k$ with $k \in \mathcal{G}$ and changing the condition $k,m,n \in \mathbb{Z}^3$ to $k,m,n \in \mathcal{G}$.

The result is a system of ODE or a control system

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n)\Pi_k \underline{q}_n - \nu|k|^2 \underline{q}_k + \underline{v}_k, \ k, m, n \in \mathcal{G}. \tag{5}$$

In the absence of forcing and for zero viscosity ν there is a conservation law for the Galerkin approximation (5).

Lemma 2.1 *The value $\sum_k \langle \underline{q}_k(t), \bar{\underline{q}}_k(t) \rangle$ is constant along each solution of the equation (5) provided that $\nu = 0, \underline{v}_k(t) \equiv 0$.* □

2.2 2D Navier-Stokes system: vorticity, spectral method, Galerkin approximations

We consider the NS system (1)-(2) in 2D case. The boundary conditions are assumed to be periodic, i.e. u is defined on the 2-dimensional torus $\mathbb{T}^2$.

Let us introduce the vorticity

$$w = \nabla^\perp \cdot u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$$

of u. Applying the operator $\nabla^\perp$ to the equation (1) we derive the following equation for w:

$$\partial w/\partial t + (u \cdot \nabla)w = \nu \Delta w + v(t,x), \tag{6}$$

where $v(t,x) = \nabla^\perp \cdot F(t,x)$.

Notice that: i) $\nabla^\perp \cdot \nabla p = 0$, ii) $\nabla^\perp \cdot \Delta u = \Delta(\nabla^\perp \cdot u)$;

$$\text{iii) } \nabla^\perp \cdot (u \cdot \nabla)u = (u \cdot \nabla)(\nabla^\perp \cdot u) + (\nabla^\perp \cdot u)(\nabla \cdot u) = (u \cdot \nabla)w,$$

for all u satisfying (2).

It is known that u satisfying (2) can be recovered in unique way (up to an additive constant) from w.

From now on in the 2D case we will deal with the equation (6). Consider the basis of eigenfunctions $\{e^{ik\cdot x}\}$ of the Laplacian on $\mathbb{T}^2$ and take the Fourier expansion $w(t,x) = \sum_k q_k(t)e^{ik\cdot x}$ and $v(t,x) = \sum_k v_k(t)e^{ik\cdot x}$. As far as w and v are real-valued, we have $\bar{w}_n = w_{-n}, \ \bar{v}_n = v_{-n}$. We assume $q_0 = 0, v_0 = 0$.

Evidently $\partial w/\partial t = \sum_k \dot{q}_k(t)e^{ik\cdot x}$. To compute $(u \cdot \nabla)w$ write the equalities $\nabla^\perp \cdot u = w, \ \nabla \cdot u = 0$ as

$$-\partial_2 u_1 + \partial_1 u_2 = w, \ \partial_1 u_1 + \partial_2 u_2 = 0.$$

From these latter we conclude (after differentiation and summation)

$$\Delta u_2 = \partial_1 w = \sum_k q_k(t)(ik_1)e^{ik\cdot x}, \ \Delta u_1 = -\partial_2 w = -\sum_k q_k(t)(ik_2)e^{ik\cdot x}.$$

Then

$$u_1 = \sum_k q_k(t)(ik_2/|k|^2)e^{ik\cdot x}, \ u_2 = -\sum_k q_k(t)(ik_1/|k|^2)e^{ik\cdot x},$$

and we obtain

$$(u \cdot \nabla)w = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n,$$

where for $m = (m_1, m_2), n = (n_1, n_2)$ the external product $m \wedge n = m_1 n_2 - m_2 n_1$.

The 2D NS system results in infinite-dimensional system of ODE for q_k:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ k, m, n \in \mathbb{Z}^2. \quad (7)$$

For any subset $\mathcal{G} \subset \mathbb{Z}^2$ we introduce Galerkin $\mathcal{G}$-approximation of the system (6) or of the system (7) by projecting them onto the linear space spanned by the harmonics $\{e^{ik \cdot x} |\ k \in \mathcal{G}\}$. The result is a finite-dimensional control system

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ k, m, n \in \mathcal{G}, \quad (8)$$

in $\mathbb{R}^N$ where N is the cardinality of $\mathcal{G}$.

Again for zero viscosity and under lack of forcing one has a conservation law.

Lemma 2.2 *If $\nu = 0$ and all $v_k(t) \equiv 0$, then every solution of the system (8) has constant norm: $\left(\sum_{k \in \mathcal{G}} |q_k(t)|^2\right)^{1/2} \equiv const.$* □

From the previous Lemma one can easily conclude that the "uncontrolled" Galerkin approximation with nonzero viscosity $\nu > 0$ and with all v_k vanishing is dissipative: its solutions tend to the origin exponentially fast.

3 Navier-Stokes equation controlled by degenerate forcing. Problem setting

A natural and standard (see [5, 6]) way to view the NS systems is to represent them as evolution equations in Hilbert spaces.

To introduce these spaces we consider Sobolev spaces $H^\ell(\mathbb{T}^s)$ with the scalar product defined as

$$\langle u, u' \rangle_\ell = \sum_{\alpha \leq \ell} \int_{\mathbb{T}^s} (\partial^\alpha u / \partial x^\alpha)(\partial^\alpha u' / \partial x^\alpha) dx;$$

the norm $\| \cdot \|_\ell$ is defined by virtue of this scalar product.

Denote by H_ℓ the closures of $\{u \in C^\infty(\mathbb{T}^s), \nabla \cdot u = 0\}$ in the norms $\| \cdot \|_\ell$ in the respective spaces $H^\ell(\mathbb{T}^s)$, $\ell \geq 0$. The norms in H_ℓ will be denoted again by $\| \cdot \|_\ell$. We will study the dynamics of the NS systems in the spaces H_0, H_1 and H_2. It will be convenient for us to redefine the norm of H_1 by putting $\|u\|_1^2 = \langle -\Delta u, u \rangle$, and the norm of H_2 by putting $\|u\|_2^2 = \langle -\Delta u, -\Delta u \rangle$.

We are interested in the case where the NS system is forced by a term $v(\cdot)$ which is *degenerate*. This means that $v(t, x) = \sum_{k \in \mathcal{K}^1} v_k e^{ik \cdot x}$, where $\mathcal{K}^1$ is a finite set. The functions $v_k(\cdot)$ with $k \in \mathcal{K}^1$ are controls at our disposal; they can be chosen freely from the space of measurable essentially bounded functions.

We introduce a finite set of *observed* modes indexed by $k \in \mathcal{K}^{obs} \subset \mathbb{Z}^j, \ j = 2, 3$. The observed modes are reunited in so-called observed projection.

We assume $\mathcal{K}^{obs} \supseteq \mathcal{K}^1$. As we will see, nontrivial controllability issues arise only if $\mathcal{K}^1$ is a proper subset of $\mathcal{K}^{obs}$. We identify the space of observed modes with $\mathbb{R}^N$ and denote by Π^{obs} the operator of projection of solutions onto the space $\mathbb{R}^N$ of the observed modes.

We will represent the controlled 2D NS equation in the following split (controlled -observed -nonobserved components) form:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ k \in \mathcal{K}^1, \tag{9}$$

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k, \ k \in \mathcal{K}^{obs} \setminus \mathcal{K}^1, \tag{10}$$

$$\dot{Q} = B(q, Q) + \nu \Delta Q. \tag{11}$$

In the latter equation $\nu \Delta Q$ and $B(q, Q)$ stay for the projections of the dissipative term and of the nonlinear term of the NS system onto the space of unobserved modes.

Galerkin approximation of the 2D NS system consists of the equations (9)-(10) under an additional condition for the summation indices:

$$m, n \in \mathcal{K}^{obs}. \tag{12}$$

In the same way the controlled 3D NS equation can be written in the form:

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n) \Pi_k \underline{q}_n - \nu |k|^2 \underline{q}_k + \underline{v}_k, \ k \in \mathcal{K}^1, \tag{13}$$

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n) \Pi_k \underline{q}_n - \nu |k|^2 \underline{q}_k, \ k \in \mathcal{K}^{obs} \setminus \mathcal{K}^1, \tag{14}$$

$$\dot{Q} = B(q, Q) + \nu \Delta Q, \tag{15}$$

where with abuse of notation we denote again by B and $\nu \Delta Q$ the projections of the nonlinear and the dissipative terms. Galerkin approximation of the 3D NS system consists of the equations (13)-(14) completed by (12).

3.1 Controllability of Galerkin approximations

Definition 3.1 *A Galerkin $\mathcal{K}^{obs}$-approximation of 2D or 3D Navier-Stokes systems is globally controllable if for any two points $\tilde{q}, \hat{q}$ in $\mathbb{R}^N$ there exists $T > 0$ and a control which steers in time T this Galerkin approximation from $\tilde{q}$ to $\hat{q}$. It is time-T globally controllable if T can be chosen the same for all $\tilde{q}, \hat{q}$.* $\square$

3.2 Controllability in observed projection

Definition 3.2 *The (2D or 3D) Navier-Stokes system is globally controllable in observed projection if for any* $\hat{q} \in \mathbb{R}^N$ *and any* $\tilde{\varphi} \in H_1$ *there exists* $T > 0$ *and a control which steers in time* T *the controlled Navier-Stokes system from* $\tilde{\varphi}$ *to some* $\hat{\varphi}$ *with* $\Pi^{obs}(\hat{\varphi}) = \hat{q}$. *The system is time-*$T$ *globally controllable if one can choose* T *the same for all* $\hat{q}, \tilde{\varphi}$. □

Remark 3.2 *In other words the system is globally controllable in observed projection if its attainable set (from each point) is projected by* Π^{obs} *onto the whole space of observed variables.* □

On the contrast with the previous definition the evolution of finite-dimensional projection of trajectories is affected by the infinite-dimensional dynamics (11) or (15) correspondingly.

3.3 L_2-approximate controllability

Definition 3.3 *The 2D Navier-Stokes system is time* T *globally* L_2*-approximately controllable, if for any two points* $\tilde{\varphi}, \hat{\varphi} \in H_1$ *and any* $\varepsilon > 0$ *there exists* $T > 0$ *and a control which steers in time* T *the controlled NS system from* $\tilde{\varphi}$ *to the* ε*-neighborhood of* $\hat{\varphi}$ *in the norm* $\| \cdot \|_0$ *(or, the same, in the norm* L_2*).* □

3.4 Problem setting

In the present paper the following questions are addressed.

Question 1. Under what conditions the finite-dimensional Galerkin approximations (9)-(10)-(12) and (13)-(14)-(12) of the 2D and 3D NS systems are globally controllable? □

Question 2. Under what conditions the systems (9)-(10)-(11) and (13)-(14)-(15) are globally controllable in observed projections? □

Question 3. Under what conditions the 2D and 3D NS systems are globally L_2-approximately controllable? □

Below we manage to answer the Question 1 for 2D and 3D controlled NS systems and the Questions 2,3 for 2D controlled NS system.

4 Relaxation of forcing for 2D NS systems: continuity and boundedness results

In this Section we establish some results on continuity and boundedness of solutions of degenerately forced 2D NS system with respect to the forcing. We assume the space of forcings to be endowed with a weak topology determined by so-called relaxation metric. These results, will be used below in Section 12 for proving controllability in observed projection; besides they provide some insight on application of relaxed controls for NS and other classes of PDE systems. We will comment more on this subject elsewhere.

4.1 Relaxation metric

Definition 4.1 *(see e.g. [12]) The relaxation pseudometric in the space* $L^1\left([0,T],\mathbb{R}^d\right)$ *of Lebesgue integrable functions* $u(t)$ *is defined by the seminorm*

$$\|u(\cdot)\|_{rx} = \max_{t\in[0,T]}\left\{\left\|\int_0^t u(\tau)d\tau\right\|_{R^d}\right\}.$$

The relaxation metric is obtained by identification of functions whose difference vanishes for almost all $\tau \in [0,T]$. □

4.2 Forcing/trajectory and forcing/observation maps

Definition 4.2 *Let us fix initial condition for trajectories of the controlled 2D or 3D NS system.*

The correspondence between the forcing (control) - $v(\cdot)$*, which can be treated as* $\mathbb{R}^d$*-valued function, and the corresponding trajectory (solution) of the NS system is established by forcing/trajectory map (*$\mathcal{F}/\mathcal{T}$*-map).*

The correspondence between the forcing (control) $v(\cdot)$ *and the observed projection* $q(t)$ *(an* $\mathbb{R}^N$*-valued function) of the corresponding trajectory is established by forcing/observation map (*$\mathcal{F}/\mathcal{O}$*-map).*

If NS system is considered on interval $[0,T]$*, then the map* $\mathcal{F}/\mathcal{O}_T : v(\cdot) \mapsto q(T)$ *will be called end-point map.* □

Remark 4.2 *i) In the terminology of control theory the first two maps would be called input/trajectory and input/output maps correspondingly.*

ii) Evidently time-T *global controllability of the NS system in observed projection is the same as surjectiveness of the end-point map* $\mathcal{F}/\mathcal{O}_T$. □

All the results we formulate and prove in this section concern 2D NS systems controlled by degenerate forcing. The time interval $[0,T]$ is supposed to be finite.

4.3 Boundedness properties of forcing/trajectory map

Consider a set **Forc** of degenerate forcings $v(t,x) = \sum_{k\in\mathcal{K}^1} v_k(t)e^{ik\cdot x}$, which we identify with $v(t) = (v_k(t)) \in L_\infty([0,T];\mathbb{R}^d)$; the index k varies in $\mathcal{K}^1$, $d = \sharp\mathcal{K}^1$. The forced 2D NS system is treated as an evolution equation in H_1.

Proposition 4.3 *Assume the set* **Forc** *of degenerate forcings to be bounded in the relaxation metric. Fix the time interval* $[0,T]$ *and the initial condition* $w(0) = w_0 \in H_1$ *for the 2D NS system. Then the forcing/trajectory map, restricted to* **Forc***, is bounded:* $\exists b$ *such that for the corresponding trajectories* w_t *of the 2D NS equation (6) there holds* $vrai\sup_{t\in[0,T]}\|w_t\|_1 \leq b$. □

The previous result can not be reduced to classical results on boundedness of solutions of 2D NS systems because the set of forcings can be bounded in the relaxation metric while being unbounded in L_∞ or L_2 metric.

Example 4.3. Consider a family of fast oscillating functions with large magnitudes: $v(t;\omega) = \omega^{1/2}\cos\omega t,\ \omega > 0$. Obviously $\|v(t;\omega)\|_{L_\infty[0,T]} = \omega^{1/2}$ and $\|v(t;\omega)\|_{L_2[0,T]} = \omega^{1/2}\sqrt{T/2} + o(\omega^{1/2})$, as $\omega \to +\infty$, i.e. the family is unbounded in both L_∞ and L_2-norms. Still this family is bounded in the relaxation metric and even tends to 0 as $\omega \to +\infty$, as far as the primitives $V(t;\omega) = \omega^{-1/2}\sin\omega t$ tend uniformly to 0. □

We will need another property which concerns equiboundedness of the variations (with respect to the time variable t) of the solutions w_t with respect to the time variable t. It is more convenient to deal with (stronger) equiboundedness of $\int_0^T \|\dot{w}_t\|_0^2 dt$.

If a set of degenerate forcings is bounded, say, in L_1-metric, then one can derive equiboundedness of the variations of the solutions w_t from classical results on NS systems (e.g. see [5]). If the set **Forc** of degenerate forcings is just bounded in the relaxation metric, and the variations of the primitives V_t are not equibounded, it is hard to expect the corresponding solutions w_t to have bounded variations. In fact it is not true. Nevertheless subtracting from the solution w_t the corresponding primitive V_t of the forcing (we identify V_t with the sum $\sum_{k\in\mathcal{K}^1} V_k(t)e^{ik\cdot x}$) we end up with functions y_t, whose variations are equibounded. This is the contents of the following Lemma, in which $\mathcal{J}$ stays for primitivization: $\mathcal{J} : v(\cdot) \mapsto V_\cdot = \int_0^\cdot v(\tau)d\tau$.

Lemma 4.3 *Assume* **Forc** *to be bounded in the relaxation metric. Fix the time interval* $[0,T]$ *and the initial condition* $w(0) = w_0 \in H_1$ *for the 2D NS system. Then the image of the map* $(\mathcal{F}/\mathcal{T} - \mathcal{J})$*, restricted to* **Forc***, consists of functions with equibounded variations with respect to t:* $\exists b'$ *such that for each* $v(\cdot) \in$ **Forc***, for its primitive* $V_t = \mathcal{J}v$*, and for the corresponding trajectory* w_t *of the 2D NS system there holds:* $\int_0^T \|\frac{\partial}{\partial t}(w_t - V_t)\|_0 dt \leq b'$*. Moreover* $\exists b$ *such that* $\int_0^T \|\frac{\partial}{\partial t}(w_t - V_t)\|_0^2 dt \leq b$*.* □

4.4 Continuity of the forcing/trajectory map

Here we establish some continuity properties of the forcing/observation and the forcing/trajectory maps for the space of degenerate forcing endowed with relaxation metric. Recall that we identify the observed projections with N-dimensional vectors.

Theorem 4.4 *Let the set* **Forc** *of (degenerate) forcings be bounded subset of* $L_\infty([0,t];\mathbb{R}^d)$*. Endow this set with the relaxation metric and endow the space of trajectories of the 2D NS equation with* $L_\infty((0,T);H_1)$*-metric. Then the restriction of the forcing/trajectory map onto* **Forc** *is continuous.* □

Corollary 4.4 *Under the conditions of the previous Theorem the forcing/observation map is continuous with respect to the relaxation metric in the space of forcings and the* C^0*-metric in the space of* $\mathbb{R}^N$*-valued observed projections.* □

For *finite-dimensional* systems the continuity of forcing/trajectory map in the metrics, mentioned in the previous Corollary, is known; see e.g. [12].

4.5 Proof of the boundedness results

Proof of the Proposition 4.3. By our assumptions the set of the primitives $V(\cdot) = \int_0^{\cdot} v(\tau)d\tau$ is bounded in the metric of $C^0([0,T],\mathbb{R}^d)$. Denote $\sum_{k\in\mathcal{K}^1} V_k(t)e^{ik\cdot x}$ by $V_t(x)$. The forced 2D NS system can be written as

$$\partial w_t/\partial t - \nu\Delta w_t = (u_t\cdot\nabla)w_t + \partial V_t/\partial t, \tag{16}$$

where

$$\nabla^\perp \cdot u_t = w_t,\ \nabla\cdot u_t = 0. \tag{17}$$

For some constant $c>0$ there holds:

$$\|u_t\|_1 \leq c\|w_t\|_0,\ \|u_t\|_2 \leq c\|w_t\|_1.$$

As far as $V_t(x)$ are trigonometric polynomials in x, then for some $B>0$ and for all $v(\cdot)\in$ **Forc** (and for respective $V_t(x)$):

$$\sup_{t\in[0,T]} \|V(t,\cdot)\|_2 \leq B. \tag{18}$$

Let $y_t = w_t - V_t$. From the two previous estimates we conclude for some $c'>0$:

$$\|u_t\|_1 \leq c'(\|y_t\|_0+1),\ \|u_t\|_2 \leq c'(\|y_t\|_1+1). \tag{19}$$

The equation (16) can be rewritten as:

$$\partial y_t/\partial t - \nu\Delta(y_t+V_t) = (u_t\cdot\nabla)(y_t+V_t). \tag{20}$$

We are going to estimate $\|y_t\|_0$ and $\|y_t\|_1$.

Multiplying both sides of the equation (20) by y_t in H_0 we obtain:

$$\langle y_t, \partial y_t/\partial t\rangle + \nu\langle -\Delta y_t, y_t\rangle = \nu\langle \Delta V_t, y_t\rangle + \langle (u_t\cdot\nabla)y_t, y_t\rangle + \langle (u_t\cdot\nabla)V_t, y_t\rangle. \tag{21}$$

The first and the second summands in the left-hand side of (21) equal $\frac{1}{2}\frac{\partial}{\partial t}\|y_t\|_0^2/2$ and $\nu\|y_t\|_1^2$ correspondingly.

The summand $\nu\langle\Delta V_t, y_t\rangle$ in (21) admits an upper estimate:

$$\nu|\langle \Delta V_t, y_t\rangle| \leq \frac{\nu}{2}\|\Delta V_t\|_0^2 + \frac{\nu}{2}\|y_t\|_0^2 \leq \frac{\nu}{2}\|V_t\|_2^2 + \frac{\nu}{2}\|y_t\|_0^2 \leq \frac{\nu}{2}B^2 + \frac{\nu}{2}\|y_t\|_0^2.$$

We used (18) to arrive to the concluding inequality.

The summand $\langle (u_t\cdot\nabla)y_t, y_t\rangle$ in (21) is known to vanish, while the summand $\langle (u_t\cdot\nabla)V_t, y_t\rangle$ admits an upper estimate (see [6, Section 6]):

$$|\langle (u_t\cdot\nabla)V_t, y_t\rangle| \leq c\|u_t\|_1\|\nabla V_t\|_1\|y_t\|_0.$$

By (18) $\|\nabla V_t\|_1 \leq B$. Involving (19) we conclude:

$$|\langle (u_t\cdot\nabla)V_t, y_t\rangle| \leq c'B(\|y_t\|_0+1)\|y_t\|_0 \leq c_2(\|y_t\|_0^2+1),$$

where c_2 is a properly chosen constant.

Substituting these estimates into (21) we arrive to the inequality

$$(\partial/\partial t)\|y_t\|_0^2 + \nu\|y_t\|_1^2 \leq a\|y_t\|_0^2 + b,$$

and after integration on an interval $[0, \tau] \subseteq [0, T]$ conclude with the inequality:

$$\|y_\tau\|_0^2 - \|y_0\|_0^2 + \nu \int_0^\tau \|y_t\|_1^2 dt \leq a \int_0^\tau \|y_t\|_0^2 dt + b\tau. \tag{22}$$

From (22) we derive

$$\|y_\tau\|_0^2 \leq \|y_0\|_0^2 + a \int_0^\tau \|y_t\|_0^2 dt + bT.$$

By application of the Gronwall inequality we conclude:

$$\|y_t\|_0^2 \leq \left(\|y_0\|_0^2 + bT\right) e^{at}, \tag{23}$$

which proves the equiboundedness of $\|y_t\|_0$. To conclude the boundedness of $\|w_t\|_0 = \|y_t + V_t\|_0$ it suffices to observe that by (18) $\|V_t\|_0$ are equibounded for all forcings from **Forc** and all $t \in [0, T]$.

Coming back to (22) we conclude:

$$\nu \int_0^\tau \|y_t\|_1^2 dt \leq a \int_0^\tau \|y_t\|_0^2 dt + bT + \|y_0\|_0^2. \tag{24}$$

Hence for some $A > 0$:

$$\int_0^\tau \|y_t\|_1^2 dt \leq A, \tag{25}$$

as long as $v \in$ **Forc** and $\tau \in [0, T]$.

To arrive to an upper estimate for $\|y_t\|_1$ let us multiply both sides of the equation (20) by $(-\Delta y_t)$ in H^0 obtaining

$$\langle -\Delta y_t, \dot{y}_t \rangle + \nu\|\Delta y_t\|_0^2 = -\nu\langle \Delta V_t, \Delta y_t \rangle - \langle (u_t \cdot \nabla)V_t, \Delta y_t \rangle - \langle (u_t \cdot \nabla)y_t, \Delta y_t \rangle. \tag{26}$$

In the left-hand side of (26) $\langle -\Delta y_t, \dot{y}_t \rangle$ can be substituted by $\frac{1}{2}\frac{\partial}{\partial t}\|y_t\|_1^2$.

At the right-hand side of (26) the summand $\nu\langle \Delta V_t, \Delta y_t \rangle$ admits an estimate:

$$|\nu\langle \Delta V_t, \Delta y_t \rangle| \leq \frac{\nu}{2}\|\Delta y_t\|_0^2 + \frac{\nu}{2}\|\Delta V_t\|_0^2 \leq \frac{\nu}{2}\|\Delta y_t\|_0^2 + \frac{\nu B^2}{2}. \tag{27}$$

The summand $\langle (u_t \cdot \nabla)V_t, \Delta y_t \rangle$ at the right-hand side of (26) admits an estimate (see [6, Section 6]) for some choice of constants $\bar{c}, c''$:

$$|\langle (u_t \cdot \nabla)V_t, \Delta y_t \rangle| \leq \bar{c}\|u_t\|_1 \|\nabla V_t\|_1 \|\Delta y_t\|_0 \leq \frac{\bar{c}^2}{\nu}\|u_t\|_1^2 \|\nabla V_t\|_1^2 +$$

$$+(\nu/4)\|\Delta y_t\|_0^2 \leq \frac{\bar{c}^2 B^2}{\nu}(\|y_t\|_0 + 1)^2 + (\nu/4)\|\Delta y_t\|_0^2 \leq c'' + (\nu/4)\|\Delta y_t\|_0^2.$$

To arrive to the concluding inequality uniform boundedness of $\|y_t\|_0$ is used.

The summand $\langle (u_t \cdot \nabla) y_t, \Delta y_t \rangle$ at the right-hand side of (26) admits an upper estimate:

$$|\langle (u_t \cdot \nabla) y_t, \Delta y_t \rangle| \leq a \|u_t\|_2 \|\nabla y_t\|_0 \|\Delta y_t\|_0 \leq a'(\|y_t\|_1^2 + 1)\|\Delta y_t\|_0 \leq$$
$$\leq \frac{2a'}{\nu}(\|y_t\|_1^2 + 1)^2 + (\nu/8)\|\Delta y_t\|_0^2 \leq a''(\|y_t\|_1^4 + 1) + (\nu/8)\|\Delta y_t\|_0^2,$$

for some choice of constants a, a', a''.

Substituting these estimates into (26) and integrating the resulting inequality on an interval $[0, \tau] \subseteq [0, T]$ we obtain for some $C > 0$:

$$\|y_\tau\|_1^2 - \|y_0\|_1^2 + \nu \int_0^\tau \|\Delta y_t\|_0^2 dt \leq C + C \int_0^\tau \|y_t\|_1^4 dt + \frac{7}{8}\nu \int_0^\tau \|\Delta y_t\|_0^2 dt. \quad (28)$$

Hence for some $C' > 0$:

$$\|y_\tau\|_1^2 \leq C' + C \int_0^\tau \|y_t\|_1^4 dt, \quad \text{or,} \quad \|y_\tau\|_1^2 \leq C' + C \int_0^\tau \gamma_t \|y_t\|_1^2 dt,$$

where $\gamma_t = \|y_t\|_1^2$. By the Gronwall inequality:

$$\|y_\tau\|_1^2 \leq C' e^{C \int_0^\tau \gamma_t dt}.$$

By (25) $\int_0^\tau \gamma_t dt = \int_0^\tau \|y_t\|_1^2 dt \leq A$ uniformly. Then

$$\|y_\tau\|_1^2 \leq C' e^{CA}. \square$$

Proof of the Lemma 4.3. To prove the equiboundedness of the variations of $y_t = w_t - V_t$ let us return to the equation (20) and multiply it by $(\partial/\partial t) y_t = \dot{y}_t$ in H_0. We obtain:

$$\|\dot{y}_t\|_0^2 + \nu \langle -\Delta y_t, \dot{y}_t \rangle = \nu \langle \Delta V_t, \dot{y}_t \rangle + \langle (u_t \cdot \nabla) y_t, \dot{y}_t \rangle + \langle (u_t \cdot \nabla) V_t, \dot{y}_t \rangle. \quad (29)$$

The summand $\nu \langle -\Delta y_t, \dot{y}_t \rangle$ at the left-hand side of (29) equals $\nu \frac{\partial}{\partial t}(\|y_t\|_1)^2$.

At the right-hand side of (29) the summand $\nu \langle \Delta V_t, \dot{y}_t \rangle$ admits an upper estimate

$$|\nu \langle \Delta V_t, \dot{y}_t \rangle| \leq \left(\nu^2 \|\Delta V_t\|_0^2/2 + \|\dot{y}_t\|_0^2/2\right) \leq \frac{\nu^2 B^2}{2} + \|\dot{y}_t\|_0^2/2.$$

To estimate the other two summands at the right-hand side of (29) we invoke once more the results of [6, Section 6] together with (19). We obtain:

$$|\langle (u_t \cdot \nabla) y_t, \dot{y}_t \rangle| \leq b \|u_t\|_2 \|\nabla y_t\|_0 \|\dot{y}_t\|_0 \leq$$
$$\leq c'^2 b^2 (\|y_t\|_1 + 1)^2 \|\nabla y_t\|_0^2 + \|\dot{y}_t\|_0/4 \leq b'' + \|\dot{y}_t\|_0^2/4.$$

(In the concluding inequality equiboundedness of $\|y_t\|_1$ is used.) Also:

$$|\langle (u_t \cdot \nabla) V_t, \dot{y}_t \rangle| \leq \beta \|u_t\|_1 \|\nabla V_t\|_1 \|\dot{y}_t\|_0 \leq \beta' B(\|y_t\|_0 + 1)\|\dot{y}_t\|_0 \leq \beta'' + \|\dot{y}_t\|_0^2/8,$$

for a proper choice of β, β', β''; here we use equiboundedness of $\|y_t\|_0$.

Substituting these estimates into (29) and integrating the resulting inequality on $[0, T]$ we conclude for some $C > 0$:

$$\int_0^T \|\dot{y}_t\|_0^2 dt + \nu\|y_T\|_1^2 - \nu\|y_0\|_1^2 \leq \int_0^T \left(C + (7/8)\|\dot{y}_t\|_0^2\right) dt,$$

from which we derive

$$(1/8) \int_0^T \|\dot{y}_t\|_0^2 dt \leq \nu\|y_0\|_1^2 + CT. \ \square \tag{30}$$

The following statement is an obvious corollary of the proofs of the Proposition 4.3 and Lemma 4.3 (see (23) and (30)). It will be used in Section 12.

Corollary 4.5 *Let the assumptions of the Proposition 4.3 hold. Then the solutions of the equation (20) are equibounded:*

$$\exists b: \sup_{t \in [0,T]} \|y_t\|_0 \leq b. \tag{31}$$

Besides if the assumptions of the Lemma 4.3 hold then these solutions have equibounded variations with respect to t, moreover:

$$\exists b', c': \int_0^T \|\dot{y}_t\|_0^2 dt \leq b'\|y_0\|_1^2 + c'T. \ \square \tag{32}$$

4.6 Proof of the continuity results via Lyapunov-Schmidt type reduction

To prove the Theorem 4.4 and the Corollary 4.4 we will use a method which reduces the whole study to a finite-dimensional situation. Following [15] we call this method Lyapunov-Schmidt type reduction.

We will consider the forced 2D NS system as an evolution equation in the Hilbert space H_1. Take the orthogonal splitting of H_1 into the sum of finite dimensional subspace $\mathcal{L}_N$ corresponding to first N harmonic modes and of the infinite-dimensional orthogonal complement $\mathcal{L}_N^{\perp}$ containing higher modes. These spaces will be coordinatized by q^N and Q^N correspondingly. We assume that the observed projection and certainly also the forced component are contained in $\mathcal{L}_N$.

Let us represent the 2D NS equation in the following concise form:

$$\dot{q}^N = f(q^N, Q^N, \nabla Q^N) + v(t), \tag{33}$$
$$\dot{Q}^N = \nu\Delta Q^N + B(q^N, Q^N). \tag{34}$$

The initial condition is fixed.

In the equation (33) f stays for the projection of the right-hand side of the unforced NS system onto $\mathcal{L}_N$ and all but the first κ_1 components of the forcing

$v(t)$ vanish identically. In the equation (34) linear operator Δ and the nonlinear operator B are projections onto $\mathcal{L}_N^{\perp}$ of the Laplacian and of the nonlinear term of the NS equation correspondingly.

The idea of the Lyapunov-Schmidt type reduction is in proving that for sufficiently large N the equation (34) can be uniquely solved with respect to Q^N and that the "implicit function" $q^N(\cdot) \mapsto Q^N(\cdot)$ is "Lipschitzian" in the sense of the Proposition which follows below.

From classical results on boundedness of solutions of 2D NS systems we know that if $v(\cdot)$ are chosen from some ball in $L_\infty([0,T];\mathbb{R}^{\kappa_1})$ then the solutions of the 2D NS system are bounded by some ball $\Omega \subset L_\infty((0,T);H_1)$. Denote by Ω_q^N, Ω_Q^N the orthogonal projections of this ball onto the spaces $L_\infty((0,T);\mathcal{L}_N)$ and $L_\infty((0,T);\mathcal{L}_N^{\perp})$ correspondingly.

Proposition 4.6 *For sufficiently large N the equation (34) defines the unique "implicit function"* $q^N(\cdot) \mapsto Q^N_{\cdot}$ *defined on* $\Omega_q^N \bigcap C^0((0,T);\mathbb{R}^N)$*. Its range is* $\Omega_Q^N \bigcap L_\infty((0,T);H_1)$*. This "implicit function" is Lipschitzian in the following sense:*

$$\forall q_2^N(\cdot), q_1^N(\cdot): \ \|Q_t^N(q_2^N(\cdot)) - Q_t^N(q_1^N(\cdot))\|_{H_1} \leq \\ \leq \ell \left\| \left(q_2^N(\cdot) - q_1^N(\cdot)\right)|_{[0,t]} \right\|_{C^0} = \ell \sup_{\tau \in [0,t]} |q_2^N(\tau) - q_1^N(\tau)|. \ \square \tag{35}$$

Assuming the claim of the Proposition 4.6 to hold we will now complete the proofs of the Theorem 4.4 and the Corollary 4.4. To simplify the notation we write q, Q in place of q^N, Q^N.

Let us substitute into the equation (33) the implicit function $Q_{\cdot}(q(\cdot))$, defined by the previous Proposition. Transforming the resulting equation into integral form we obtain the integral-functional equation

$$q(t) = q^0 + \int_0^t f\left(q(\tau), Q_\tau\left(q(\cdot)\right), \nabla Q_\tau\left(q(\cdot)\right)\right) d\tau + \int_0^t v(\tau) d\tau. \tag{36}$$

Denote $\int_0^t v(\tau)d\tau$ by $V(\tau)$.

Consider two solutions $q''(\cdot), q'(\cdot)$ of the equation (36) which correspond to V_t'', V_t'. Denote $\delta q(\cdot) = q''(\cdot) - q'(\cdot), \delta V(\cdot) = V''(\cdot) - V'(\cdot)$ and let

$$\zeta(t) = \sup_{\tau \in [0,t]} \|\delta q(\tau)\|.$$

The equation for $\delta q(\cdot)$ is:

$$\delta q(t) = \delta V(t) + \\ \int_0^t \left(f\left(q''(\tau), Q_\tau\left(q''(\cdot)\right), \nabla Q_\tau\left(q''(\cdot)\right)\right) - \right. \\ \left. - f\left(q'(\tau), Q_\tau\left(q'(\cdot)\right), \nabla Q_\tau\left(q'(\cdot)\right)\right)\right) d\tau. \tag{37}$$

The vector-function f is polynomial, or more precisely, linear+quadratic with respect to the components of $q(\tau), Q\left(q(\tau)\right), \nabla Q\left(q(\tau)\right)$. As far as $q(\cdot) \in$

$\Omega_q, Q. \in \Omega_Q$, these components are bounded in H_1-norm. By (35) $Q(q(\cdot))$ is Lipschitzian and we can estimate the difference under the integral in (37) as:

$$\begin{aligned} &\| f\left(q''(\tau), Q_\tau\left(q''(\cdot)\right), \nabla Q_\tau\left(q''(\cdot)\right)\right) - \\ &- f\left(q'(\tau), Q_\tau\left(q'(\cdot)\right), \nabla Q_\tau\left(q'(\cdot)\right)\right) \| \leq L''\zeta(t). \end{aligned} \tag{38}$$

Hence from (37) we arrive to the inequality

$$\zeta(t) \leq L'' \int_0^t \zeta(\tau) d\tau + \delta V(t).$$

Applying the Gronwall inequality we conclude

$$\zeta(t) \leq L'' \int_0^t e^{L''(t-\tau)} \delta V(\tau) d\tau + \delta V(t). \tag{39}$$

Obviously this implies continuity of the map $V(\cdot) \mapsto q(\cdot)$ in C^0-metrics and hence the continuity of the map $v(\cdot) \mapsto q(\cdot)$ in the relaxation metric for $v(\cdot)$. Thus we complete the proof of the Corollary 4.4.

As long as $\|Q_\tau\left(q''(\cdot)\right) - Q_\tau\left(q'(\cdot)\right)\|$ can be estimated via $\zeta(t)$ according to (35), while $\zeta(t)$ in its turn can be estimated via $\delta V(\tau)$ according to (39), we arrive to the conclusion of the Theorem 4.4. □

4.7 Proof of the Proposition 4.6

The proof of the solvability of the equation (34) and of the Lipschitzian property of the corresponding implicit function is based on a variant of fixed point theorem for contractions.

Theorem 4.7 (fixed point theorem: parametric version) *Let X, Σ be metric spaces and X be complete. Let $T : X \times \Sigma \to X$ be a map which is uniform contraction with respect to the first argument:*

$$\exists \beta < 1: \ \rho_X\left(T(x^2, \sigma), T(x^1, \sigma)\right) \leq \beta \rho_X(x^2, x^1), \ \forall \sigma \in \Sigma,$$

and uniformly Lipschitzian with respect to the second argument:

$$\exists L \geq 0: \ \rho_X\left(T(x, \sigma^2), T(x, \sigma^1)\right) \leq L \rho_\Sigma(\sigma^2, \sigma^1), \ \forall x \in X.$$

Then for each $\sigma \in \Sigma$ the map T possess a unique fixed point $x_\sigma : \ T(x_\sigma, \sigma) = x_\sigma$, and the map $\sigma \mapsto x_\sigma$ is Lipschitzian. □

One can find the 'parametric version' of standard fixed point theorem, for example, in [12] where T is assumed to be continuous with respect to the parameter σ and then the dependence $\sigma \mapsto x_\sigma$ is proven to be continuous as well. The proof of the existence and uniqueness of fixed point is standard. To prove the Lipschitzian property evaluate $\rho_X(x_{\sigma^2}, x_{\sigma^1})$ as follows:

$$\begin{aligned} \rho_X(x_{\sigma^2}, x_{\sigma^1}) &= \rho_X\left(T(x_{\sigma^2}, \sigma^2), T(x_{\sigma^1}, \sigma^1)\right) \leq \\ \leq \rho_X\big(T(x_{\sigma^2}, \sigma^2), T(x_{\sigma^1}, \sigma^2)\big) &+ \rho_X\left(T(x_{\sigma^1}, \sigma^2), T(x_{\sigma^1}, \sigma^1)\right) \leq \\ &\leq \beta \rho_X(x_{\sigma^2}, x_{\sigma^1}) + L \rho_\Sigma(\sigma^2, \sigma^1). \end{aligned}$$

Rearranging terms we obtain

$$(1-\beta)\rho_X(x_{\sigma^2}, x_{\sigma^1}) \leq L\rho_\Sigma(\sigma^2, \sigma^1),$$

which implies the Lipschitzian property.

To reduce our argument to the above formulated theorem let us introduce a map $\mu : (Q_\cdot, q(\cdot)) \mapsto Q'_\cdot$, where $Q'_\cdot$ is the solution of the linear PDE

$$\dot{Q}'_t = \nu(\Delta Q')_t + B(q+Q)_t, \tag{40}$$

and $q(\cdot)$ is treated as a functional parameter. We assume the initial (for $t=0$) condition for Q'_t and Q_t to be the same and fixed.

Evidently $Q_\cdot$ is (for a given $q(\cdot)$) the unique solution of the equation (34) if and only if it is the unique "fixed point" of the map μ.

The following Proposition claims that for sufficiently large N μ is a uniform contraction with respect to the pair $(Q_\cdot, q(\cdot))$. It is even more than is needed for the assumptions of the Theorem 4.7 to hold.

Proposition 4.7 *For sufficiently large N the map $Q_\cdot \mapsto \mu(Q_\cdot, q(\cdot))$ maps Ω_Q^N into itself and is a contraction with respect to the pair $(Q_\cdot, q(\cdot)) \in \Omega_Q^N \times \Omega_q^N$:*

$$\exists \beta < 1 : \sup_{t\in[0,T]} \|\mu\left(Q^2_\cdot, q(\cdot)\right)|_t - \mu\left(Q^1_\cdot, q(\cdot)\right)|_t\|_1 \leq$$
$$\leq \beta \sup_{\tau\in[0,t]} \left(\|q^2(\tau) - q^1(\tau)\| + \|Q^2_\tau - Q^1_\tau\|_1\right). \ \square$$

4.8 Proof of the Proposition 4.7

We start with two technical results which provide some estimates for solutions of linear PDE.

Consider the equation:

$$\dot{\eta}_t = \nu(\Delta\eta)_t + f_t. \tag{41}$$

Let the initial data η_0 and f_t belong to $\mathcal{L}_N^\perp$. Then η_t remains in $\mathcal{L}_N^\perp$ for all $t \geq 0$. Besides the following fact holds

Lemma 4.8 *Let λ_N be the minimal eigenvalue of the restriction of $-\Delta$ onto the subspace $\mathcal{L}_N^\perp$. Then*

$$\left(\|(-\Delta\eta)_t\|_0\right)^2 \geq \lambda_N \left(\|\eta_t\|_1\right)^2. \tag{42}$$

In addition $\lim_{N\to+\infty} \lambda_N = +\infty$. $\square$

Assuming that $t \mapsto \|f_t\|_0$ is of class L_∞ on $[0,T]$ let us estimate $\|\eta_t\|_1$.

Proposition 4.8 *The solution of the equation (41) admits an estimate*

$$\|\eta_t\|_1^2 \leq \|\eta_0\|_1^2 e^{-\nu\lambda_N t} + t(f_t^\infty)^2/(\nu^2\lambda_N), \tag{43}$$

where $f_t^\infty = \sup_{\tau\in[0,t]} \|f_\tau\|_0$. $\square$

Proof of the Proposition 4.8. Multiplying both parts of the equation (41) by $-\Delta\eta_t$ and rearranging the terms we obtain

$$(1/2)d\left(\|\eta_t\|_1\right)^2/dt + \nu\left(\|(\Delta\eta)_t\|_0\right)^2 = \langle f_t, (-\Delta\eta)_t\rangle.$$

By Young inequality

$$|\langle f_t, (\Delta\eta)_t\rangle| \leq \nu\left(\|(\Delta\eta)_t\|_0\right)^2/2 + \left(\|f_t\|_0\right)^2/2\nu,$$

and hence we obtain

$$d\left(\|\eta_t\|_1\right)^2/dt = -\nu\left(\|(\Delta\eta)_t\|_0\right)^2 + \left(\|f_t\|_0\right)^2/\nu. \tag{44}$$

Substituting (42) in (44) we conclude

$$d\|\eta_t\|_1^2/dt + \nu\lambda_N\|\eta_t\|_1^2 \leq \|f_t\|_0^2/\nu. \tag{45}$$

We derive (43) from (45) by application of the Gronwall inequality .

Observe that taking $\eta_0 = 0$ in (43) we end up with the estimate:

$$\|\eta_t\|_1 \leq t^{1/2}(f_t^\infty)/(\nu\sqrt{\lambda_N}).\ \square \tag{46}$$

Returning to the *proof of the Proposition 4.7* recall that the map μ is defined via the linear PDE (40).

The equation (40) is a particular case of the equation (41) with $\eta_t = Q'_t$ and $f_t = B(q+Q)_t$, where B is the projection of the nonlinear term of the 2D NS system onto $\mathcal{L}_N^\perp$. The estimate (43) holds and we conclude:

$$\|Q'_t\|_1^2 \leq \|Q'_0\|_1^2 e^{-\nu\lambda_N t} + t\sup_{\tau\in[0,t]}\|B(q(\tau)+Q_\tau)\|_0^2/(\nu\sqrt{\lambda_N})^2. \tag{47}$$

For large N, λ_N is large. As far as $\|B(u)\|_0$ is bounded on Ω we are able to conclude from the latter inequality, that for large N and $(q(\cdot), Q_\cdot) \in \Omega_q \times \Omega_Q$ the corresponding $Q'_\cdot \in \Omega_Q$.

To prove that μ is uniform contraction with respect to the pair $(q(\cdot), Q_\cdot)$ let us now denote by η_t the difference $\eta_t = \mu(Q^2_\cdot, q^2(\cdot))|_t - \mu(Q^1_\cdot, q^1(\cdot))|_t$. Obviously η_t belongs to $\mathcal{L}_N^\perp$ and satisfies the equation (41) where f_t is now defined as:

$$f_t = B(q^2+Q^2)_t - B(q^1+Q^1)_t; \tag{48}$$

besides $\eta_0 = 0$. Again B is the projection of the nonlinear term of the 2D NS system onto $\mathcal{L}_N^\perp$. Once again the estimate (46) is valid and we only have to estimate f_t^∞.

As long as the nonlinearity term of the Navier-Stokes system equals $B(u) = (u\cdot\nabla)u$, one can represent the difference $B(u^2) - B(u^1)$ as:

$$B(u^2) - B(u^1) = \left((u^2-u^1)\cdot\nabla\right)u^2 + \left(u^1\cdot\nabla\right)(u^2-u^1).$$

Assuming $u^i = q^i + Q^i,\ i = 1,2$, we may estimate from above the function f_t defined by (48) as:

$$\|f_t\|_0 \leq c_1\|u_t^2 - u_t^1\|_1 \leq c_2\left(\|q^2(t) - q^1(t)\| + \|Q_t^2 - Q_t^1\|_1\right),$$

with c_1 , which can be chosen the same for all $u^2, u^1 \in \Omega$.

Therefore

$$f_t^\infty \le c_3 \sup_{\tau\in[0,t]} \left(\|q^2(\tau) - q^1(\tau)\| + \|Q_\tau^2 - Q_\tau^1\|_1\right), \tag{49}$$

and recalling the definition of η_t we obtain from (46) and (49)

$$\begin{aligned} &\|\mu(Q^2_\cdot, q^2(\cdot))|_t - \mu(Q^1_\cdot, q^1(\cdot))|_t\|_1 \le \\ \le c\nu^{-1}(\lambda_N)^{-1/2}t^{1/2} &\sup_{\tau\in[0,t]} \left(\|q^2(\tau) - q^1(\tau)\| + \|Q_\tau^2 - Q_\tau^1\|_1\right). \end{aligned} \tag{50}$$

Again if N, and therefore λ_N, are sufficiently large, then (50) implies that $\mu : (q(\cdot), Q(\cdot)) \mapsto Q'_t$ is a contraction. □

Corollary 4.8 *For the implicit function $q^N(\cdot) \mapsto Q^N_\cdot$ introduced in the Proposition 4.6 there holds (provided that N is large enough):*

$$\sup_{\tau\in[0,t]} \|Q_\tau\|_1^2 \le \|Q_0\|_1^2 + c\nu^{-2}\lambda_N^{-1} t \sup_{\tau\in[0,t]} \|(q(\tau)\|^2, \tag{51}$$

where c does not depend on N. □

Proof. Indeed $Q_\cdot$ is a fixed point of the map μ defined by the linear PDE (40). Therefore an analogous to (47) estimate holds for $Q_\cdot$:

$$\|Q_t\|_1^2 \le \|Q_0\|_1^2 e^{-\nu\lambda_N t} + t \sup_{\tau\in[0,t]} \|B(q(\tau) + Q_\tau)\|_0^2 / \left(\nu\sqrt{\lambda_N}\right)^2. \tag{52}$$

There holds

$$\sup_{\tau\in[0,t]} \|B(q(\tau) + Q_\tau)\|_0 \le c_4 \sup_{\tau\in[0,t]} (\|q(\tau)\| + \|Q_\tau\|_1).$$

Then from (52) we obtain (51) for large N. □

5 Geometric nonlinear control: controllability via extension of control systems

In this section we briefly survey some controllability results and the methods of geometric nonlinear control theory by which they can be obtained.

We will consider *real-analytic nonlinear control systems* $\dot{x} = f(x, u)$, or, in other words, collections $\mathcal{F}$ of real-analytic vector fields $f(\cdot, u)$ in $\mathbb{R}^N$ parameterized by $u \in U \subset \mathbb{R}^r$. Our admissible controls $u(t)$ are measurable essentially bounded functions of time.

5.1 Attainable sets and global controllability

Definition 5.1 *A point $\tilde{x}$ is attainable from $\hat{x}$ in time T for the system $\dot{x} = f(x,u)$ if for some admissible control $\tilde{u}(\cdot)$ the corresponding trajectory, which starts at $\hat{x}$ at $t = 0$, attains $\tilde{x}$ at $t = T$. A point $\tilde{x}$ is attainable from $\hat{x}$ if it is attainable from $\hat{x}$ in some time $T \geq 0$. The set of points attainable from $\hat{x}$ in time T is called time-T attainable set from $\hat{x}$ and is denoted by $\mathcal{A}^T_{\mathcal{F}}(\hat{x})$. The set of points attainable from $\hat{x}$ is called attainable set from $\hat{x}$ and is denoted by $\mathcal{A}_{\mathcal{F}}(\hat{x})$. We say that the system is globally controllable (globally controllable in time T) from $\hat{x}$ if its attainable set $\mathcal{A}_{\mathcal{F}}(\hat{x})$ (attainable set $\mathcal{A}^T_{\mathcal{F}}(\hat{x})$ in time T) from $\hat{x}$ coincides with the whole state space.* $\square$

5.2 Extension of control systems

We define (loosely following terminology of [14, Ch. 3]) the *extension* or, alternatively *completion* or, *saturation* of a control system.

Definition 5.2 *The family $\mathcal{F}'$ of real analytic vector fields is an extension of $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$ and the closures of the attainable sets $\mathcal{A}_{\mathcal{F}}(\tilde{x})$ and $\mathcal{A}_{\mathcal{F}'}(\tilde{x})$ coincide. The family $\mathcal{F}'$ of real analytic vector fields is a fixed-time extension of $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$ and $\forall T > 0$ the closures of the time-T attainable sets $\mathcal{A}^T_{\mathcal{F}}(\tilde{x})$ and $\mathcal{A}^T_{\mathcal{F}'}(\tilde{x})$ coincide.* $\square$

The inclusions $\mathcal{A}_{\mathcal{F}}(\tilde{x}) \subset \mathcal{A}_{\mathcal{F}'}(\tilde{x})$, $\mathcal{A}^T_{\mathcal{F}}(\tilde{x}) \subset \mathcal{A}^T_{\mathcal{F}'}(\tilde{x})$ are obvious as is the following Lemma.

Lemma 5.2 *If an extension $\mathcal{F}'$ of a system $\mathcal{F}$ is globally controllable, then the attainable set $\mathcal{A}_{\mathcal{F}}(\tilde{x})$ of $\mathcal{F}$ is dense in the state space.* $\square$

Our idea is to proceed with a series of extensions of a control system in order to arrive to a system for which the controllability can be verified.

It looks like this method can at its best ensure only "approximate controllability" meaning that the attainable set of the original system is dense in the state space $\mathbb{R}^N$. To overcome this problem we formulate at the end of this Section the condition under which the approximate controllability implies controllability.

5.3 Extension by convexification

There are different ways of extension of a control system; we refer to [2] and to the references therein for more details. Here we will use two methods: the first one is classical and underlies the theory of relaxed controls (see [12, 13, 17]).

Let

$$\text{co}\mathcal{F} = \{\sum_{i=1}^{m} \beta_i f_i, \ f_i \in \mathcal{F}, \ \beta_i \in C^\omega(\mathbb{R}^N), \ \beta_i \geq 0, \ \sum_{i=1}^{m} \beta_i \equiv 1, \ i = 1, \ldots, m\},$$

where $C^\omega(\mathbb{R}^N)$ is algebra of real-analytic functions in $\mathbb{R}^N$.

Proposition 5.3 *For the systems co$\mathcal{F}$ and $\mathcal{F}$ the closures of their time-T attainable sets coincide.* □

Proof of this result and of its modifications can be found in [2, Chapter 8], [14, Chapter 3],[12, Chapters II,III].

5.4 Extension by reduction

Another method arises from our previous work [3] where it is called *reduction* of a *control-affine* system. Though reduction sounds like something opposite to extension this is only a seeming contradiction, as far as it is the state space not the system which is being reduced.

Consider control-affine nonlinear system:

$$\dot{q} = f(q) + G(q)v(t),\ q \in \mathbb{R}^N,\ v \in \mathbb{R}^r, \tag{53}$$

where $G(q) = \left(g^1(q), \ldots, g^r(q)\right)$, and $f(q), g^1(q), \ldots, g^r(q)$ are complete real-analytic vector fields in $\mathbb{R}^N$; $v(t) = (v_1(t), \ldots, v_r(t))$ is a control.

We will use the notation $\overrightarrow{\exp} \int_0^t X_\tau d\tau$ introduced in [1] and called right chronological exponent. It denotes the flow generated by the time-variant vector field X_τ or, the same, by the time-variant ODE $\dot{x}(\tau) = X_\tau\left(x(\tau)\right)$. If the vector field is time-invariant $X_\tau \equiv X$, then the corresponding flow is denoted by e^{tX}. If P is a diffeomorphism, then $P_*^{-1}X$ stays for the pullback of a vector field X by (the differential of) the diffeomorphism P^{-1}.

Proposition 5.4 (see [3]) *Assume that the vector fields $g^1(q), \ldots, g^r(q)$, are mutually commuting: $\left[g^i, g^j\right] = 0,\ \forall i, j$. Then the flow of the system (53) can be represented as a composition of flows:*

$$\overrightarrow{\exp} \int_0^t \left(f + Gv(\tau)\right) d\tau = \overrightarrow{\exp} \int_0^t (e^{-GV(\tau)})_* f d\tau \circ e^{GV(t)}, \tag{54}$$

where $V(t) = \int_0^t v(s)ds$. □

The equation for the first factor of the composition in the right-hand side is

$$\dot{x} = \left(\left(e^{-GV(\tau)}\right)_* f\right)(x), \tag{55}$$

and is called *reduced control system* for (53).

When studying the forced Navier-Stokes equation we deal with *constant* controlled vector fields $g^1, \ldots, g^r$, for which the commutativity assumption holds automatically. Besides in this particular case the formula (54) takes a simpler form and can be easily proven.

Indeed, assuming $G(q) \equiv G$ to be a constant matrix, we proceed with a time-variant substitution of variable $q = y + GV(t) = y + G\int_0^t v(s)ds$ in the equation (53). Then we arrive to an equation for y:

$$\dot{y} = f\left(y + GV(t)\right). \tag{56}$$

It is an elementary exercise to verify that the latter equation coincides with (55), if G is constant.

The result which will be instrumental in our reasoning is contained in [3, Propositions 1,1'] and is based on the formulae (54) and (56) and on the results on continuous dependence of flows on the right-hand side of ODE. It says that one can reduce the study of controllability of the system (53) to the study of controllability of the reduced control system (55) on the quotient space $\mathbb{R}^N/\mathcal{G}$, where $\mathcal{G}$ is the linear span of the (constant) values of the vector fields $g^1, \ldots, g^r$.

Define

$$\mathcal{F}^r = \{f(\cdot + V)|\ V \in \mathbb{R}^r\}. \tag{57}$$

Theorem 5.4 *(cf. [3, Propositions 1,1']) If the controlled vector fields in (53) are constant then $\forall T > 0$ the closure of the attainable set $\mathcal{A}^T_{\mathcal{F}}(\tilde{x})$ coincides with the closure of $\mathcal{A}^T_{\mathcal{F}^r}(\tilde{x}) + span\{GV|V \in \mathbb{R}^r\}$ where the reduced control system $\mathcal{F}^r$ is defined by (57) or, the same, by (55).* □

Evidently the fact of system being control-affine is important for the validity of the formula (54) and therefore of the previous Theorem.

Remark 5.4 *The reduction procedure can be interpreted as a particular type of Lie extension (see [14, Ch.3]). The advantage of this particular type of extension is in explicit formula (54). This will be helpful in our treatment of the (nontruncated) 2D NS system.* □

5.5 Bracket generating+approximate controllability ⇒ global controllability

We still need to eliminate the gap between the eventual approximate controllability and controllability of a control affine system (53). To accomplish it we invoke "full Lie rank" or, "bracket generating" property.

Definition 5.5 *The system (53) possesses full Lie rank or, equivalently, is bracket generating if for every point $x^0 \in \mathbb{R}^N$ the iterated Lie brackets (evaluated at x^0) of the vector fields $f, g^1, \ldots, g^r$ span the whole $\mathbb{R}^N$.* □

This condition is related to *accessibility property*, which means nonvoidness of the interior of attainable set from every point.

Theorem 5.5 *(see [14, 2] and references therein). If the control system (53) is real-analytic then it is accessible if and only if it is bracket generating.*

If the control system (53) is C^∞-smooth and is bracket generating then it is accessible. □

It is known that for a bracket generating system approximate controllability implies controllability (see [14, Ch.3, §1.1]).

Proposition 5.5 *If a system is bracket generating and its attainable set is dense in $\mathbb{R}^N$, then this attainable set coincides with $\mathbb{R}^N$.* □

6 Extension for Galerkin approximations of the controlled 2D Navier-Stokes equations

We shall use the extension techniques surveyed in the previous section for establishing global controllability.

6.1 Reduction of the 2D NS system

Let us start with the reduction of the control-affine system (9)-(10).

The controlled vector fields are constant $g_k = \partial/\partial q_k,\ k \in \mathcal{K}^1$, and hence we may apply the formula (57) and the Theorem 5.4. The "drift" (or uncontrolled) vector field f is quadratic+linear:

$$f = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k.$$

There are two summands - $(m \wedge n)|m|^{-2} q_m q_n$ and $(n \wedge m)|n|^{-2} q_m q_n$ - under the summation sign, which contain the product $q_m q_n$. They annihilate if $|m| = |n|$. Grouping these terms in one we rewrite the expression for the drift term as:

$$f = \sum_{m+n=k, |m|>|n|} (m \wedge n) \left(|m|^{-2} - |n|^{-2}\right) q_m q_n - \nu |k|^2 q_k.$$

According to (57) the right-hand side of the reduced system is:

$$\dot{q}_k = -\nu |k|^2 \left(q_k - \chi(k) v_k\right) + \tag{58}$$
$$+ \sum_{m+n=k, |m|>|n|} (m \wedge n) \left(|m|^{-2} - |n|^{-2}\right) \left(q_m - \chi(m) v_m\right) \left(q_n - \chi(n) v_n\right),$$

where $\chi(\cdot)$ is the characteristic function of $\mathcal{K}^1$: $\chi \equiv 1$ on $\mathcal{K}^1$ and vanishes outside $\mathcal{K}^1$.

Besides according to the Theorem 5.4 we can move freely along the directions $e_k, k \in \mathcal{K}^1$.

Let us enumerate the elements $m \in \mathcal{K}^1$ in some order and form the vector $v = (v_{m^1}, \ldots, v_{m^{\kappa_1}}),\ \kappa_1 = \sharp \mathcal{K}^1$.

The right-hand side of the reduced system (58) is a second-degree polynomial map with respect to (the components of) V with coefficients depending on q. Let us represent this polynomial map as $\mathcal{V}(v) = \mathcal{V}^{(0)} + \mathcal{V}^{(1)} v + \mathcal{V}^{(2)}(v)$, where $\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \mathcal{V}^{(2)}$ stay for the free, the linear and the quadratic terms respectively. Evidently $\mathcal{V}^{(0)}$ is the projection of the right-hand side of the unforced Navier-Stokes equation onto the quotient space.

We are not able to apply immediately another reduction to the system (58), as we would wish, because this system is not control-affine. Instead we will proceed with an extension and then extract from the extended system a control-affine subsystem which is similar to (9)-(11).

Let us first demonstrate, that certain constant vector fields are contained in the image of the control-quadratic term $\mathcal{V}^{(2)}$.

Proposition 6.1 *Let $\mathcal{K}^{(2)}$ be the set of $k \in \mathbb{Z}^2$ for which there exist $m, n \in \mathcal{K}$ such that $m \wedge n \neq 0 \bigwedge \ \|m\| \neq \|n\| \ \bigwedge \ m + n = k$. Then the image of $\mathcal{V}^{(2)}$ contains all the vectors $\{\pm e_k | \ k \in \mathcal{K}^{(2)}\}$ from the standard base.* □

Proof. The projection of the vector-valued quadratic form $\mathcal{V}^{(2)}(V)$ onto e_k equals

$$\mathcal{V}_k^{(2)}(v) = \sum_{m+n=k, |m|>|n|} (m \wedge n) \left(\|m\|^{-2} - \|n\|^{-2}\right) \chi(m)\chi(n) v_m v_n.$$

For this projection to be nonvanishing there must be some $m, n \in \mathcal{K}^1$ such that $m + n = k$ and $|m| > |n|$. Construct two vectors v^+, v^- by taking $v_s^{\pm} = 0$ for $s \neq k \bigwedge s \neq m$, and then taking $v_m = v_n = 1$ for v^+ and $v_m = -v_n = 1$ for v^-.

A direct calculation shows that

$$\mathcal{V}^{(2)}(v^+) = -\mathcal{V}^{(2)}(v^-) = (m \wedge n) \left(|m|^{-2} - |n|^{-2}\right) e_k. \ \square$$

6.2 Convexification of the 2D NS system

Now we will proceed with an extension by convexification.

Lemma 6.2 *The convex hull of the image of $\mathcal{V}^{(1)} + \mathcal{V}^{(2)}$ contains the (independent of q) linear space E^2 spanned by $\{e_k | \ k \in \mathcal{K}^{(2)}\}$.* □

Proof. From the previous Proposition for each $k \in \mathcal{K}^{(2)}$ there exists v such that $\mathcal{V}^{(2)}(v) = e_k$, . Obviously $\mathcal{V}^{(2)}(-v) = e_k$, while $\mathcal{V}^{(1)}(v) = -\mathcal{V}^{(1)}(-v)$. Hence

$$(1/2)\left(\left(\mathcal{V}^{(1)} + \mathcal{V}^{(2)}\right)(v) + \left(\mathcal{V}^{(1)} + \mathcal{V}^{(2)}\right)(-v)\right) = e_k.$$

We can apply the same argument to $-e_k$ and to all $\pm e_k, \ k \in \mathcal{K}^{(2)}$, arriving to the conclusion of the corollary. □

6.3 Extraction of a control-affine subsystem

We have established that the convex hull of the right-hand side (evaluated at q) of the reduced system (58) contains the affine space $\mathcal{V}^{(0)}(q) + E^2$. We consider this affine space as the right-hand side (evaluated at q) of a new control-affine system. The latter can be written as:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ k \in \mathcal{K}^2, \tag{59}$$

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k, \ k \in \mathcal{K}^{obs} \setminus \left(\mathcal{K}^1 \bigcup \mathcal{K}^2\right). \tag{60}$$

Recall that we can move freely in the directions $e_k, \ k \in \mathcal{K}^1$.

If the image of the attainable set of this latter system under the canonical projection $\mathbb{R}^N \to \mathbb{R}^N/\mathcal{G}$ coincides with $\mathbb{R}^N/\mathcal{G}$ or, in other words, the (linear) sum of this attainable set with the linear subspace $\mathcal{G}$ coincides with $\mathbb{R}^N$, then according to the Theorem 5.4 the attainable set of the original system will be dense in $\mathbb{R}^N$ and hence, by the Proposition 5.5, will coincide with $\mathbb{R}^N$.

Therefore we managed to reduce the study of controllability of the system (9)-(11) to the study of a similar system with smaller state space.

Remark 6.3 *Observe that all the results of this Section remain valid for $\nu = 0$, i.e. for controlled 2D Euler system describing motion of ideal fluid.* □

7 Extension for Galerkin approximations of the controlled 3D Navier-Stokes equations

In this section we repeat the "reduction+extension+extraction" procedure for the control system (13)-(14)

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n)\Pi_k \underline{q}_n - \nu|k|^2 \underline{q}_k + \underline{v}_k.$$

7.1 Reduction

As in the previous Section the controlled vector fields are constant $g_m = \partial/\partial \underline{q}_m$, $m \in \mathcal{K}^1$. The "drift" vector field f in (13)-(14) is quadratic+linear:

$$f = -i \sum_{m+n=k} (\underline{q}_m \cdot n)\Pi_k \underline{q}_n - \nu|k|^2 \underline{q}_k.$$

The result of the reduction is the system:

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} \Big((\underline{q}_m - \chi(m)\underline{v}_m) \cdot n \Big) \Pi_k \Big(\underline{q}_n - \chi(n)\underline{v}_n \Big) - \nu|k|^2 \underline{q}_k, \tag{61}$$

where $\chi(\cdot)$ is the characteristic function of $\mathcal{K}^1$.

Let us enumerate the elements $m \in \mathcal{K}^1$ in some order and form the vector $\underline{v} = (\underline{v}_{m^1}, \ldots \underline{v}_{m^{\kappa_1}})$, $\kappa_1 = \sharp\mathcal{K}^1$.

The right-hand side of the reduced system (61) is a second degree polynomial map with respect to (the components of) $\underline{v}$ and with coefficients depending on $\underline{q}$. We represent this polynomial map as $\mathcal{V}(\underline{v}) = \mathcal{V}^{(0)} + \mathcal{V}^{(1)}\underline{v} + \mathcal{V}^{(2)}$ where $\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \mathcal{V}^{(2)}$ are the free, the linear and the quadratic terms respectively.

We proceed as in the previous section in order to arrive to an affine-control system with constant controlled vector fields. We start with the control-quadratic term $\mathcal{V}^{(2)}(\underline{v})$.

The k-th component of the vector-valued quadratic form $\mathcal{V}^{(2)}(\underline{v})$ equals

$$\mathcal{V}_k^{(2)}(\underline{v}) = -i \sum_{m+n=k} (\chi(m)\underline{v}_m \cdot n)\, \Pi_k\, (\chi(n)\underline{v}_n)\,.$$

Let us fix some $m, n \in \mathcal{K}^1$ such that $m + n = k$ and pick the corresponding summand $(\underline{v}_m \cdot n)\,\Pi_k\,(\underline{v}_n)$ in the expression for $\mathcal{V}_k^{(2)}(\underline{v})$. Recall that by definition $\underline{v}_m \in m^\perp$, $\underline{v}_n \in n^\perp$. Let us consider the 4-dimensional linear space L_{mn} of all $\underline{v}$ defined by the relations $\underline{v}_k = 0$, for $k \neq m \bigwedge k \neq n$.

First observation is that, similarly to the 2D case, if m and n are *collinear*, then the (two) terms containing simultaneously $\underline{v}_m$ and $\underline{v}_n$ vanish. Indeed, these terms include either the factor $\underline{v}_m \cdot n$, or the factor $\underline{v}_n \cdot m$, and if m and n are collinear, then $\underline{v}_m \cdot n = \underline{v}_m \cdot m = 0$, $\underline{v}_n \cdot m = \underline{v}_n \cdot n = 0$.

Assuming now $m \wedge n \neq 0$ let us study the restriction of the quadratic map $\mathcal{V}^{(2)}(\underline{v})$ onto the space L_{mn}. As far as $\underline{v}_m \in m^\perp \bigwedge \underline{v}_n \in n^\perp$ are independent we may consider instead the bilinear $(m+n)^\perp$-valued form on $m^\perp \times n^\perp$ or the corresponding linear operator $\bar{\mathcal{V}}^{(2)}$ which maps the tensor product $m^\perp \otimes n^\perp$ into $(m+n)^\perp$.

Choose the base in the 2-dimensional space $m^\perp$: $e_{1m} = m \wedge n$, $e_{2m} = \Pi_m n$ and also the base in $n^\perp$: $e_{1n} = m \wedge n, e_{2n} = \Pi_n m$. Note that $e_{1n} \perp e_{2n}$ and $e_{1m} \perp e_{2m}$. Obviously $e_{\alpha m} \otimes e_{\beta n}$, $\alpha, \beta = 1, 2$, form the basis of the tensor product $m^\perp \otimes n^\perp$.

A direct computation shows that $\bar{\mathcal{V}}^{(2)}(e_{1m} \otimes e_{1n}) = 0$, while both vectors $\bar{\mathcal{V}}^{(2)}(e_{1m} \otimes e_{2n})$ and $\bar{\mathcal{V}}^{(2)}(e_{2m} \otimes e_{1n})$ are collinear to the vector $(m \wedge n) \in (m+n)^\perp$. What rests is to compute

$$\bar{\mathcal{V}}^{(2)}(e_{2m} \otimes e_{2n}) = \Pi_{m+n}\left((\Pi_m n \cdot n)\, e_{2n} + (\Pi_n m \cdot m)\, e_{2m}\right).$$

First observe that $\forall x, y \in \mathbb{R}^3 : \; \Pi_x y = \|x\|^{-2}\,(x \wedge (y \wedge x))$. Applying this formula to $\Pi_m n, \Pi_n m$ we easily establish that

$$\Pi_m n \cdot n = \|m \wedge n\|^2 / \|m\|^2, \; \Pi_n m \cdot m = \|m \wedge n\|^2 / \|n\|^2. \tag{62}$$

After omission of the common factor $\|m \wedge n\|^2$ we are left with the vector $\Pi_{m+n}\left(\|m\|^{-2} e_{2n} + \|n\|^{-2} e_{2m}\right)$. Our goal is to verify whether (when) this vector belonging to the plane $(m+n)^\perp$ is linearly independent from the vector $(m \wedge n)$ belonging to the same plane.

It happens if and only if the vectors $(m+n), (m \wedge n), \left(\|m\|^{-2} e_{2n} + \|n\|^{-2} e_{2m}\right)$ are linearly independent in $\mathbb{R}^3$. We invoke again (62) to compute their mixed product equal to

$$\|m \wedge n\|^2 \|m\|^{-2} \|n\|^{-2} (m+n) \cdot (n-m).$$

Obviously $(m+n) \cdot (n-m) = |n|^2 - |m|^2$, i.e. the mixed product vanishes if and only if $|m| = |n|$, therefore revealing once more similarity with the 2D case.

Remark 7.1 *If $m \wedge n \neq 0 \bigwedge \|m\| \neq \|n\|$ then we can proceed with an extension by two constant vector fields e_k, e_k' $k = m + n$. This possibility repeated for all combinations of m, n results in the Proposition 7.1 below. If $m \wedge n \neq 0$ but $\|m\| = \|n\|$, one still can proceed with an extension by one constant vector field $e_k = m \wedge n$.* $\square$

The following Proposition is similar to the Proposition 6.1.

Proposition 7.1 *Let $\mathcal{K}^{(2)}$ be the set of $k \in \mathbb{Z}^3$ for which there exist $m, n \in \mathcal{K}^1$ such that $\|m\| \neq \|n\| \bigwedge m \wedge n \neq 0 \bigwedge m + n = k$. Then the image $Im\mathcal{V}^{(2)}$ of $\mathcal{V}^{(2)}$ contains all the vectors $\{e_k | \ k \in \mathcal{K}^{(2)}\}$ from the standard base together with their opposites.* □

7.2 Convexification and extraction of control-affine subsystem

Reasoning as in the previous Section we conclude with

Lemma 7.2 *The convex hull of the image of $\mathcal{V}^{(1)} + \mathcal{V}^{(2)}$ contains the (independent of q) linear space E^2 spanned by $\{e_k | \ k \in \mathcal{K}^{(2)}\}$.* □

Therefore after convexification we may extract from the convexified system a control affine "subsystem" of the following form:

$$\begin{aligned}
\dot{\underline{q}}_k &= -i \sum_{m+n=k} (\underline{q}_m \cdot n)\Pi_k \underline{q}_n - \nu|k|^2 \underline{q}_k + \underline{v}_k, \ k \in \mathcal{K}^2, \\
\dot{\underline{q}}_k &= -i \sum_{m+n=k} (\underline{q}_m \cdot n)\Pi_k \underline{q}_n - \nu|k|^2 \underline{q}_k, \ k \in \mathcal{K}^{obs} \setminus \left(\mathcal{K}^1 \bigcup \mathcal{K}^2\right).
\end{aligned}$$

Remark 7.2 *Observe that all the conclusions of this Section remain valid for $\nu = 0$, i.e. for controlled 3D Euler system describing motion of ideal fluid.* □

8 Controllability of the Galerkin approximations of 2D and 3D Navier-Stokes and Euler systems

We will formulate global controllability criterion for finite-dimensional Galerkin approximations of the 2D and 3D Navier-Stokes systems controlled by degenerate forcing. The criterion is based on the evolution of the "sets of forcing modes" resulting from the consequent reduction+extension+extraction procedures.

Let $\mathcal{K}^1 \subset \mathbb{Z}^s, \ (s = 2, 3)$ be the set of controlled forcing modes. The assumptions we have imposed on the forcing imply that (here and in what follows) $\mathcal{K}^1 \subset \mathbb{Z}^s \setminus \{0\}, \ (s = 2, 3)$. Define the sequence of sets $\mathcal{K}^j \subset \mathbb{Z}^s, \ j = 2, \ldots,$ by:

$$\mathcal{K}^j = \{m + n | \ m, n \in \mathcal{K}^{j-1} \bigwedge \|m\| \neq \|n\| \bigwedge m \wedge n \neq 0\}. \square \qquad (63)$$

Theorem 8 *Let $\mathcal{K}^1$ be the set of controlled forcing modes. Define iteratively sequence of sets $\mathcal{K}^j, \ j = 2, \ldots,$ by (63) and assume that for some $M : \ \bigcup_{j=1}^M \mathcal{K}^j$ coincides with the set of observed modes $\mathcal{K}^{obs}$. Then for any $T > 0$ the Galerkin approximations (9)-(10)-(12) and (13)-(14)-(12) of the 2D and 3D NS systems are time-T globally controllable. The result is valid under lack of viscosity ($\nu = 0$), i.e. it holds for Galerkin approximations of 2D and 3D Euler systems.* □

Proof. The proofs for the 2D and the 3D cases are similar; we consider the 2D case. We proceed by induction on M.

For $M = 1$, when all the observed modes are controlled, global controllability of the Galerkin approximation is almost trivial fact. Actually we are not just able to attain arbitrary points but can design arbitrary Lipschitzian trajectories.

Indeed for $M = 1$, the Galerkin approximation is just the equation (9), which in a concise form can be written as: $\dot{q} = f(q) + v$. Picking any Lipschitzian function $\tilde{q}(t)$ and taking the control $\tilde{v}(t) = \dot{\tilde{q}}(t) - f(\tilde{q}(t))$ we conclude that $\tilde{q}(t)$ is the trajectory corresponding to this control.

Assume that the statement of the Theorem is proven for all $M \leq \bar{M} - 1$. Let $\bigcup_{j=1}^{\bar{M}} \mathcal{K}^j = \mathcal{K}^{obs}$. Applying to the system (9)-(10)-(11) one step of reduction+convexification+extraction procedure (see Section 6) we arrive to the control system (59)-(60).

As far as $\bigcup_{j=2}^{\bar{M}} \mathcal{K}^j = \mathcal{K}^{obs} \setminus (\mathcal{K}^1 \setminus \mathcal{K}^2)$, then for the reduced system the assumption of our Theorem holds with $M = \bar{M} - 1$. Hence by induction assumption this system is globally controllable in its state space $\mathbb{R}^N / \mathcal{G}$, where $\mathcal{G} = \mathrm{span}\{e_k |\ k \in \mathcal{K}^1 \setminus \mathcal{K}^2\}$. Then the original system is globally controllable according to the Theorem 5.4 and to the Proposition 5.5. □

As far as we know these are new results regarding controllability of Galerkin approximations of NS systems controlled by *degenerate* forcing. We would like to mention paper [8] by W. E and J.C. Mattingly, where *bracket generation* property for 2D NS system with few forced modes has been established. This property guarantees the nonvoidness of interior of attainable set but in general is not sufficient for controllability. We learned recently about the result of M.Romito ([16]), who proved global controllability for Galerkin approximations of 3D NS system controlled by degenerate stochastic forcing.

9 Global controllability in observed projection for 2D Navier-Stokes system

Theorem 9 *[see Definition 3.2] Let $\nu > 0$ and $\mathcal{K}^1$ be the set of controlled forcing modes. Define iteratively sequence of sets $\mathcal{K}^j$, $j = 2, \ldots,$ by (63) and assume that for some M : $\bigcup_{j=1}^{M} \mathcal{K}^j$ contains all the observed modes: $\bigcup_{j=1}^{M} \mathcal{K}^j \supseteq \mathcal{K}^{obs}$. Then for any $T > 0$ the 2D NS system (9)-(10)-(11) is time-T globally controllable in observed projection.* □

This Theorem is proven in Section 12.

10 2D Navier-Stokes system: approximate controllability

Another controllability result regards L_2-approximate controllability.

Theorem 10 *[see Definition 3.3] Consider the 2D Navier-Stokes equation (with $\nu > 0$) controlled by degenerate forcing. Let $\mathcal{K}^1$ be the set of controlled forc-*

ing modes. Define iteratively sequence of sets $\mathcal{K}^j$, $j = 2, \ldots$, by (63) and assume that for each bounded (finite) set $\mathcal{K} \subset \mathbb{Z}^2$ there exists $M(\mathcal{K})$ such that: $\bigcup_{j=1}^{M(\mathcal{K})} \mathcal{K}^j \supseteq \mathcal{K}$. Then for any $T > 0$ the system (9)-(10)-(11) is time-T globally L_2-approximately controllable. $\square$

The Theorem 10 is proven in Section 13.

11 Saturating sets of forcing modes

Definition 11 *A set $\mathcal{K}^1$ of forcing modes is called saturating if for any bounded (finite) subset $\mathcal{K}$ of $\mathbb{Z}^s$, $s = 2, 3$, there exists M such that $\mathcal{K} \subseteq \bigcup_{j=1}^{M} \mathcal{K}^j$, where $\mathcal{K}^j$ are defined by (63).* $\square$

Lemma 11 *The set $\mathcal{K}^1 = Sq_3 = \{k = (k_1, k_2) | \ |k_1| \leq 3 \bigwedge |k_2| \leq 3\} \subset \mathbb{Z}^2$ is saturating. Moreover the sets $\mathcal{K}^j$, defined iteratively by (63) are growing monotonously: $\mathcal{K}^j \subset \mathcal{K}^{j+1}$, $j \geq 1$.* $\square$

Proof. To verify the monotonous growth it is enough to prove that $\mathcal{K}^1 \subseteq \mathcal{K}^2$. This can be verified by direct computation. We prove by induction that

$$\mathcal{K}^j \supseteq \mathrm{Sq}_{j+2} = \{k = (k_1, k_2) | \ |k_1| \leq (j+2) \bigwedge |k_2| \leq (j+2)\}, \ j \geq 1. \quad (64)$$

For $j = 1$ (64) is the assumption of the Proposition. Assume (64) to be proven for $j = M - 1$, $M \geq 2$. Consider the border of the square Sq_{M+1}:

$$S_{M+1} = \{k = (k_1, k_2) \in \mathrm{Sq}_{M+1} | \ |k_1| = (M+1) \bigvee |k_2| = (M+1)\},$$

which is contained in $\mathcal{K}^{M-1}$ by induction assumption. By the same assumption the square Sq_2 is contained in $\mathcal{K}^{M-1}$. A direct computation suffices to verify that the set

$$\{m + n | \ m \in S_{M+1} \bigwedge n \in \mathrm{Sq}_2 \bigwedge |m| \neq |n| \bigwedge m \wedge n \neq 0\}$$

contains the border S_{M+2} of Sq_{M+2}. Then the needed conclusion follows from the monotonicity. $\square$

Another example of a saturating set is provided by the following corollary.

Corollary 11 *The set $\mathcal{K}^1 = \{k = (k_1, k_2) | \ |k_1| + |k_2| \leq 2\} \subset \mathbb{Z}^2$ which can be also described as $\mathcal{K}^1 = \{k \in \mathbb{Z}^2 | \ \|k\| \leq 2\}$ is saturating.* $\square$

Proof. First we observe that $\mathcal{K}^1 \not\subseteq \mathcal{K}^2$; for example $(2, 0) \in \mathcal{K}^1 \setminus \mathcal{K}^2$. The growth becomes monotonous from the third term of the sequence: one concludes by a direct computation that $\mathcal{K}^2 \subseteq \mathcal{K}^3$. Besides $\mathcal{K}^3$ contains the set Sq_3 from the previous example. Thus we got another example of a saturating set. $\square$

A "minimalist" example of a set in $\mathbb{Z}^3$, which is not saturating according to the Definition 11, but still is sufficient for guaranteeing global controllability (see

Remark 7.1) of *Galerkin approximations of 3D NS systems*, has been provided by M.Romito in [16]. He has proven that the set of controlled forcing modes $\mathbb{K}^1 = \{(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{Z}^3$ is sufficient to guarantee global controllability of any finite-dimensional Galerkin approximation of 3D NS system.

The inclusion (64) proves that the growth of $\mathcal{K}^j$ is at least linear with respect to j. One can show that in fact it is exponential.

12 Proof of controllability in observed projection for 2D Navier-Stokes system

12.1 Solid controllability

Theorem 9 can be derived from the following result which claims that a finite-dimensional family of controls suffices for providing the controllability in observed projection. It will be convenient to endow the finite-dimensional spaces of controlled and observed modes and of control parameters with the l_1-norm:

$$\|x\|_{l_1} = \sum_{j=1}^{n} |x_i|, \ \forall x \in \mathbb{R}^n.$$

Theorem 12.1 *Let $\mathcal{K}^1$ be the set of non-vanishing modes of the forcing. Define the sequence of sets $\mathcal{K}^j$, $j = 2, \ldots$ according to (63) and assume that for some M : $\bigcup_{j=1}^{M} \mathcal{K}^j$ contains all the observed modes: $\bigcup_{j=1}^{M} \mathcal{K}^j \supset \mathcal{K}^{obs}$.*

Then for each starting point $\tilde{\varphi} \in H_1$, for each $R > 0$ and for all sufficiently small $T > 0$ there exists a family of controls $v(\cdot, b)$, $b \in \mathcal{B}_R$, parameterized continuously in L_1-metric by an open bounded subset $\mathcal{B}_R$ of a finite-dimensional linear space, such that the projection onto the space of observed modes of the attainable set from $\tilde{\varphi}$ contains the "ball"

$$\mathcal{C}_R = \{x \in \mathbb{R}^N | \ \|x\|_{l_1} \leq R\}. \ \square$$

The only additional restriction in the claim of the latter result is smallness of time. To deal with large T we can apply zero control on the interval $[0, T - \theta]$ with θ small and then apply the result of the Theorem 12.1.

We will yet strengthen the property we are going to establish.

Definition 12.1 *Let $\Phi^0 : M^1 \mapsto M^2$ be a continuous map between two (finite-dimensional) C^0-manifolds, $\Omega \subset M^1$ be an open set with compact closure, and $S \subseteq M^2$ be any subset. We say that $\Phi^0(\Omega)$ covers S solidly, if $S \subseteq \Phi^0(\Omega)$ and this inclusion is stable with respect to C^0-small perturbations of Φ^0, i.e. for some C^0-neighborhood of $\Phi^0|_{clos\ \Omega}$ and for each map Φ, belonging to this neighborhood, there holds: $S \subseteq \Phi(\Omega)$.* $\square$

Recall that $\mathcal{F}/\mathcal{O}_T$ is the end-point map introduced in the Definition 4.2. We have explained in the Subsection 4.2 that surjectiveness of the end-point map means controllability. The statement we are going to formulate is stronger than the claims of the Theorem 12.1 and of the Theorem 9.

Proposition 12.1 *If the assumptions of the Theorem 12.1 hold then, in addition to its claim, the corresponding family of controls $v(t,b),\ b \in \mathcal{B}_R$ can be chosen in such way that the map $b \mapsto \mathcal{F}/\mathcal{O}_T\,(v(\cdot;b))$ covers the cube $Q(R)$ solidly, i.e. the system is solidly controllable by means of this family. Besides one can choose the controls $v(t,b),\ b \in \mathcal{B}_R$ uniformly (with respect to t,b) bounded:*

$$\forall t,b: \ \|v(t,b)\|_{l_1} \leq A(T,R).$$

The bound $A(T,R)$ depends only on T,R and for sufficiently small $T>0$: $A(T,R) \leq \gamma R T^{-1}$, where $\gamma > 0$ is a fixed constant. □

By assumption the set $\mathcal{K}^{obs}$ of observed modes is contained in the union $\bigcup_{j=1}^{M} \mathcal{K}^j$. We will proceed by induction on M. To simplify our presentation we will make an additional assumption that the sets $\mathcal{K}^j,\ j=1,\dots M$, which appear in the formulation of the Theorem 12.1, "grow monotonously", i.e. $\mathcal{K}^{j-1} \subset \mathcal{K}^j,\ \forall j=2,\dots,M$. This property is satisfied for all starting sets $\mathcal{K}^1$ which are "sufficiently symmetric" (see Section 11).

12.2 Proof of the Proposition 12.1: first induction step

The first induction step ($M=1$) is the contents of the following Proposition.

Lemma 12.2 *Let $M=1$, i.e. the system is split in the subsystems (9) and (11). Then the conclusion of the Proposition 12.1 holds for (9)-(11).* □

Proof. Let us write the system (9)-(11) in a concise form as:

$$dq_1/dt = f_1(q_1,Q) + v, \ dQ/dt = F(q_1,Q). \tag{65}$$

Without lack of generality we may assume the initial condition for the observed projection to be $q_1(0) = 0_{R^{\kappa_1}}$.

Fix $\gamma > 1$. Take the interval $[0,\tau]$; the value of *small* $\tau > 0$ will be specified later on. Recall that $\mathcal{C}_R = \{x \in \mathbb{R}^N |\ \|x\|_{l_1} \leq R\}$. For each $p \in \gamma\mathcal{C}_R$ take $v(t;p,\tau) = \tau^{-1}p$ - a constant control. Obviously $\gamma\mathcal{C}_R \supset \mathcal{C}_R$ and:

$$\int_0^\tau v(t;p,\tau)dt = p. \tag{66}$$

For fixed $\tau,\ p \mapsto v(t;p,\tau)$ is continuous in L_1-metric.

We claim that $\exists \tau_0 > 0$ such that for all positive $\tau \leq \tau_0$ the family of controls $v(t;p,\tau),\ p \in \gamma\mathcal{C}_R$ is the one we are seeking for in the Lemma, so we may take $b = p,\ \mathcal{B}_R = \gamma\mathcal{C}_R$.

Denote by $\Phi(p,\tau)$ the restriction of the end-point map $\mathcal{F}/\mathcal{O}_\tau$ onto the family $\{v(\cdot;p,\tau)\}$: $\Phi(p,\tau) = \mathcal{F}/\mathcal{O}_\tau\,(v(\cdot;b,\tau))$. As far as $v(\cdot;p,\tau)$ depends continuously in L_1-metric on p and the end-point map is continuous in L_1-metric as well, we conclude that $p \mapsto \Phi(p;\tau)$ is continuous.

Restricting the equations (65) to the interval $[0,\varepsilon^2]$ let us proceed with time substitution $t = \tau\xi,\ \xi \in [0,1]$. The equations take form:

$$dq_1/d\xi = \tau f_1(q_1,Q) + \bar{v}(t;p), \ dQ/d\xi = \tau F(q_1,Q), \ \xi \in [0,1], \tag{67}$$

where $\bar{v}(t;p,\tau) = \tau v(t;p,\tau)$.

For $\tau = 0$ the latter system becomes

$$dq_1/d\xi = \bar{v}(t;p), \ dQ/d\xi = 0, \ \xi \in [0,1]. \tag{68}$$

According to (66) the end-point map $\Phi(p,0)$ of (68) is the identity map: $\Phi(p,0) = Id$.

From classical results on continuity of solutions of 2D Navier-Stokes system with respect to the data we conclude that the q_1-component of the solution of the system (67) deviates from the similar component of the solution (with the same initial condition) of the system (68) by a quantity $\leq C\tau$, where the constant C can be chosen independent of p and ε for sufficiently small $\tau > 0$.

Then $\|\Phi(\cdot;\tau) - Id\| \leq C\tau$ and by degree theory argument there exists τ_0 such that $\forall \tau \leq \tau_0$ the image of $\Phi(b;\tau)$ covers $\mathcal{C}_R$ solidly.

To complete the proof note that $\|v(t,b)\|_{l_1}$ are uniformly bounded by γR. $\square$

12.3 Generic induction step: solid controllability of the reduced system

Let us proceed further with the induction. Assume that the statement of the Proposition 12.1 has been proven for all $M \leq (N-1)$; we are going to prove it for $M = N$.

Coming back to the system (9)-(10)-(11) let us proceed with one step of reduction. As a result the equations (9)-(10) change to the equations (59)-(60). Recall that by our additional assumption the set $\mathcal{K}^2$ contains $\mathcal{K}^1$. Hence the system (59)-(60)-(11) corresponds to the 2D NS system with an extended set $\mathcal{K}^2$ of controlled forcing modes.

Obviously this "reduced" system satisfies the conditions of the Theorem 12.1, moreover the observed modes are contained in the union $\bigcup_{j=2}^{M} \mathcal{K}^j$ of $(M-1)$ sets. This means that $(M-1)$ steps of reduction+convexification+extraction steps suffice for establishing controllability of the Galerkin approximation of this "reduced" system.

Then by induction hypothesis the "reduced" system is solidly controllable in observed projection: there exists a continuous in L_1-metric family of controls $v(t;b)$ which satisfies the conclusion of the Proposition 12.1. This family of controls is uniformly bounded; assume that

$$\|v(t;b)\|_{l_1} \leq A, \ \forall b \in B, \ \forall t \in [0,T]. \tag{69}$$

the values of $v(t;b)$ belong to $\mathbb{R}^{\kappa_2}$, where $\kappa_2 = \#\mathcal{K}^2$.

Let us enumerate the vectors $k \in \mathcal{K}^2$, and take $e^+_{k^i} = (0,\ldots,1_i,\ldots 0), \ i = 1,\ldots,\kappa_2$, together with their opposites $e^-_{k^i}$. Multiply each of vectors $e^-_{k^i}, e^+_{k^i}$ by A and denote the set of these $2\kappa_2$ vectors by E^A_2. The convex hull $\mathrm{conv}E^A_2$ of E^A_2 contains the values of $v(t;b)$.

Now we will approximate the family of functions $v(t;b)$ which take their values in $\mathrm{conv}E^A_2$ by E^A_2-valued functions. Such a possibility is a central result of relaxation theory.

Definition 12.3 *Define δ-pseudometric ρ_δ in the space $L^\infty([0,T],\mathbb{R}^\kappa)$ of measurable functions in the following way:*

$$\rho_\delta\left(u^1(\cdot),u^2(\cdot)\right) = meas\{t\in[0,T] \mid u^1(t)\neq u^2(t)\}.$$

Identifying those functions, which coincide beyond a set of zero measure, we obtain δ-metric. □

Remark 12.3 *The δ-metric is a restriction onto the set of ordinary (=nonrelaxed) controls of strong metric of the space of relaxed controls (see [12]).* □

We will apply R.V.Gamkrelidze Approximation Lemma (see [12, Ch.3],[13, p.119]). According to it given a δ-continuous family of conv E_2-valued functions and $\varepsilon>0$ one can construct a δ-continuous family of E_2-valued functions which ε-approximates the family $\{v(t;b)|b\in B\}$ in the relaxation metric uniformly with respect to $b\in B$. Moreover the functions of the family can be chosen piecewise-constant and the number L of the intervals of constancy can be chosen the same for all $b\in B$. Actually the Approximation Lemma in [12, Ch.3] regards relaxed controls (Young measures). If one applies it to nonrelaxed controls, (or just functions, or to the families of Dirac δ-measures) the result can be easily strengthened to the following one.

Theorem 12.3 *(Approximation Lemma; [12, Ch.3]). Let $\{v(t;b)|b\in B\}$ be a family of (conv E_2^A)-valued functions, which depends on $b\in B$ continuously in L_1 metric. Then for each $\varepsilon>0$ one can construct a δ-continuous (and hence L_1-continuous) equibounded family $\{z(t;b)|\ b\in B\}$ of E_2^A-valued functions which ε-approximates the family $\{v(t;b)|b\in B\}$ in the relaxation metric uniformly with respect to $b\in B$. Moreover the functions $z(t;b)$ can be chosen piecewise-constant and the number L of the intervals of constancy can be chosen the same for all $b\in B$.* □

We omit the proof, which is a slight variation of the proof in [12, Ch.3].

Coming back to the generic induction step we observe that: i) the reduced system is solidly controllable by means of δ-continuous family $\{v(t;b)|\ b\in B\}$; ii) δ-continuous family $\{z(t;b)|\ b\in B\}$ of E_2^A-valued functions approximates the family $\{v(t;b)|\ b\in B\}$ uniformly in the relaxation metric; iii) according to the Theorem 4.2 the end-point map $\mathcal{F}/\mathcal{O}_t$ is continuous in the relaxation metric. Therefore we obtain the following intermediate result.

Proposition 12.3 *There exist a number L and a δ-continuous family of piecewise-constant E_2^A-valued controls $\{z(t;b)|\ b\in B\}$ (with at most L intervals of constancy) such that the reduced system is solidly controllable by means of this family.* □

12.4 Generic induction step: solid controllability of the original system

Let us come back to the original system (9)-(10)-(11) and compare it with the reduced system we were treating in the previous subsection.

In the reduced and the original system the equations for the coordinates q_k, indexed by $k \in \mathcal{K}^1$, coincide and are:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ k \in \mathcal{K}^1. \tag{70}$$

We collect these coordinates into the vector denoted by q^1. In the original system the equations for the variables $q_k, \ k \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right)$ are:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k, \ k \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right). \tag{71}$$

They differ from the corresponding equations of the reduced system, which are:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ k \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right). \tag{72}$$

We collect $q_k, \ k \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right)$ into the vector denoted by q^2.

Finally the equation for the component Q_t which recollects the higher modes $e^{ik \cdot x}, \ k \notin \mathcal{K}^2$, does not contain controls and is the same in both systems. It suffices for our goals to write this equation in a concise form as:

$$\dot{Q} = h(q, Q). \tag{73}$$

Roughly speaking our task is to design a family of "small-dimensional" controls for the equations (70)-(71)-(73), which "produce approximately the same effect" as the family of E_2^A-valued controls $z(t, b)$ constructed in the previous Subsection for the "reduced" system (70)-(72)-(73).

Let us recall that the the values of the piecewise-constant controls $z(t; b)$ are the vectors $\pm A e_k, \ k \in \mathcal{K}^2$. The intervals of constancy vary continuously with $b \in B$.

If on some interval of constancy the value of $z(t; b)$ is $\pm A e_k$ with $k \in \mathcal{K}^1$, then we may just take take the control in (70) coinciding with $z(\cdot; b)$ on this interval.

The problem arises when on some interval of constancy $z(t; b)$ takes value $\pm A e_{\bar{k}}$ with $\bar{k} \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right)$. Since there are no controls accessible in the equation (71) for $q_{\bar{k}}$ we will "affect" its evolution via the variables $q_k, \ k \in \mathcal{K}^1$ which enter this equation.

First we prove that one is able to design any (Lipschitzian with respect to time) "evolution" of q^1 by choosing proper control in (70).

Lemma 12.4 *Let $\tilde{q}^1(t) \in W_{1,\infty}([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$ be a Lipschitzian function; $\tilde{q}^1(\underline{t}) = \tilde{q}^1_0$. Choose a point $u_0 = \left(q_0^1, q_0^2, \ldots, q_0^M, Q_0\right)$, such that $q_0^1 = \tilde{q}_0^1$. Consider the controlled system (70)-(71)-(73).*

There exists a control $\tilde{v}(t) \in L_\infty([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$ for this system, such that $\tilde{q}^1(t)$ coincides with the q^1-component of the corresponding trajectory, which starts at u_0 at the moment $\underline{t}$. The control $\tilde{v}(t)$ depends on $\tilde{q}^1(t)$ and on $\left(q_0^2, \ldots, q_0^M, Q_0\right)$. It varies continuously in $L_1([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$, whenever $\left(q_0^2, \ldots, q_0^M, Q_0\right)$ varies continuously in H_1 and $\tilde{q}^1(t)$ varies continuously in $W_{1,1}([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$. □

Proof. Substituting $q^1(t) = \tilde{q}^1(t)$ into the equations (71)-(73) we obtain a closed system which can be uniquely solved (for given initial conditions). Let its solution be $\left(\tilde{q}^2(t), \ldots, \tilde{q}^M(t), \tilde{Q}(t)\right)$; denote $\tilde{q}(t) = \left(\tilde{q}^1(t), \ldots, \tilde{q}^M(t)\right)$. Denote by $f^1(q,Q) + v$ the right-hand side of the equation (70). The control

$$\tilde{v}(t) = \dot{\tilde{q}}^1(t) - f^1\left(\tilde{q}(t), \tilde{Q}(t)\right) \tag{74}$$

satisfies the statement of the Lemma.

To prove the continuity observe first that $\dot{\tilde{q}}^1(t)$ in (74) varies continuously in $L_1([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$ as $\tilde{q}^1(t)$ varies continuously in $W_{1,1}([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$.

By classical results $\left(\tilde{q}(\cdot)), \tilde{Q}(\cdot)\right)$ varies continuously in $L_\infty((0,T); H_1)$-metric with continuous variation of its initial point in H_1 (see [5]).

Invoking the Proposition 4.6 we can prove (the way we proved the Theorem 4.4) that $\left(\tilde{q}(\cdot), \tilde{Q}(\cdot)\right)$ varies continuously in $L_\infty((0,T); H_1)$-metric as $\tilde{q}^1(t)$ varies continuously in $C^0([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$ (and even more so if the latter varies continuously in $W_{1,1}([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$). From this we can conclude in a similar way as we did in (39) that $f^1\left(\tilde{q}(t), \tilde{Q}(t)\right)$ varies continuously, say, in $L_\infty([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$-metric with the variation of $\tilde{q}^1(t)$. □

Remark 12.4 *The component q^1 in the formulation of the previous Lemma may include only part of the coordinates q_k $(k \in \mathcal{K}^1)$). In fact in further application of this Lemma q^1 will be 2-dimensional: $q^1 = (q_m, q_n)$, $m, n \in \mathcal{K}^1$. In this case we may choose corresponding control $v(t)$ with only 2 nonzero components v_m, v_n to satisfy the equality (74).* □

We will now use this freedom of designing any evolution for $q^1(t)$ in order to affect in a suitable way the evolution of q_k, $k \in (\mathcal{K}^2 \setminus \mathcal{K}^1)$. We want to "imitate" the action of the controls $z(t;b)$ we constructed in the Proposition 12.3 for the "reduced" system (70)-(72)-(73).

Once again if on some interval of constancy the value of $z(t;b) = \pm Ae_k$, $k \in \mathcal{K}^1$, then we just take $v(t;b) = z(t;b)$ on this interval, called the interval of the 1st kind.

Assume now that $z(t;b) = Ae_{\bar{k}}$, $\bar{k} \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right)$ on some interval of constancy $[\underline{t}, \bar{t}]$, called the interval of the 2nd kind. Consider the trajectory $\bar{y}_t$ of the reduced system (70)-(72)-(73) driven by this control on $[\underline{t}, \bar{t}]$.

On the other side pick m, n, such that $m + n = \bar{k}$, $(m \wedge n) \neq 0$ and $|m| \neq |n|$. This means that the right-hand side of the equation for $q_{\bar{k}}$ contains the term $(m \wedge n)(|m|^{-2} - |n|^{-2})q_m q_n$.

For each $\omega > 0$ consider a family of functions $\phi(t; \underline{t}, \bar{t}, \omega) \in W_{1,\infty}([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$ such that for some $\gamma > 0$ not depending on ω: a) the distance in δ-metric between $\phi(t; \underline{t}, \bar{t}, \omega)$ and $\sin(\omega t)$ is $\leq \gamma\omega^{-1}$; b) $\|\phi(t; \underline{t}, \bar{t}, \omega)\|_{C^0} \leq 1$; c) all $\phi(t; \underline{t}, \bar{t}, \omega)$ vanish at the end-points $\underline{t}, \bar{t}$; d) for fixed ω the function $\phi(t; \underline{t}, \bar{t}, \omega)$ varies continuously in $W_{1,1}([\underline{t}, \bar{t}], \mathbb{R}^{\kappa_1})$ with the variation of $\underline{t}, \bar{t}$.

The family $\{\sin \omega t\}$ would suit, but it fails to satisfy the condition c). We provide in the next subsection an example of a family of functions $\phi(t; \underline{t}, \bar{t}, \omega)$, which satisfies a)-d).

Now we construct a *continuous* trajectory y_t^ω, $t \in [0, T]$, which starts at the same initial point $\bar{y}_0$ as $\bar{y}_t$. It will be defined piecewise on each interval of constancy $[\underline{t}, \bar{t}]$ of the control $z(t; b)$ according to the following procedure.

Procedure 12.4 *1) If the interval of constancy $[\underline{t}, \bar{t}]$ of the control $z(t; b)$ is of the first kind, i.e. $z(t; b) = e_k$ with $k \in \mathcal{K}^1$ then we define y_t^ω on this interval as the trajectory of the system (70)-(71)-(73) driven by $z(t; b)$;*

2) If the interval of constancy $[\underline{t}, \bar{t}]$ of the control $z(t; b)$ is of the second kind, i.e. $z(t; b) = Ae_k$ with $k \in \mathcal{K}^2 \setminus \mathcal{K}^1$, then:

i) we take the trajectory of the system (70)-(71)-(73) driven by zero control $v \equiv 0$ on $[\underline{t}, \bar{t}]$.

ii) denoting by $\tilde{q}^1(t)$ the q^1-component of this trajectory we choose constants A_m, A_n such that

$$|A_m| = |A_n| \bigwedge A_m A_n (m \wedge n)(|m|^{-2} - |n|^{-2}) = 2A. \qquad (75)$$

and add the functions $A_m \phi(t; \underline{t}, \bar{t}, \omega), A_n \phi(t; \underline{t}, \bar{t}, \omega)$ to the respective components q_m, q_n of $\tilde{q}^1(t)$, leaving other components of $\tilde{q}^1(t)$ unaltered;

iii) denoting the result by $\tilde{q}^1(t; \omega)$ (observe that $\tilde{q}^1(t; \omega)$ takes at the endpoints of the interval the same values as $\tilde{q}^1(t)$.) We substitute $\tilde{q}^1(t; \omega)$ into the equations (71)-(73), complement its solution by $\tilde{q}^1(t; \omega)$ and denote the result by y_t^ω its solution.

iv) calculate by application of the Lemma 12.4 the control $v(t; b)$ which provides necessary evolution for $q_m(t), q_n(t)$ on $[\underline{t}, \bar{t}]$. □

We will prove that y_t^ω and $\bar{y}_t$ match asymptotically as $\omega \to \infty$. We will compare all the components of these trajectories but q_m ($m \in \mathcal{K}^1$). Let P_2 be the projection onto the orthogonal complement to the modes $\{e^{im \cdot x} | \ m \in \mathcal{K}^1\}$.

Proposition 12.4 *For any $\varepsilon > 0$ there exists $\delta > 0$ and ω_0 such that, if $\omega \geq \omega_0$ and at the initial moment $\|y_{\underline{t}}^\omega - \bar{y}_{\underline{t}}\|_0 \leq \delta$, then $\|P_2 (y_t^\omega - \bar{y}_t)\|_0 \leq \varepsilon$ on $[0, T]$.* □

Assuming the claim of this proposition (which is proven in the next Subsection) to hold true let us complete our generic induction step.

As we know the reduced (extended) system (70)-(72)-(73) is solidly controllable in observed projection by means of the family $z(t; b)$. Basing on the approximation property established by the Proposition 12.4 and using the degree theory argument we conclude that the original system (70)-(71)-(73) is solidly controllable in observed projection by means of the family $v(t, b)$ we have defined above, provided that $\varepsilon > 0$ is chosen sufficiently small.

According to the Remark 12.4 one is able to choose the control $\tilde{v}(t)$ with just two nonvanishing components v_m and v_n on each interval $[\underline{t}, \bar{t}]$. These controls are equibounded.

As far as the number of the intervals of constancy of $z(t;b)$ is bounded by L for all b, we can choose the same $\varepsilon > 0$ and the same ω for all these intervals and proceed with the previous construction for all of them. As a result we obtain a family of controls, parameterized by $b \in B$.

These controls depend continuously (in L_1-metric) on $b \in B$. Indeed the lengths of the intervals of constancy depend continuously on b and hence the functions $\phi(t;\underline{t},\bar{t},\omega)$ vary continuously in $W_{1,1}([\underline{t},\bar{t}],\mathbb{R}^{\kappa_1})$. Since $z(t;b)$ depend continuously in L_1-metric on b, then by Theorem 4.4 or by classical continuity results the intermediate values $\left(\tilde{q}^1(\underline{t}),\tilde{q}^2(\underline{t}),\ldots,\tilde{q}^M(\underline{t}),\tilde{Q}(\underline{t})\right)$ vary continuously with the variation of b. Then by Lemma 12.4 the controls we construct on the intervals of the second kind depend continuously in L_1-metric on $b \in B$. □

12.5 Proof of the Proposition 12.4

We start with an example of a family of functions $\phi(t;\underline{t},\bar{t},\omega)$ whose properties have been described in the previous subsection.

Example 12.5 Take a C^1 function $\alpha(t)$ defined on $[0,2\pi]$, vanishing at 0 and 2π and positive elsewhere. Assume $\alpha'(0) \neq 0,\ \alpha'(2\pi) \neq 0$ and $\alpha(t) < t$ on $(0,2\pi]$. Continue this function onto $\mathbb{R}$ 2π-periodically.

For each pair $T > 0, \omega > 0$ introduce a continuous function $\phi(t;0,T,\omega)$ on $[0,T]$, which coincides with $\sin\omega t$ on $[0,T-\omega^{-1}\alpha(\omega T)]$, vanishes at T, and is linear on $[T-\omega^{-1}\alpha(\omega T),T]$. The (constant) derivative of its linear piece equals $-\omega\sin\left(\omega T-\alpha(\omega T)\right)/\alpha(\omega T)$. The conditions we imposed on $\alpha(T)$ guarantee that for fixed ω these (constant) derivatives are uniformly (with respect to $T > 0$) bounded. The function $\phi(t;0,T,\omega)$ varies continuously in $W_{1,1}([\underline{t},\bar{t}],\mathbb{R}^{\kappa_1})$ with the variation of T. Define on $[-T,0]$ the function $\phi(t;-T,0,\omega)$ by: $\phi(t;-T,0,\omega) = \phi(-t;0,T,\omega)$.

Take $\phi(t;\underline{t},\bar{t},\omega) \equiv 0$, if none of the points $2s\pi/\omega,\ s \in \mathbb{Z}$ belong to $[\underline{t},\bar{t}]$ (meaning in particular that $\bar{t}-\underline{t} < 2\pi/\omega$). Otherwise choose any point $t_0 = 2s\pi/\omega,\ s \in \mathbb{Z}$ in $[\underline{t},\bar{t}]$, take the (defined on $[\underline{t}-t_0,\bar{t}-t_0]$) concatenation $\phi(t)$ of two functions $\phi(t;\underline{t}-t_0,0,\omega)$ and $\phi(t;0,\bar{t}-t_0,\omega)$.

Define $\phi(t;\underline{t},\bar{t},\omega) = \phi(t-t_0)$ for all $t \in [\underline{t},\bar{t}]$. □

Lemma 12.5 *The functions $\phi(t;\underline{t},\bar{t},\omega)$, where $\omega \in \mathbb{R}_+$, tend to zero in the relaxation metric, as $\omega \to +\infty$. Besides if for a family $\{r_\beta(t)|\beta \in \mathcal{B}\}$ of absolutely-continuous functions their $W_{1,2}$-norms are equibounded:*

$$\exists C,\rho : \int_{\underline{t}}^{\bar{t}} (\dot{r}_\beta(\tau))^2 d\tau \leq C,\ |r_\beta(0)| \leq \rho,\ \forall \beta \in \mathcal{B},$$

then the relaxation seminorms (see the Definition 4.1) $\|r_\beta(t)\phi(t;\underline{t},\bar{t},\omega)\|_{rc}$ are $O(\omega^{-1})$, as $\omega \to +\infty$, uniformly with respect to β. □

If we take $\sin\omega t$ instead of $\phi(t;\underline{t},\bar{t},\omega)$ then the *proof* is direct:

$$\int_s^t \sin(\omega\tau) r_\beta(\tau) d\tau = \omega^{-1} \int_s^t r_\beta(\tau) d(1-\cos(\omega\tau))$$

$$= \omega^{-1}\int_s^t (\cos(\omega\tau) - \cos(\omega t))\dot{r}_\beta(\tau)d\tau.$$

Since $|(\cos(\omega\tau) - \cos(\omega t))| \leq 2$, then by application of Cauchy-Schwarz inequality

$$\left|\int_s^t \sin(\omega\tau) r_\beta(\tau)d\tau\right| \leq 2\omega^{-1}(t-s)^{1/2}\left(\int_s^t (\dot{r}_\beta(\tau))^2 d\tau\right)^{1/2}. \tag{76}$$

On the other side for any $[s,t] \subseteq [\underline{t}, \bar{t}]$:

$$\int_s^t (\phi(\tau;\underline{t},\bar{t},\omega)d\tau - \sin(\omega\tau))\, dt \leq 2\min((t-s), \gamma\omega^{-1}),$$

where γ is the constant appearing in the definition of the functions $\phi(t;\underline{t},\bar{t},\omega)$. This proves the conclusion of the Lemma.

Besides $\min((t-s),\gamma\omega^{-1}) \leq (t-s)^{1/2}\gamma^{1/2}\omega^{-1/2}$ and hence

$$\left|\int_s^t \phi(\tau;\underline{t},\bar{t},\omega) r_\beta(\tau)d\tau\right| \leq \tag{77}$$

$$\leq 2\omega^{-1/2}(t-s)^{1/2}\min\left(\omega^{-1/2}\left(\int_s^t (\dot{r}_\beta(\tau))^2 d\tau\right)^{1/2} + \gamma^{1/2}\right). \square$$

Proof of the Proposition 12.4.

1. Denote $e^{i\ell\cdot x}$ by e_ℓ, $\phi(t;\underline{t},\bar{t},\omega)$ by $\phi(t;\omega)$; denote $V_t^\omega = A_m\phi(t;\omega)e_m + A_m\phi(t;\omega)e_n$.

Let $\mathcal{V}_t^\omega, \mathcal{Y}_t^\omega, \bar{\mathcal{Y}}_t$ be divergence-free solutions of the corresponding equations:

$$\nabla^\perp\cdot\mathcal{V}_t^\omega = V_t^\omega,\ \nabla^\perp\cdot\mathcal{Y}_t^\omega = y_t^\omega,\ \nabla^\perp\cdot\bar{\mathcal{Y}}_t = \bar{y}_t,$$

with periodic boundary conditions.

Let $\mathcal{V}_\ell$ be the divergence-free solution of the equation

$$\nabla^\perp\cdot\mathcal{V}_\ell = e_\ell,$$

with periodic boundary conditions. Obviously

$$\mathcal{V}_t^\omega = A_m\phi(t;\omega)\mathcal{V}_m + A_n\phi(t;\omega)\mathcal{V}_n. \tag{78}$$

2. On an interval of the first kind the equations for y_t^ω $\bar{y}_t$ coincide and have form

$$\partial_t y_t^\omega - \nu\Delta y_t^\omega = (\mathcal{Y}_t^\omega\cdot\nabla)\, y_t^\omega + Ae_k, \tag{79}$$

$$\partial_t\bar{y}_t - \nu\Delta\bar{y}_t = \left(\bar{\mathcal{Y}}_t\cdot\nabla\right)\bar{y}_t + Ae_k,\ k\in\mathcal{K}^1. \tag{80}$$

On an interval of the second kind the equations for y_t^ω and $\bar{y}_t$ are correspondingly:

$$\partial_t y_t^\omega - \nu\Delta y_t^\omega = ((\mathcal{Y}_t^\omega + \mathcal{V}_t^\omega)\cdot\nabla)\,(y_t^\omega + V_t^\omega), \tag{81}$$

$$\partial_t\bar{y}_t - \nu\Delta\bar{y}_t = \left(\bar{\mathcal{Y}}_t\cdot\nabla\right)\bar{y}_t + Ae_{m+n},\ m,n\in\mathcal{K}^1, \tag{82}$$

where V_t^ω is defined by (78).

3. Introducing the notation $\eta_t^\omega = y_t^\omega - \bar{y}_t$, we obtain the equations for η_t^ω on the intervals of first and second kind by subtracting (80) from (79) and (82) from (81).

On an interval of the first kind the equation is:

$$\partial_t \eta_t^\omega - \nu \Delta \eta_t^\omega = (\mathcal{Y}_t^\omega \cdot \nabla)\, \eta_t^\omega + (\mathcal{H}_t^\omega \cdot \nabla)\, \bar{y}_t, \tag{83}$$

where $\mathcal{H}_t^\omega = \mathcal{Y}_t^\omega - \bar{\mathcal{Y}}_t$.

On an interval of the second kind we obtain

$$\partial_t \eta_t^\omega - \nu \Delta \eta_t^\omega = ((\mathcal{Y}_t^\omega + \mathcal{V}_t^\omega) \cdot \nabla)\, y_t^\omega - (\bar{\mathcal{Y}}_t \cdot \nabla)\, \bar{y}_t + ((\mathcal{V}_t^\omega \cdot \nabla)\, V_t^\omega - Ae_{m+n})\,.$$

Subtracting and adding $((\mathcal{Y}_t^\omega + \mathcal{V}_t^\omega) \cdot \nabla)\, \bar{y}_t$ to the right-hand side of the latter equation we transform it into

$$\begin{aligned} \partial_t \eta_t^\omega - \nu \Delta \eta_t^\omega = ((\mathcal{Y}_t^\omega + \mathcal{V}_t^\omega) \cdot \nabla)\, \eta_t^\omega + (\mathcal{H}_t^\omega \cdot \nabla)\, \bar{y}_t + \\ + (\mathcal{V}_t^\omega \cdot \nabla)\, \bar{y}_t + ((\mathcal{V}_t^\omega \cdot \nabla)\, V_t^\omega - Ae_{m+n})\,, \end{aligned} \tag{84}$$

4. Evaluation of the evolution of $\|\eta_t^\omega\|_0$ according to the equation (83) is a standard computation.

Multiplying both parts of this equation by η_t^ω we observe at once that as long as $\nabla \cdot \mathcal{Y}_t^\omega = 0$, then by standard argument: $\langle (\mathcal{Y}_t^\omega \cdot \nabla)\, \eta_t^\omega, \eta_t^\omega \rangle = 0$. Also $\langle -\Delta \eta_t^\omega, \eta_t^\omega \rangle > 0$.

After integration we obtain

$$\|\eta_\tau^\omega\|_0^2 \leq \|\eta_0^\omega\|_0^2 + \int_0^\tau \langle (\mathcal{H}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle. \tag{85}$$

5. Let us repeat this argument for an interval of the second kind.

We multiply both parts of the latter equation by η_t^ω, observing that:

$$\langle ((\mathcal{Y}_t^\omega + \mathcal{V}_t^\omega) \cdot \nabla)\, \eta_t^\omega, \eta_t^\omega \rangle = 0.$$

Again $\langle -\Delta \eta_t^\omega, \eta_t^\omega \rangle > 0$. After integration one obtains

$$\begin{aligned} \|\eta_\tau^\omega\|_0^2 \leq \|\eta_0^\omega\|_0^2 + \int_0^\tau \langle (\mathcal{H}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle + \\ + \int_0^\tau \langle (\mathcal{V}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle + \int_0^\tau \langle (\mathcal{V}_t^\omega \cdot \nabla)\, V_t^\omega - Ae_{m+n}, \eta_t^\omega \rangle. \end{aligned} \tag{86}$$

Observe that (85) can be obtained from (86) by taking vanishing V_t^ω and $\mathcal{V}_t^\omega$.

Therefore we can unify our treatment of the intervals of the first and the second kind by defining function V_t^ω (and respective $\mathcal{V}_t^\omega$) piecewise on the whole interval $[0, T]$. It is already defined on the intervals of the second kind, and we take it zero on the intervals of the first kind. The formula (86) will refer from now on to the interval $[0, T]$.

6. Let us evaluate the right-hand side of (86).

The summand $\langle (\mathcal{H}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle$ can be estimated by means of standard inequality (see [6, Section 6]) as:

$$|\langle (\mathcal{H}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle| \le C \|\mathcal{H}_t^\omega\|_1 \|\nabla \bar{y}_t\|_1 \|\eta_t^\omega\|_0 \le C' \|\bar{y}_t\|_2 (\|\eta_t^\omega\|_0)^2. \qquad (87)$$

What for the factor $\|\bar{y}_t\|_2$ at the right-hand side of (87) then it is known (see [5, Chapter 1, Section 6]), that if the initial data of the system (70)-(72)-(73), or all the same, of the equation (80), belongs to H_1, then the solution $\bar{y}_t$ belongs to H_2 for $t > 0$ and for some $C'' > 0$:

$$\|\bar{y}_t\|_2 \le C'' t^{-1/2}. \qquad (88)$$

Hence the second summand $\int_0^\tau \langle (\mathcal{H}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle dt$ can be estimated from above by

$$C_1 \int_0^\tau t^{-1/2} (\|\eta_t^\omega\|_0)^2 dt.$$

The third summand at the right-hand side of (86) can be represented as

$$\int_0^\tau \langle (\mathcal{V}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle dt = \phi(t;\omega) \left(A_m \langle (\mathcal{V}_m \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle + A_n \langle (\mathcal{V}_n \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle \right).$$

The last summand in the right-hand side of (86) can be represented as

$$\int_0^\tau \left((m \wedge n) \left(|m|^{-2} - |n|^{-2} \right) A_m A_n (\phi(t;\omega))^2 - A \right) \langle e_{m+n}, \eta_t^\omega \rangle dt.$$

As long as by construction $(m \wedge n) \left(|m|^{-2} - |n|^{-2} \right) A_m A_n = 2A$, then it can be transformed into $\int_0^\tau (2(\phi(t;\omega))^2 - 1) A \langle e_{m+n}, \eta_t^\omega \rangle dt$. Observe that $2(\phi(t;\omega))^2 - 1$ coincides with $-\cos(2\omega t)$ on $[0, T]$ beyond a subset (union of two subintervals) of measure $\le 4\pi/\omega$.

We will invoke the Lemma 12.5 to estimate this term in which $\phi(t;\omega)$ under integral is multiplied by $(A_m \langle (\mathcal{V}_m \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle + A_n \langle (\mathcal{V}_n \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle)$. Let us estimate

$$\int_0^\tau |\partial_t \langle (\mathcal{V}_m \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle|^2 dt \le$$

$$\le 2 \int_0^\tau |\langle (\mathcal{V}_m \cdot \nabla)\, \partial_t \bar{y}_t, \eta_t^\omega \rangle|^2 dt + 2 \int_0^\tau |\langle (\mathcal{V}_m \cdot \nabla)\, \bar{y}_t, \partial_t \eta_t^\omega \rangle|^2 dt.$$

From standard estimates (see [5, Chapter 1, Section 6]) for $\partial_t \bar{y}_t$ and from the estimate (31) of Corollary 4.5 applied to η_t^ω we conclude that for some constants c_1 the first integral in the right-hand side can be estimated from above by $c_1 \tau$. Applying Cauchy-Schwarz inequality to the second integral we estimate it from above by

$$2 \left(\int_0^\tau \| (\mathcal{V}_m \cdot \nabla)\, \bar{y}_t \|_0^2 dt \right)^{1/2} \left(\int_0^\tau \|\partial_t \eta_t^\omega\|_0^2 dt \right)^{1/2}.$$

The first factor can be estimated from above by $c_2\tau^{1/2}$ and $\int_0^\tau \|\partial_t \eta_t^\omega\|_0^2$ can be estimated from above by a constant c_3 by virtue of the Corollary 4.5. The constants c_1, c_2, c_3 do not depend on ω.

Hence by virtue of the Lemma 12.5 $\int_0^\tau |\langle (\mathcal{V}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega\rangle| dt = O(\omega^{-1})$ as $\omega \to \infty$ and by virtue of the formula (77)

$$\left| \int_0^\tau \langle (\mathcal{V}_t^\omega \cdot \nabla)\, \bar{y}_t, \eta_t^\omega \rangle dt \right| \leq c_4 \tau \omega^{-1/2} |A_m|; \tag{89}$$

recall that $|A_m| = |A_n|$.

According to the Corollary 4.5 $\int_0^\tau |\partial_t \langle e_{m+n}, \eta_t^\omega\rangle|^2 dt$ is bounded by a constant (not depending on ω). Then reasoning as in the proof of the Lemma 12.5 we obtain for the last summand in the right-hand side of (86) an upper estimate

$$c_5 A \tau^{1/2} \omega^{-1/2}, \tag{90}$$

where c_5 depends neither on A, nor on τ, nor on ω.

In particular $\langle (\mathcal{V}_t^\omega \cdot \nabla)\, V_t^\omega - A e_{m+n}, \eta_t^\omega \rangle$ tends to 0 in the relaxation metric as $\omega \to \infty$.

7. Hence from the inequality (86) we arrive to

$$\|\eta_\tau\|_0^2 \leq \|\eta_0\|_0^2 + c \int_0^\tau t^{-1/2} \|\eta_t^\omega\|_0^2 dt +$$
$$c_4 \tau \omega^{-1/2} |A_m| + c_5 A \tau^{1/2} \omega^{-1/2}.$$

Recall that we know a priori that $\tau \mapsto \|\eta_\tau^\omega\|_0^2$ is essentially bounded and hence the (improper) integral containing $\tau^{-1/2}$ converges.

By assumption $\|\eta_0\|_0^2 \leq \delta^2$ and $\forall \varepsilon > 0$ one can chose ω_0 in such a way that for $\omega > \omega_0$ the sum of the last two summands in the right-hand of the latter inequality is $\leq \varepsilon$ for all τ.

Thus we obtain for $\omega > \omega_0$:

$$\|\eta_\tau\|_0^2 - c \int_0^\tau t^{-1/2} \|\eta_t^\omega\|_0^2 dt \leq (\delta^2 + \varepsilon).$$

Multiplying both parts of this inequality by $\tau^{-1/2} e^{-2c\tau^{1/2}}$ we can represent the result as

$$\partial_\tau \left(e^{-2c\tau^{1/2}} \int_0^\tau t^{-1/2} \|\eta_t^\omega\|_0^2 dt \right) \leq (\delta^2 + \varepsilon) \tau^{-1/2} e^{-2c\tau^{1/2}}.$$

Integrating the latter inequality on the interval $[0, t]$ we conclude

$$e^{-2ct^{1/2}} \int_0^t \tau^{-1/2} \|\eta_\tau^\omega\|_0^2 d\tau \leq (\delta^2 + \varepsilon) c^{-1} \left(1 - e^{-2ct^{1/2}} \right),$$

and therefore

$$\|\eta_t\|_0^2 \leq (\delta^2 + \varepsilon) + c \int_0^t \tau^{-1/2} \|\eta_\tau^\omega\|_0^2 d\tau \leq$$
$$\leq (\delta^2 + \varepsilon) + c(\delta^2 + \varepsilon) c^{-1} \left(e^{2ct^{1/2}} - 1 \right) = (\delta^2 + \varepsilon) e^{2ct^{1/2}}.$$

This implies the statement of the Proposition 12.4. □

13 Proof of the Theorem 10

Let us fix $\tilde{\varphi}, \hat{\varphi} \in H_1$ and $\varepsilon > 0$ and assume that we want to steer the 2D controlled NS system from $\tilde{\varphi}$ to the ε-neighborhood of $\hat{\varphi}$ in the metric of H_0.

Consider Fourier series for $\tilde{\varphi}, \hat{\varphi}$. Denote by Π_N the projection of $\varphi \in H_1$ onto the space of modes $|k| \leq N$; $\Pi_N(\tilde{\varphi}) \to \tilde{\varphi}, \Pi_N(\hat{\varphi}) \to \hat{\varphi}$ in H_0 as $N \to \infty$.

Choose N in such a way that the norms $\| \cdot \|_0$ of $\Pi_N(\tilde{\varphi}) - \tilde{\varphi}, \Pi_N(\hat{\varphi}) - \hat{\varphi}$ are $\leq \varepsilon/3$. We assume that all the controlled modes satisfy $|k| \leq N$.

The set $\mathcal{K}^1$ of controlled modes is saturating, i.e. for $\mathcal{K}^1$ defined by (63) there exists M such that

$$\bigcup_{j=1}^{M} \mathcal{K}^j \supseteq \{k| \ |k| \leq N\}. \tag{91}$$

According to the Theorem 9, there exists a degenerate controlled forcing v, which steers the 2D NS system from $\tilde{\varphi}$ to some point $\bar{\varphi} \in \Pi_N^{-1}((\Pi_N(\hat{\varphi})))$.

If we coordinatize the space of modes $\{|k| \leq N\}$ by q and its orthogonal complement by Q, then we may say that we control the finite-dimensional component $q(\cdot)$ exactly. We are going to prove that our control can be cleverly chosen in a way that it guides the NS system to the ε-neighborhood of $\hat{\varphi}$.

Recall that in the proof of the Theorem 9 (Section 12) we start with a "full-dimensional" set of controlled modes and then construct successively diminished sets of controlled modes indexed by $\mathcal{K}^j, \ j = M, M-1, \ldots, 1$. At the end we arrive to a set of controlled modes indexed by $\mathcal{K}^1$. In other words we start with a large set of extended controlled modes and simulate its actuation by means of small-dimensional controls. Since the component q is controlled exactly it suffices to follow the evolution of the infinite-dimensional component $Q_\cdot$.

Assume that we are at the first induction step under the conditions of the Lemma 12.2, i.e. that all the coordinates of q are controlled. Then we pick the control from the family, constructed in Lemma 12.2; this control steers the component q from $\Pi_N(\tilde{\varphi})$ to the point $\Pi_N(\hat{\varphi})$. The control is defined on an interval of length $\tau > 0$, which can be chosen arbitrarily small. From (67) and (68) we can conclude for some constant $C > 0$:

$$\|Q_t\|_0 \leq \|Q_0\|_0 + C\tau, \ t \in [0, \tau].$$

Recall that $\|Q_0\|_0 \leq \varepsilon/3$. We choose $\tau \leq \varepsilon/6C$, so that $\|Q_\tau\|_0 \leq \varepsilon/2$.

Let us check what happens with the component $Q_\cdot$ at each induction step of the proof of the Theorem 9. At the first stage of this step (Subsection 12.3) we proceeded with "deconvexification", applying the Approximation Lemma (Theorem 12.3). At this stage the trajectories are approximated up to arbitrary small (uniformly on $[0, \tau]$) error $\delta > 0$. We can choose $\delta \leq \varepsilon/(12M)$.

At the second stage of each induction step we apply Procedure 12.4. According to the Proposition 12.4 the component $Q_\cdot$ (which belongs to the image of the projection Π_2 at each induction step) suffers arbitrarily small alteration. We can make it (uniformly on $[0, \tau]$) smaller than $\varepsilon/(12M)$.

Therefore at each induction step the component $Q.$ suffers alteration by value $\leq \varepsilon/(6M)$; total alteration is $\leq \varepsilon/6$. Hence after the induction procedure $\|Q_\tau\|_0 \leq \varepsilon/2 + \varepsilon/6 = 2\varepsilon/3$. Therefore

$$\|Q_\tau - (\Pi_N(\hat{\varphi}) - \hat{\varphi})\|_0 \leq \varepsilon$$

and as far as $q(\tau) = \Pi_N(\hat{\varphi})$ we conclude that $(q(\tau), Q_\tau)$ belongs to the ε-neighborhood of the point $\hat{\varphi}$. $\square$

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