Control of Diffeomorphisms and Densities

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We consider a control system:

$$\dot{q} = f_u(q), \quad q \in M, \quad u \in U \subset N,$$

where M, N are smooth manifolds and the map $(u, q) \mapsto f_u(q)$ is smooth on $\overline{U} \times M$.

Classical controls are measurable bounded functions $\quad t \mapsto u(t) \in U, \ t \geq 0$.

We assume that time-varying vector field $t \mapsto f_{u(t)}$ is complete $\forall u(\cdot)$ and therefore generates a flow

$$Q_t : M \to M, \quad Q_t \in \mathrm{Diff}_0 M,$$

where

$$\frac{\partial Q_t(q)}{\partial t} = f_{u(t)}\left(Q_t(q)\right), \quad Q_0 = Id.$$

Time-varying feedback *controls* are smooth w. r. t. $q \in M$ and measurable bounded w. r. t. t mappings $(t, q) \mapsto u(t, q) \in U$.

We set $\mathbf{u} : t \mapsto u(t, \cdot)$; then the time-varying field $f_{\mathbf{u}}$ is complete and generates a flow. These are *admissible flows* of our control system on the group of diffeomorphisms. We denote by $Q_t^{\mathbf{u}}$ the flow associated to the control $\mathbf{u}$.

Attainable sets:

$$\mathcal{A}_t(q) = \left\{ Q_t^{u(\cdot)}(q) : u(\cdot) \text{ is a classical control} \right\} \subset M,$$

$$\mathbf{A}_t = \{ Q_t^{\mathbf{u}} : \mathbf{u} \text{ is a control} \} \subset \mathrm{Diff}_0 M.$$

Special classes of control systems:

- Homogeneous w. r. t. control: U is a cone in $\mathbb{R}^k$ and

$$f_{\alpha u} = \alpha f_u, \quad \forall \alpha > 0,\ u \in U;$$

- Homogeneous symmetric: $U = -U$ and

$$f_{\alpha u} = \alpha f_u, \quad \forall \alpha \in \mathbb{R},\ u \in U;$$

- Affine w. r. t. control: $U = \mathbb{R}^k$ and

$$f_{\alpha u + (1-\alpha) v} = \alpha f_u + (1-\alpha) f_v,$$

$\forall \alpha \in \mathbb{R},\ u, v \in U.$

Control system is *bracket generating* if

$$span\,\{[f_{u_1},[\cdot\cdots,f_{u_l}]\cdots](q): u_i\in U,\ l\in\mathbb{N}\}=T_qM,$$

$\forall q\in M$, where $[f,g]$ is the Lie bracket of vector fields f and g.

Theorem 1 (Rashevskii-Chow-Krener) *If the system is homogeneous w. r. t. control and bracket generating, then $\mathcal{A}_t(q)\subset\overline{int\mathcal{A}_t(q)}$, $\forall q\in M,\ t>0$. If, additionally, the system is symmetric, then $\mathcal{A}_t(q)=M,\ \forall q\in M,\ t>0$.*

Theorem 2 (Nagano) *If the map $q\mapsto f_u(q)$ is real analytic $\forall u\in U$ and*

$$\dim span\,\{[f_{u_1},[\cdot\cdots,f_{u_l}]\cdots](q): u_i\in U,\ l\in\mathbb{N}\}=m,$$

then $\mathcal{A}_q(t)$ is contained in a m-dimensional immersed submanifold of M.

Theorem 3 *If the system is homogeneous w. r. t. control and bracket generating, then* $\mathbf{A}_t \subset \overline{int \mathbf{A}_t}$*,* $\forall t > 0$*. If, additionally, the system is symmetric, then* $\mathbf{A}_t = \text{Diff}_0 M$*,* $\forall t > 0$*.*

Classical optimal control problem:

given a smooth function $(q, u) \mapsto \varphi(q, u)$ and points $q_0, q_1 \in M$, find a control $u(\cdot)$ that minimizes the integral functional:

$$J(u(\cdot)) = \int_0^1 \varphi\left(Q_t^{u(\cdot)}(q_0), u(t)\right) dt$$

under constraint $Q_1^{u(\cdot)}(q_0) = q_1$.

Let μ be a probability measure on M (it indicates "weights" of the states). Optimal control problem on the group of diffeomorphisms: given $Q_1 \in \mathrm{Diff}_0 M$, find a control $\mathbf{u}$ that minimizes the functional:

$$\int_0^1 \int_M \varphi\left(Q_t^{\mathbf{u}}(q), u(t, Q_t^{\mathbf{u}}(q))\right)\, d\mu dt$$

under constraint $Q_1^{\mathbf{u}} = Q_1$.

The problem is reduced to the family of fixed endpoints problems and we need generalized solutions to attain the minimum.

Now we relax the constraint and state optimal control version of the Monge transportation problem: given a probability measure ν, try to minimize the above functional under constraint $Q_{1*}^{\mathbf{u}}(\mu) = \nu$.

Let

$$c(q_0, q_1) = \min\{J(u(\cdot)) : Q_1^{u(\cdot)}(q_0) = q_1\},$$

optimal cost of the classical fixed endpoints problem. We have to find a map $\Phi : M \to M$ that minimizes the functional

$$\int\limits_M c(q, \Phi(q))\, d\mu$$

under constraint $\Phi_*(\mu) = \nu$.

Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian of the classical problem,

$$H(\psi) = \max_{u\in U}(\langle \psi, f_u(q)\rangle - \varphi(q, u)),$$

where $q \in M$, $\psi \in T_q^*M$.

We assume that H is a well-defined C^2-function, the associated to H Hamiltonian vector field on T^*M is complete and generates a Hamiltonian flow

$$\mathcal{H}_t : T^*M \to T^*M, \quad t \in \mathbb{R}.$$

Theorem 4 *Assume that measures μ, ν have compact supports and the cost function c is Lipschitz in a neighborhood of $supp(\mu) \times supp(\nu)$. If μ is absolutely continuous, then there exists a unique up to μ-measure zero optimal map Φ. Moreover, there exists a Lipschitz function p on M such that*

$$\Phi(q) = \pi \circ \mathcal{H}_1(d_q p),$$

for μ-almost all $q \in M$, where $\pi : T^*M \to M$ is the standard projection.

Assume that M is compact and μ is a volume form. We have:

$$T_{Id}\mathsf{Diff}\,M = \mathsf{Vec}M, \quad T^*_{Id}\mathsf{Diff}\,M = \Lambda^1 M,$$

where $\mathsf{Vec}M$ is the space of smooth vector fields and $\Lambda^1 M$ is the space of differential 1-forms; they are paired as follows:

$$(\omega, X) \mapsto \int_M \langle \omega, X \rangle \, d\mu, \quad \omega \in \Lambda^1 M, \ X \in \mathsf{Vec}M.$$

Left translations on the group define isomorphisms:

$$T_Q\mathsf{Diff}\,M \cong \mathsf{Vec}M, \quad T^*_Q\mathsf{Diff}\,M \cong \Lambda^1 M,$$

$\forall Q \in \mathsf{Diff}\,M$.

Hamiltonian of the optimal control problem on $\mathsf{Diff}_0 M$:

$$\mathsf{H}(\omega, Q) = \int\limits_M H\left(\omega_{Q(q)}\right)\, d\mu.$$

Given $\psi \in T_q^*M$, we define a *vertical derivative*:

$$H^v(\psi) = d_\psi(H\Big|_{T_q^*M}) \in T_qM.$$

Hamiltonian system for H has a form:

$$\frac{\partial Q(q)}{\partial t} = H^v(\omega_{Q(q)}), \qquad \frac{\partial \omega}{\partial t} = -i_{H^v(\omega)} d\omega - dH(\omega).$$

Second equation is an intrinsic expression of the Burgers equation with Hamiltonian H. Here

$$H(\omega) : q \mapsto H(\omega_q), \qquad (i_X d\omega)(\cdot) = d\omega(X, \cdot).$$

For the Monge transportation problem, the Hamiltonian system is supplemented by the "transversality condition" $\omega = dp$, and the Burgers equation is reduced to the Hamilton–Jacobi one: $\frac{\partial p}{\partial t} = -H(dp)$.

Affine w. r. t. control systems with a positive quadratic w. r. t. control functional can be locally presented as follows:

$$f_u(q) = f_0(q) + \sum_{i=1}^{k} u^i f_i(q),$$

$$\varphi(q,u) = \frac{1}{2}|u|^2 - V(q).$$

Then $H(\psi) = \frac{1}{2}\sum_{i=1}^{k}\langle\psi, f_i(q)\rangle^2 + \langle\psi, f_0(q)\rangle + V(q)$, $\psi \in T_q^*M$.

Theorem 5 *If, for any $q \in M$,*

$$span\{f_i(q),\ [f_j, f_i](q) : 0 \le j < i \le k\} = T_qM,$$

then the cost

$$(q_0, q_1) \mapsto c(q_0, q_1)$$

is a Lipschitz function on $M \times M$.

The group of volume preserving diffeomorphisms:

$$\text{Diff}^{\mu} M = \{Q \in \text{Diff} M : Q^* \mu = \mu\}.$$

We have:

$$T_{Id}(\text{Diff}^{\mu} M) = \{X \in \text{Vec} M : \text{div}_{\mu} X = 0\},$$

$$T^*_{Id}(\text{Diff}^{\mu} M) = \Lambda^1 M / \{\text{exact forms}\}.$$

Now consider the problem with a "state constraint" $Q \in \text{Diff}^{\mu} M$. We assume that $\text{div}_{\mu} f_0 = 0$ and linear part of the system $\sum_{i=1}^{k} u^i f_i$ is bracket generating.

Let $\mathbf{H}^{\mu}$ be the Hamiltonian of the problem on Diff$^{\mu}M$; it is invariant w. r. t. left translations and has a form:

$$\mathbf{H}^{\mu}(\omega, Q) = \int_M H(\omega) - H(dp_\omega)\, d\mu,$$

where $\Delta_H(p_\omega) = \mathrm{div} H^v(\omega)$. Here

$$\Delta_H : p \mapsto \mathrm{div} H^v(dp)$$

is a second order hypo-elliptic differential operator, the "Laplacian" associated to the Hamiltonian H. Hamiltonian system has a form:

$$\frac{\partial Q(q)}{\partial t} = X_\omega(Q(q)), \qquad \frac{\partial \omega}{\partial t} = -i_{X_\omega} d\omega,$$

where $X_\omega = H^v(\omega - dp_\omega)$. Second equation is the "Euler equation" associated to the Hamiltonian H.